Supply Function Equilibria:
Step functions and continuous representations[1]

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Abstract

In most wholesale electricity markets generators must submit step-function offers of supply to a uniform price auction, and the market is cleared at the price of the most expensive offer needed to meet realised demand. Such markets can most elegantly be modelled as the pure-strategy, Nash Equilibrium of continuous supply functions, in which each supplier has a unique profit maximising choice of supply function given the choices of other suppliers. This approach has been criticised by von der Fehr and Harbord, who argue that the discreteness and discontinuity of the required steps rule out pure-strategy equilibria and result in price instability. This paper argues that if prices must be selected from a finite set the resulting step function converges to the continuous supply function as the number of steps increases, reconciling the apparently very disparate approaches to modelling electricity markets.

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1 INTRODUCTION

Electricity liberalisation creates wholesale electricity markets, whose prices determine the returns to generation and thus the incentives to offer capacity to meet demand and to invest in new plant and retire obsolete plant. A fully liberalised market needs a number of interdependent markets. The balancing market is needed to secure real-time balancing services, to ensure that supply and demand can be instantaneously matched, taking account of the physical constraints of the network. The day-ahead market is the main spot market, providing hourly or half-hourly prices for adjusting contract positions, which themselves are traded in over-the-counter (OTC) or futures markets.

If the markets are competitive and liquid, there should be a close relationship between the contract, spot and balancing prices, otherwise profitable arbitrage would be possible. In such cases one can talk about a wholesale spot price, and contract (and futures) prices will be close to the expected value of the spot prices. The wholesale prices will guide investment decisions and hence determine the reserve margin and the degree of security of supply, measured by the Loss of Load Probability (LOLP). The market design problem is to design a set of such markets and market rules (as well as payments for various ancillary services, access to the transmission system, and possibly capacity payments for availability to encourage adequate capacity and hence reliability). The aim of market design is to deliver an efficient, competitive and sustainably reliable supply of electricity, with adequate opportunities to hedge risk. National regulatory authorities (NRAs), alone or in conjunction with competition authorities and transmission system operators, monitor the markets to detect and prevent abuse, examine proposed rule changes, mergers and acquisitions, and requests for permission to build new plant or retire old plant. Most of these oversight or regulatory functions require an ability to predict market behaviour and wholesale price formation under various counterfactuals – what might happen if the rules are changed, capacity payments introduced or abolished, renewables supported by some new mechanism, a merger waived through, or an interconnector constructed.

The aim of this paper is to discuss the problem of modelling the wholesale electricity market, to assess progress to date, and to reconcile two apparently very different approaches to finding equilibrium prices. The issue is important – huge sums of money are devoted to constructing and running market equilibrium models. Some are quite rudimentary in their conception (typically Nash Cournot) although their implementation may be quite complex.[5] Many are content to solve for a competitive equilibrium, and then adopt various fixes to simulate the effects of market power (e.g. Redpoint's model for the British Department of Trade and Industry, Redpoint, 2007). The more sophisticated models take full account of the non-convexities associated with start-up costs, ramp rates, minimum loads, as well as transmission constraints (e.g. Powersym developed by the Tennessee Valley Authority). Yet others try and develop intellectually coherent imperfectly competitive models in the spirit of this paper (e.g. Frontier Economic's Spark model described in e.g. Burns et al, 2004).

The design problem is non-trivial, as there are several distinct approaches on offer, with a large variety of alternatives. In a centrally despatched gross pool, as with the former English Electricity Pool, into which all available generating sets above 50 MW were required to bid, the day-ahead and balancing markets were combined, and the System Marginal Price (SMP) was

[5] The intellectual basis of residual Demand Analysis as practiced, for example by the Californian Independent System Operator, is a Cournot dominant firm facing a fringe, see Sheffrin (2002a,b).

determined from the bid of the last or marginal plant required to meet forecast demand. The Pool price formed the natural base for contracts, typically purely financial Contracts for Differences (CfDs).[6] The depth of the Pool (100% of all generation called upon) provided the potential liquidity for pricing and trading such contracts, although the small number of producers for most of its life limited actual liquidity. Spain, Australia, several US states, and some Latin American countries have adopted this pool model, sometimes as a compulsory centrally despatched market, sometimes voluntary with self-dispatch. Without exception pools have prices determined in a uniform price auction, although the form and durability of the bids vary considerably.

The spot price in the pool determines the value of uncontracted sales, as the contracted volume will effectively be sold at the contract price (e.g. the strike price in a CfD). With efficient arbitrage the contract price will be equal to the average spot price, but contracting a large fraction of supply ahead of time substantially reduces the incentive to bid above the competitive price. Indeed the optimal offer price for a fully contracted generator is just the short-run marginal cost (Green, 1996; Newbery, 1995; 1998a) so that a fully contracted market would behave competitively. The focus of this paper is on determining the spot price and so the effective offer into the spot market is just the capacity available after meeting contractual obligations.

Most other wholesale markets are voluntary power exchanges in which bids to buy and offers to sell are matched to determine a day-ahead hourly price, and are primarily used to adjust the existing portfolio of contracts to the desired hourly patterns of demand and supply in the light of better near-real time information. As such they often transact a small fraction of total demand (3-15%), with the bulk of demand contracted ahead on OTC and futures markets. Again such markets typically determine a Market Clearing Price (MCP) or SMP as the intersection of the bids and offers, as in a uniform price auction. To settle deviations between real-time production/demand and day-ahead positions, the day-ahead market needs to be supplemented by a balancing market (or, in Great Britain, a balancing mechanism) that determines prices to be paid for being short and having to buy the shortfall, or payments for being long and selling into the market or to the mechanism. Such balancing markets may clear at a single price (SMP) as in a uniform price auction, or be paid-as-bid as in a discriminatory auction. In addition there may be a single price for buying and selling, or a wedge between the System Buy and Sell Prices. In Britain, if the market as a whole is short, the System Sell Price is set equal to the day-ahead or 'spot' price and the System Buy Price is then the system average price for offers to supply additional power, which can be far below the marginal offer accepted but well above the System Sell Price.

Modelling wholesale electricity markets is, however, problematic for a number of reasons. From the description above, the market is best thought of as an auction, and for present purposes we only consider uniform price auctions.[7] Most wholesale markets are supplied by a significant number of separately dispatched generating sets (termed gensets; in England and Wales under the Pool the number was over 200). Each genset has its own variable costs,[8] and may therefore be bid in as a separate unit. The resulting offers to the market constitute a supply

[6] A CfD for amount M MWhr at strike price s requires the generator to pay the holder $(p-s)M$ when the Pool price is p, and if negative to receive $(s-p)M$ from the holder. This is similar to a standard futures contract.

[7] Pay-as-bid auction models of the electricity market have been analysed by Fabra et al (2006) and Holmberg (2005a).

[8] We shall ignore the complexities of modeling start-up costs and constraints on the speed with which generation

schedule, with the SMP determined by its instantaneous intersection with the demand schedule, with a different price each hour or half-hour (or 5 minutes in the Australian New Electricity Market, or NEM). Offers are submitted ahead of time (typically the day before) and may have to be valid for an extended period (e.g. 48 half-hour periods in the English Pool) during which demand can vary significantly. Plant may fail suddenly, requiring replacement at short notice, so the residual demand (i.e. the total demand *less* the supply accepted at each price from other generators) may shift suddenly with an individual failure, again increasing the range of supplies required.

Green and Newbery (1992) argued that the natural way to model such a market was to adapt Klemperer and Meyer's (1989) supply function equilibrium (SFE) formulation, in which firms make offers before the realization of demand is revealed. Units of electricity are assumed to be divisible, so firms offer continuous supply functions (SFs) to the auction. Accordingly, residual demand is differentiable and firms have a well-defined continuous marginal revenue, which offers the prospect of a well-defined best response function at each point. With a uniform price auction and a continuous SF the effect of lowering the price to capture the marginal unit lowers the price for the large quantity of inframarginal units (the 'price' effect) while only capturing an infinitesimal sale (the 'quantity' effect). As a result very collusive supply function equilibria can be supported. An equilibrium is such that each firm ensures that given the supplies offered by all other firms, it is maximising its profits for each realization of demand. The first order conditions for the Nash equilibrium for each demand realization satisfy a set of linked differential equations, which under various simplifying assumptions can be solved analytically, although for realistic specifications of costs numerical integration is normally required. This approach opened the way for a large number of papers deriving solutions under various assumptions. Analytical solutions can be found for the case of equal and constant marginal costs and linear marginal costs.[9] Closed form solutions are available for symmetric firms and perfectly inelastic demand (Rudkevich et al, 1998; Anderson and Philpott, 2002). Finally, numerical algorithms can solve for SFE of markets with asymmetric firms and general cost functions (Holmberg, 2005b; Anderson and Hu, 2006). The SFE model has also been extended to account for transmission constraints (Wilson, 2008).

In theory, quantities and prices in offers to divisible good auctions are chosen from a continuum of allowed prices and quantities (Wilson, 1979; Klemperer and Meyer, 1989). In practice, however, prices and quantities in electricity auctions are chosen from a finite set. Offers are piece-wise linear in a few auctions, but most pools and power exchanges require offers to take the form of a series of steps or a ladder. The successive offers specify a quantity that would be available at a fixed per unit price. The smallest step in the ladder is given by the number of allowed decimals in the offer. Thus all prices and quantities in an offer have to be a multiple of the price tick size and quantity multiple, respectively. In most markets there is also a limit to the number of steps to the ladder.[10] Figure 1 is taken from the Amsterdam Power Exchange, which

can be increased and the minimum load at which it can operate.

[9] In general there is a continuum of equilibria bounded above and below, although these collapse to a unique equilibrium under certain conditions, such as free entry or limited capacity (Newbery, 1998, Holmberg, 2007). For the case of linear marginal costs, there is a unique linear SF equilibrium, (Klemperer and Meyer, 1989; Green, 1996; Baldick et al., 2004) although the general analytic solution can still be characterised as a closed form solution and solved numerically (Newbery, 2002).

[10] The limit in Australia is ten steps per genset, more generous than the three in the English Pool.

requires stepped bids, and shows the determination of the market clearing price (MCP) for hour 12 on 26 June 2007, illustrating a part of the ladders of offers and bids and showing that 1,942.4 MW was traded at a price of 58.83 Euros/MWh.[11]

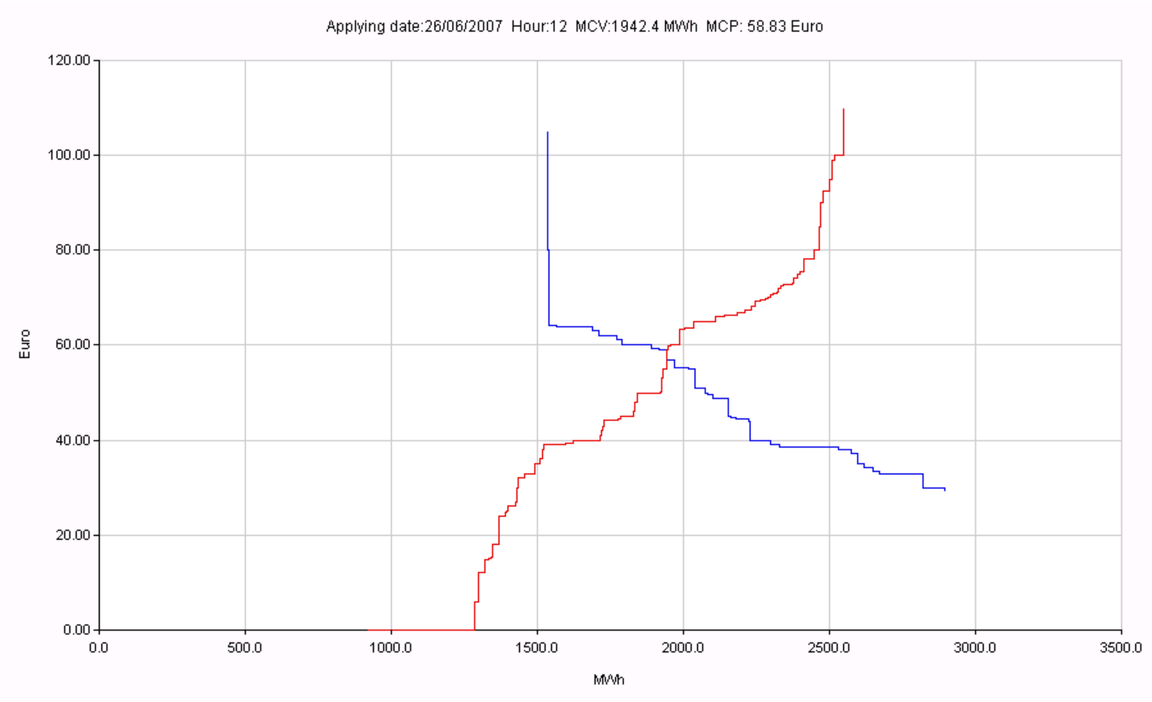

Figure 1 Market clearing price detail from APX Hour 12, 26 June 2007[12]

Green and Newbery (1992) argued that the large number of possible steps meant that, given the uncertainty about, and variability of, demand, such steps could reasonably be approximated by continuous and piecewise differentiable functions.[13] von der Fehr and Harboard (1993), however, argued that the ladders were step functions that were not continuously differentiable, and it would be inappropriate to assume that they were. Instead, their paper models the electricity market as a multiple-unit auction. Costs were assumed to be common knowledge. Each genset could submit a single bid, which is selected from a continuum of prices (although the set of prices is finite in practice). Demand was perfectly inelastic and drawn from a probability distribution with finite support, and the market price was set at the bid of the marginal unit called to meet demand, as in a uniform-price auction.

The authors specifically contrasted this with the Green and Newbery supply function approach. The contrast was sharp - a step function (or ladder) of bids combined with inelastic demand gives rise to a residual demand schedule facing any bidder that is also a step function, and whose marginal revenue is either at the residual demand price or is discontinuous at the steps. Competition is therefore almost everywhere in prices, with winner takes all over the whole step. Thus the ‘price’ effect, which can be made infinitesimally small in their model, of stealing some market is no longer larger than the now significant ‘quantity’ effect. Not surprisingly such

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[11] Typical capacity connected to the Dutch system would be over 15,000 MW and the APX covers a wider area than just the Netherlands, so a relatively small fraction of capacity is traded on the APX.

[12] At http://www.apxgroup.com/marketdata/powernl/public/aggregated_curves/curves.html.

[13] In the English Electricity Pool, each set could offer up to three incremental prices, giving over 600 possible steps.

Bertrand competition often destroys any pure strategy, and if demand uncertainty is sufficiently large the only equilibrium has mixed strategies in which the firms randomise over a distribution of possible prices. As these equilibria are hard to solve, the examples typically only have one or two steps, so the step lengths are large, as are the supports of the price distributions. Solving for the mixed strategy equilibrium with a more realistic number of steps proved extremely difficult, so the result was partly destructive: existing supply function models were claimed to be flawed but suitable auction models were intractable.

In a subsequent paper, Fabra, von de Fehr and Harbord (2006) extended their analysis in various important directions, although (for the most part) under an extremely strong restriction on the timing of demand realization.[14] Whereas the 1993 model was, plausibly, one in which the bids were submitted *before* the realization of demand, in this later paper the bids are made *after* the realization of demand. With two firms with variable costs $c_1 \leq c_2 = c$ facing a price cap P, and each submitting a single bid for the whole of their capacity, the pure strategy equilibrium is readily found. If demand is low enough for either firm to supply the entire market, the equilibrium is Bertrand (price $p = c$). If both firms are required to meet demand, one of them offers its supply at the price cap, $p = P$, while the other supplier submits an offer price sufficiently low so as to make undercutting unprofitable. This simple model is extended to allow multiple bids (b_{in}, k_{in}), where b_{in} is the n-th bid of generator i for an amount k_{in}. This allows a step function bid for each generator (as in figure 1) that might be expected to more closely match a smooth supply function. The authors also extend the model to allow long-lived but single bids with varying demand. Not surprisingly, this has an effect only when both low and high demand realizations occur with positive probability (i.e. cover the range where either the capacity of only one or both firms are required to meet demand). In such cases demand variability or uncertainty destroys all pure strategy equilibria, leaving a unique mixed-strategy equilibrium in which both suppliers submit bids that strictly exceed c. It is possible (but difficult) to compute the mixed strategy equilibrium when both suppliers have the same capacity (but possibly different costs).

Choosing a mixed strategy in prices means that prices will be inherently volatile or unstable, even if exactly the same demand is realised each day at the same time with the same generating sets available for dispatch and the same level of contract cover. It is clearly the case that spot prices are indeed very volatile, even at the same level of realised demand. Figure 2 shows the scatter of points for varying levels of generation in the Netherlands at hour 12 over the year 2005.

[14] As the title of their paper suggests, the emphasis is on comparing pay-as-bid with uniform-price auctions, but our concern here is restricted to comparing step function with continuous supply functions, and so we ignore this important aspect of auction design.

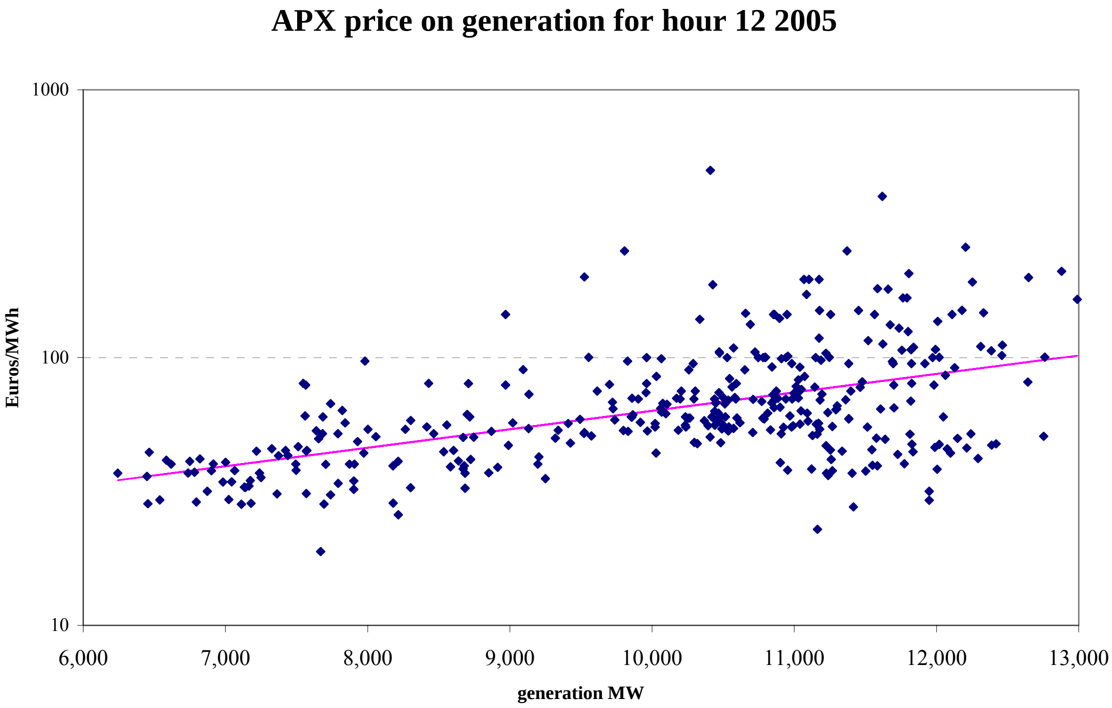

Figure 2 Scatter of spot prices in the Netherlands, hour 12 for 2005

The prices are plotted logarithmically, so the fitted line relating price to generation is exponential, and prices can vary by a factor of 10 for the same level of output. Nevertheless, there are many explanations for such volatility apart from suppliers randomising over price offers. The APX is essentially a residual market in which contract portfolio positions are adjusted to expected supply and demand. Thus a generator may sell 1,000 MW on a year-long base-load contract, and 500 MW for peak hours for the next month, but clearly actual demand will not remain constant at 1,000MW from 8pm to 8am and at 1,500 MW from 8am to 8pm each day. The generator may thus make short-term contracts for particular hours in the OTC market and then offer additional amounts into the day-ahead market, and possibly bid to buy from the APX to meet a short-fall in its generation relative to its contracted deliveries. As contract positions, demand, imports and exports, as well as plant availability, vary over short periods of time, so will the necessity of buying and selling in the APX and hence so would the position of a smooth SF (if such were allowed). One can see this visually by looking at successive days' bid and offer ladders, of which fig 1 is just a single hour's example.

Despite the theoretical problem pointed out by von der Fehr and colleagues, three empirical studies of the balancing market in Texas (ERCOT) suggest that the continuous representation is approximately correct in describing the behaviour of the largest producers in this market (Niu et al., 2005; Hortacsu and Puller, 2008; Sioshansi and Oren, 2007). Sweeting (2007) similarly estimates best responses to realizations of a smoothed residual demand schedule in the English Electricity Pool and is able to convincingly characterise the various phases of market evolution and the exercise of market power. He finds periods in which each generator was behaving as though maximising its own profits given the behaviour of other generators, and a period (during which plant was being sold) in which prices appeared to be supported at higher levels than would be individually rational (although this was evidently collectively rational and allowed plant to be sold at prices substantially higher than their subsequent sales prices). Wolak (2001) has also used observed bidding behaviour to back out the unobserved underlying cost and contract positions of generators bidding into the Australian market. He notes that continuity of the SF allows each price quantity pair to be a best response and hence does not depend on the distribution of shocks, whereas the choice of an optimal step function will depend on the distribution of the shocks, and can only be an approximation to the continuous representation, but nevertheless Wolak is content to treat the profit maximisation problem as smooth with differentials. Thus all these empirical papers are based on the assumption that competitors play pure strategies when bidding in electricity auctions and that there is a well-defined best response to realizations of a random smoothed residual demand schedule. Accordingly, they implicitly accept the SFE characterisation.

The central question raised by these criticisms and empirical applications is whether smoothing and/or increasing the number of steps in the ladder, combined with the need to bid before demand is realised, can reconcile the discrete and continuous approaches of modelling electricity markets. Do markets with uncertain demand and sufficiently finely graduated bidding ladders converge to supply function equilibria, or do they remain resolutely and significantly different? The central claim of this paper is that under well-defined conditions, convergence can be assured, providing an intellectually solid basis for accepting the SFE approach. We also conjecture that there may be a wider class of cases in which convergence can be established, but leave that for further investigation.

Fabra et al (2006) argue that the difference between the two approaches derives from the finite benefit of infinitesimal price undercutting in the ladder model. But this argument assumes that prices can be infinitely finely varied. In practice, as mentioned earlier, the price tick size cannot be less than the smallest unit of account (e.g. 1 US cent, 1 Eurocent, 1 pence, normally per MWh). In this case, the undercutting strategy is not necessarily profitable, because the price reduction cannot be made arbitrarily small. Whereas von der Fehr and Harbord (1993) considered the extreme case when the set of quantities is finite and the set of prices is infinite, this paper considers the other extreme when the set of quantities is infinite and the set of prices is finite. We provide a simple example which shows that pure strategy equilibria can exist for any number of steps. We also show that, with sufficiently many allowed steps in the bid curves, the step function and the market-clearing price (MCP) generally converge to the supply functions and price predicted by the SFE model. The results suggest that with a negligible quantity multiple and sufficiently many steps, discrete supply functions are inherently stable (and hence so is the price for each realization, *cet. par.*) and a continuous supply function equilibrium is a valid approximation of bidding in such electricity auctions.

Our model has parallels in the theoretical work by Anderson and Xu (2004). They analyse a duopoly model that reflects two important features of the Australian electricity market, in which prices and quantities are specified separately. They assume demand is random but inelastic, with an elastic outside supply at some price, P, which effectively sets a price ceiling. At the day-ahead stage, each of two generators simultaneously chooses ten prices, which are then published. Subsequently (nearer to the time of dispatch) each generator decides how much to offer at each of its chosen prices. Demand is then realised and both generators are paid the MCP. Anderson and Xu are able to show that, under certain conditions, the second stage has a pure strategy equilibrium in quantities, although the first stage only has mixed strategies in the choice of prices. The second stage of the game has similarities with our model, because prices are discrete in both models. On the other hand, generators' chosen price vectors generally differ as the declared prices are chosen by randomising over a continuous range of prices. In our paper, however, the available price levels are given by the market design and are accordingly the same for all firms. Moreover, Anderson and Xu (2004) do not compare their discrete equilibrium with a continuous SFE. Another model of the Australian market is developed by Wolak (2004). It is very similar to the Anderson & Xu model, but Wolak derives a best response rather than an equilibrium and a producer is assumed to know both competitors' selected price grid and their offers when making its own offer. This model is applied empirically to successfully recover the cost function of a producer from observed bids. The same model is used by Gans and Wolak (2007) to assess vertical integration between a large electricity retailer and a large electricity generator in the Australian market.

More generally, the convergence problem under study is related to the seminal paper by Dasgupta & Maskin (1986) on games with discontinuous profits. They show that if payoffs are discontinuous, then Nash equilibria in games with finite approximations of the strategy space of a limit game may not necessarily converge to Nash equilibria of the limit game. Later Simon (1987) showed that convergence may depend on how the strategy space is approximated. This intuitively explains why NE in the model by von der Fehr and Harbord (1993), in which payoffs are discontinuous, do not necessarily converge to continuous SFE, and also why it is not surprising that NE in our discrete model, in which payoffs are continuous, converge to continuous SFE. However, Dasgupta & Maskin (1986) and Simon (1987) derive their results for a limit game in which the strategy space has a finite dimension. Accordingly their results are not directly applicable to our problem, where the strategy space of the limit game has infinitely many dimensions (a continuous supply function has infinitely many price/quantity pairs).

1.1 Bid constraints and rationing in electricity auctions

The two key markets that we wish to model are the day-ahead market (sometimes called the spot market) and the balancing market (in the English Electricity Pool they were combined). In most such markets there is a separate auction for each delivery period, which is typically a half-hour or hour. Normally, the British balancing mechanism being the exception, the markets are organised as uniform price auctions. Thus all accepted bids and offers pay or are paid the market clearing price (MCP), which is set by the marginal bid. Producers submit non-decreasing step function offers to the auction. With its offer the producer states how much power it is willing to generate at each price. Table 1 shows that the form of the bids/offers are often constrained by a price tick size, a quantity multiple, a price cap, a price floor, and a restriction on the maximum number of steps. In particular it is worth noting that most electricity markets have significantly more possible quantity levels compared with possible price levels. In that sense, the quantity multiple is small relative to the price tick size, which concurs with our modelling assumption that only the price tick size is significant. However, often bidding is more likely to be constrained by the maximum number of allowed steps per bidder or per unit rather than by the tick size and quantity multiple. It should be noted, however, that the number of units or gensets can be large in electricity markets, of the order of 100 or more, so even if only 3-5 steps per unit is allowed, there can still be many steps in the market.

Table 1: Constraints on the supply functions in various electricity markets.

Market	*Installed capacity*	*Max steps*	*Price range*	*Price tick size*	*Quantity multiple*	*No. quantities/ No. prices*
Nord Pool spot	90,000 *MW*	64 *per bidder*	0-5,000 *NOK/MWh*	0.1 *NOK/MWh*	0.1 *MWh*	18
ERCOT balancing	70,000 *MW*	40 *per bidder*	-$1,000/*MWh*- $1,000/*MWh*	*$0.01/MWh*	0.01 *MWh*	35
PJM	160,000 *MW*	10 *per genset*	*0-$1,000/MWh*	$0.01/*MWh*	0.01 *MWh*	160
UK (NETA)	80,000 *MW*	5 *per genset*	-£9,999/*MWh*- £9,999/*MWh*	£0.01/*MWh*	0.001 *MWh*	4
Spain Intra-day market	46,000 *MW*	5 *per genset*	*Yearly cap on revenues*	€0.01/*MWh*	0.1 *MWh*	—

In electricity auctions all purchase bids with a price limit higher than the MCP and all sales offers with a price limit lower than the MCP are executed. However, rationing of excess supply at the clearing price may be necessary since the step sizes are discrete. Different solutions to this problem exist (Madlener and Kaufmann, 2002). Some exchanges, like Nord Pool and Powernext, make a linear interpolation of volumes between each adjacent pair of submitted price steps. This ensures that demand and supply has only one intersection point, and the rationing problem is avoided. The APX (Netherlands) and OMEL (Spain) accepts incremental supply at the MCP in proportion to a firm's incremental supply at this price. This rationing mechanism is sometimes called pro-rata on-the-margin (Kremer and Nyborg, 2004) and is the one modeled below.

2 THE MODEL

Consider a uniform price auction and assume that excess supply is rationed by means of the pro-rata on-the-margin mechanism. We calculate a pure strategy Nash equilibrium of a one-shot game, in which each risk-neutral electricity producer, *i,* chooses a step supply function to maximise its expected profit, $\mathrm{E}(\pi_i)$. There are M equidistant price levels p_j, $j=1,2,\ldots M$, i.e. the price tick size is $\Delta p = p_j - p_{j-1}$.[15] The minimum quantity increment is assumed to be zero, so that quantities can be continuously varied.

Generator i submits a supply vector $\mathbf{s_i}$ consisting of maximum quantities $\{s_i^1,\ldots,s_i^M\}$ it is willing to produce at each price level $\{p_1,\ldots,p_M\}$. Offer curves must be non-decreasing in price. Let $\mathbf{s}=\{\mathbf{s_1},\ldots,\mathbf{s_N}\}$ and $\Delta s_i^j = s_i^j - s_i^{j-1}$. Denote competitors' collective quantity offers at

[15] Price levels are equidistant in all electricity markets we know of and this assumption simplifies our notation. However, it is our belief that in most of our derivations "equidistant" can be relaxed to the requirement that the sequence of steps $\{p_i\}_{i=1}^M$ is dense in $[0,p]$ as $M\to\infty$.

price p_j by s_{-i}^j and the total market offer at price p_j by s^j. The cost function of firm i, $C_i(s_i)$, is a smooth, increasing and convex function. Costs are common knowledge.

Electricity consumers are assumed to be non-strategic. Their demand is stepped and the minimum demand at each price is given by $D(p^j, \varepsilon) = d^j + \varepsilon$, where ε is a demand shock. Define the decremental demand by $\Delta d^j = d^j - d^{j-1} \leq 0$. Assume that $\Delta d^j > \Delta d^{j+1}$, which would correspond to a concave demand curve in the continuous case. The demand shock has a continuous probability density, $g(\varepsilon)$, which has support on the interval $[\underline{\varepsilon}, \overline{\varepsilon}]$. Let $\tau^j = s^j - d^j$, the incremental total net supply at price p_j, and define $\Delta\tau^j = \tau^j - \tau^{j-1}$. Similarly, let $\tau_{-i}^j = s_{-i}^j - d^j$ and $\Delta\tau_{-i}^j = \tau_{-i}^j - \tau_{-i}^{j-1}$.

As in Baldick and Hogan (2001) and Holmberg (2008), it is assumed that the market design is such that the lowest price is chosen whenever demand and supply intersect at multiple prices. Thus the equilibrium price as a function of the demand shock is left continuous, and the MCP equals p_j if $\varepsilon \in (\tau^{j-1}, \tau^j]$.

3 ANALYSIS

In this section we analyze the general case as in Klemperer and Meyer (1989), ignoring capacity constraints. With a pro-rata on-the-margin mechanism all supply offers below the clearing price are accepted, while offers at p_j are rationed on a pro-rata basis. Thus for $\varepsilon \in [\tau^{j-1}, \tau^j]$, $\varepsilon - \tau^{j-1}$ is excess demand at p_{j-1} so the accepted supply of a generator i is given by:

$$s_i(\varepsilon) = s_i^{j-1} + \frac{\Delta s_i^j(\varepsilon - \tau^{j-1})}{\Delta\tau^j} = \varepsilon - \tau_{-i}^{j-1} - \frac{\Delta\tau_{-i}^j(\varepsilon - \tau^{j-1})}{\Delta\tau^j}, \tag{1}$$

which makes use of the fact that $\tau^j = \tau_{-i}^j + s_i^j$ (and $\Delta\tau^j = \Delta\tau_{-i}^j + \Delta s_i^j$). Hence, the contribution to the expected profit of generator i from realizations $\varepsilon \in (\tau^{j-1}, \tau^j]$ is:

$$\begin{aligned} E_i^j &= \int_{\tau^{j-1}}^{\tau^j} [p_j s_i - C_i(s_i)] g(\varepsilon) d\varepsilon = \\ &\int_{s_i^{j-1}+\tau_{-i}^{j-1}}^{s_i^j+\tau_{-i}^j} \left[p_j\left(\varepsilon - \tau_{-i}^{j-1} - \frac{\Delta\tau_{-i}^j(\varepsilon - \tau^{j-1})}{\Delta\tau^j}\right) - C_i\left(\varepsilon - \tau_{-i}^{j-1} - \frac{\Delta\tau_{-i}^j(\varepsilon - \tau^{j-1})}{\Delta\tau^j}\right)\right] g(\varepsilon) d\varepsilon, \end{aligned} \tag{2}$$

where again $\tau^j = \tau_{-i}^j + s_i^j$. The total expected profit is:

$$\mathrm{E}(\pi_i(\mathbf{s})) = \sum_{j=1}^{M} E_i^j(s_i^j, s_i^{j-1}). \tag{3}$$

The Nash equilibrium is found by deriving the best response of each firm given its competitors' chosen stepped supply functions. We differentiate the expected profit in (3) to get the first-order condition for each firm.

Proposition 1. With discrete supply function offers, the first-order condition for the supply of firm i at the price level j is given by:

$$\frac{\partial \mathrm{E}(\pi_i(\mathbf{s}))}{\partial s_i^j} = -\Delta p s_i^j g(\tau^j) + \int_{\tau^{j-1}}^{\tau^j} \left[p_j - C_i'(s_i(\varepsilon)) \right] \left(\frac{\Delta \tau_{-i}^j (\varepsilon - \tau^{j-1})}{(\Delta \tau^j)^2} \right) g(\varepsilon) d\varepsilon +$$

$$+ \int_{\tau^j}^{\tau^{j+1}} \left[p_{j+1} - C_i'(s_i(\varepsilon)) \right] \Delta \tau_{-i}^{j+1} \left(\frac{\Delta \tau^{j+1} - (\varepsilon - \tau^j)}{(\Delta \tau^{j+1})^2} \right) g(\varepsilon) d\varepsilon = 0,$$

where

$$s_i(\varepsilon) = \varepsilon - \tau_{-i}^{j-1} - \frac{\Delta \tau_{-i}^j (\varepsilon - \tau^{j-1})}{\Delta \tau^j} \text{ if } \varepsilon \in \left[\tau^{j-1}, \tau^j\right].$$

Proof: See Appendix.

First, we note that $\frac{\partial \mathrm{E}(\pi_i(\mathbf{s}))}{\partial s_i^j}$ is always well-defined. Because from our definitions and assumed restrictions on the bids and offers it follows that $\Delta \tau^j \geq \Delta \tau_{-i}^j \geq 0$ and that $\Delta \tau^j \geq \varepsilon - \tau^{j-1} \geq 0$ if $\varepsilon \in \left[\tau^{j-1}, \tau^j\right]$. Thus the expected profit is continuous in the strategy variables. The first-order condition can be intuitively interpreted as follows. When calculating $\frac{\partial \mathrm{E}(\pi_i(\mathbf{s}))}{\partial s_i^j}$, we increase the supply at the price level p_j, while we hold the supply at all other price levels constant. This implies that the offer price of one (infinitesimally small) unit of power is decreased from p_{j+1} to p_j. This decreases the MCP for the event when the unit is price-setting, i.e. when $\varepsilon=\tau^j$. This event brings a negative contribution to the expected profit, which corresponds to the first term in the first-order condition. On the other hand, because of the rationing mechanism, decreasing the price of one unit (weakly) increases the supply for demand outcomes $\varepsilon \in \left[\tau^{j-1}, \tau^{j+1}\right]$. This brings a positive contribution to the expected profit, which corresponds to the two integrals in the first-order condition. The first integral represents $\varepsilon \in \left(\tau^{j-1}, \tau^j\right]$ when the MCP is p_j, and the other integral represents $\varepsilon \in \left(\tau^j, \tau^{j+1}\right]$ when the MCP is p_{j+1}.

The lemma below states that given that firms' supplies at price levels p_j and p_{j-1} satisfy certain properties and if Δp is sufficiently small, then there exists a set of supplies at the price level p_{j+1} that satisfy the system of first-order conditions in Proposition 1. Moreover, supplies at the price level p_{j+1} can be uniquely determined if we rule out that producers bid below their marginal cost. This is a non-restrictive constraint, because profit maximizing producers with a non-negative output would never bid below their marginal cost, and solutions with prices below the marginal cost would never constitute Nash equilibria.

Lemma 1. For every pair of vectors $\left\{s_i^j\right\}_{i=1}^N$ and $\left\{s_i^{j-1}\right\}_{i=1}^N$, for which one can find positive constants L and δ, such that $0 \le s_i^j - s_i^{j-1} \le L\Delta p$ and $p_{j+1} - C_i'\left(s_i^j\right) \ge \delta > 0 \; \forall i = 1 \ldots N$, there exists at least one vector $\left\{s_i^{j+1}\right\}_{i=1}^N$ that satisfy the first-order condition in Proposition 1 if Δp is sufficiently small. Moreover, the vector is unique if we add the constraint that $p_{j+1} - C_i'\left(s_i(\varepsilon)\right) \ge 0 \; \forall i = 1 \ldots N$.

Definition 1. Let $\left\{\hat{s}_i^j\right\}_{i=1}^N$ be a set of discrete solutions in the range $[p_1,p_M]$ to a system of equations as in Proposition 1.

Below we analyse convergence of a solution of the system of discrete first-order conditions to that of continuous conditions given by Klemperer and Meyer (1989):

$$-s_i(p) + \left[p - C_i'\left(s_i(p)\right)\right]\left(s_{-i}'(p) - d'(p)\right) = 0 \tag{4}$$

As shown by Baldick and Hogan (2001), the system of differential equations can be written on the standard form.

$$s_i'(p) = \frac{d'(p)}{N-1} - \frac{s_i(p)}{p - C_i'\left(s_i(p)\right)} + \frac{1}{N-1}\sum_k \frac{s_k(p)}{p - C_k'\left(s_k(p)\right)}. \tag{5}$$

Definition 2. Let $\left\{\hat{s}_i(p)\right\}_{i=1}^N$ be a set of continuous solutions to a system of equations as in (4).

The first step in proving convergence is to verify that the discrete system of equations is consistent with the continuous system.

Lemma 2. *The difference equation in Proposition* 1 *is consistent with the continuous equation in (*5*) if* $\left\{\hat{s}_i(p)\right\}_{i=1}^N$ is non-decreasing and $p - C_i'\left(\hat{s}_i(p)\right) \ge \delta > 0 \; \forall i = 1 \ldots N$.
Proof: See Appendix.

That the discrete system is a consistent approximation of the continuous one implies that the system itself will converge to the continuous first-order system in (5). However, this does not ensure that the discrete solution will converge to the continuous one, because if the error increases at each step, it could explode when the number of steps becomes infinitely large, i.e. the discrete solution would be unstable. The next proposition states that the solution of the discrete first-order system indeed converges to the solution of the continuous first-order system, as the number of steps increases. Whenever a solution exists in the continuous and discrete case it may be one of an uncountably infinite set of solutions, and each solution is indexed by initial values. Note that the convergence result is valid for general cost functions, asymmetric producers and general probability distributions of the demand shock. From Proposition 1 we know that the latter influences the first-order condition for a finite number of steps, but apparently this dependence disappears in the limit.

Proposition 2. *Assume that a set of continuous solutions* $\{\hat{s}_i(p)\}_{i=1}^N$ *exists for some set of initial-values* $\{k_i\}$*. Moreover, assume that the solution is increasing in the interval [a,b] and that the solution satisfies the property* $p - C_i'(\hat{s}_i(p)) \geq \delta > 0 \; \forall i = 1 \ldots N$ *in this interval. For a sufficiently large number of steps M, there will exist at least one discrete solution* $\{\hat{s}_i^j\}_{i=1}^N$*, such that* $\hat{s}_i^1 = k_i$*,* $\hat{s}_i^2 = k_i + W_i \Delta p$*, where* $W_i \geq 0$*,* $\lim_{M\to\infty} p_1 = a$*, and* $\lim_{M\to\infty} p_M = b$*. As the number of steps grows (*$M \to \infty$*),* $\{\hat{s}_i^j\}_{i=1}^N$ *will be non-decreasing and converge to* $\{\hat{s}_i(p)\}$ *in the interval [a,b].*
Proof: See Appendix.

In both the discrete and continuous case, only non-decreasing solutions of the first-order system can constitute valid supply function equilibria. In the continuous case it can be proven that increasing solutions is a sufficient condition for supply function equilibrium. [16]

Proposition 3. *Consider a set of continuous solutions* $\{\hat{s}_i(p)\}_{i=1}^N$ *to some set of initial-values* $\{k_i\}_{i=1}^N$*. Assume that the solutions are increasing and satisfy* $p - C_i'(\hat{s}_i(p)) \geq \delta > 0 \; \forall i = 1 \ldots N$ in the price range $p \in [a,b]$. *If the market price is in this range for all possible realizations of the concave demand curve, then the set of solutions is a continuous SFE.*
Proof: See Appendix.

The proposition below makes a similar statement for the discrete case if the number of price levels is sufficiently large.

Proposition 4. *Consider a set of discrete solutions* $\{\hat{s}_i^j\}_{i=1}^N$*. As the number of steps grows* ($M \to \infty$), *if the solution of the discrete first-order system is non-decreasing for each firm everywhere in the region of possible realizations of the residual demand curve, then it is a discrete SFE.*
Proof: See Appendix.

Thus as the number of steps approaches infinity ($M \to \infty$), any set of non-decreasing solutions to the discrete first-order condition is a discrete supply function equilibrium. Now consider a SFE given by a set of initial values $\{k_i\}_{i=1}^N$ and a corresponding set of continuous solutions $\{\hat{s}_i(p)\}_{i=1}^N$, which are increasing and satisfy $p - C_i'(\hat{s}_i(p)) \geq \delta > 0$ for all realized prices. We realize from Lemma 2 and Proposition 2 that as the number of steps grows ($M \to \infty$), the corresponding discrete solution $\{\hat{s}_i^j\}_{i=1}^N$ is non-decreasing and converges to $\{\hat{s}_i(p)\}$ in

[16] Note that this result is more general than the corresponding result at pp. 1254-1255 in Klemperer and Meyer (1989), because symmetry is not required.

the range of realized prices. Thus we can conclude from Proposition 4 that $\{\hat{s}_i^j\}$ is a discrete SFE when the number of steps is sufficiently large.

Corollary 1. *Consider a continuous SFE given by a set of initial values* $\{k_i\}_{i=1}^N$ *and a corresponding set of continuous solutions* $\{\hat{s}_i(p)\}$, *which are increasing and that satisfy* $p - C_i'(\hat{s}_i(p)) \geq \delta > 0 \; \forall i = 1 \ldots N$ and all realized prices. *As the number of steps grows* ($M \to \infty$), *there is a corresponding discrete solution, which is a discrete SFE that converges to the continuous one.*

One implication of Corollary 1 is that with a sufficient number of steps, existence of discrete SFE is ensured if a corresponding continuous SFE exists. As an example, Klemperer and Meyer (1989) establish the existence of SF equilibria if firms are symmetric, ε has strictly positive density everywhere on its support $[\underline{\varepsilon}, \overline{\varepsilon}]$, the cost function is C_2 and convex, and if the demand function $D(p, \varepsilon)$ is also C_2, but concave and with a negative first derivative.

3.1 Example 1

Due to singularities at points where the price equals the marginal cost of some firm, we were not able to prove convergence near these points. However, as illustrated by the following example convergence at these points is not necessarily a problem in practice. Consider a market in which there are M=100 allowed price levels including the end-points 0 and 5, so that Δp = 0.05. There are two symmetric firms in the market. Each producer has linear marginal costs $C_i' = cs_i$, where c=1. Demand at each price level is by assumption given by $D(p_j, \varepsilon) = \varepsilon - 0.5 p_j$. The demand shock, ε, is assumed to be uniformly distributed on the interval [0,2.5], i.e. $g(\varepsilon) = 0.4$ in this range.

In the continuous case, there is a continuum of symmetric solutions to the first-order condition in (4). The chosen solution depends on the end-condition. Klemperer and Meyer (1989) and Green and Newbery (1992) show that in the continuous case, the symmetric solution slopes upwards between the marginal cost curve and the Cournot schedule, while it slopes downwards outside this wedge. The Cournot schedule is the set of Cournot solutions that would result for all possible realizations of the demand shock, and the continuous SFE is vertical at this line. In the other extreme, when price equals marginal cost the solution becomes horizontal. Thus a continuous symmetric solution constitutes a SFE if and only if the solution is within the wedge for all realized prices. Fig. 3 plots the most and least competitive SFE. All solutions in-between them are also continuous SFE.

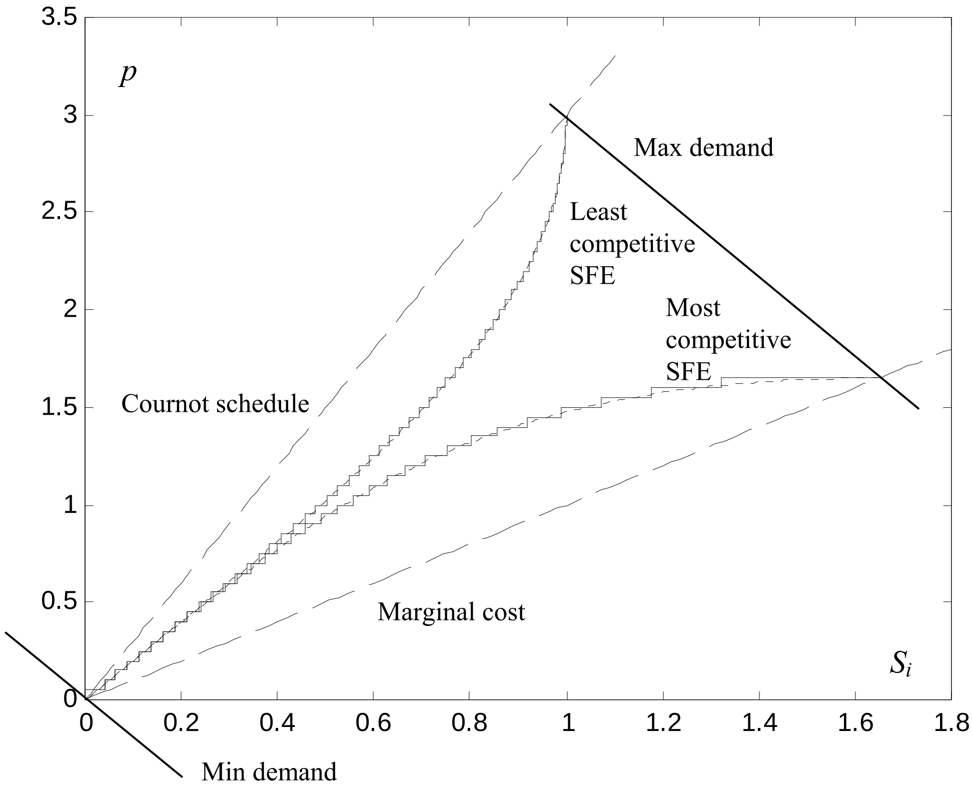

Figure 3. The most and least competitive SFE and their discrete approximations.

For the marginal cost and demand curves assumed in this example, the discrete first-order condition in Proposition 1 can be simplified to:

$$-\Delta p s_i^j + \frac{1}{2}\left(p_j - \frac{c}{3}\left(s_i^{j-1} + 2s_i^j\right)\right)\Delta\tau_{-i}^j + \frac{1}{2}\left(p_{j+1} - \frac{c}{3}\left(2s_i^j + s_i^{j+1}\right)\right)\Delta\tau_{-i}^{j+1} = 0. \quad (6)$$

In a symmetric, duopoly equilibrium with $\Delta d = -0.5\Delta p$ we have $\Delta\tau_{-i}^j = s_i^j - s_i^{j-1} + 0.5\Delta p$. Thus the first-order condition can be written:

$$-\Delta p s_i^j + \frac{1}{2}\left(p_j - \frac{c}{3}\left(s_i^{j-1} + 2s_i^j\right)\right)\left(s_i^j - s_i^{j-1} + 0.5\Delta p\right) +$$
$$\frac{1}{2}\left(p_{j+1} - \frac{c}{3}\left(2s_i^j + s_i^{j+1}\right)\right)\left(s_i^{j+1} - s_i^j + 0.5\Delta p\right) = 0.$$

With the assumed demand curve and its assumed shock distribution, the profit is maximized if $s_i^1 = 0$, because any positive capacity offered at zero price would be offered below marginal cost. This pins down one of the unknown parameters in the second-order difference equation. The other parameter is given by the end-condition at max demand, i.e. $D\left(p_j, \bar{\varepsilon}\right) = \bar{\varepsilon} - 0.5 p_j$. Thus in this example there is a one-to-one correspondence between discrete and continuous solutions that have the same end-condition. In Fig. 3 the discrete solutions with the same end-conditions as the most and least competitive SFE, respectively are plotted. With a sufficiently small Δp these solutions will be discrete SFE according to Corollary 1, and so will all discrete solutions in-between them.

3.2 Example 2

From the analysis section, we know that discrete solutions to the first-order condition constitute SFE as long as the solution is upward sloping and the step-size is sufficiently small. The following example establishes that a pure-strategy step SFE can exist for any (not necessarily large) number of steps. Moreover, the example shows that unique discrete SFE can be found if one considers that the loss of load probability (LOLP) is positive in power systems. Consider a market in which there are M allowed price levels including the end-points 0 and 1. There are two symmetric firms in the market. Each producer has constant marginal costs, which can be normalised to zero, and has unit capacity.[17] The uncertain demand is assumed to be perfectly inelastic so that $D(p_j,\varepsilon)=\varepsilon$. The demand shock, ε, is uniformly distributed on the interval [0,2], i.e. $g(\varepsilon)$ = ½. In case of excess demand at the maximum price, p_M, the system operator clears the market at this price by forced disconnection of consumers. This assumption is reasonable for most electricity markets. It also ensures that no production is withheld from the auction, because with a positive LOLP it is always better to offer capacity at p_M than to withhold it. Under the assumed market conditions (c = 0), the discrete first-order condition in (6) can be simplified to:

$$p_j\left(2s_i^j+\Delta\tau_{-i}^j\right)+p_{j+1}\left(-2s_i^j+\Delta\tau_{-i}^{j+1}\right)=0. \tag{7}$$

In a symmetric equilibrium with perfectly inelastic demand we have $\Delta\tau_{-i}^j=s_i^j-s_i^{j-1}$.
Thus whenever $1\le j\le M$-1 the first-order condition becomes:[18]

$$p_{j+1}s_i^{j+1}+3\left(p_j-p_{j+1}\right)s_i^j-p_js_i^{j-1}=0\,. \tag{8}$$

This is a difference equation of the second order with two boundary conditions, $s_i^M=k_i$, because no capacity is withheld, and $s_i^0=0$, i.e. the lowest supply offered at zero price is zero. As stated below the difference equation has a unique solution:

Proposition 5. *The difference equation* $p_{j+1}s_i^{j+1}+3\left(p_j-p_{j+1}\right)s_i^j-p_js_i^{j-1}=0$ *has a unique and non-decreasing solution if the boundary conditions are* $s_i^0=0$ and $s_i^M=k_i$. Moreover, the solution is a NE.

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[17] For the positive marginal cost case define p as the spot price less the marginal cost.

[18] Note that the first-order condition can be written as $p_{j+1}\Delta s_i^{j+1}-\Delta ps_i^j+p_j\Delta s_i^j-\Delta ps_i^j=0$. In the limit $M\to\infty$ this is equivalent to $p_j\Delta s_i^j-\Delta ps_i^j=0$, the first-order condition of continuous SFE for zero costs and perfectly inelastic demand.

Proof: See Appendix

The step SFE can be compared to the continuous SFE where Holmberg (2007) proves that under the assumed market conditions there is a unique SFE given by

$$S_i(p) = p \, .$$

As shown in Fig. 5 the discrete case converges to the continuous case as the number of price levels increases.

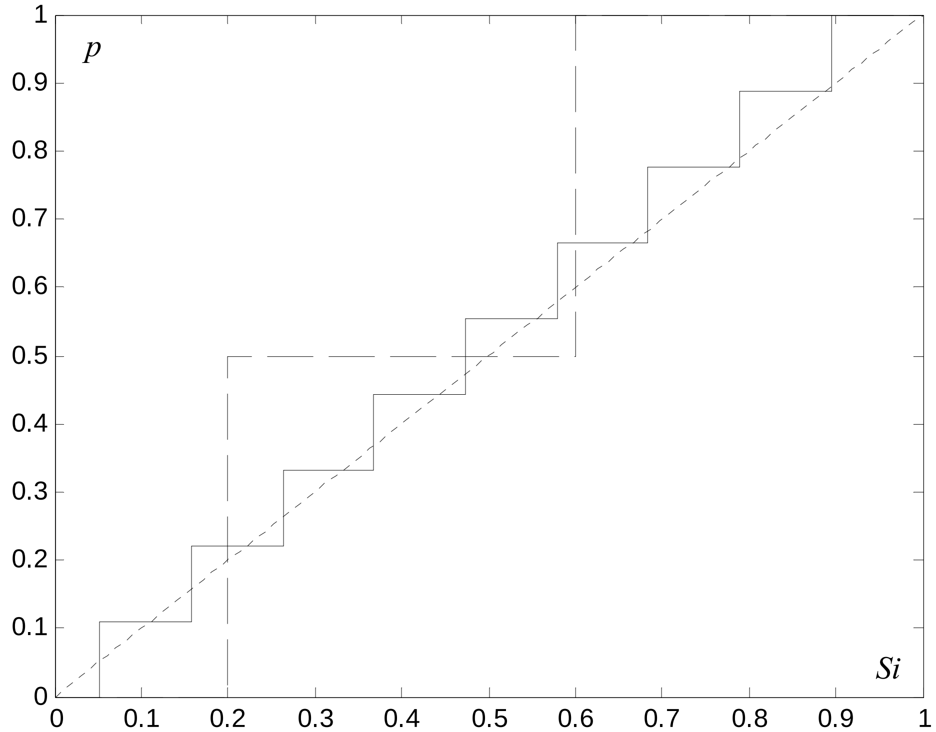

Figure 5. *Step-SFE with 3 and 10 steps are compared to the continuous SFE.*

3.3 Convergence when the price tick size is small.

Anderson and Xu (2007) only solve for a very simple example with two firms each choosing one price in the first stage of the Australian market, noting that to solve for the mixed strategy in prices would be challenging. For similar reasons von der Fehr and Harbord (1993) only consider mixed equilibria in which each firm chooses one price. An interesting conjecture is that if firms can choose a large but finite number of prices, then the range over which each price is sampled may shrink as the number of possible price choices increases, particularly if the prices themselves must be discrete. It may then be possible to demonstrate convergence of step SFEs to the continuous SFEs even when the possible price steps are smaller than the quantity steps. If so, the price instability at any level of demand would be small, and errors in using continuous representations also small.

It follows from classical existence results that NE in finite approximations of a limit game converge to the NE in the limit game if the strategy space is convex, compact and payoffs are continuous and quasi-concave (Dasgupta and Maskin, 1986). As the strategy space in the von der Fehr and Harbord model is convex and compact, it is our belief that their equilibrium fails to converge to a continuous SFE because of the payoff discontinuity; payoffs in their model can be significantly increased by slightly undercutting competitors' offers. Thus we argue that the risk of price instability would be mitigated if payoffs could be made continuous. For example, if costs are private information to some extent as in Parisio and Bosco (2003), then uncertainty about competitors' offers would make expected profits continuous. In spite of this additional uncertainty, we believe that pure-strategy equilibria in such a market can be approximated by a continuous SFE if demand uncertainty dominates uncertainty about competitor's production costs. Further, it would be helpful if the market design did not require stepped offers. For example, Nord Pool (in the Nordic countries) and Powernext (in France) make a linear interpolation of volumes between each adjacent pair of submitted price steps. This also makes payoffs continuous, unless producers choose to make stepped offers. Continuous payoffs, because of piece-wise linear offers or uncertainty about competitors' production costs, are helpful but only guarantee convergence to SFE in the limit. To ensure price stability, an SFE must exist in the limit game, the quantity multiple needs to be sufficiently small, and the allowed number of steps sufficiently large.

4 CONCLUSIONS

Green and Newbery (1992), and Newbery (1998) assume that the allowed number of steps in the supply function bids of electricity auctions is so large that equilibrium bids can be approximated by continuous SFE. This is a very attractive assumption, because it implies that a pure-strategy equilibrium can be calculated analytically for simple cases and numerically for general cost functions and asymmetric producers. The pure-strategy equilibrium and that prices are inherently stable also justifies empirical approaches that enable observers to deduce contract positions, marginal costs and the price-cost mark-up from observed bids, as in Wolak (2001).

von der Fehr and Harbord (1993), however, argue that as long as the number of steps is finite, then continuous SFE are not a valid representation of bidding in electricity auctions. Under the extreme assumption that prices can be chosen from a continuous distribution so that the price tick size is negligible, von der Fehr and Harbord (1993) show that uniform price electricity auctions have an inherent price instability (if demand variation is sufficiently large); there are no pure strategy Nash equilibria, only mixed strategy Nash equilibria. The intuition behind the non-existence of pure strategy Nash equilibria is that producers slightly undercut each others' step bids until mark-ups are zero. Whenever producers are pivotal they have profitable deviations from such an outcome.

We claim that the von der Fehr and Harbord result is not driven by the stepped form of the supply functions, but rather by their discreteness assumption. We consider the other extreme in which the price tick size is significant and the quantity multiple is negligible. We show that in this case step equilibria converge to continuous supply function equilibria. The intuition for the existence of pure strategy equilibria is that with a significant price tick size, it is not necessarily profitable to undercut perfectly elastic segments in competitors' bids.

Our results imply that the concern that electricity auctions have an inherent price instability and that they cannot be modelled by continuous SFE is not necessarily correct. We also claim that this potential problem can be avoided if tick sizes are such that the number of price levels is small compared to the number of quantity levels, which is the case in most electricity markets. To avoid price instability, we also recommend that restrictions in the number of steps should be as lax as possible, even if some restrictions are probably administratively necessary. Restricting the number of steps increases each producer's incremental supply offered at each step, encouraging price randomisation. Our recommendation to have small quantity multiples contrasts with that of Kremer and Nyborg (2004b) who recommend a large minimum quantity increment relative to the price tick size to encourage competitive bidding. We believe that their recommendation is correct for markets in which bidders are not pivotal, because in such markets pure strategy equilibria with very low mark-ups are possible. In markets with pivotal producers, however, encouraging producers to undercut competitors' bids leads to non-existence of pure strategy Nash equilibria and not necessarily lower average mark-ups (von der Fehr and Harbord, 1993). As undercutting incentives are only problematic when producers are pivotal, it is possible that an optimal market design would have a price tick-size that increases with the price. This could be achieved by limiting the number of digits rather than the number of decimals in the bids, or by requiring a minimum percentage increment in successive prices. If this turns out be an attractive option, it should be noted that the first-order condition in Proposition 1 is valid even if the tick size varies with the price.

Circumstances under which NE in a game with a finite approximation of the strategy space converge to NE in the limit game with an infinite strategy space have been studied by Dasgupta and Maskin (1986) and Simon (1987). With the caveat that the strategy space of the limit game has a finite dimension in their case and it is infinitely dimensional in our case, their results suggest that the risk of non-convergence and price instability would be lower if payoffs were continuous. Expected payoffs would be continuous if there is some uncertainty about competitors' costs as in Parisio and Bosco (2003). Further, payoffs would be continuous if offers were not required to be stepped, but instead piece-wise linear as in Nord Pool (in the Nordic countries) and Powernext (in France). Continuous payoffs are helpful, but only ensure convergence to SFE in the limit, i.e. if the minimum quantity increment is sufficiently small. Moreover, convergence only ensures price stability if the limit game has a pure strategy equilibrium. For example, continuous SFE do not exist if the demand curve is sufficiently convex or if production costs are sufficiently non-convex.

If an electricity market would fail to have a pure-strategy NE due to large quantity increments, then problems caused by instability might not be too severe for levels of demand when no generator is pivotal and the MCP were close to system marginal cost. We also conjecture that if mixed strategy equilibria occur, then the price instability at any level of demand would be small if there are many available price and quantity levels.

Recently, it has been empirically verified that large producers in the balancing market of Texas (ERCOT) approximately bid in accordance with the first-order condition for continuous supply functions (Niu et al., 2005; Hortascu and Puller, 2007; Sioshansi and Oren, 2007). It is possible that the new discrete model could improve the accuracy of such empirical studies, because the new first-order condition considers the influence by the demand uncertainty on stepped offers. This effect has previously been considered by Wolak (2004) in an empirical analysis of the Australian market, but this market is quite different from most other markets, as producers choose their own price grid in Australia.

A possible spin-off from our work is that discrete NE might be numerically more well-behaved than continuous NE, whose differential equations are ill-conditioned and difficult to solve numerically (Baldick and Hogan, 2002). An advantage with the discrete NE is that the first-order condition is a second-order difference equation and its solution can be determined by a boundary value problem, one initial condition and one end-condition. This should also stabilize the solution for a small number of steps. Hence, when confronting the computer-intensive problem of numerically calculating asymmetric SFE of real electricity markets it is possible that solving for discrete NE would be a more attractive option than numerically integrating the differential equations of continuous SFE. When calculating approximate equilibria, the assumed price tick size does not necessarily have to correspond to the tick size of the studied auction. In a numerically efficient solver, it may also be of interest to vary the tick size with the price.

Finally, we would not claim that the apparent tension between tractable but unrealistic continuous SFEs and realistic but intractable step SFEs is the only, or even the main, problem in modeling electricity markets. First, under reasonable conditions, there can be a continuum of continuous SFE bounded by (in the short run) a least and most profitable SFE. Second, the position of the SFEs depends on the contract position of all the generators, and determining the choice of contracts and their impact on the spot market is a hard and important problem. The greater the extent of contract cover, the less will be the incentive for spot market manipulation (Newbery, 1995), and as electricity demand is very inelastic and markets typically concentrated, this is an important determinant of market performance. Newbery (1998) argued that these can be related, in that incumbents can choose contract positions to keep both the contract and average spot price at the entry-deterring level, thus simultaneously solving for prices, contract positions, and embedding the short-run SFE within a longer run investment and entry equilibrium. A full long-run model of the electricity market should also be able to investigate whether some market power is required for (or inimical to) adequate investment in reserve capacity to maintain adequate security of supply. With such a model one could also make a proper assessment of how many competing generators are needed to deliver a workably competitive but secure electricity market.

REFERENCES

Anderson, E.J. and A. B. Philpott (2002). 'Using supply functions for offering generation into an electricity market', *Operations Research* 50 (3), pp. 477-489.

Anderson, E. J. and X. Hu (2007), *Finding Supply Function Equilibria with Asymmetric Firms*, Workshop on Supply Function Equilibrium Modelling of Electricity Markets, April 2006, Auckland, New Zealand.

Anderson, E. J. and H. Xu (2004). 'Nash equilibria in electricity markets with discrete prices', *Mathematical Methods of Operations Research* 60, pp. 215–238.

Baldick, R., and W. Hogan (2002), *Capacity constrained supply function equilibrium models for electricity markets: Stability, non-decreasing constraints, and function space iterations*, POWER Paper PWP-089, University of California Energy Institute.

Baldick, R., R. Grant, and E. Kahn (2004). 'Theory and Application of Linear Supply Function Equilibrium in Electricity Markets', *Journal of Regulatory Economics* 25 (2), pp. 143-67.

Burns, P., M Huggins and R Lydon (2004) *Generation strategies in the England and Wales electricity market – a synthesis of simulation modelling and econometric analysis*, University of Bath Centre for the Study of Regulated Industries Technical Paper 16.

Dasgupta P. and Maskin E. (1986) 'The existence of equilibrium in discontinuous economic games, I: Theory.' *Review of Economic Studies* 53, pp. 1-27.

Fabra, Natalia & Nils-Henrik M. von der Fehr & David Harbord, (2006) "Designing Electricity Auctions," RAND Journal of Economics. The earlier working paper (2004) is available as CSEM WP 122 at www.ucei.berkely.edu/pubs-csemwp.html.

Gans, J.S. and Wolak, F.A. (2007), 'A Comparison of Ex Ante versus Ex Post Vertical Market Power: Evidence from the Electricity Supply Industry'…

Green, R.J. and D.M. Newbery (1992). `Competition in the British Electricity Spot Market' *Journal of Political Economy* 100 (5), October, pp 929-53.

Green, R.J. (1996), 'Increasing competition in the British Electricity Spot Market', *Journal of Industrial Economics* 44 (2), pp. 205-216.

Holmberg, P. (2005a). *Comparing Supply Function Equilibria of Pay-as-Bid and Uniform-Price Auctions*, Working Paper 2005:17, Department of Economics, Uppsala University.

Holmberg, P. (2005b). *Numerical calculation of asymmetric supply function equilibrium with capacity constraints*, Working Paper 2005:12, Department of Economics, Uppsala University.

Holmberg, P. (2007), 'Supply Function Equilibrium with Asymmetric Capacities and Constant Marginal Costs', *Energy Journal* 28 (2), pp. 55-82.

Hortacsu, A. and S. Puller (2008), 'Understanding Strategic Bidding in Multi-Unit Auctions: A Case Study of the Texas Electricity Spot Market', *Rand Journal of Economics* 39 (1), pp. 86-114.

Klemperer, P. D. and M.A. Meyer, (1989). `Supply Function Equilibria in Oligopoly under Uncertainty', *Econometrica*, 57 (6), pp. 1243-1277.

Kremer, I and K.G. Nyborg (2004a). 'Divisible Good Auctions: The Role of Allocation Rules.' *RAND Journal of Economics* 35, pp. 147–159.

Kremer, I. and K.G. Nyborg (2004b). 'Underpricing and Market Power in Uniform Price Auctions', *The Review of Financial Studies* 17 (3), pp. 849-877.

LeVeque, R. (2007), *Finite Difference Methods for Ordinary and Partial Differential Equations,* Philadelphia: Society for Industrial and Applied Mathematics.

Madlener, R. and M. Kaufmann (2002), *Power exchange spot market trading in Europe: theoretical considerations and empirical evidence*, OSCOGEN Report D 5.1b, Centre for Energy Policy and Economics, Switzerland.

Newbery, D.M. (1995) `Power Markets and Market Power', *Energy Journal* **16**(3), 41-66

Newbery, D. M. (1998a). 'Competition, contracts, and entry in the electricity spot market', *RAND Journal of Economics* 29 (4) pp. 726-749.

Newbery, D.M., (1998b) 'Pool Reform and Competition in Electricity', ch.5, pp117-166 in M. Beesley (ed.) *Regulating Utilities: Understanding the Issues*, London: Institute of Economic Affairs

Newbery, D.N. (2002) 'Supply Function Equilibria', mimeo, Department of Applied Economics, November

Niu, H., R. Baldick, and G. Zhu (2005), "Supply Function Equilibrium Bidding Strategies With Fixed Forward Contracts", *IEEE Transactions on power systems* 20 (4), pp. 1859-1867.

Parisio, L. and B. Bosco (2003), "Market power and the power market: Multi-unit bidding and (in)efficiency in electricity auctions", *International tax and public finance* 10 (4), pp. 377-401.
Redpoint Energy (2007) *The dynamics of GB electricity generation investment*, 18 May, available at http://www.dti.gov.uk/files/file38972.pdf
Rudkevich, A., M. Duckworth and R. Rosen (1998). 'Modeling electricity pricing in a deregulated generation industry: The potential for oligopoly pricing in poolco', *The Energy Journal* 19 (3), pp. 19-48, 1998.
Sheffrin, A. (2002a) "Predicting Market Power Using the Residual Supply Index" Presented to FERC Market Monitoring Workshop December 3-4, 2002.
Sheffrin, A. (2002b) "Empirical evidence of strategic bidding in California ISO real-time market", in *Electric Pricing In Transition*, Norwell, MA: Kluwer 2002, p.267-281.
Simon, L.K. (1987), 'Games with discontinuous payoffs.' *Review of Economic Studies* 54, pp. 569-97.
Sioshansi, R. and S. Oren (2007), 'How Good are Supply Function Equilibrium Models: An Empirical Analysis of the ERCOT Balancing Market', *Journal of Regulatory Economics* 31 (1), pp. 1-35
Spiegel, M.R. (1968). *Schaum's Mathematical Handbook of Formulas and Tables,* New York: McGraw-Hill, Inc.
Sweeting, A. (2007) 'Market Power in the England and Wales Wholesale Electricity Market 1995-2000' *Economic Journal*, Vol. 117, Issue 520, 654-85.
von der Fehr, N-H. M. and D. Harbord (1993). `Spot Market Competition in the UK Electricity Industry', *Economic Journal* 103 (418), pp. 531-46.
Wilson, R. (1979), 'Auctions of Shares', *Quarterly Journal of Economics*, Vol. 93, No. 4 , pp. 675-689.
Wilson, R. (2008), 'Supply Function Equilibrium in a Constrained Transmission System', *Operations Research* 56, pp. 369-382.
Wolak, F. A. (2001), 'Identification and Estimation of Cost Functions Using Observed Bid Data: An Application to Electricity Markets', Cambridge, MA. : National Bureau of Economic Research.
Wolak, F.A. (2004), 'Quantifying the Supply-Side Benefits from Forward Contracting in Wholesale Electricity Markets', available from http://www.stanford.edu/~wolak.

APPENDIX - PROOFS OF PROPOSITIONS

Proposition 1. With discrete supply function offers, the first-order condition for the supply of firm i at the price level j is given by:

$$\frac{\partial \mathrm{E}(\pi_i(\mathbf{s}))}{\partial s_i^j} = -\Delta p s_i^j g(\tau^j) + \int_{\tau^{j-1}}^{\tau^j} \left[p_j - C_i'(s_i(\varepsilon))\right]\left(\frac{\Delta \tau_{-i}^j(\varepsilon - \tau^{j-1})}{(\Delta \tau^j)^2}\right) g(\varepsilon)d\varepsilon +$$

$$+\int_{\tau^j}^{\tau^{j+1}} \left[p_{j+1} - C_i'(s_i(\varepsilon))\right]\Delta \tau_{-i}^{j+1}\left(\frac{\Delta \tau^{j+1} - (\varepsilon - \tau^j)}{(\Delta \tau^{j+1})^2}\right) g(\varepsilon)d\varepsilon = 0,$$

where

$$s_i(\varepsilon) = \varepsilon - \tau_{-i}^{j-1} - \frac{\Delta \tau_{-i}^j(\varepsilon - \tau^{j-1})}{\Delta \tau^j} \text{ if } \varepsilon \in \left[\tau^{j-1}, \tau^j\right].$$

Proof: In order to find an equilibrium we want to determine the best response of firm

i given the bids of its competitors. The best response necessarily satisfies a first-order condition for each price level. To obtain this condition we start by differentiating (2) with respect to s_i^j and s_i^{j-1}. Applying Leibniz's rule for differentiating when the limits are functions of the relevant variable, because $\tau^j = \tau_{-i}^j + s_i^j$, gives (Spiegel, 1968):

$$\frac{\partial E_i^j}{\partial s_i^j} = \int_{\tau^{j-1}}^{\tau^j} \left(p_j - C_i'(\cdot) \right) \frac{\Delta \tau_{-i}^j \left(\varepsilon - \tau^{j-1} \right)}{\left(\Delta \tau^j \right)^2} g(\varepsilon) d\varepsilon + \left(p_j s_i^j - C_i\left(s_i^j\right) \right) g\left(\tau^j\right) \tag{9}$$

and

$$\frac{\partial E_i^j}{\partial s_i^{j-1}} = \int_{\tau^{j-1}}^{\tau^j} \left(p_j - C_i'(\cdot) \right) \Delta \tau_{-i}^j \frac{\left(\tau^j - \varepsilon \right)}{\left(\Delta \tau^j \right)^2} g(\varepsilon) d\varepsilon - \left[p_j s_i^{j-1} - C_i\left(s_i^{j-1}\right) \right] g\left(\tau^{j-1}\right).$$

From the last expression it follows that:

$$\frac{\partial E_i^{j+1}}{\partial s_i^j} = \int_{\tau^j}^{\tau^{j+1}} \left[p_{j+1} - C_i'(\cdot) \right] \Delta \tau_{-i}^{j+1} \frac{\left(\tau^{j+1} - \varepsilon \right)}{\left(\Delta \tau^{j+1} \right)^2} g(\varepsilon) d\varepsilon - \left[p_{j+1} s_i^j - C_i\left(s_i^j\right) \right] g\left(\tau^j\right). \tag{10}$$

Combining (9) and (10) gives the first-order condition for step supply functions:

$$\begin{aligned} \frac{\partial \, \mathrm{E}(\pi_i(\mathbf{s}))}{\partial s_i^j} &= \frac{\partial E_i^j}{\partial s_i^j} + \frac{\partial E_i^{j+1}}{\partial s_i^j} = \\ &= -\Delta p s_i^j g\left(\tau^j\right) + \int_{\tau^{j-1}}^{\tau^j} \left[p_j - C_i'\left(s_i(\varepsilon)\right) \right] \frac{\Delta \tau_{-i}^j \left(\varepsilon - \tau^{j-1} \right)}{\left(\Delta \tau^j \right)^2} g(\varepsilon) d\varepsilon + \\ &+ \int_{\tau^j}^{\tau^{j+1}} \left[p_{j+1} - C_i'\left(s_i(\varepsilon)\right) \right] \Delta \tau_{-i}^{j+1} \frac{\left(\tau^{j+1} - \varepsilon \right)}{\left(\Delta \tau^{j+1} \right)^2} g(\varepsilon) d\varepsilon = 0, \end{aligned} \tag{11}$$

where

$$s_i(\varepsilon) = \varepsilon - \tau_{-i}^{j-1} - \frac{\Delta \tau_{-i}^j \left(\varepsilon - \tau^{j-1} \right)}{\Delta \tau^j} \text{ if } \varepsilon \in \left[\tau^{j-1}, \tau^j \right]. \square$$

Lemma 1. For every pair of vectors $\left\{ s_i^j \right\}_{i=1}^N$ and $\left\{ s_i^{j-1} \right\}_{i=1}^N$, for which one can find positive constants L and δ, such that $0 \le s_i^j - s_i^{j-1} \le L \Delta p$ and $p_{j+1} - C_i'\left(s_i^j\right) \ge \delta > 0 \; \forall i = 1 \ldots N$, there exists at least one vector $\left\{ s_i^{j+1} \right\}_{i=1}^N$ that satisfy the first-order condition in Proposition 1 if Δp is sufficiently small. Moreover, the vector is unique if we add the constraint that $p_{j+1} - C_i'\left(s_i(\varepsilon)\right) \ge 0 \; \forall i = 1 \ldots N$.

Proof:

The property that $0 \le s_i^j - s_i^{j-1} \le L \Delta p$, implies that the integral

$\int_{\tau^j}^{\tau^{j+1}} \left[p_{j+1} - C_i'(s_i(\varepsilon)) \right] \Delta \tau_{-i}^{j+1} \frac{(\tau^{j+1} - \varepsilon)}{(\Delta \tau^{j+1})^2} g(\varepsilon) d\varepsilon$ must be of the order Δp if a vector $\{s_i^{j+1}\}_{i=1}^N$ is to satisfy the first-order condition in Proposition 1. If we add the constraint that $p_{j+1} - C_i'(s_i(\varepsilon)) \geq 0 \; \forall i = 1 \ldots N$, we realize that any solution vector $\{s_i^{j+1}\}_{i=1}^N$ must have the property that differences $s_i^{j+1} - s_i^j$ are of the order Δp, otherwise the first-order condition cannot be satisfied for each firm. Moreover, we realize that the unique solution is $s_i^{j+1} = s_i^j$ when $\Delta p = 0$.

The next step is to use the implicit function theorem to prove that a unique solution $\{s_i^{j+1}\}_{i=1}^N$ exists also for sufficiently small Δp. In our case the implicit function is defined by:

$$\Gamma_i(\tau^{j-1}, \tau^j, \tau^{j+1}, \tau_{-i}^{j-1}, \tau_{-i}^j, \tau_{-i}^{j+1}) = -\Delta p (\tau^j - \tau_{-i}^j) g(\tau^j) +$$
$$+ \int_{\tau^{j-1}}^{\tau^j} \left[p_j - C_i'(s_i(\varepsilon)) \right] \frac{\Delta \tau_{-i}^j (\varepsilon - \tau^{j-1})}{(\Delta \tau^j)^2} g(\varepsilon) d\varepsilon + \int_{\tau^j}^{\tau^{j+1}} \left[p_{j+1} - C_i'(s_i(\varepsilon)) \right] \Delta \tau_{-i}^{j+1} \frac{(\tau^{j+1} - \varepsilon)}{(\Delta \tau^{j+1})^2} g(\varepsilon) d\varepsilon.$$

We know from the first-order condition in Proposition 1 that $\Gamma_i(\tau^{j-1}, \tau^j, \tau^{j+1}, \tau_{-i}^{j-1}, \tau_{-i}^j, \tau_{-i}^{j+1}) \equiv 0$ for any combination of vectors $\{s_i^j\}_{i=1}^N$ and $\{s_i^{j-1}\}_{i=1}^N$ and solutions $\{s_i^{j+1}\}_{i=1}^N$. It is straightforward to show that:

$$\frac{\partial \Gamma_i}{\partial \tau_{-i}^{j+1}} = \int_{\tau^j}^{\tau^{j+1}} C_i''(s_i(\varepsilon)) \frac{\Delta \tau_{-i}^{j+1} (\varepsilon - \tau^j)(\tau^{j+1} - \varepsilon)}{(\Delta \tau^{j+1})^3} g(\varepsilon) d\varepsilon +$$
$$+ \int_{\tau^j}^{\tau^{j+1}} \left[p_{j+1} - C_i'(s_i(\varepsilon)) \right] \frac{(\tau^{j+1} - \varepsilon)}{(\Delta \tau^{j+1})^2} g(\varepsilon) d\varepsilon.$$

Evaluating the integrals in the limit when $\Delta p \to 0$ yields,

$$\lim_{\Delta p \to 0} \frac{\partial \Gamma_i}{\partial \tau_{-i}^{j+1}} = C_i''(s_i^j) \frac{\Delta \tau_{-i}^{j+1}}{2 \Delta \tau^{j+1}} g(\tau_i^j) + \frac{\left[p_{j+1} - C_i'(s_i^j) \right] g(\tau_i^j)}{2} > 0. \tag{12}$$

Similarly,

$$\frac{\partial \Gamma_i}{\partial \tau^{j+1}} = -\int_{\tau^j}^{\tau^{j+1}} C_i''(s_i(\varepsilon)) \Delta \tau_{-i}^{j+1} \frac{\Delta \tau_{-i}^{j+1} (\varepsilon - \tau^j)(\tau^{j+1} - \varepsilon)}{(\Delta \tau^{j+1})^4} g(\varepsilon) d\varepsilon +$$
$$+ \int_{\tau^j}^{\tau^{j+1}} \left[p_{j+1} - C_i'(s_i(\varepsilon)) \right] \Delta \tau_{-i}^{j+1} \frac{(\varepsilon - \tau^{j+1} + \varepsilon - \tau^j)}{(\Delta \tau^{j+1})^2} g(\varepsilon) d\varepsilon$$

and

$$\lim_{\Delta p \to 0} \frac{\partial \Gamma_i}{\partial \tau^{j+1}} = 0. \tag{13}$$

For convenience we introduce $\alpha_i = \frac{\partial \Gamma_i}{\partial \tau_i^{j+1}}$ and $\beta_i = \frac{\partial \Gamma_i}{\partial \tau_i^{j+1}} + \frac{\partial \Gamma_i}{\partial \tau_{-i}^{j+1}}$. We know that

$\frac{\partial \Gamma_i}{\partial s_i^{j+1}} = \frac{\partial \Gamma_i}{\partial \tau_i^{j+1}} = \alpha_i$ and that $\frac{\partial \Gamma_i}{\partial s_k^{j+1}} = \frac{\partial \Gamma_i}{\partial \tau_i^{j+1}} + \frac{\partial \Gamma_i}{\partial \tau_{-i}^{j+1}} = \beta_i \; \forall k \neq i$. Accordingly, if Δp is sufficiently small, then it follows from (12) and (13) that

$$\beta_i > \alpha_i \text{ and } \beta_i > 0. \tag{14}$$

The Jacobian matrix of the functions $\Gamma_1 \ldots \Gamma_N$ is:

$$J_1 = \begin{bmatrix} \alpha_1 & \beta_1 & \cdots & \beta_1 \\ \beta_2 & \alpha_2 & \cdots & \beta_2 \\ \vdots & \vdots & & \vdots \\ \beta_N & \beta_N & \cdots & \alpha_N \end{bmatrix}.$$

To verify that the matrix is invertible, we want to prove that its determinant is non-zero. The non-zero property of the determinant is unaltered if we divide each row *i* by the factor $\beta_i > 0$.

$$J_2 = \begin{bmatrix} \alpha_1/\beta_1 & 1 & \cdots & 1 \\ 1 & \alpha_2/\beta_2 & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & \alpha_N/\beta_N \end{bmatrix}.$$

The determinant cannot get (but may lose) its non-zero property if one row is replaced by a linear combination of the rows. In the next step, each row (except for the last row) is subtracted by the row below.

$$J_3 = \begin{bmatrix} \alpha_1/\beta_1 - 1 & 1-\alpha_2/\beta_2 & 0 & \cdots & 0 \\ 0 & \alpha_2/\beta_2 - 1 & 1-\alpha_3/\beta_3 & \cdots & 0 \\ 0 & 0 & \alpha_3/\beta_3 - 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & \cdots & \alpha_N/\beta_N \end{bmatrix}$$

It can now be shown that $|J_3| = \sum_{j=1}^{N} \prod_{k \neq j} (\alpha_k/\beta_k - 1)$. By means of (14) we are now ready to conclude that $|J_3| \neq 0$. Consequently, we can also conclude that $|J_2| \neq 0$ and that $|J_1| \neq 0$. Thus the Jacobian matrix of the functions $\Gamma_1 \ldots \Gamma_N$ with respect to s_k^{j+1} is invertible for sufficiently small Δp. Moreover, it is straightforward to verify that the functions $\Gamma_1 \ldots \Gamma_N$ are continuously differentiable in all of their parameters. We have already shown that we get a unique solution $\{s_i^{j+1}\}_{i=1}^{N}$ when $\Delta p = 0$. Thus we can conclude from the Implicit Function Theorem that there is a unique solution to the discrete equation in Proposition 1 for sufficiently small Δp. □

Lemma 2. *The difference equation in Proposition* 1 *is consistent with the continuous* *equation in (5) if* $\{\hat{s}_i(p)\}_{i=1}^N$ is non-decreasing and $p - C_i'(\hat{s}_i(p)) \geq \delta > 0\ \forall i = 1\ldots N$.

Proof: A discrete approximation of an ordinary differential equation is consistent if the local truncation error is infinitesimally small when the step length is infinitesimally small (LeVeque, 2007). The local truncation error is the discrepancy between true slope and its discrete estimate when discrete values s_i^j are replaced with samples of the true continuous solution $s_i(p_j)$. We realize from the continuous first-order condition in (5) and the constraint $p - C_i'(\hat{s}_i(p)) \geq \delta > 0$, that $\{\hat{s}_i'(p)\}_{i=1}^N$ are bounded. Thus differences $\hat{s}_i(p_{j+1}) - \hat{s}_i(p_j)$ will be of the order Δp. This can be used to approximate the discrete first-order condition in (11) by means of a Taylor series extension in Δp. The first-order condition in (11) can be written:

$$\frac{\partial \mathrm{E}(\pi_i(\mathbf{s}))}{\partial s_i^j} = -\Delta p s_i^j g(\tau^j) + \int_{\tau^j}^{\tau^{j+1}} \left[p_{j+1} - C_i'(\cdot)\right] \frac{\Delta \tau_{-i}^{j+1}}{\Delta \tau^{j+1}} g(\varepsilon) d\varepsilon + \int_{\tau^{j-1}}^{\tau^j} \left[p_j - C_i'(\cdot)\right] \frac{\Delta \tau_{-i}^j (\varepsilon - \tau^{j-1})}{(\Delta \tau^j)^2} g(\varepsilon) d\varepsilon - \int_{\tau^j}^{\tau^{j+1}} \left[p_{j+1} - C_i'(\cdot)\right] \frac{\Delta \tau_{-i}^{j+1} (\varepsilon - \tau^j)}{(\Delta \tau^{j+1})^2} g(\varepsilon) d\varepsilon = 0. \tag{15}$$

In the limit we have

$$\lim_{M \to \infty} \int_{s_i^j + \tau_{-i}^j}^{s_i^{j+1} + \tau_{-i}^{j+1}} \left[p_{j+1} - C_i'(\cdot)\right] \frac{\Delta \tau_{-i}^{j+1}}{\Delta s_i^{j+1} + \Delta \tau_{-i}^{j+1}} g(\varepsilon) d\varepsilon = \left[p_{j+1} - C_i'(\cdot)\right] \Delta \tau_{-i}^{j+1} g(s_i^j + \tau_{-i}^j). \tag{16}$$

Each of the terms below are of the first-order:

$$\int_{s_i^{j-1} + \tau_{-i}^{j-1}}^{s_i^j + \tau_{-i}^j} \left[p_j - C_i'(\cdot)\right] \left(\frac{\Delta \tau_{-i}^j (\varepsilon - s_i^{j-1} - \tau_{-i}^{j-1})}{(\Delta s_i^j + \Delta \tau_{-i}^j)^2} \right) g(\varepsilon) d\varepsilon$$
$$- \int_{s_i^j + \tau_{-i}^j}^{s_i^{j+1} + \tau_{-i}^{j+1}} \left[p_{j+1} - C_i'(\cdot)\right] \frac{\Delta \tau_{-i}^{j+1} (\varepsilon - s_i^j - \tau_{-i}^j)}{(\Delta s_i^{j+1} + \Delta \tau_{-i}^{j+1})^2} g(\varepsilon) d\varepsilon,$$

but their difference is of the second-order, and is negligible in the limit when $M \to \infty$. This result together with (16) implies that in the limit, the first-order condition (15) can be written as

$$-\Delta p s_i^j g(s_i^j + \tau_{-i}^j) + \left[p_{j+1} - C_i'(s_i^j)\right] \Delta \tau_{-i}^{j+1} g(s_i^j + \tau_{-i}^j) + O(\Delta p^2) = 0, \tag{17}$$

or equivalently

$$-\Delta p s_i^j + \left[p_{j+1} - C_i'(s_i^j)\right] (\Delta s_{-i}^{j+1} - \Delta d^{j+1}) + O(\Delta p^2) = 0. \tag{18}$$

This lemma considers prices for which $p_{j+1} \neq C_i'(s_i^j)$. Then (18) can be rewritten as:

$$\frac{-\Delta p s_i^j + O(\Delta p^2)}{p_{j+1} - C_i'(s_i^j)} + \Delta s_{-i}^{j+1} - \Delta d^{j+1} = 0 \,. \tag{19}$$

Summing the corresponding expressions of all firms and then dividing by N-1 yields:

$$\Delta s^{j+1} - \frac{N}{N-1}\Delta d^{j+1} - \frac{1}{N-1}\sum_k \frac{\Delta p s_k^j + O(\Delta p^2)}{p_{j+1} - C_k'(s_k^j)} = 0 \,. \tag{20}$$

By subtracting (19) from (20) followed by some rearrangements we obtain:

$$\frac{s_i^{j+1} - s_i^j}{\Delta p} = \frac{\Delta d^{j+1}}{\Delta p(N-1)} - \frac{s_i^j + O(\Delta p)}{p_{j+1} - C_i'(s_i^j)} + \frac{1}{N-1}\sum_k \frac{s_k^j + O(\Delta p)}{p_{j+1} - C_k'(s_k^j)} \,. \tag{21}$$

Let $\lim_{\Delta p \to 0} \frac{\Delta d^{j+1}}{\Delta p} = d'(p)$ and $\lim_{\Delta p \to 0} p^j = p$. Now, if s_k^j are samples of the solution of the continuous system, then it follows from (21) that

$$\lim_{\Delta p \to 0} \frac{s_i^{j+1} - s_i^j}{\Delta p} = \frac{d'(p)}{N-1} - \frac{s_i(p)}{p - C_i'(s_i(p))} + \frac{1}{N-1}\sum_k \frac{s_k(p)}{p - C_k'(s_k(p))} \,. \tag{22}$$

Comparing (22) and (5) we can conclude that

$$\lim_{\Delta p \to 0} \frac{s_i^{j+1} - s_i^j}{\Delta p} = s_i'(p), \tag{23}$$

which proves that the discrete system is a consistent approximation of the continuous system. □

Proposition 2. *Assume that a set of continuous solutions $\{\hat{s}_i(p)\}_{i=1}^N$ exists for some set of initial-values $\{k_i\}$. Moreover, assume that the solution is increasing in the interval [a,b] and that the solution satisfies the property $p - C_i'(\hat{s}_i(p)) \geq \delta > 0 \; \forall i = 1 \ldots N$ in this interval. For a sufficiently large number of steps M, there will exist at least one discrete solution $\{\hat{s}_i^j\}_{i=1}^N$, such that $\hat{s}_i^1 = k_i$, $\hat{s}_i^2 = k_i + W_i \Delta p$, where $W_i \geq 0$, $\lim_{M \to \infty} p_1 = a$, and $\lim_{M \to \infty} p_M = b$. As the number of steps grows ($M \to \infty$), $\{\hat{s}_i^j\}_{i=1}^N$ will be non-decreasing and converge to $\{\hat{s}_i(p)\}$ in the interval [a,b].*

Proof: We already know from Lemma 2 that the discrete equation is a consistent approximation of the continuous equation. In order to show that the discrete solution converges to the continuous solution, we need to prove that the discrete solution exits and that it is stable, i.e. the error does not explode after infinitesimally steps. Our convergence proof closely follows LeVeque's **(**2007) convergence proof for general one-step methods. Denote the vector of global errors at se price p_j by

$$E^j = s^j - \hat{s}(p_j).$$

The corresponding vector for the local truncation error is

$$\upsilon^j = \frac{s(p_{j+1}) - s(p_j)}{\Delta p} - s'(p_j).$$

We know that $p - C_i'[\hat{s}_i(p)] \geq \delta > 0 \; \forall i = 1\ldots N$. Combining this inequality with (5) and the fact that the continuous solution is increasing, implies that

$$0 < \hat{s}_i'(p) \leq \eta < \infty. \tag{24}$$

If Δp is sufficiently small, so that the initial errors $\|E^1\|_\infty$ and $\|E^2\|_\infty$ are small enough, then (24) implies that $0 \leq s_i^2 - s_i^1 = \hat{s}_i(p_2) - \hat{s}_i(p_1) \leq L\Delta p$, and the constraint $p - C_i'[\hat{s}_i(p)] \geq \delta > 0$ implies that $p_3 - C_i'(s_i^2) \geq \delta > 0 \; \forall i = 1\ldots N$.[19] Thus it follows from Lemma 1 that $\{s_i^3\}_{i=1}^N$ can be uniquely determined if Δp is sufficiently small. Singularities are avoided as $p_3 - C_i'(s_i^2) \geq \delta > 0 \; \forall i = 1\ldots N$. Accordingly, it follows directly from (21) that we can find a Lipschitz constant L_2 such that:

$$\|E^3\|_\infty = \|s^3 - \hat{s}(p_3)\|_\infty < \|E^2\|_\infty + L_2\Delta p \|s^2 - \hat{s}(p_2)\|_\infty + \Delta p\|\upsilon^2\|_\infty = (1 + L_2\Delta p)\|E^2\|_\infty + \Delta p\|\upsilon^2\|_\infty.$$

Thus if Δp is sufficiently small, so that the initial error $\|E^2\|_\infty$ and the truncation error $\|\upsilon^2\|$ are small enough, then $\|E^3\|_\infty$ is sufficiently small, which implies that $0 \leq s_i^3 - s_i^2 \leq L\Delta p$, due to (24). If Δp is sufficiently small, then the argument for the vector s^3 can be repeated iteratively to prove that the vector $s^k \; \forall k = 4 \ldots M$ can be uniquely determined and that

$$\|E^k\|_\infty = \|s^k - \hat{s}(p_k)\|_\infty < (1 + L_{k-1}\Delta p)\|E^{k-1}\|_\infty + \Delta p \|\upsilon^{k-1}\|_\infty. \tag{25}$$

Let $L_k^{\max} = \max\{L_p\}_{p=2}^k$ and $\upsilon_{\max}^k = \max\{\|\upsilon^p\|_\infty\}_{p=2}^k$. From the inequality in (25), we can show by induction that

$$\begin{aligned} \|E^k\|_\infty &< (1 + L_{k-1}^{\max}\Delta p)^{k-2}\|E^2\|_\infty + \Delta p \sum_{m=2}^{k} \upsilon_{\max}^{k-1}(1 + L_{k-1}^{\max}\Delta p)^{k-m} < \\ &(1 + L_{k-1}^{\max}\Delta p)^{k-2}\left(\|E^2\|_\infty + (k-2)\Delta p\,\upsilon_{\max}^{k-1}\right) < (1 + L_{M-1}^{\max}\Delta p)^{M-2}\left(\|E^2\|_\infty + (M-2)\Delta p\,\upsilon_{\max}^{M-1}\right). \end{aligned} \tag{26}$$

In the limit when $\Delta p \to 0$ we have that $\|E^2\|_\infty \to 0$, $\upsilon_{\max}^{M-1} \to 0$ (because of Lemma 2), $(M-2)\Delta p \to b - a$, and that $(1 + L_{M-1}^{\max}\Delta p)^{M-2} \to e^{L_{M-1}^{\max}(b-a)}$. Thus it follows from (26) that $\|E^k\|_\infty \to 0$ when $\Delta p \to 0$, which proves that the discrete solution converges to the continuous one. □

[19] Note that $\|\cdot\|_\infty$ is the max-norm, i.e. $\|E^j\|_\infty = \max_{1\leq i\leq N}|E_i^j|$ (LeVeque, 2007).

Proposition 3. *Consider a set of continuous solutions* $\{\hat{s}_i(p)\}_{i=1}^N$ *to some set of initial-values* $\{k_i\}_{i=1}^N$. *Assume that the solutions are increasing and satisfy* $p - C_i'(\hat{s}_i(p)) \geq \delta > 0 \; \forall i = 1 \ldots N$ in the price range $p \in [a,b]$. *If the market price is in this range for all possible realizations of the concave demand curve, then the set of solutions is a continuous SFE.*

Proof:

Let X be a potential equilibrium, i.e. all supply functions are non-decreasing and satisfy the continuous first-order condition in (4) in the whole price-setting region, $[a,b]$. Consider an arbitrary firm i. Assume that its competitors follow the potential equilibrium strategy. The question now is whether it will be a best response of firm i to do the same. The profit of producer i for the outcome ε is given by

$$\pi_i(p,\varepsilon) = (\varepsilon + D(p) - S_{-i}^X(p))p - C_i(\varepsilon + D(p) - S_{-i}^X(p)).$$

Hence

$$\frac{\partial \pi_i(p,\varepsilon)}{\partial p} = [D'(p) - S_{-i}'^X(p)][p - C_i'(\varepsilon + D(p) - S_{-i}^X(p))] + \varepsilon + D(p) - S_{-i}^X(p) \; . \qquad (27)$$

From the first-order condition in (4) it is known that

$$[D'(p) - S_{-i}'^X(p)][p - C_i'(S_i^X(p))] + S_i^X(p) = 0 \quad \forall p \in [a,b].$$

Subtracting this expression from (27) yields:

$$\frac{\partial \pi_i(\varepsilon,p)}{\partial p} = [S_{-i}'^X(p) - D'(p)]\left[C_i'\left(\underbrace{\varepsilon + D(p) - S_{-i}^X(p)}_{S_i}\right) - C_i'(S_i^X(p))\right] + \left(\underbrace{\varepsilon + D(p) - S_{-i}^X(p)}_{S_i} - S_i^X(p)\right) \; \forall p \in [p^X(\underline{\varepsilon}), p^X(\bar{\varepsilon})], \qquad (28)$$

Due to monotonicity of the supply functions we know that:

$$S_{-i}^{X\,\prime}(p) - D'(p) \geq 0$$

and that

$$p(S_i) \leq p(S_i^X) \Leftrightarrow S_i \geq S_i^X \Leftrightarrow C_i'(S_i) \geq C_i'(S_i^X).$$

Thus for every $p(S_i) \in [p^X(\underline{\varepsilon}), p^X(\bar{\varepsilon})]$ we can conclude from (28) that

$$\frac{\partial \pi_i(\varepsilon, S_i)}{\partial p} \geq 0 \text{ if } p(S_i) \leq p(S_i^X)$$

and

$$\frac{\partial \pi_i(\varepsilon, S_i)}{\partial p} \le 0 \text{ if } p(S_i) \ge p(S_i^X).$$

Hence, given $S_{-i}^X(p)$ and ε, the profit of firm *i* is pseudoconcave in the price range $[a,b]$ and the profit maximum is given by the first-order condition if prices are restricted to this range. The next step in the proof is to rule out profitable deviations outside this price range, where potential equilibrium strategies do not necessarily satisfy the first-order condition. To accomplish our proof we choose these strategies to be as supportive as possible of the equilibrium, i.e. they should discourage deviations. To make it impossible to increase the price above *b* we choose all equilibrium strategies to be perfectly elastic at this price. To rule out profitable deviations below the price *a*, we choose all supply functions of he otential equilibrium to be perfectly inelastic below this price. This assumption and concavity of the demand curve implies that $\frac{d\{D'(p) - S'^X_{-i}(p)\}}{dp} \le 0$ if $p \le a$.

Thus

$$\frac{\partial^2 \pi_i(p,\varepsilon)}{\partial p^2} = \underbrace{\left[D''(p) - S''^X_{-i}(p)\right]}_{\le 0}\underbrace{\left[p - C_i'\left(\varepsilon + D(p) - S_{-i}^X(p)\right)\right]}_{\ge 0} +$$

$$+\underbrace{\left[D'(p) - S'^X_{-i}(p)\right]}_{\le 0}\left[\underbrace{1 - \underbrace{C_i''\left(\varepsilon + D(p) - S_{-i}^X(p)\right)}_{\ge 0}\underbrace{\left[D'(p) - S'^X_{-i}(p)\right]}_{\le 0}}_{\ge 0}\right] + \underbrace{\left[D'(p) - S'^X_{-i}(p)\right]}_{\le 0} \le 0 \;\forall p \in \left[C_i'(S_i), a\right].$$

Hence, given that competitors stick to their potential equilibrium strategies $S_{-i}^X(p)$, the profit function is concave in the range $\left[C_i'(S_i), a\right]$. Offering supply below marginal cost can never be profit maximizing. Thus we can conclude that $S_i^X(p)$ must be a best response to $S_{-i}^X(p)$. This is true for any firm and we can conclude that *X* is an equilibrium. □

Proposition 4. *As the number of steps approaches infinity* ($M \to \infty$), *any set of solutions to the discrete first-order condition that are non-decreasing everywhere in the region of possible realizations of the residual demand curve are part of a discrete SFE.*

Proof:

Consider a set of non-decreasing solutions for some price range, $p_n \ldots p_m$, to a system of discrete first-order conditions as in Proposition 1. Denote the solution by $\bar{\mathbf{s}} = \{\bar{\mathbf{s}}_1, \ldots, \bar{\mathbf{s}}_N\}$. In what follows it will be shown that an arbitrary chosen firm *i* has no incentives to unilaterally deviate from the supply schedule $\bar{\mathbf{s}}_i = \{\bar{s}_i^1, \ldots, \bar{s}_i^M\}$ given that competitors stick to $\bar{\mathbf{s}}_{-i} = \{\bar{s}_{-i}^1, \ldots, \bar{s}_{-i}^M\}$. Thus $\bar{\mathbf{s}} = \{\bar{\mathbf{s}}_1, \ldots, \bar{\mathbf{s}}_N\}$ constitutes a Nash equilibrium. Assume that in the potential equilibrium, the market price is in the range $p_n \ldots p_m$ for all realizations of the demand curve, where $1 \le n < m \le M$. Now, assume that competitors stick to $\bar{\mathbf{s}}_{-i} = \{\bar{s}_{-i}^1, \ldots, \bar{s}_{-i}^M\}$ and calculate the total differential of the expected profit of firm i for some supply schedule $\mathbf{s}_i$.

$$d\,\mathrm{E}(\pi_i(\mathbf{s}_i)) = \sum_{j=1}^{M} \frac{\partial\,\mathrm{E}(\pi_i(\mathbf{s}_i))}{\partial s_i^j} ds_i^j .$$

In the limit when the number of price levels approaches infinity, we get from (17)

$$\lim_{M\to\infty} d\,\mathrm{E}(\pi_i(\mathbf{s}_i)) = \sum_{j=1}^{M} \left\{ -\Delta p s_i^j + \left[p_{j+1} - C_i'\left(s_i^j\right)\right] \Delta \bar{\tau}_{-i}^{j+1} \right\} g\left(s_i^j + \bar{\tau}_{-i}^j\right) ds_i^j . \tag{29}$$

For the solution $\bar{\mathbf{s}}_i = \{\bar{s}_i^1, \ldots, \bar{s}_i^M\}$ we know from (18) that

$$\lim_{M\to\infty} -\Delta p \bar{s}_i^j + \left[p_{j+1} - C_i'\left(\bar{s}_i^j\right)\right] \Delta \bar{\tau}_{-i}^{j+1} = 0, \forall j \in n \ldots m.$$

As this expression is zero it can be subtracted without changing the result. Hence,

$$-\Delta p s_i^j + \left[p_{j+1} - C_i'\left(s_i^j\right)\right] \Delta \bar{\tau}_{-i}^{j+1} = \Delta p\left(\bar{s}_i^j - s_i^j\right) + \left[C_i'\left(\bar{s}_i^j\right) - C_i'\left(s_i^j\right)\right] \Delta \bar{\tau}_{-i}^{j+1} \;\; \forall j \in n \ldots m . \tag{30}$$

The cost function is increasing and convex by assumption. It now follows from (29) that for any supply schedule $\mathbf{s}_i$ that differs from $\bar{\mathbf{s}}_i$ at some price $j \in n \ldots m$, the expected profit can be increased by the following adjustment of the supply schedule: 1) Marginally increase the supply at each price level $j \in n \ldots m$, for which $\bar{s}_i^j > s_i^j$. Because for this case (30) implies that $-\Delta p s_i^j + \left[p_{j+1} - C_i'\left(s_i^j\right)\right] \Delta \bar{\tau}_{-i}^{j+1} \ge 0$. 2) Marginally decrease the supply at each price level $j \in n \ldots m$, for which $\bar{s}_i^j < s_i^j$. Because for this case (30) implies that $-\Delta p s_i^j + \left[p_{j+1} - C_i'\left(s_i^j\right)\right] \Delta \bar{\tau}_{-i}^{j+1} \le 0$. This analysis applies to any firm and it implies that there are no profitable unilateral deviations at the price levels $j \in n \ldots m$ from the equilibrium candidate $\bar{\mathbf{s}} = \{\bar{\mathbf{s}}_1, \ldots, \bar{\mathbf{s}}_N\}$.

The next step is to prove that there are no profitable unilateral deviations for the other price levels either. As these price levels are not realized in the potential equilibrium, we have some freedom to choose the producers strategies at these price levels. We choose the strategies to be supportive of the potential equilibrium, in order to ensure its existence. First assume that

$$\lim_{M\to\infty} -\Delta p s_i^j + \left[p_{j+1} - C_i'\left(s_i^j\right)\right] \Delta \bar{\tau}_{-i}^{j+1} \ge 0, \text{for all } j > m, \text{such that } g\left(s_i^j + \bar{\tau}_{-i}^j\right) > 0. \tag{31}$$

This is for example the case if all firms offer all of their (infinite) capacity at the price m+2, so that $\Delta\overline{\tau}_{-i}^{m+2}$ is infinite and, for every conceivable deviation, $g\left(s_i^j+\overline{\tau}_{-i}^j\right)$ is zero for *j>m+2*. The probability density $g\left(s_i^j+\overline{\tau}_{-i}^j\right)$ is positive for some *j>m* only if $\overline{s}_i^j > s_i^j$. It now follows from (29) and (31) that for any supply schedule $\mathbf{s}_i$ that differs from $\overline{\mathbf{s}}_i$ at some price $j>m$, such that $g\left(s_i^j+\overline{\tau}_{-i}^j\right)>0$, i.e. $\overline{s}_i^j > s_i^j$, the expected profit can be increased by marginally increasing s_i^j. This analysis applies to any firm and it accordingly implies that there are no profitable unilateral deviations at price levels *j>m* from the equilibrium candidate $\overline{\mathbf{s}}=\left\{\overline{\mathbf{s}}_1,\ldots,\overline{\mathbf{s}}_N\right\}$.

Next assume that

$$\lim_{M\to\infty} -\Delta p s_i^j+\left[p_{j+1}-C_i'\left(s_i^j\right)\right]\Delta\overline{\tau}_{-i}^{j+1}\le 0, \text{for all } j<n, \text{such that } g\left(s_i^j+\overline{\tau}_{-i}^j\right)>0. \qquad (32)$$

Bearing in mind that $\Delta d^j > \Delta d^{j+1}$, this would for example be the case if each firm *k,* in the potential equilibrium, offers the quantity $\overline{s}_k^n$ at every price *j≤n,* so that $\Delta\overline{\tau}_{-i}^{j+1}=-\Delta d^{j+1}$ for *j<n.* The probability density $g\left(s_i^j+\overline{\tau}_{-i}^j\right)$ is positive for some *j<n* only if $\overline{s}_i^j < s_i^j$. It now follows from (29) and (32) that for any supply schedule $\mathbf{s}_i$ that differs from $\overline{\mathbf{s}}_i$ at some price $j<n$, such that $g\left(s_i^j+\overline{\tau}_{-i}^j\right)>0$, i.e. $\overline{s}_i^j < s_i^j$, the expected profit can be increased by marginally decreasing s_i^j. This analysis applies to any firm and it accordingly implies that there are no profitable unilateral deviations at price levels $j<n$ from the equilibrium candidate $\overline{\mathbf{s}}=\left\{\overline{\mathbf{s}}_1,\ldots,\overline{\mathbf{s}}_N\right\}$. We have now shown this result for every price level. Accordingly, we can conclude that $\overline{\mathbf{s}}_i=\left\{\overline{s}_i^1,\ldots,\overline{s}_i^M\right\}$ globally maximizes the expected profit of firm *i* and that $\overline{\mathbf{s}}=\left\{\overline{\mathbf{s}}_1,\ldots,\overline{\mathbf{s}}_N\right\}$ constitutes a Nash equilibrium. □

Proposition 5. *The difference equation* $p_{j+1}s_i^{j+1}+3\left(p_j-p_{j+1}\right)s_i^j-p_js_i^{j-1}=0$ *has a unique and non-decreasing solution if the boundary conditions are* $s_i^0=0$ and $s_i^M=k_i$.

Proof: First we show that with regard to uniqueness and monotonicity, the boundary conditions $s_i^0=0$ and $s_i^M=k_i$ are equivalent to the initial conditions $s_i^0=0$ and $s_i^1=1$. An easy forward induction argument, using the positivity of p_{j+1} and $p_{j+1}-p_j$, and the initial conditions, gives a unique and positive sequence $\hat{s}_i^2,\ldots,\hat{s}_i^M$. As the difference equation is linear in s_i, a (unique) solution of the original problem is thus constructed by scaling each of $\hat{s}_i^2,\ldots,\hat{s}_i^M$ by the positive quantity $k_i/\hat{s}_i^M$. This also implies that monotonicity of the latter (boundary value) solution is equivalent to monotonicity of the former (initial value) solution.

We now show that the solution of the initial value problem is non-decreasing, by induction. The induction hypothesis is

$$\frac{p_{j+1}-3(p_{j+1}-p_j)}{p_j}s_i^j \le s_i^{j-1} \le s_i^j \;.$$

The hypothesis clearly holds for j = 1, because $s_i^0 = 0$ and $p_1 = 0$. Assume it holds for some $j \geq 1$; we will show it holds for j+1. First, by using (10) to write

$$s_i^{j+1} = \frac{3(p_{j+1}-p_j)s_i^j + p_j s_i^{j-1}}{p_{j+1}},$$

it can immediately be verified that $s_i^j \le s_i^{j+1}$, which is one of the two induction conditions. To show the other let $\Delta p = p_{j+1} - p_j$, which is independent of j, and observe that

$$\begin{aligned}\frac{p_{j+2}-3(p_{j+2}-p_{j+1})}{p_{j+1}}s_i^{j+1} &= \frac{p_{j+1}-2\Delta p}{p_{j+1}}\frac{3(p_{j+1}-p_j)s_i^j + p_j s_i^{j-1}}{p_{j+1}} \\ &\le \frac{p_{j+1}-2\Delta p}{p_{j+1}}\frac{3(p_{j+1}-p_j)+p_j}{p_{j+1}}s_i^j \\ &= \frac{p_{j+1}-2\Delta p}{p_{j+1}}\frac{p_{j+1}+2\Delta p}{p_{j+1}}s_i^j \\ &= \left(1-2\frac{\Delta p}{p_{j+1}}\right)\left(1+2\frac{\Delta p}{p_{j+1}}\right)s_i^j = \left(1-4\frac{\Delta p^2}{p_{j+1}^{\;2}}\right)s_i^j \le s_i^j\end{aligned}$$

as required, where the first inequality uses the bound $s_i^{j-1} \le s_i^j$.

Finally, we want to show that the solution is a NE. By evaluating the integrals in Proposition **1** and differentiating the result with respect to s_i^j it is straightforward to show that

$$\frac{\partial^2 \,\mathrm{E}(\pi_i(\mathbf{s}))}{\partial s_i^{j\,2}} = \frac{-\Delta p}{2} < 0$$

and that

$$\frac{\partial^2 \,\mathrm{E}(\pi_i(\mathbf{s}))}{\partial s_i^j \partial s_i^k} = 0 \;\text{ if } j \neq k. \tag{33}$$

This ensures that the first-order condition gives a best response and that the symmetric solution implied by (8) is a symmetric NE. □