Questions on: The basic model with identical agents

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Exercise 1. Linear utility function with exogenous housing consumption

We assume that the city is *linear* and *monocentric*. This means that the city is described by a line in which all jobs and all firms (which are assumed to be identical) are located in the Central Business District (CBD hereafter), which is normalized to zero for simplicity, and all workers/consumers endogenously decide their residential location between 0 and the city fringe x_f. Landlords allocate the land to the highest bids in the city. All workers/consumers are employed and are identical in all respect. There are exactly N identical workers. There are neither mobility costs within the city nor migration costs between outside the city and the city. However, individuals do incur commuting costs to go to work.

Part 1. The individual location choice

All individuals consume the same amount of land, which is normalized to 1. This means that $h_L = 1$. This assumption implies that the utility function of each individual can be rewritten as:

$$\Gamma(z_L, 1) = z_L \tag{0.1}$$

They have the following commuting costs

$$T(x) = \tau\, x$$

where τ is the commuting cost per unit of distance and x is the distance to the CBD. Each individual has a wage of w_L and pay a rent $R(x)$ at a distance x from the CBD.

(1a) Write the budget constraint.

The budget constraint is:

$$w_L - \tau x = R(x) + z_L \tag{0.2}$$

(1b) Determine the bid rent function.

Solving (0.2) and replacing z_L into (0.1) yields the following indirect utility function:

$$\Gamma(w_L - \tau x - R(x), 1) \equiv W_L = w_L - \tau x - R(x) \tag{0.3}$$

which is quite intuitive since it expresses a net income, i.e. wage minus commuting costs minus housing cost. In this context, the bid rent can be written as:

$$\Psi_L(x, W_L) = w_L - \tau x - W_L \tag{0.4}$$

(1c) Show how the bid rent vary with x, w_L, τ and the utility W_L. Give the intuition.

It is easily checked that

$$\frac{\partial \Psi_L(x, W_L)}{\partial x} < 0 \qquad \frac{\partial \Psi_L(x, W_L)}{\partial w_L} > 0 \qquad \frac{\partial \Psi_L(x, W_L)}{\partial \tau} < 0 \qquad \frac{\partial \Psi_L(x, W_L)}{\partial W_L} < 0$$

(1d) Determine the Alonso-Muth condition

We are now able to determine the Alonso-Muth condition. By differentiating the utility function (0.3) with respect to x, we have:

$$\frac{\Gamma(w_L - \tau x - R(x), 1)}{\partial x} = -\tau - R'(x) = 0$$

or equivalently

$$R'(x) = -\tau$$

which is the Alonso-Muth condition.

Part 2: The urban land use equilibrium

Assume now that there are N identical individuals in the city.

$(2a)$ Write the equilibrium conditions

Let us determine the urban-land use equilibrium. The two conditions (population constraint and continuity of the land rent) are:

$$x_f = N$$

$$w_L - \tau x - W_L = R_A$$

$(2b)$ Consider a closed city with absentee landlords. Determine the equilibrium values of the city-fringe x_f^*, the utility W_L^* and the equilibrium land rent $R^*(x)$. How x_f^* vary with N? How W_L^* vary with w_L, τ, N and R_A? Explain. Also, how $R^*(x)$ vary with N? Explain.

Using the equilibrium conditions, we obtain:

$$x_f^* = N \tag{0.5}$$

$$W_L^* = w_L - \tau x - R_A \tag{0.6}$$

with

$$\frac{\partial x_f^*}{\partial N} > 0$$

$$\frac{\partial W_L^*}{\partial w_L} > 0 \qquad \frac{\partial W_L^*}{\partial \tau} < 0 \qquad \frac{\partial W_L^*}{\partial N} < 0 \qquad \frac{\partial W_L^*}{\partial R_A} < 0$$

When wages increase or commuting costs decrease, workers are richer and their utility level increases. Concerning N, the effect is less obvious. When N rises, the city becomes larger (since the city-fringe is equal to N) and workers are, on average, further away from jobs. This means that their commuting costs increase, implying a reduction in their utility level. In this very simple model, larger cities imply lower levels of utility compared to smaller cities. This is because only commuting costs are taken into account. However, it is well known that large cities offer more diversity and more amenities than smaller cities. For example, one can think of the variety of restaurants, the nice theatres, the fine architecture of monuments that is offered by big cities. If we introduce these elements in the model, then there will obviously a trade-off between commuting costs and amenities so that big cities do not always imply lower utility levels.

We can now calculate the equilibrium land rent in the city. For that we plug the value of (0.6) into (0.4) and easily obtain:

$$R^*(x) = \begin{cases} \tau(N-x)+R_A & \text{if} \quad x \le N \\ R_A & \text{if} \quad x > N \end{cases} \tag{0.7}$$

A comparative statics analysis shows that, within the city (i.e. $x \le N$), land rent linearly decreases from the city center ($x = 0$) to the city-fringe ($x_f = N$) at a rate of τ. The interesting result here is that N is positively correlated with $R(x)$, which means that in big cities land prices are higher than in small cities. In this model, the intuition runs as follows. When N increases, the size of the city increases so that everybody is further away from jobs and thus incurs more commuting costs (even though the commuting per unit of distance τ stays the same). Now, in order for all workers to obtain the same utility level, it must be the land rent has to decrease.

($2c$) Consider now an open city with absentee landlords. Determine the equilibrium values of the city-fringe x_f^*, the population size N^* and the equilibrium land rent $R^*(x)$. How x_f^* and N^* vary with w_L, τ, W_L and R_A? Explain. Also, how $R^*(x)$ vary with w_L, τ, W_L and R_A? Explain.

Let us now study the *open-city case with absentee landlords*, that is utility W_L is now given by N is endogenous. Solving the equilibrium equations leads to:

$$x_f^* = N^* = \frac{w_L - W_L - R_A}{\tau} \tag{0.8}$$

with

$$\frac{\partial x_f^*}{\partial w_L} = \frac{\partial N^*}{\partial w_L} > 0 \qquad \frac{\partial x_f^*}{\partial W_L} = \frac{\partial N^*}{\partial W_L} < 0 \qquad \frac{\partial x_f^*}{\partial \tau} = \frac{\partial N^*}{\partial \tau} < 0$$

Therefore, when wages increase and/or commuting costs decrease, the city become larger because more individuals are attracted to the city. However, when the (exogenous) utility level W_L outside the city increases, more workers are induced to stay outside the city and thus the city-size is reduced.

Using (0.4), we can calculate the equilibrium land rent in the open-city case:

$$R^*(x) = \begin{cases} w_L - \tau x - W_L & \text{if} \quad x \le N \\ R_A & \text{if} \quad x > N \end{cases} \tag{0.9}$$

We have the same type of figure as for the closed city case (linearly decreasing in x with a slope of τ). However, different parameters affect the equilibrium land rent. Higher wages imply higher land rents (the willingness to pay is higher) and higher outside utility implies lower land rents (this is a direct consequence of the definition of the bid rent function where bid rents and utility are negatively correlated).

(2d) Consider now a closed city with resident landlords. Determine the equilibrium values of the city-fringe x_f^*, the utility W_L^* and the equilibrium land rent $R^*(x)$.

We now study the case of a *closed-city with resident landlords*. Using the equilibrium conditions, the total differential rent, TDR^*, is equal to:

$$\begin{aligned} TDR^* &= \int_0^{x_f} R^*(x)dx - R_A\, x_f^* \\ &= \int_0^{x_f} \left[\tau(N-x) + R_A\right] dx - R_A\, N \end{aligned}$$

and thus

$$TDR^* = \frac{\tau N^2}{2} \tag{0.10}$$

In this equilibrium, we thus have:

$$\begin{aligned} W_L^* &= w_L + \frac{TDR^*}{N} - \tau N - R_A \\ &= w_L - \frac{\tau N}{2} - R_A \end{aligned} \tag{0.11}$$

while $x_f^* = N$ and the equilibrium land rent $R^*(x)$ is still given by (0.7).

(2e) Consider now an open city with resident landlords. Determine the equilibrium values of the city-fringe x_f^*, the population size N^* and the equilibrium land rent $R^*(x)$.

In the case of an *open-city with resident landlords*, by using the equilibrium conditions, we have:

$$\begin{aligned} TDR^* &= \int_0^{x_f} R^*(x)dx - R_A\, x_f^* \\ &= \int_0^{x_f} \left[w_L - \tau x - W_L\right] dx - R_A\, \frac{w_L - W_L - R_A}{\tau} \end{aligned}$$

and thus

$$TDR^* = \frac{(w_L - W_L - R_A)^2}{2\tau} \tag{0.12}$$

In that case, we have the following equilibrium values:

$$\begin{aligned} x_f^* &= N^* = \frac{w_L + TDR^*/N - W_L - R_A}{\tau} \\ &= (w_L - W_L - R_A)\left(\frac{1}{\tau} + \frac{w_L - W_L - R_A}{2N}\right) \end{aligned} \tag{0.13}$$

$$R^*(x) = \begin{cases} w_L - W_L + \frac{(w_L - W_L - R_A)^2}{2\tau N} - \tau x & \text{if} \quad x \leq N \\ R_A & \text{if} \quad x > N \end{cases} \tag{0.14}$$

It is easy to see that the four different cases are totally equivalent.