Effect of Transmission Constraints on a Nodal Price Electricity Market

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Abstract

The allocation of transmission rights is crucial determining equilibrium outcome when transmission line is congested (Joskow and Tirole, 2000). Moreover, as Fabra et al. (2006) have shown, the type of auction implemented modifies equilibrium outcome in electricity markets. In this paper, I characterize the equilibrium in an electricity market for different transmission rights allocations when uniform and discriminatory auctions are implemented. When financial transmission rights are assigned to the grid operator, equilibrium price is lower than when transmission rights are assigned to the firm that submits the lower bid in the spot electricity market. Moreover, when the transmission line is congested, uniform and discriminatory price auction perform equally with independence of the transmission rights allocation rule implemented. Hence, in a nodal electricity market, the allocation rule of transmission rights, instead of the type of auction implemented, determines equilibrium outcome.

KEYWORDS: electricity auctions, transmission constraint, market design.

1 Introduction

When electricity transmission lines are congested, equilibrium prices differ across markets. The increase in prices and the difference in equilibrium prices across markets due to transmission constraint can be substantially large. Transmission line congestion increases equilibrium prices around 5% (Offer 1995; Bohn et al., 1999). Financial transmission rights provide its owner the possibility to buy electricity in the low market price and sell it in the high market price. Joskow and Tirole (2000) shown that the type of transmission right implemented in an electricity market determines crucially equilibrium outcome.

When firms are asymmetric in generation cost and transmission line is not congested,[1] uniform price auction generates higher equilibrium prices but the most efficient firm is

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[1] The term "congested" is used throughout this article in the electrical engineering sense: a line is congested when the flow of power is equal to the line's capacity, as determined by engineering standards.

dispatched first in the auction with certainty. By contrast, the discriminatory price auction generates lower equilibrium prices but the most efficient firm is dispatched last in the auction with some positive probability Fabra et al. (2006). However, as I have exposed in the first chapter of my thesis, when the transmission line is congested and there are losses during the transmission process, a new criteria to evaluate the performance of auctions need to be introduced. In particular, the auction that prioritizes in the dispatch the firm located in the high demand market performs better because less electricity flow through the grid (and so less losses). Under this new criteria, when the firms are symmetric in generation cost and capacity, the transmission line is congested and the electricity market is a single price market,[2] discriminatory price auction performs better because it generates lower equilibrium price and the firm located in the high demand market is dispatched first with positive probability, therefore, less electricity flows through the grid and less losses. However, in this paper, I show that in a nodal electricity market, uniform and discriminatory price auction perform equally with independence of the transmission rights allocation rule implemented. Hence, in a nodal electricity market, the allocation rule of transmission rights, instead of the type of auction implemented, determines equilibrium outcome.

I this paper, I characterize the equilibrium in a nodal price electricity market for uniform and discriminatory auctions when different transmission rights allocations are implemented.

As in the first chapter of my thesis, my analysis proceeds by first extending a simple duopoly model similar to the one in Fabra et al. (2006), which is then varied in several directions. In the basic set up, two suppliers with symmetric capacities and (marginal) costs, are allocated in two different markets (North and South) connected by a transmission line. The two firms face a demand in each market that is assumed to be perfectly inelastic and known with certainty when suppliers submit their offer prices. Each supplier must submits a single price offer for its entire capacity. The assumption of price-inelastic demand can be justified by the fact that the vast majority of consumers purchase electricity under regulated tariffs that are independent of the prices negotiated in the wholesale market, at least in the short run.

The assumption that suppliers have perfect information concerning market demand is reasonable when applied to markets in which offers are "short lived", such as in Spain, where there are 24 hourly day-ahead markets each day. In such markets suppliers can be assumed to know the total demand they face in any period with a high degree of certainty. In markets in which offer prices remain fixed for longer periods, e.g., a whole day, like in Australia and in the former markets in England and Wales, on the other hand, it is more accurate to assume that suppliers face some degree of demand uncertainty, or volatility, at the time they submit their offers.

In addition, the transmission rights are assigned according to two allocation rules. First, transmission rights are assigned to the grid operator (e.g., NordPool or EMCC). Second, transmission rights are assigned to the firm that submits the lower bid in the spot

[2] A single price electricity market is the one in which the equilibrium price is the same for the whole market even when the transmission line is congested. By contrast, a nodal price electricity market is the one in which the equilibrium price differs across markets when the transmission line is congested.

electricity market. I have introduced this transmission right allocation, because under it, firms in spot electricity markets compete not only for electricity demand, but also for financial transmission rights. I would like to analyze if the introduction of competition for transmission rights exacerbate competition in the spot electricity market, or by the contrary, the firms behave less aggressively.

I have found that when the transmission line is congested and financial transmission rights are assigned to the grid operator, the equilibrium price is lower than when the transmission rights are assigned to the firm that submits the lower bid in the spot electricity auction. When transmission line is congested, the equilibrium is in mixed strategies for both allocation rules. However, when the financial transmission rights are assigned to the firm that submits the lower bid in the spot electricity auction, the firm located in the low demand market faces lower demand in its own market and so after satisfies the demand in its own market, it has a large residual capacity to be sold in the high demand market, therefore, in the equilibrium it has incentives to submits lower bids. The firm located in the high demand market anticipates that in equilibrium it will be dispatched last the majority of the times and so randomizes assigning a large probability to the maximum bid allowed by the auctioneer. The overall effect is an increase on equilibrium price.

When the transmission line is congested, uniform and discriminatory price auctions perform equally in equilibrium. When transmission rights are assigned to the grid owner, it is the grid owner, instead of the firm that submits the lower bid in the auction, the one that captures congestion rents. This allocation rules closes the gap between profits in uniform and discriminatory auction, transforming the first one into the second one.[3] By contrast, when transmission rights are assigned to the firm that submits the lower bid in the spot electricity auction, it is the firm that submits the the lower bid the one that captures congestion rents. This allocation rule closes the gap between profits in uniform and discriminatory auction, transforming the second one into the first one. Consequently, when the transmission line is congested, it is transmission rights allocation rule, instead of the type of auction, the one that determines equilibrium outcome.

Pioneering research on electricity markets in which transmission lines are congested was done by Schweppe et al. (1988). They concluded that the short-term price of transmission services between any two locations is the difference of spot prices between those two points. Hogan (1992) introduces the concept of contract network which provides a mechanism for allocating long-term transmission capacity rights subject to maintaining short-run price efficiency. Chao and Peck (1996) use the physical rights approach to incorporate network externality impacts into the competitive trading mechanism. When competition in the spot electricity market is perfect, the mechanisms proposed by Hogan (1992) and Chao and Peck (1996) generates the efficient equilibrium predicted by Schweppe et al. (1988). However, as Joskow and Tirole (2000) have shown, when competition in the spot electricity market is imperfect, the way in which transmission rights are assigned modifies the equilibrium outcome on the spot electricity market.

Joskow and Tirole (2000) assume in their analysis that the equilibrium price in one of the markets is a parameter. Under this assumption, they work out the equilibrium in the

[3] I will explain in detail within each section the payoff function for each type of auction and its equivalence.

other market using two types of transmission rights, financial and physical. Borenstein et al. (2000) work out the equilibrium when the firms compete in quantities and financial transmission rights are assigned to the grid operator.

The previous models work out the equilibrium assuming congestion in the transmission line and nodal pricing. By contrast, Fabra et al. (2006) characterize the equilibrium in a single price electricity market when uniform and discriminatory auctions are run by the auctioneer and the firms are asymmetry in capital and generation costs and the transmission line is not congested. In this paper, I characterize the equilibrium in a nodal price electricity market for uniform and discriminatory auction when firms are symmetric and different transmission rights allocations are implemented.

The article proceeds as follows, in section two I describe the model. In section three, I characterize the equilibrium when the transmission rights are assigned to the grid operator. In section four, I characterize the equilibrium when the transmission rights are assigned to the firm that submits the lower bid in the spot electricity market. Section five concludes. Proofs are in the Appendix.

2 The model

Set up of the model. There exist two electricity markets, market North and market South, that are connected by a transmission line with capacity T.

There exist two duopolists with capacities k_n and k_s, where subscript n means that the firm is located in market North and subscript s means that the duopolist is located in market South. Suppliers' marginal costs of production are c_n and c_s. In this paper I analyze the effect that asymmetries in the access to demand has on equilibrium. In order to focus in this effect, I will assume that firms are symmetric in capital $k_n = k_s = k > 0$ and symmetric in costs $c_n = c_s = c = 0$. The level of demand in any period, θ_n in region North and θ_s in region South, is a random variable uniformly distributed that is independent across markets[4] and independent of the market price, i.e., perfectly inelastic. In particular, $\theta_i \in [\underline{\theta}_i, \overline{\theta}_i] \subseteq [0, k+T]$ is distributed according to some known distribution function $G(\theta_i)$, $i = n, s, i \neq j$

The capacity of the transmission line is lower than the installed capacity in each market $T \leq k$, i.e. the transmission line could be congested for some realization of demands (θ_s, θ_n). The transmission line is congested when the firm that is dispatched first in the auction, after satisfy the demand in its own market, can not sell its remain residual generation capacity in the other market because of the transmission constraint.

Timing of the game. Having observed the realization of demands $\theta \equiv (\theta_s, \theta_n)$, each supplier simultaneously and independently submits a bid specifying the minimum price at which it is willing to supply up to its capacity, $b_i \leq P$, $i = n, s$, where P denotes

[4]In the majority of electricity markets, demand in one market is higher than in the other market. Moreover, there exists the possibility of some type of correlation between demands across markets. In this paper, I assume uniform distribution and independence of demand. However, the model can be easily modified to introduce different distributions of demand and correlation between demands across markets.

the "market reserve price", possibly determined by regulation.[5] Let $b \equiv (b_s, b_n)$ denote a bid profile. On the basis of this profile, the auctioneer calls suppliers into operation. If suppliers submit different bids, the lower-bidding supplier's capacity is dispatched first. Without loss of generality, assume that $b_n < b_s$. If the capacity of supplier n is not sufficient to satisfy the total demand $(\theta_s + \theta_n)$ in the case of the transmission line not congested, or $(\theta_n + T)$ in the case of the transmission line congested,[6] the higher-bidding supplier's capacity, firm s, is then dispatched to serve residual demand, $(\theta_s + \theta_n - k)$ if $(\theta_s > k - \theta_n$ and $\theta_n \in [k - T, k])$, or $(\theta_s - T)$ if $(\theta_s > T$ and $\theta_n \in [0, k - T])$. If the two suppliers submit equal bids, then supplier i is ranked first with probability ρ_i, where $\rho_n + \rho_s = 1$, $\rho_i = 1$ if $\theta_i > \theta_j$, and $\rho_i = \frac{1}{2}$ if $\theta_i = \theta_j$, $i = n, s$, $i \neq j$. The tie breaking rule implemented is such that if the bids of both firms are equal and the demand in market i is greater than the demand in market j, the auctioneer dispatches first the supplier located in market i.

The output allocated to supplier $i, i = n, s$, denoted by $q_i(\theta, b)$, is given by

$$q_i(b; \theta, T) = \begin{cases} min\{\theta_i + \theta_j, \theta_i + T, k_i\} & \text{if } b_i < b_j \\ \rho_i min\{\theta_i + \theta_j, \theta_i + T, k_i\} + & \\ \qquad [1 - \rho_i] max\{0, \theta_i - T, \theta_i + \theta_j - k_j\} & \text{if } b_i = b_j \\ max\{0, \theta_i - T, \theta_i + \theta_j - k_j\} & \text{if } b_i > b_j \end{cases} \tag{1}$$

The output function has an important role determining the equilibrium, therefore I will explain it in greater detail. Below, I describe the construction of firm n's output function, the one for firm s is symmetric.

The total demand that can be satisfied by firm n when it submits the lower bid $(b_n < b_s)$ is defined by $min\{\theta_n + \theta_s, \theta_n + T, k\}$. The realization of (θ_s, θ_n) determines three different areas (left panel in figure 1).

$$min\{\theta_n + \theta_s, \theta_n + T, k\} = \begin{cases} \theta_s + \theta_n & \text{if } \theta_n \leq k - \theta_s \text{ and } \theta_s < T \\ \theta_n + T & \text{if } \theta_n < k - T \text{ and } \theta_s > T \\ k & \text{if } \theta_n > k - \theta_s;\ \theta_s \in [0, T] \\ & \text{or if } \theta_n > k - T;\ \theta_s \in [T, k + T] \end{cases}$$

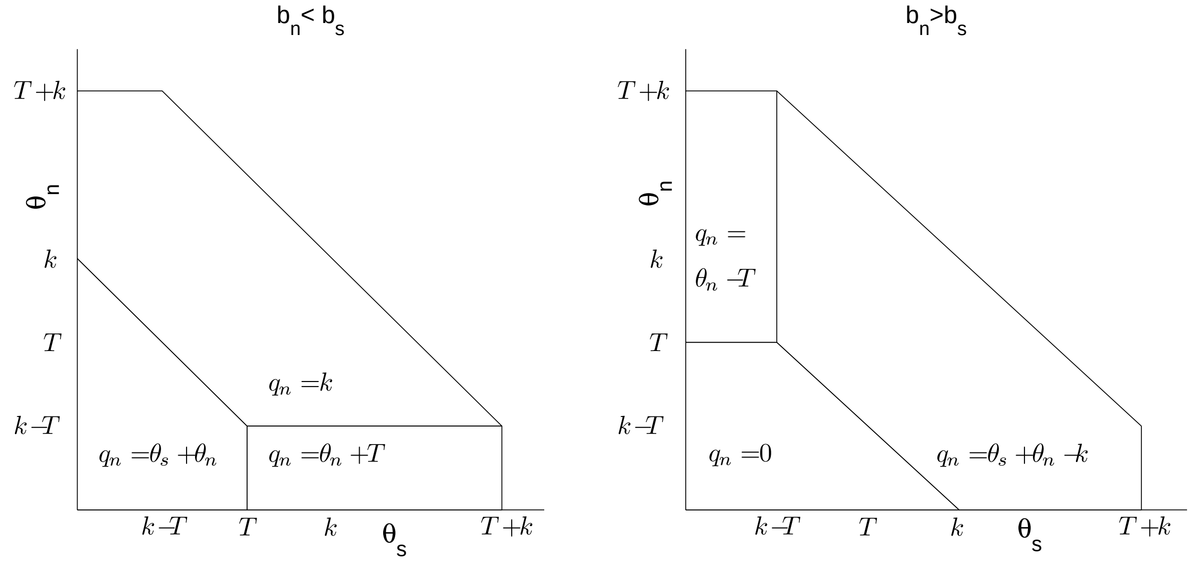
When demand in both markets is low, firm n can satisfy the total demand $(\theta_s + \theta_n)$. If the demand in market South is greater than the transmission capacity $\theta_s > T$, firm n cannot satisfy the demand in market South even when it has enough generation capacity to do so, therefore the total demand that firm n can satisfy is $(\theta_n + T)$. Finally, if the demand is big enough the total demand that firm n can satisfy is its own generation capacity.

The residual demand that firm n satisfies when it submits the higher bid $(b_n > b_s)$ is defined by $max\{0, \theta_n - T, \theta_s + \theta_n - k\}$. The realization of (θ_s, θ_n) determines three

[5] P can be interpreted as the price at which all consumers are indifferent between consuming and not consuming, or a price cap imposed by the regulatory authorities. See von der Fehr and Harbord (1993, 1998).

[6] When the demand in market South is larger than the transmission line capacity $\theta_s > T$, firm n can only satisfy the demand in its own region and T units of demand in market South $(\theta_n + T)$. Below in this section, I will explain with detail the total demand that can be satisfied by each firm and the residual demand that can be satisfied by each firm.

Figure 1: Output function for firm n. ($k_n = k_s = 60, T = 40$)



different cases (right panel in figure 1).

$$max\,\{0, \theta_n - T, \theta_s + \theta_n - k\} = \begin{cases} 0 & \text{if } \theta_n < T;\ \theta_s \in [0, k-T] \\ & \text{or } \theta_n < k - \theta_s;\ \theta_s \in [k-T, k] \\ \theta_n - T & \text{if } \theta_n > T \text{ and } \theta_s \in [0, k-T] \\ \theta_s + \theta_n - k & \text{if } \theta_n > k - \theta_s;\ \theta_s \in [k-T, T+k] \end{cases}$$

When demand in both markets is low, firm s satisfies the total demand, therefore the residual demand that remains to firm n is zero. When the total demand is large enough, firm s cannot satisfy the total demand and some residual demand ($\theta_s + \theta_n - k$) remains to firm n. Due to the transmission constraint, the total demand that firm s can satisfy diminishes. As soon as demand in market North is greater than the transmission capacity ($\theta_n > T$), firm s cannot satisfy it, therefore some residual demand ($\theta_n - T$) remains to firm n.

Finally, the payments are worked out by the auctioneer. Depending on the transmission rights allocation rule, the profit function varies. Therefore, I will specify the payoff function for each allocation rule in sections three and four.

3 Transmission rights assigned to the grid operator.

In this section, I characterize the equilibrium in a spot electricity market when the financial transmission rights are assigned to the grid operator. I also run different comparative statics analysis, focusing mainly on the effect that a reduction in transmission capacity has on equilibrium.

3.1 The model

The same that in section two.

The payments are worked out by the auctioneer. I will work out the payoff when the auctioneer runs a uniform price auction and a discriminatory price auction and the transmission rights are assigned to the grid owner.[7]

When the auctioneer runs a uniform price auction, the price received by a supplier for any positive quantity dispatched by the auctioneer is equal to the higher accepted bid in the auction in each market. Hence, for a given realization of $\theta \equiv (\theta_s, \theta_n)$ and a bid profile $b \equiv (b_s, b_n)$, supplier n's profits, $i = n, s$, can be expressed as

$$\pi_i^u(b;\theta,T) = \begin{cases} (b_j - c_i)k & \text{if } b_i < b_j \text{ and } \theta_i + \theta_j \geq k_i \text{ and } k - \theta_i \leq T \\ (b_i - c_i)q_i(b;\theta,T) & \text{otherwise} \end{cases} \tag{2}$$

As in the case of the production function, the payoff function has an important role determining the equilibrium, therefore I will explain it in greater detail. Below, I describe the construction of firm n's payoff function, the one for firm s is symmetric.

If $b_n < b_s$ and $\theta_n + \theta_s \geq k$ and $k - \theta_s \leq T$. Firm n submits the lower bid, has not enough capacity to satisfy the demand in both markets and after satisfy the demand in its own market, the remaining of its generation capacity is lower than the transmission line capacity (the transmission line is not congested). In such a case, firm s sets the price in the auction and the transmission line is not congested, therefore the equilibrium price is the same in both markets. Hence, the payoff function for firm n is equal to $\pi_n^u(b;\theta,T) = (b_s - c_n)k$. In the rest of the cases, the payoff for firm n is its own bid multiply by its dispatch, i.e., the payoff in case of discriminatory auction.

The previous paragraph explains firm n's payoff function. However, due to its economic relevance, I will explain in detail firm n's payoff function when the transmission line is congested. When $b_n < b_s$, $\theta_s \geq T$ and $k - \theta_n \leq T$.[8] Firm n is dispatched first and sets the price in market North, however due to the transmission constraint, it can not sell its remaining generation capacity $k - \theta_n$ in market South, therefore firm s sets the price in market South. Hence, firm n's satisfy the demand in market North at price b_n, therefore its payoff in market North is $b_n\theta_n$; after satisfy the demand in its own market, firm n serves demand in market South up to the transmission line capacity at price b_s, therefore its payoff in market South is b_sT; finally, given that transmission rights are assigned to the grid owner, firm n payback to the grid owner for the use of transmission line. Hence, firm n's payoff is $\pi_n^u = b_n\theta_n + b_sT - (b_s - b_n)T = b_n(\theta_n + T) = \pi_n^d$. Consequently, when the transmission line is congested uniform and discriminatory perform equally.[9] It is impor-

[7]The owner of the transmission rights can buy electricity in the low price market and sell it in the high price market. In this section of the paper, I assume that the transmission rights are assigned to the grid owner, as is the case in the majority of electricity markets in the world, e.g., NordPool, EMCC.

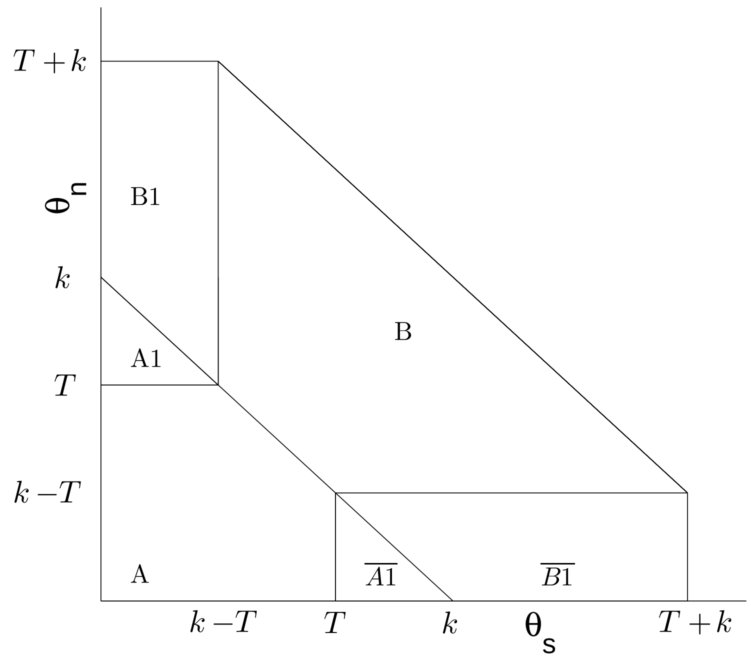
[8]If $k - \theta_n \leq T$, then $k - T \leq \theta_n$. Plug in this last equation into $\theta_n + \theta_s \geq k$. Then, the constraint defined in equation 2 reduces to $\theta_s \geq T$ and $k - \theta_n \leq T$.

[9]When the the electricity market is a single price market instead of a nodal price market, firm i's payoff function is defined by

$$\pi_i^u(b;\theta,T) = \begin{cases} (b_j - c_i)min\{\theta_i + T, k\} & \text{if } b_i < b_j \text{ and } \theta_i + \theta_j \geq k_i \text{ or } \theta_j \geq T \\ (b_i - c_i)q_i(b;\theta,T) & \text{otherwise} \end{cases} \tag{3}$$

In case of single price market and uniform auction. When the transmission line is congested firm n's

Figure 2: Payoff function and equilibrium areas.



tant to remark that when the transmission rights are assigned to the grid operator, the uniform price auction is "transformed" into a discriminatory price auction. By contrast, in the next section, I will show that when the transmission rights are assigned to the firm that submits the lower bid in the auction, it is the discriminatory auction the one that is "transformed" into a uniform price auction. These "transformations" will have important effects on market outcome.

When the auctioneer runs a discriminatory price auction, the price received by a supplier for any positive quantity dispatched by the auctioneer is equal to its own bid. Hence, for a given realization of $\theta \equiv (\theta_s, \theta_n)$ and a bid profile $b \equiv (b_s, b_n)$, supplier n's profits, $i = n, s$, can be expressed as

$$\pi_i^d(b; \theta, T) = (b_i - c_i) q_i(b; \theta, T) \quad (4)$$

To conclude this section, I explain firms' payoff function for both types of auctions. Equations 2 and 4 define firms' payoff function for both types of auctions. Depending of the realization of demand and agents' bids, these equations define different areas (figure 2). In particular, area A is the area in which the realization of demand in each market is lower than the transmission line capacity and both firms have enough generation capacity to satisfy the total demand in both markets, I named this area "low demand"; areas $A1$ and $B1$ (or its symmetric $\overline{A1}$ and $\overline{B1}$) are the areas in which the transmission line is congested when the firm located in the high demand market submits the higher bid, I named this area "intermediate"; finally area B is the area in which the realization of

payoff function is $\pi_n^u = b_s(\theta_n + T)$, which is different of firm n's payoff function in case of nodal price market and discriminatory auction $\pi_n^d = b_n(\theta_n + T)$.

Table 1: Nodal pricing. Uniform and discriminatory auction. Transmission Rights assigned to the grid operator. Payoff function

Area	$b_n < b_s$	$b_n > b_s$
Area A (low)	$\pi_n^u = b_n(\theta_n + \theta_s)$ $\pi_n^d = b_n(\theta_n + \theta_s)$	$\pi_n^u = b_n 0$ $\pi_n^d = b_n 0$
Area $\overline{A1}$ (intermediate)	$\pi_n^u = b_n\theta_n + b_sT - (b_n - b_s)T = b_n(\theta_n + T)$ $\pi_n^d = b_n(\theta_n + T)$	$\pi_n^u = b_n 0$ $\pi_n^d = b_n 0$
Area $\overline{B1}$ (intermediate)	$\pi_n^u = b_n\theta_n + b_sT - (b_n - b_s)T = b_n(\theta_n + T)$ $\pi_n^d = b_n(\theta_n + T)$	$\pi_n^u = b_n(\theta_s + \theta_n - k)$ $\pi_n^d = b_n(\theta_s + \theta_n - k)$
Area B (high)	$\pi_n^u = b_s k$ $\pi_n^d = b_n k$	$\pi_n^u = b_n(\theta_s + \theta_n - k)$ $\pi_n^d = b_n(\theta_s + \theta_n - k)$

demand in higher than the installed generation capacity of the firms and the transmission line is not congested, I named this area "high demand". Based on the realization of demand, table 1 defines firm n's payoff function. As can be observed in table 1 firm n's payoff function coincide for uniform and discriminatory set up in all areas except in area B.

3.2 Equilibrium analysis

Lemma 1. When the realization of demands (θ_s, θ_n) is low (Area A), the equilibrium is in pure strategies for both types of auctions. When the realization of demands (θ_s, θ_n) is high (Area B), a pure strategy equilibrium exists for the uniform price auction, but not for the discriminatory price auction. When the realization of demands (θ_s, θ_n) is intermediate (areas $A1$, $B1$, $\overline{A1}$, $\overline{B1}$) a pure strategy equilibrium does not exist nor for uniform, neither for discriminatory price auction (figure 2).

Proof. When the realization of demands (θ_s, θ_n) is low (area A), the two producers have enough capacity to satisfy the demand in both regions. Therefore, they compete fiercely to be dispatched first in the auction. Hence, the equilibrium is the typical Bertrand equilibrium in which both firms submit bids equal to their marginal cost.

When the realization of demands (θ_s, θ_n) is high (area B) and the auction is uniform. One of the firms (without loss of generality, I assume that is firm i) submits a bid equal to the maximum price allowed by the auctioneer and the other firm (firm j) submits a bid that made firm i be indifferent between submits the maximum bid allowed by the auctioneer and satisfy the residual demand or undercut firm j and satisfy the total demand. By the way in which the equilibrium have been constructed, firm i has no incentive to deviate. Firm j has no incentive to deviate because in the equilibrium, it is dispatched first and sells its generation capacity at the maximum price allowed by the auctioneer. Therefore, when the auction is uniform, a pure strategies equilibrium exists in which one of the firms submit the maximum price[10] allowed by the auctioneer.

[10]In pure strategies equilibrium, the firm that satisfies the residual demand always submits the maximum price allowed by the auctioneer. Otherwise, it can increases its expected payoff by increasing its bid.

When the realization of demands (θ_s, θ_n) is high (area B) and the auction is discriminatory, a pure equilibrium does not exist. First, an equilibrium such that $b_i = b_j = c$ does not exist because at least one firm has incentive to deviate and satisfy the residual demand. Second, an equilibrium such that $b_i = b_j > c$ does not exist because at least one firm has incentive to undercut the other to be dispatched first. Finally, an equilibrium such that $b_j > b_i > c$ does not exist because firm i has incentive to shade the bid submitted by firm j.

When the realization of demands (θ_s, θ_n) is intermediate (areas $A1$, $B1$, $\overline{A1}$, $\overline{B1}$) a pure equilibrium does not exist not for uniform price auction neither for discriminatory auction. As I have shown in the previous subsection, when demand is intermediate, uniform and discriminatory auction generate the same payoff function. Therefore, the same procedure applied in area B to show that a pure equilibrium does not exist can be applied in areas $A1$, $B1$, $\overline{A1}$, $\overline{B1}$ $\square$

When the auction is discriminatory and the demand is high or for any type of auction when the demand is intermediate, a pure strategies equilibrium does not exist. However, the model satisfies the properties[11] established by Dasgupta and Maskin (1986) that guarantees that a mixed strategy equilibrium exists.

Lemma 2. In a mixed strategy equilibrium none firm submits a bid lower than bid $(\underline{b}_i)$ such that $\underline{b}_i min\{\theta_i + \theta_j, \theta_i + T, k\} = Pmax\{0, \theta_i - T, \theta_i + \theta_j - k\}$. Moreover, the support for the mixed strategies equilibrium for both firms is $S = \left[max\left\{\underline{b}_i, \underline{b}_j\right\}, P\right]$.

Proof. Each firm can guarantees to itself the payoff $Pmax\{0, \theta_i - T, \theta_i + \theta_j - k\}$, because each firm always can submit the highest bid and satisfies the residual demand. Therefore, in a mixed strategies equilibrium, none firm submits a bid that generate a payoff equilibrium lower than $Pmax\{0, \theta_i - T, \theta_i + \theta_j - k\}$. Hence, none firm submits a bid lower than $\underline{b}_i$, where $\underline{b}_i$ solves $\underline{b}_i min\{\theta_i + \theta_j, \theta_i + T, k\} = Pmax\{0, \theta_i - T, \theta_i + \theta_j - k\}$.

None firm can rationalize submit a bid lower than $\underline{b}_i, i = n, s$. In the case that $\underline{b}_i = \underline{b}_j$, the mixed strategy equilibrium and the support is symmetric. In the case that $\underline{b}_i < \underline{b}_j$, firm i knows that firm j never submits a bid lower than $\underline{b}_j$. Therefore, in a mixed strategy equilibrium, firm i never submits a bid b_i such that $b_i \in \left(\underline{b}_i, \underline{b}_j\right)$, because firm i can increases its expected payoff choosing a bid b_i such that $b_i \in \left[\underline{b}_j, P\right]$. Hence, the equilibrium strategies support for both firms is $S = \left[max\left\{\underline{b}_i, \underline{b}_j\right\}, P\right]$ $\square$

Using Lemmas one and two, I establish the main result of this section.

Proposition 1. When the auction is uniform, the characterization of the equilibrium strategies fall into one of the next three categories (figure 2).

i Low demand (area A). The equilibrium strategies pair is in pure strategies.

ii Intermediate demand (areas $A1$, $B1$, $\overline{A1}$, $\overline{B1}$). The equilibrium strategies pair is in mixed strategies.

[11] In the Annex, Proposition 1. I proof that the model satisfies the properties established by Dasgupta and Maskin that guarantee that a mixed strategy equilibrium exists.

iii High demand (area B). The equilibrium strategies pair is in pure strategies.

When the auction is discriminatory, the characterization of the equilibrium fall into one of the next two categories (figure 2).

i Low demand (area A). The equilibrium strategies pair is in pure strategies.

ii Intermediate demand (areas $A1$, $B1$, $\overline{A1}$, $\overline{B1}$) or high demand (area B). The equilibrium strategies pair is in mixed strategies.

When the realization of demand is low or high, the transmission line is not congested. Therefore, the equilibrium price is the same in both markets and the results of chapter one of my thesis applies. In particular, when the realization of demand is low, the equilibrium is in pure strategies for both types of auctions and both firms submit a bid equal to its marginal cost, the equilibrium price is equal to zero and no electricity flows through the grid. When the realization of demand is high and the auction is uniform the equilibrium is in pure strategies and one of the firms submit the maximum bid allowed by the auctioneer. However, when the auction is discriminatory the equilibrium pair is in mixed strategies and the expected equilibrium price is lower than the maximum bid allowed by the auctioneer.

When the realization of demand is intermediate, the transmission line is congested and equilibrium price differ across markets. In such a case, uniform and discriminatory price auctions perform equally and the equilibrium is an asymmetric mixed strategy equilibrium in which the firm located in the high demand market assigns higher probability to the maximum bid allowed by the auctioneer. Therefore, the expected equilibrium price in the high demand market is higher than in the low demand market.

In the rest of this section, I analyze the effect that an increase in demand in each market and the effect that an increase in transmission capacity have on equilibrium. The analysis coincide with the one presented in chapter one of my thesis when the auction is discriminatory and the realization of demand is intermediate. However, I have decided to introduce those results here to present a complete analysis of the equilibrium.

Corollary 1. An increase in θ_n increases the lower bound of the support $\underline{b}$, the payoff and the expected bids for both firms. When the realization of demands (θ_s, θ_n) belongs to area $B1$, an increase in θ_n does not change the probability that the firm located in the high demand market assigns to the maximum bid allowed by the auctioneer. However when the realization of demands (θ_s, θ_n) belongs to area $A1$, an increase in θ_n modifies in a non-monotonic pattern the probability that the firm located in the high demand market assigns to the maximum bid allowed by the auctioneer (table 2 and figure 3).

When the realization of demands (θ_s, θ_n) belongs to areas $A1$ or $B1$, an increase in the demand in the high demand market increases the lower bound of the support. An increase of θ_n increases the residual demand and so the bid that made the firms be indifferent between satisfy the residual demand at the maximum price allowed by the auctioneer and satisfy the total demand at a bid equal to the lower bound of the support. Consequently the equilibrium expected price and the payoff increase.

Table 2: Numerical example: increase in θ_n ($\theta_s = 5, k = 60, T = 40, c = 0, P = 7$)

Area	θ_n	$\underline{b}$	$F_n(P)$	$\overline{\pi}_s$	$\overline{\pi}_n$	$E(b_s)$	$E(b_n)$
Area A1	41	0.15	0.746	6.84	7	0.59	2.22
	45	0.7	0.736	31.5	35	1.79	3.16
	50	1.27	0.736	57.2	70	2.65	3.79
	55	1.75	0.75	78.7	105	3.23	4.17
Border A1-B1	60	2.33	0.75	105	140	3.84	4.63
Area B1	61	2.45	0.75	110.25	147	3.95	4.71
	65	2.91	0.75	131.25	175	4.37	5.03
	70	3.5	0.75	157.5	210	4.85	5.38

The cumulative distribution function for firm n is discontinuous at P, $1 - F_n(P)$ represents the probability that firm n assigns to the maximum bid allowed by the auctioneer.

When the realization of demands (θ_s, θ_n) belongs to area $B1$, the probability that the firm located in the high demand market assigns to the maximum bid allowed by the auctioneer does not change. In area $B1$, the probability that the firm located in the high demand market assigns to the maximum bid allowed by the auctioneer represents the opportunity cost of submit high bids for the firm located in the low demand market. When θ_n increases and θ_s remain fix, the total demand that firm s can satisfy does not change and so the opportunity cost of submit high bids for the firm located in the low demand market does not change. However, in area $A1$, the probability that the firm located in the high demand market assigns to the maximum bid allowed by the auctioneer changes in a non-monotonic pattern. In area $A1$, an increase in θ_n modifies the total and the residual demand that firm n can satisfy. Therefore, an increase in θ_n modifies the opportunity cost of submit high bids for the firm located in the high demand market.

Corollary 2. When the realization of the demands (θ_s, θ_n) belongs to area $A1$, an increase in θ_s reduces the lower bound of the support $\underline{b}$; increases the probability that the firm located in the high demand market assigns to high bids; does not modify the payoff of the firm located in the high demand market, increases the payoff of the firm located in the low demand market and decreases the expected bid of both firms. When the realization of demands (θ_s, θ_n) belongs to area $B1$, an increase in θ_s does not modify the lower bound of the support $\underline{b}$, the probability that the firm located in the high demand market assign to high bids and the payoff of the firm located in the high demand market; increases the payoff of the firm located in the low demand market; decreases the expected bid of the firm located in the high demand market and does not modify the expected bid of the firm located in the low demand market (table 3 and figure 4).

When the realization of the demands (θ_s, θ_n) belongs to Area $A1$. An increase in θ_s increases the total demand that firm n can satisfy in case of be dispatched first in the auction and so decreases the lower bound of the support. The probability that the firm located in the high demand market assigns to the maximum bid allowed by the auctioneer represents the opportunity cost of submit high bids for firm s. An increase in θ_s increases the residual demand for firm s and so reduces the probability that firm n assigns to the maximum bid allowed by the auctioneer. An increase in θ_s increases the total demand

Figure 3: Effect of an increase in θ_n on the main variables. Area $A1, (\theta_n < 55)$ and area $B1, (\theta_n > 55)$

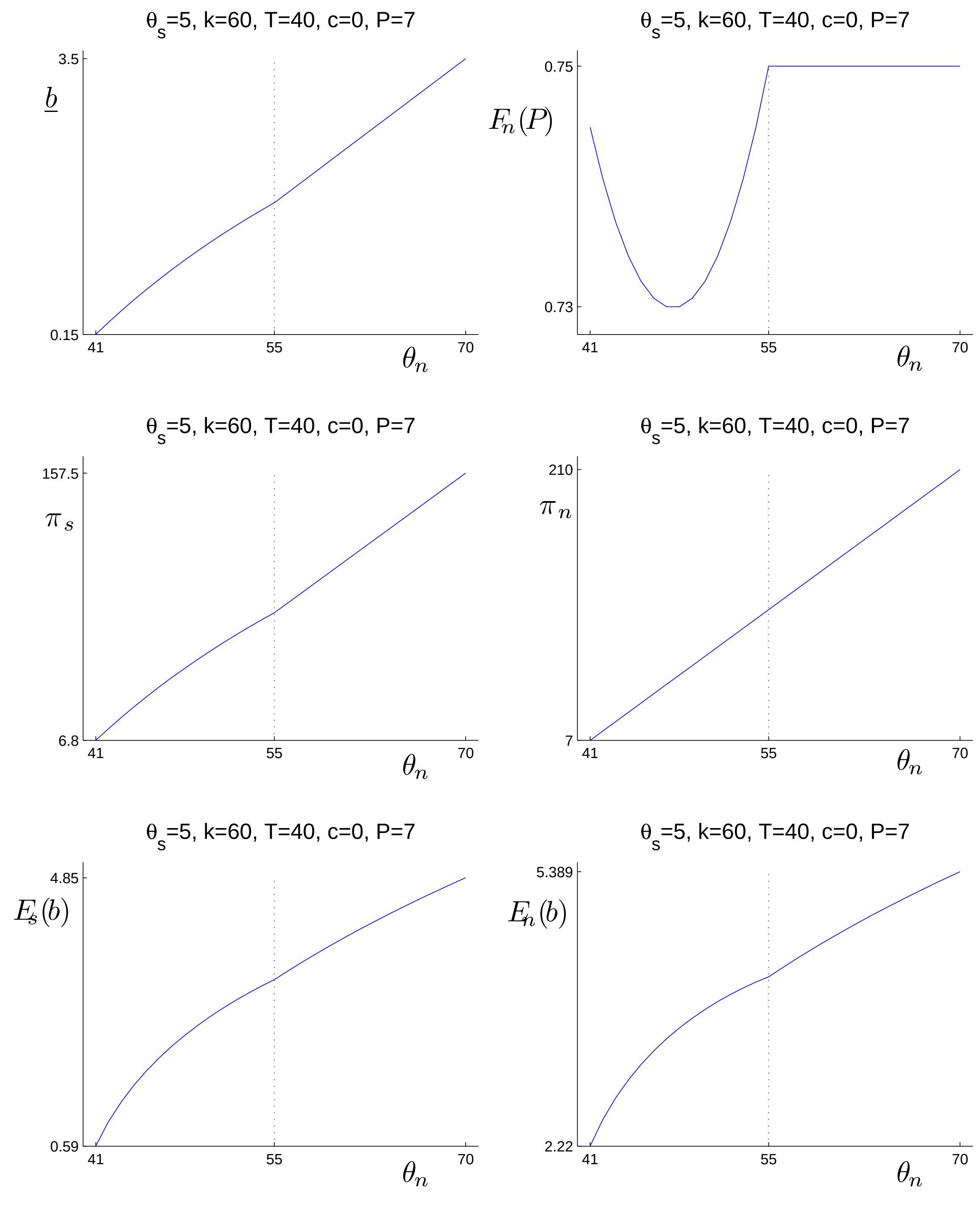


Table 3: Numerical example: increase in θ_s ($\theta_n = 45, k = 60, T = 40, c = 0, P = 7$)

Area	θ_s	$\underline{b}$	$F_n(P)$	$\overline{\pi}_s$	$\overline{\pi}_n$	$E(b_s)$	$E(b_n)$
Area A1	1	0.76	0.66	31.99	35	1.89	3.6
	5	0.7	0.73	31.5	35	1.79	3.16
	10	0.63	0.82	31.81	35	1.67	2.6
	14	0.59	0.89	32.03	35	1.59	2.14
Border A1-B1	15	0.58	0.91	32.08	35	1.58	2.03
Area B1	16	0.58	0.93	32.66	35	1.58	1.94
	20	0.58	1	35	35	1.58	1.58

The cumulative distribution function for firm n is discontinuous at P, $1 - F_n(P)$ represents the probability that firm n assigns to the maximum bid allowed by the auctioneer.

that both firms can satisfy. In the case of firm n, this increase is compensated with the decrease in expected bids due to the decrease in the lower bound of the support, therefore, its payoff does not change. By contrast, the increase in the demand for firm s dominates the decrease in expected bids and so its payoff increases. Finally, an increase in θ_s induces a decrease in expected bids for both firms because the lower bound of the support decreases and the probability that the firm located in the high demand market assigns to high bids decreases.

When the realization of the demands (θ_s, θ_n) belongs to Area $B1$. An increase in θ_s does not modify the total demand that firm n can satisfy in case of be dispatched first in the auction and so the lower bound of the support does not change. As in Area $A1$, an increase in θ_s increases the residual demand for firm s and so reduces the probability that firm n assign to the maximum bid allowed by the auctioneer. An increase in θ_s does not change the total demand that firm n can satisfy and so its payoff does not change, by contrast, an increase in θ_s increases the total demand that firm s can satisfy and so its payoff increases. Finally, an increase in θ_s reduces the expected bids for firm n because the lower bound of the support decreases and the probability that the firm located in the high demand market assigns to high bids decreases. By contrast the expected bid for firm s does not change because its cumulative distribution function and the lower bound of the support does not change.

Corollaries two and three describe the effect that a change in demand has on equilibrium outcome. In this paper, I have assumed uniform distribution of the demand. However, in real world the demand in one market is usually higher than in the other market. In such a case, analyze the effect that an increase in demand in each region has on equilibrium outcome is relevant from an economy policy perspective.

Proposition 2. An increase in T reduces the lower bound of the support $\underline{b}$ and reduces the probability that the firm located in the high demand market assigns to high bids. Moreover, an increase in T reduces the payoff of the firm located in the high demand market. However, an increase in T modifies in a non-monotonic pattern the payoff of the firm located in the low demand market. Finally, an increase in T reduces the expected bids for both firms.

Figure 4: Effect of an increase in θ_s on the main variables. Area $A1, (\theta_s < 15)$ and area $B1, (\theta_s > 15)$

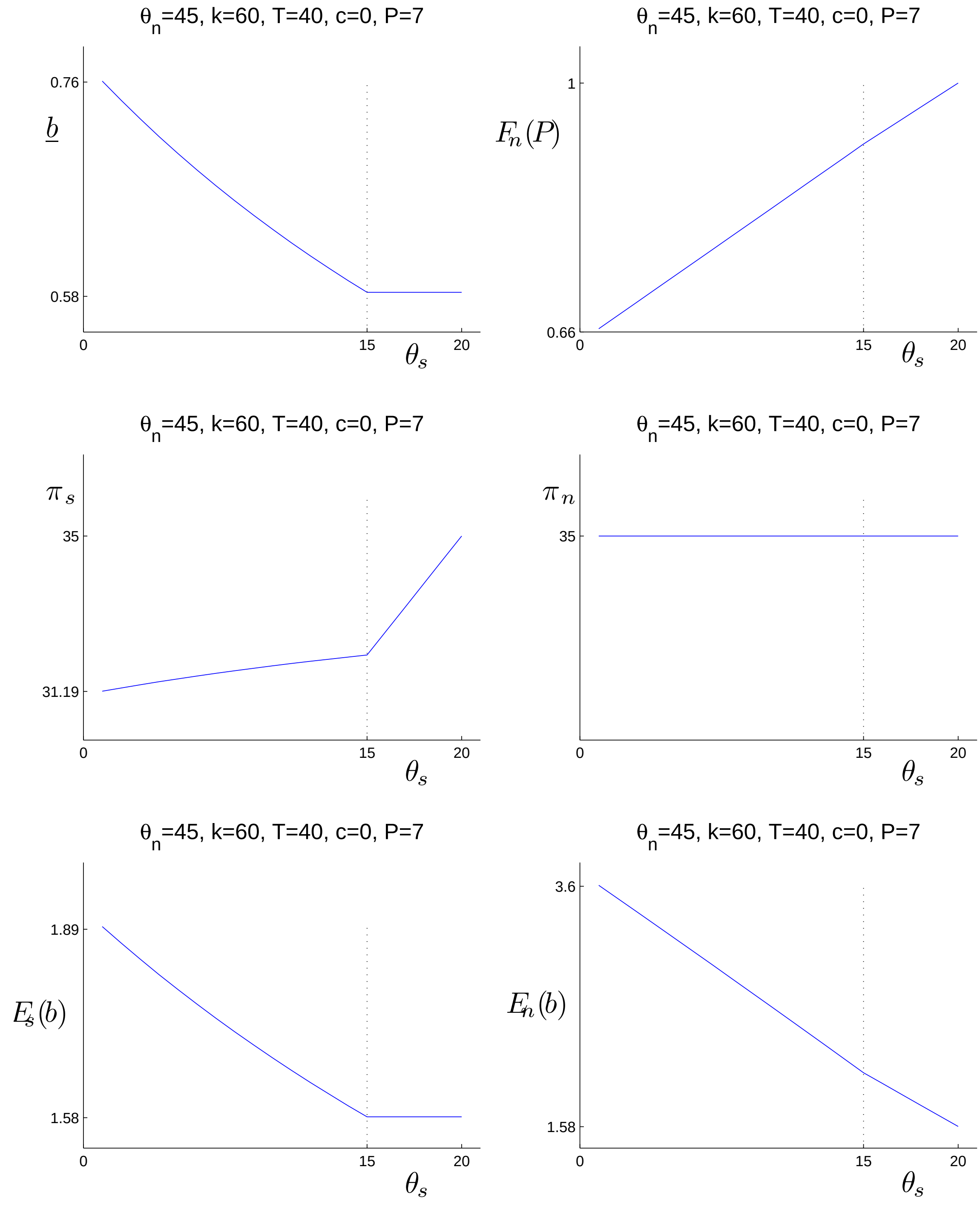


Table 4: Numerical example: increase in T ($\theta_n = 55, \theta_s = 5, k = 60, c = 0, P = 7$)

T	$\underline{b}$	$F_n(P)$	$\overline{\pi}_s$	$\overline{\pi}_n$	$E(b_s)$	$E(b_n)$
0	7	0	35	385	7	7
10	5.25	0.25	78.75	315	6.04	6.76
20	4.08	0.41	102.08	245	5.28	6.28
30	2.91	0.58	102.08	175	4.37	5.47
40	1.75	0.75	78.75	105	3.23	4.17
50	0.58	0.91	32.083	35	1.58	2.03
55	0	1	0	0	0	0

The cumulative distribution function for firm n is discontinuous at P, $1 - F_n(P)$ represents the probability that firm n assigns to the maximum bid allowed by the auctioneer.

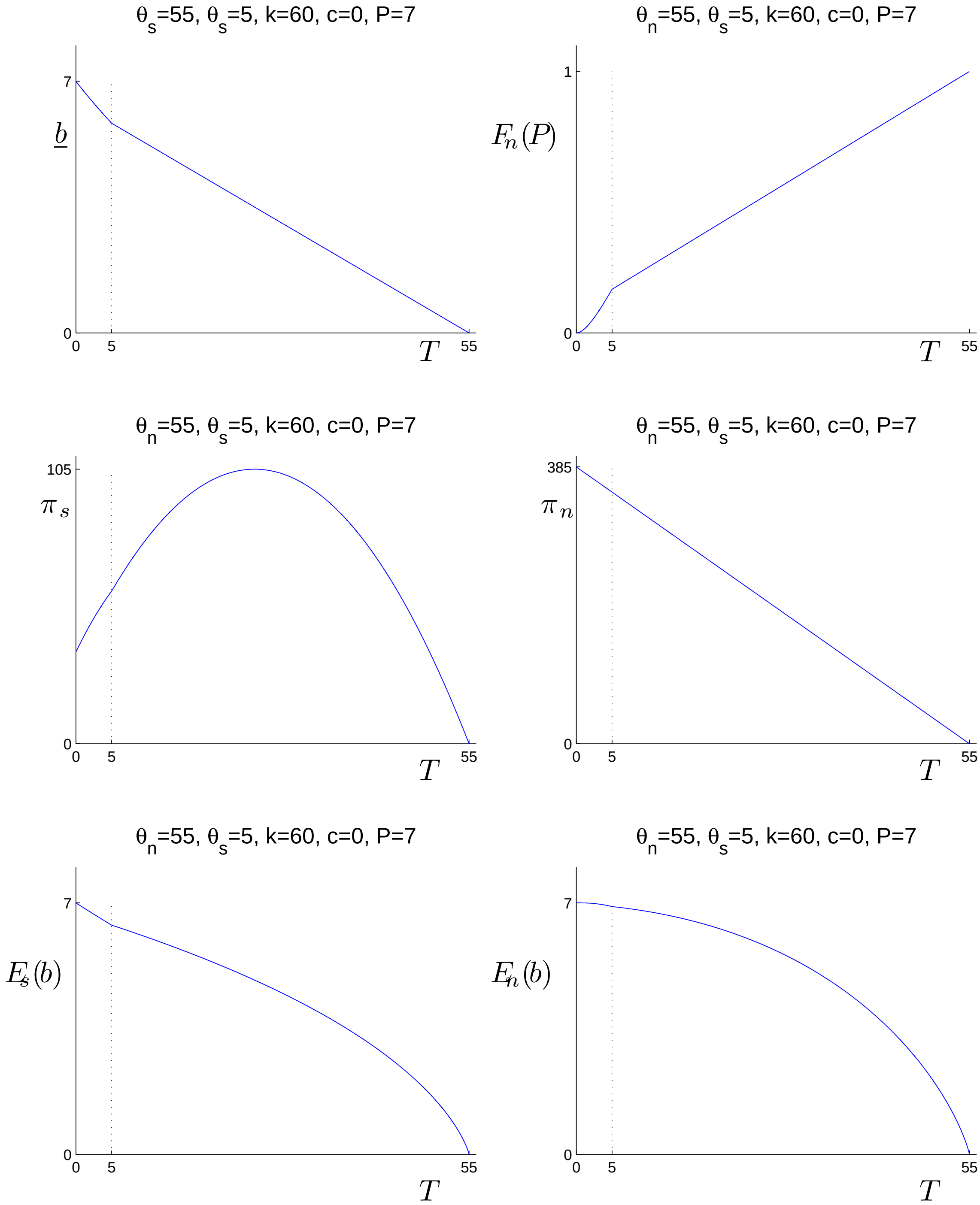
An increase in T reduces the residual demand that each firm faces in case of be dispatched last in the auction and so reduces the lower bound of the support. Moreover, an increase in T increases the total demand that each firm can satisfy in case of be dispatched first in the auction and so reduces the probability that the firm located in the high demand region assigns to high bids. Consequently, the expected bid for both firms and the payoff of the firm located in the high demand market decreases. By contrast, the payoff of the firm located in the low demand market changes in a non-monotonic pattern. When the transmission capacity is low, an increase in transmission capacity increases the total demand that the firm located in the low demand market faces. The increase in demand effect dominates the decrease in expected bid effect and the payoff increases. However, when the transmission capacity is big enough the decrease in bids effect dominates and the payoff decreases.

The literature that analyzes the effect that an increase in transmission capacity has on equilibrium outcomes usually focus in the effect on competition between regions. However, as I have shown in proposition two, an increase in transmission capacity modifies the payoff of the firm located in the low demand market. This could have important implications in generation capacity investment decisions. To motivate the argument, I introduce the next example: imagine that small hydro-power plant that faces a fix entry cost would like to install some generation capacity in the low demand market. When there is no transmission capacity between markets, due to the reduced size of the market, the firm can not cover its fix entry cost. However, if the transmission line increases, the size of the market increases and the firm could enter in the low demand market. This entry could increase the competition within the low demand market.

4 Transmission rights assigned to the firm that submits the lower bid in the spot electricity market

When the financial transmission rights are assigned to the firm that submits the lower bid in the spot electricity auction, firms compete not only for electricity demand, but also for financial transmission rights. In this section, I characterize the equilibrium when the financial transmission rights are assigned to the firm that submits the lower bid in the spot electricity market and I analyze if the introduction of competition for transmission

Figure 5: Effect of an increase in T on the main variables. Border $A1 - B1$



rights exacerbates competition in the spot electricity market, or by the contrary, firms behave less aggressively.

4.1 The model

The same that in section two.

The payments are worked out by the auctioneer. I work out the payoff when the auctioneer runs a uniform price auction and a discriminatory price auction and the transmission rights are assigned to the firm that submits the lower bid in the auction.

When the auctioneer runs a uniform price auction, the price received by a supplier for any positive quantity dispatched by the auctioneer is equal to the higher accepted bid in the auction in each market. Hence, for a given realization of $\theta \equiv (\theta_s, \theta_n)$ and a bid profile $b \equiv (b_s, b_n)$, supplier n's profits, $i = n, s$, can be expressed as

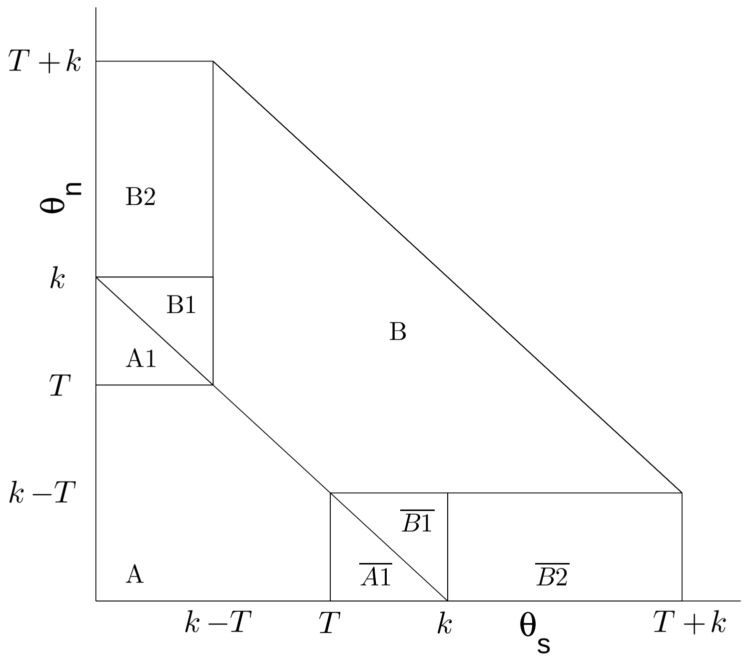
$$\pi_i^u(b;\theta,T) = \begin{cases} (b_i - c_i)(\theta_i + \theta_j) & \text{if } b_i < b_j \text{ and } \theta_i + \theta_j \leq k_i \text{ and } \theta_j \leq T \\ (b_i - c_i)min\{\theta_i, k_i\} + & \\ (b_j - c_i)max\{0, min\{T, k_i - \theta_i\}\} & \text{if } b_i < b_j \text{ and } k - \theta_i < T \text{ and } \theta_j > T \\ (b_j - c_i)k & \text{if } b_i < b_j \text{ and } \theta_i + \theta_j \geq k \text{ and } k - \theta_i \leq T \\ (b_i - c_i)max\{0, \theta_i - T, \theta_i + \theta_j - k_j\} & \text{if } b_i > b_j \end{cases} \quad (5)$$

As in the previous section, the payoff function has an important role determining the equilibrium, therefore I will explain it in greater detail. Below, I describe the construction of firm n's payoff function, the one for firm s is symmetric.

Equation 5 defines four possible cases. First, when $b_n < b_s$ and the realization of demand is low, firm n has enough capacity to satisfy the demand ($\theta_n + \theta_s < k$) and the transmission line is not congested ($\theta_s < T$), then firm n is dispatched first, sets the price and satisfies the demand in both markets. Second, when $b_n < b_s$ and the demand is intermediate (the transmission line is congested), firm n satisfies demand in market North, sets the price in market North and sells the rest of its generation capacity in market South at the price set in market South by firm s (financial transmission rights are assigned to firm n). Third, when $b_n < b_s$ and the demand is high (the transmission line is not congested), firm s is dispatched last in the auction, sets the price in both markets (the equilibrium price is unique in both markets because the transmission line is not congested), in such a case firm n's payoff is $b_s k$. Finally, when firm n submits the higher bid in the auction, its payoff is its own bid multiply by its residual demand.

When the auctioneer runs a discriminatory price auction, the price received by a supplier for any positive quantity dispatched by the auctioneer is equal to its own bid. In addition, when the transmission line is congested, the transmission rights are assigned to the firm that submits the lower bid in the auction. Hence, for a given realization of $\theta \equiv (\theta_s, \theta_n)$ and a bid profile $b \equiv (b_s, b_n)$, supplier n's profits, $i = n, s$, can be expressed as

Figure 6: Payoff function and equilibrium areas.



$$\pi_i^d(b;\theta,T) = \begin{cases} (b_i-c_i)(\theta_i+\theta_j) & \text{if } b_i<b_j \text{ and } \theta_i+\theta_j\le k_i \text{ and } \theta_j \le T \\ (b_i-c_i)min\{\theta_i,k_i\}+ & \\ (b_i-c_i)max\{0,min\{T,k_i-\theta_i\}\}+ & \\ (b_j-b_i)max\{0,min\{T,k_i-\theta_i\}\} & \text{if } b_i<b_j \text{ and } k-\theta_i<T \text{ and } \theta_j>T \\ (b_i-c_i)k & \text{if } b_i<b_j \text{ and } \theta_i+\theta_j\ge k \text{ and } k-\theta_i\le T \\ (b_i-c_i)max\{0,\theta_i-T,\theta_i+\theta_j-k_j\} & \text{if } b_i>b_j \end{cases} \quad (6)$$

Equation 6 defines four possible cases. First, when $b_n < b_s$ and the realization of demand is low, firm n has enough capacity to satisfy the demand ($\theta_n+\theta_s<k$) and the transmission line is not congested ($\theta_s<T$), then firm n is dispatched first, sets the price and satisfies the demand in both markets. Second, when $b_n<b_s$ and the demand is intermediate (the transmission line is congested), firm n satisfies demand in market North and sells the rest of its generation capacity in market South at a price equal to its own bid, in addition it receives the financial transmission rights, hence firm n satisfies the demand in its own market at the price set by itself and sells the rest of its generation capacity in market South at the price set by firm s. Third, when $b_n<b_s$ and the demand is high (the transmission line is not congested), the equilibrium price is unique in both markets, in such a case firm n's payoff is its own bid multiply by the capacity that it sells. Finally, when firm n submits the higher bid in the auction, its payoff is its own bid multiply by its residual demand.

To conclude this section, I explain firms' payoff function for both types of auctions. Equations 5 and 6 define firms' payoff function for both types of auctions. Depending of

Table 5: Nodal Pricing. Uniform and discriminatory auction. Transmission Rights assigned to the firm that submits the lower bid in the spot electricity market. Payoff function

Area	$b_n < b_s$	$b_n > b_s$
Area A (low)	$\pi_n^u = b_n(\theta_n + \theta_s)$ $\pi_n^d = b_n(\theta_n + \theta_s)$ $\pi_s^u = b_s(0)$ $\pi_s^d = b_s(0)$	$\pi_n^u = b_n 0$ $\pi_n^d = b_n 0$ $\pi_s^u = b_s(\theta_n + \theta_s)$ $\pi_s^d = b_s(\theta_n + \theta_s)$
Area $\overline{A1}$ (intermediate)	$\pi_n^u = b_n\theta_n + b_s T$ $\pi_n^d = b_n(\theta_n + T) + (b_n - b_s)T =$ $= b_n\theta_n + b_s T$ $\pi_s^u = b_s(\theta_s - T)$ $\pi_s^d = b_s(\theta_s - T)$	$\pi_n^u = b_n 0$ $\pi_n^d = b_n 0$ $\pi_s^u = b_s(\theta_n + \theta_s)$ $\pi_s^d = b_s(\theta_n + \theta_s)$
Area $\overline{B1}$ (intermediate)	$\pi_n^u = b_n\theta_n + b_s T$ $\pi_n^d = b_n(\theta_n + T) + (b_s - b_n)T =$ $= b_n\theta_n + b_s T$ $\pi_s^u = b_s(\theta_s - T)$ $\pi_s^d = b_s(\theta_s - T)$	$\pi_n^u = b_n(\theta_s + \theta_n - k)$ $\pi_n^d = b_n(\theta_s + \theta_n - k)$ $\pi_s^u = b_s\theta_s + b_n(k - \theta_s)$ $\pi_s^d = b_s\theta_s + b_s(k - \theta_s) +$ $(b_n - b_s)(k - \theta_s) = b_s\theta_s + b_n(k - \theta_s)$
Area $\overline{B2}$ (intermediate)	$\pi_n^u = b_n\theta_n + b_s T$ $\pi_n^d = b_n(\theta_n + T) + (b_s - b_n)T =$ $= b_n\theta_n + b_s T$ $\pi_s^u = b_s(\theta_s - T)$ $\pi_s^d = b_s(\theta_s - T)$	$\pi_n^u = b_n(\theta_s + \theta_n - k)$ $\pi_n^d = b_n(\theta_s + \theta_n - k)$ $\pi_s^u = b_s k$ $\pi_s^d = b_s k$
Area B (high)	$\pi_n^u = b_s k$ $\pi_n^d = b_n k$ $\pi_s^u = b_s(\theta_s - T)$ $\pi_s^d = b_s(\theta_s - T)$	$\pi_n^u = b_n(\theta_s + \theta_n - k)$ $\pi_n^d = b_n(\theta_s + \theta_n - k)$ $\pi_s^u = b_n k$ $\pi_s^d = b_s k$

the realization of demand and agent's bids, these equations define different areas (figure 6). As in the case in which the transmission rights are assigned to the grid operator, uniform and discriminatory price auction generate the same payoff when the demand is low or intermediate. However, when the transmission rights are assigned to the firm that submits the lower bid in the auction, the discriminatory auction is "transformed" in a uniform price auction (table 5). When the transmission rights are assigned to the firm that submits the lower bid in the auction, the payoff of the firm that submits the lower bid in the auction is equal to its own bid multiplied by the total demand, in addition it receives the transmission rights. Hence the firm that submits the lower bid, satisfy the demand in its own market at the price set by itself in its own market and sells the rest of its generation capacity in the other market at the price set by the other firm. This "transformation" will have important effects on market outcome.

4.2 Equilibrium analysis

I this section, I characterize the equilibrium when the transmission rights are assigned to the firm that submits the lower bid in the spot electricity auction.

As in the previous section, lemma one can be applied to show that when the demand is low, a pure strategies equilibrium exists for both firms; when the demand is intermediate a pure strategies equilibrium does not exist nor for uniform, neither for discriminatory auction; finally, when the demand is high a pure strategies equilibrium exists for uniform, but not for discriminatory auction.

Lemma 3. In a mixed strategy equilibrium none firm submits a bid lower than bid[12] ($\underline{b}_i^{II}$) such that

$$\underline{b}_i^{II} min\{\theta_i, k_i\} + E(b_j \mid b_j \geq \underline{b}^{II}) max\{0, min\{T, k_i - \theta_i\}\} = \\ Pmax\{0, \theta_i - T, \theta_i + \theta_j - k\}.$$

Moreover, the support for the mixed strategy equilibrium for both firms is $S^{II} = \left[max\left\{\underline{b}_i^{II}, \underline{b}_j^{II}\right\}, P\right]$. Furthermore, $\underline{b}_i^{II} \leq \underline{b}_i^{I}, \forall i = n, s$.

The proof of the two first statements of the lemma use same logic that the one used in lemma one.

The proof for the latest statement is as follows. When $\theta_i > k$, the lower bound of the support is defined by $\underline{b}_i k = Pmax\{0, \theta_i - T, \theta_i + \theta_j - k\}$, both for the model that assigns the transmission rights to the grid operator and for the model that assigns the transmission rights to the firm that submits the lowest bid in the spot electricity auction. Hence, if $\theta_i > k$, $\underline{b}_i^{II} = \underline{b}_i^{I}, \forall i = n, s$. Instead, when $\theta_i < k$, the lower bound of the support is defined by $\underline{b}_i^{I} k = Pmax\{0, \theta_i - T, \theta_i + \theta_j - k\}$ for the model that assigns the transmission rights to the grid operator and by $\underline{b}_i^{II} min\{\theta_i, k_i\} + E(b_j \mid b_j \geq \underline{b}^{II}) max\{0, min\{T, k_i - \theta_i\}\} = Pmax\{0, \theta_i - T, \theta_i + \theta_j - k\}$ for the model that assigns the transmission rights to the firm that submits the lower bid in the spot electricity auction. The right hand side of the equations is equal in both models. However, in the latest model, the firm that submits the lower bid in the auction sells part of its capacity in the other market at a higher price. Therefore, it can lower its bid, $\underline{b}_i^{II}$, because it compensate a reduction in profit in its own market with an increase in profit in the other market. Hence, if $\theta_i < k$, $\underline{b}_i^{II} < \underline{b}_i^{I}, \forall i = n, s$ □

Using lemmas one and three, I establish the main result of this section.

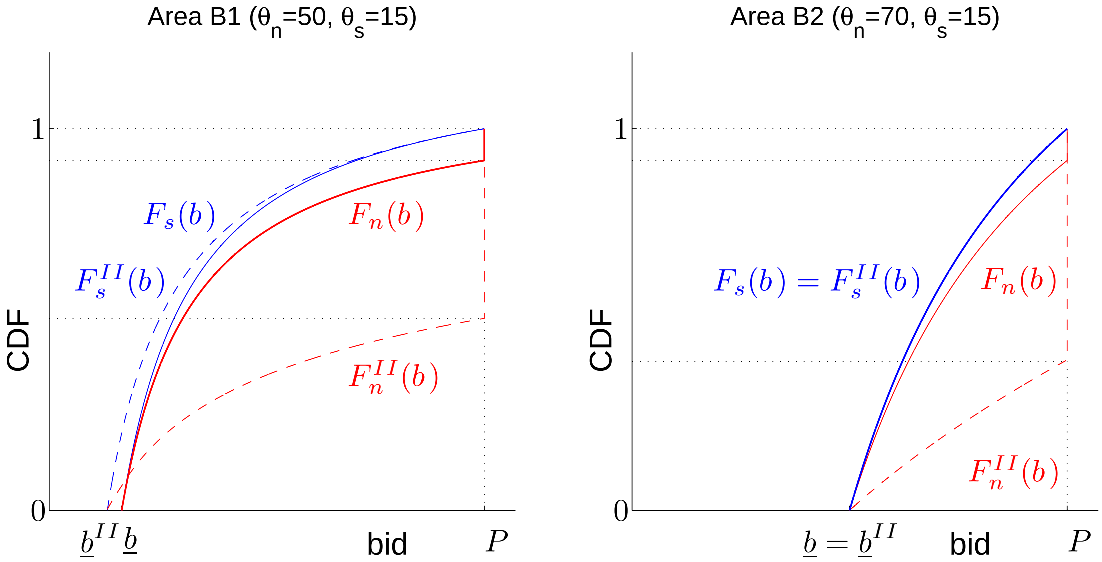
Proposition 3. When the auction is uniform, the characterization of the equilibrium strategies falls into one of the next three categories (figure 6).

i Low demand (area A). The equilibrium strategies pair is in pure strategies.

ii Intermediate demand (areas $A1$, $B1$, $B2$, $\overline{A1}$, $\overline{B1}$, $\overline{B2}$). The equilibrium strategies pair is in mixed strategies.

[12] Superscript II denotes support, profits, prices and strategies referred to model II, the one that assigns financial transmission rights to the firm that submits the lower bid in the spot electricity auction.

Figure 7: Numerical example: equilibrium strategies ($k_n = k_s = 60, T = 40, c_n = c_s = 0, P = 7$)



iii High demand (area B). The equilibrium strategies pair is in pure strategies.

When the auction is discriminatory, the characterization of the equilibrium fall into one of the next two categories (figure 6).

i Low demand (Area A). The equilibrium strategies pair is in pure strategies.

ii Intermediate demand (areas $A1$, $B1$,$B2$, $\overline{A1}$, $\overline{B1}$, $\overline{B2}$) or high demand (area B). The equilibrium strategies pair is in mixed strategies.

As in the case in which transmission rights are assigned to the grid operator, when the transmission rights are assigned to the firm that submits the lower bid in the auction, the same type of equilibrium emerges. However, there are important differences between the performance of both types of auctions depending of the allocation of financial transmission rights. Proposition four summarizes these differences.

Proposition 4. If financial transmission rights are assigned to the firm that submits the lower bid in the auction (set up II), instead of been assigned to the transmission grid operator:

i Intermediate demand (area $B2$) (figure 6). The lower bound of the strategies support does not change (figure 7). The cumulative distribution function of firm s does not change and the cumulative distribution function of firm n is lower for all the bids in the support, i.e., $F_n^{II}(b)$ stochastic dominates $F_n(b)$. Moreover, the expected value of the bids in market North increases, the expected value of the bids in market South decreases, the expected payoff of firm n does not change and the expected payoff of firm s increases.

ii Intermediate demand (areas B, $B1$) (figure 6). The lower bound of the strategies support decreases (figure 7). The cumulative distribution function of firm s is higher

Table 6: $(k_n = k_s = 60, T = 40, c_n = c_s = 0, P = 7)$

Model	Area	$\underline{b}$	$F_n(P)$	$E(b_n)$	$E(b_s)$	$\overline{\pi}_n$	$\overline{\pi}_s$
	Area B1						
	$(\theta_n = 50, \theta_s = 15)$	1.17	0.91	2.88	2.51	70	64.17
II	$(\theta_n = 50, \theta_s = 15)$	0.93	0.50	4.89	2.32	70	230.4
	Area B2						
	$(\theta_n = 70, \theta_s = 10)$	3.5	0.83	5.03	4.85	210	175
II	$(\theta_n = 70, \theta_s = 15)$	3.5	0.91	5.21	4.85	210	192.5

The values of the variables for model II are worked out using the algorithm that I have described in the annex.

for all the bids in the support, i.e., $F_s^{II}(b)$ stochastic dominates $F_s(b)$. The cumulative distribution function of firm n $F_n^{II}(b)$ is higher than $F_n(b)$ for low bids, but no rank between $F_n^{II}(b)$ and $F_n(b)$ can be established for high bids, therefore none stochastic dominance rank of the cumulative distribution function can be established for firm n. Moreover, the expected value of the bids in market South decreases, the expected payoff of firm n does not change and no rank can be established neither on the expected value of the bids in market North, nor on the expected payoff of firm s.

Proposition four establishes the main differences on equilibrium performance when different type of transmission rights allocation rule is implemented. In particular, as can be observed in figure 7 and table 6, when the transmission rights are assigned to the firm that submits the lower bid in the auction, the equilibrium price[13] and the expected payoff of the firm located in the low demand region are higher than when the transmission rights are assigned to the grid operator. The intuition behind this result is as follows, when the transmission rights are assigned to the firm that submits the lower bid in the spot electricity market, the firm that faces lower demand in its own market has incentives to submits lower bids for two reasons: first, because it faces a low residual demand, therefore, in case of be dispatched last, its payoff is very low; second, because in the case of be dispatched first, after satisfies the demand in its own market, it has high residual generation capacity to sell in the other market for a higher price. These two mechanisms induce an increase in the weight that the firm located in the low demand market assigns to low bids. The firm located in the high demand market anticipates that in equilibrium it will be dispatched last the majority of the times, therefore it increases the weight that assign to the maximum bid allowed by the auctioneer (if the firm located in the high demand market is dispatched last, it prefers to set a price equal to the maximum price allowed by the auctioneer). The overall effect is an increase in prices.

As I have explained in the model section, when the transmission rights are assigned to the grid operator, the uniform price auction's payoff is "transformed" into a discriminatory price auction's payoff. By contrast, when the transmission rights are assigned to the firm that submits the lower bid in the spot electricity auction, the discriminatory price auction's payoff is "transformed" into a uniform price auction's payoff. It is precisely

[13]In area $B1$ the equilibrium price in the low demand area decreases, however, given that the demand is distributed uniformly across markets, when the realization of demand belongs to area $\overline{B1}$, the equilibrium price is higher. Therefore, in average the equilibrium price increases.

this fact, the one that is driving the results exposed in proposition four. As Fabra et al. (2006) and as I have shown in the first chapter of my thesis, discriminatory price auctions generates lower equilibrium prices than uniform price auction. As proposition four shows, these results can be extended to the case in which the transmission line is congested and the electricity market is a nodal price market.

5 Conclusions

I this paper, I have characterized the equilibrium in a nodal price electricity market for uniform and discriminatory auctions when different transmission rights allocations are implemented.

I have found that when the transmission line is congested, discriminatory and uniform price auctions perform equally. However, the way in which transmission rights are allocated modifies the equilibrium outcome. In particular, when financial transmission rights are assigned to the grid operator, the equilibrium price is lower than when the transmission rights are assigned to the firm that submits the lower bid in the spot electricity auction.

My analysis, however, does not take into consideration other possible assignments of transmission rights. In the next future, I would like to characterize the equilibrium when the transmission and the spot electricity markets run sequentially; first, firms acquire transmission rights in the transmission rights market and later they compete in the spot electricity market.

Finally, I have found that when transmission rights are assigned to the grid operator, an increase in transmission capacity increases competition between markets. Moreover, an increase in transmission capacity induces changes on firms' payoff functions that could have important implications in generation capacity investment decisions. Hence, in the next future I would like to use analyze the effect that an increase in transmission capacity could have on generation capacity investment decisions.

Annex

Proposition 1. Uniform Price Auction (pure strategies equilibrium). Using lemma one, the proof is straight forward.

When the auction is uniform. Using lemma one, the proof is straight forward.

When the demand is low: $b_n = b_s = c = 0$. The equilibrium payoff is zero for both firms. No electricity flows through the grid.

When the demand is high:

The pure strategies equilibrium is defined by

$$b_i = P;\ \ b_j = \frac{Pmax\left\{\theta_i - T, \theta_j + \theta_i - k\right\}}{min\left\{\theta_i + T, k\right\}}\ \ \forall i, j = s, n$$

The equilibrium price is P.

The payoff function is defined by either

$$\overline{\pi}_i = Pmax\left\{\theta_i - T, \theta_j + \theta_i - k\right\};\ \ \overline{\pi}_j = Pk\ \ \forall i, j = s, n$$

The probability that firm i submits the lower bid depend of which type of equilibrium emerge and can not be determined a priori.

Uniform Price Auction and Discriminatory Price Auction (mixed strategies equilibrium).

Proof:

The model presented in section two satisfies the properties established by Dasgupta and Maskin (1986), that guarantee that a mixed strategy equilibrium exists. In particular, the discontinuities of $\pi_i, \forall i, j$ are restricted to the strategies such that $b_i = b_j$. Furthermore, it is simple to confirm that by lowering its price from a position where $b_i = b_j$, a firm discontinuously increases its profit. Therefore, $\pi_i(b_i, b_j)$ is everywhere left lower semi-continuous in b_i, and hence weakly lower semi-continuous. Obviously $\pi_i(b_i, b_j)$ is bounded. Finally, $\pi_i(b_i, b_j) + \pi_j(b_i, b_j)$ is continuous, because discontinuous shifts in clientele from one firm to another occur only where both firms derive the same profit per customer. Therefore, theorem five in Dasgupta and Maskin (1986) applies, hence a mixed strategy equilibrium exists. However, Dasgupta and Maskin (1986) did not provide an algorithm to work out the equilibrium. Nevertheless, using the approach proposed by (Karlin, 1959; Beckmann, 1965; Shapley, 1957; Shilony, 1977; Varian, 1980; Deneckere and Kovenock, 1986; Osborne and Pitchik, 1986; Fabra et al., 2006), the equilibrium characterization is guaranteed by construction. I will use the approach proposed by this branch of the literature to work out the mixed strategy equilibrium.

First, I work out the cumulative distribution function.

First step, the payoff function for any firm is:

$$\begin{aligned}\pi_i(b) &= b\left[F_j(b) max\left\{0, \theta_i - T, \theta_i + \theta_j - k\right\} + (1 - F_j(b)) min\left\{\theta_i + \theta_j, \theta_i + T, k\right\}\right] = \\ &= -bF_j(b)\left[min\left\{\theta_i + \theta_j, \theta_i + T, k\right\} - max\left\{0, \theta_i - T, \theta_i + \theta_j - k\right\}\right] + \\ & \quad bmin\left\{\theta_i + \theta_j, \theta_i + T, k\right\}\end{aligned} \tag{7}$$

Second step, $\pi_i(b) = \overline{\pi}_i \forall b \in S_i, i = n, s$, where S_i is the support of the mixed strategies. Then,

$$\begin{aligned}\overline{\pi}_i &= -bF_j(b)\left[min\left\{\theta_i + \theta_j, \theta_i + T, k\right\} - max\left\{0, \theta_i - T, \theta_i + \theta_j - k\right\}\right] + \\ & \quad bmin\left\{\theta_i + \theta_j, \theta_i + T, k\right\} \Rightarrow \\ F_j(b) &= \frac{bmin\left\{\theta_i + \theta_j, \theta_i + T, k\right\} - \overline{\pi}_i}{b\left[min\left\{\theta_i + \theta_j, \theta_i + T, k\right\} - max\left\{0, \theta_i - T, \theta_i + \theta_j - k\right\}\right]}\end{aligned} \tag{8}$$

Third step, at $\underline{b}$, $F_i(\underline{b}) = 0 \forall i = n, s$. Then,

$$\overline{\pi}_i = \underline{b}min\left\{\theta_i + \theta_j, \theta_i + T, k\right\} \tag{9}$$

Fourth step, Plug in 9 into 8, I obtain the mixed strategies for both firms.

$$\begin{aligned}F_j(b) &= \frac{bmin\left\{\theta_i + \theta_j, \theta_i + T, k\right\} - \underline{b}min\left\{\theta_i + \theta_j, \theta_i + T, k\right\}}{b\left[min\left\{\theta_i + \theta_j, \theta_i + T, k\right\} - max\left\{0, \theta_i - T, \theta_i + \theta_j - k\right\}\right]} = \\ &= \frac{min\left\{\theta_i + \theta_j, \theta_i + T, k\right\}}{min\left\{\theta_i + \theta_j, \theta_i + T, k\right\} - max\left\{0, \theta_i - T, \theta_i + \theta_j - k\right\}} \frac{b - \underline{b}}{b} \quad \forall i = n, s\end{aligned} \tag{10}$$

For further reference:
$L_i(b) = bmin\left\{\theta_i + \theta_j, \theta_i + T, k\right\}$ and
$H_i(b) = bmax\left\{0, \theta_i - T, \theta_i + \theta_j - k\right\}$.

It is easy to verify that equation $F_j(b) \forall i, j$ is indeed a cumulative distribution function. First, in the third step, I have established that $F_j(\underline{b}) = 0$. Second, $F_j(b)$ is an increasing function in b. At $\underline{b}$, $L_i(\underline{b}) = H_i(b)$, for any $b > \underline{b}$, $L_i(\underline{b}) < H_i(b)$; moreover, $\frac{\partial L_i(b)}{\partial b} > 0$, $\frac{\partial L_i(\underline{b})}{\partial b} = 0$ and $\frac{\partial H_i(b)}{\partial b} > 0$, therefore, $\frac{\partial\left(L_i(b) - L_i(\underline{b})\right)}{\partial b} > \frac{\partial\left(L_i(b) - H_i(b)\right)}{\partial b}$. Third, $F_j(b) \leq 1 \forall b \in S_i$. Finally, $F_j(b)$ is continuous in the support because $L_i(b) - L_i(\underline{b})$ and $L_i(b) - H_i(b)$ are continuous functions in the support.

In area $A1$, using equation 10, the cumulative distribution function for both firms is defined by:

$$F_s(b) = \begin{cases} 0 & \text{if } b < \underline{b} \\ \frac{\theta_n + \theta_s}{\theta_s + T}\frac{b - \underline{b}}{b} & \text{if } b \in (\underline{b}, P) \\ 1 & \text{if } b = P \end{cases}$$

$$F_n(b) = \begin{cases} 0 & \text{if } b < \underline{b} \\ \frac{\theta_s + T}{k + T - \theta_n}\frac{b - \underline{b}}{b} & \text{if } b \in (\underline{b}, P) \\ 1 & \text{if } b = P \end{cases}$$

Moreover,

$$
\begin{aligned}
F_s(P) &= \frac{\theta_n+\theta_s}{\theta_s+T}\frac{P-\dfrac{P(\theta_n-T)}{\theta_n+\theta_s}}{P}=1 \\
F_n(P) &= \frac{\theta_s+T}{k+T-\theta_n}\frac{P-\dfrac{P(\theta_n-T)}{\theta_n+\theta_s}}{P}=\frac{(\theta_s+T)^2}{(\theta_n+\theta_s)(k+T-\theta_n)}<1
\end{aligned}
$$

In area $B1$, using equation 10, the cumulative distribution function for both firms is defined by:

$$
F_s(b)=\begin{cases} 0 & \text{if } b<\underline{b} \\ \dfrac{k}{T+k-\theta_n}\dfrac{b-\underline{b}}{b} & \text{if } b\in(\underline{b},P) \\ 1 & \text{if } b=P \end{cases}
$$

$$
F_n(b)=\begin{cases} 0 & \text{if } b<\underline{b} \\ \dfrac{\theta_s+T}{T+k-\theta_n}\dfrac{b-\underline{b}}{b} & \text{if } b\in(\underline{b},P) \\ 1 & \text{if } b=P \end{cases}
$$

Moreover,

$$
\begin{aligned}
F_s(P) &= \frac{k}{T+k-\theta_n}\frac{P-\dfrac{P(\theta_n-T)}{k}}{P}=1 \\
F_n(P) &= \frac{\theta_s+T}{T+k-\theta_n}\frac{P-\dfrac{P(\theta_n-T)}{k}}{P}=\frac{\theta_s+T}{k}<1
\end{aligned}
$$

Using equation 10, the cumulative distribution function in area B is defined by:

$$
F_i(b)=\begin{cases} 0 & \text{if } b<\underline{b} \\ \dfrac{k}{2k-\theta_i-\theta_j}\dfrac{b-\underline{b}}{b} & \text{if } b\in(\underline{b},P) \quad \forall i=s,n \\ 1 & \text{if } b=P \end{cases} \tag{11}
$$

Second, I work out the support of the mixed strategy equilibrium.

In the border between areas $B1$ and B, $\theta_s=k-T$. In the border, $\underline{b}_n$ solves $\underline{b}_n min\{\theta_n+\theta_s,\theta_n+T,k\}=Pmax\{0,\theta_n-T,\theta_s+\theta_n-k\}$, therefore $\underline{b}_n=\dfrac{P(\theta_n-T)}{k}$ and $\underline{b}_s$ solves $\underline{b}_s min\{\theta_n+\theta_s,\theta_s+T,k\}=Pmax\{0,\theta_s-T,\theta_s+\theta_n-k\}$, therefore $\underline{b}_s=\dfrac{P(\theta_n+\theta_s-k)}{\theta_s+T}$. Plug in the value of θ_s in the border between these areas into $\underline{b}_s$ formula, I obtain $\underline{b}_s=\dfrac{P(\theta_n+k-T-k)}{k-T+T}=\dfrac{P(\theta_n-T)}{k}=\underline{b}_n$. Therefore, in the border between these areas, $\underline{b}_s = \underline{b}_n = \frac{P(\theta_n - T)}{k}$.

In areas $A1$ and $B1$, $\underline{b}_n > \underline{b}_s$. In area $A1$, taking partial derivatives $\frac{\partial \underline{b}_n}{\partial \theta_s} = \frac{-P(\theta_n - T)}{(\theta_n + \theta_s)^2} < 0$ and $\frac{\partial \underline{b}_s}{\partial \theta_s} = \frac{P(k + T - \theta_n)}{(\theta_s + T)^2} > 0$. In area $B1$, taking partial derivatives $\frac{\partial \underline{b}_n}{\partial \theta_s} = 0$ and $\frac{\partial \underline{b}_s}{\partial \theta_s} = \frac{P(k + T - \theta_n)}{(\theta_s + T)^2} > 0$. Therefore, in areas $A1$ and $B1$, $\underline{b}_n > \underline{b}_s$. Hence, $S = [max\{\underline{b}_n, \underline{b}_s\}, P] = [\underline{b}_n, P]$. In particular, in area $A1$, $S = \left[\frac{P(\theta_n - T)}{(\theta_n + \theta_s)}, P\right]$ and in area $B1$, $S = \left[\frac{P(\theta_n - T)}{k}, P\right]$.

In area B, $\underline{b}_s = \underline{b}_n = \frac{P(\theta_s + \theta_n - k)}{k}$. Therefore, in area B, $S = [max\{\underline{b}_n, \underline{b}_s\}, P] = \left[\frac{P(\theta_s + \theta_n - k)}{k}, P\right]$.

Third, I work out the expected bid.

In Area $A1$,

$$
\begin{aligned}
f_s(b) &= \frac{\partial F_s(b)}{\partial b} = \frac{\theta_n + \theta_s}{\theta_s + T} \frac{\underline{b}}{b^2} \\
f_n(b) &= \frac{\partial F_n(b)}{\partial b} = \frac{\theta_s + T}{k + T - \theta_n} \frac{\underline{b}}{b^2}
\end{aligned}
$$

$$
\begin{aligned}
E(b_s) &= \int_{\underline{b}}^{P} b f_s(b_s) \partial b = \int_{\underline{b}}^{P} \frac{\theta_n + \theta_s}{\theta_s + T} \frac{\underline{b}}{b} \partial b = \frac{\theta_n + \theta_s}{\theta_s + T} \underline{b} \left[ln(b)\right]_{\underline{b}}^{P} \\
E(b_n) &= \int_{\underline{b}}^{P} b f_n(b_n) \partial b = \int_{\underline{b}}^{P} \frac{\underline{b}}{b^2} \partial b = \frac{\theta_s + T}{k + T - \theta_n} \underline{b} \left[ln(b)\right]_{\underline{b}}^{P} + (1 - F_n(P)) P
\end{aligned}
$$

In area $B1$,

$$
\begin{aligned}
f_s(b) &= \frac{\partial F_s(b)}{\partial b} = \frac{k}{T + k - \theta_n} \frac{\underline{b}}{b^2} \\
f_n(b) &= \frac{\partial F_n(b)}{\partial b} = \frac{\theta_s + T}{T + k - \theta_n} \frac{\underline{b}}{b^2}
\end{aligned}
$$

$$
\begin{aligned}
E(b_s) &= \int_{\underline{b}}^{P} b f_s(b_s) \partial b = \int_{\underline{b}}^{P} \frac{k}{T + k - \theta_n} \frac{\underline{b}}{b} \partial b = \frac{k}{T + k - \theta_n} \underline{b} \left[ln(b)\right]_{\underline{b}}^{P} \\
E(b_n) &= \int_{\underline{b}}^{P} b f_n(b_n) \partial b = \int_{\underline{b}}^{P} \frac{\theta_s + T}{T + k - \theta_n} \frac{\underline{b}}{b} \partial b = \\
&\quad \frac{\theta_s + T}{T + k - \theta_n} \underline{b} \left[ln(b)\right]_{\underline{b}}^{P} + (1 - F_n(P)) P
\end{aligned}
$$

In Area B.

$$f_i(b) = \frac{\partial F_i(b)}{\partial b} = \frac{k}{2k - \theta_i - \theta_j} \frac{\underline{b}}{b^2} \;\; \forall i = s, n$$

$$E(b_i) = \int_{\underline{b}}^{P} b f_i(b_i) \partial b = \int_{\underline{b}}^{P} \frac{k}{2k - \theta_n - \theta_s} \frac{\underline{b}}{b} \partial b = \frac{k}{2k - \theta_n - \theta_s} \underline{b} \left[ln(b)\right]_{\underline{b}}^{P}$$

Fourth, I work out the payoff function.

Using 9, the payoff function in area $A1$ is:

$$\begin{aligned} \overline{\pi}_n &= \underline{b}(\theta_n + \theta_s) \\ \overline{\pi}_s &= \underline{b}(\theta_s + T) \end{aligned}$$

Using 9, the payoff function in area $B1$ is:

$$\begin{aligned} \overline{\pi}_n &= \underline{b}k \\ \overline{\pi}_s &= \underline{b}(\theta_s + T) \end{aligned}$$

Using 9, the payoff function in area B is:

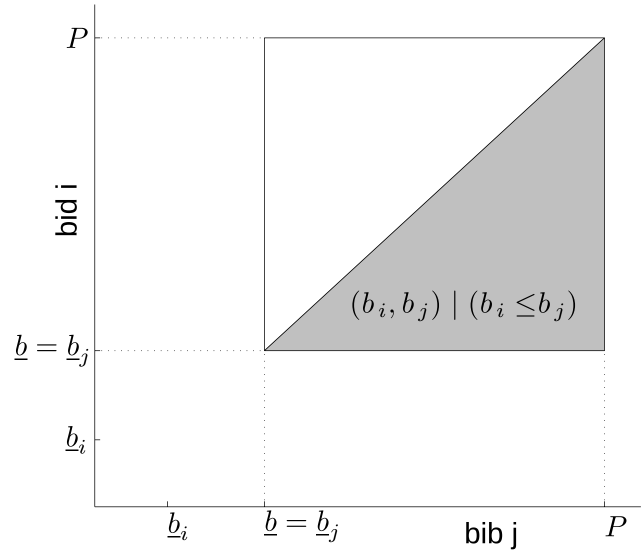
$$\overline{\pi}_i = \underline{b}k \;\; \forall i = s, n$$

Finally, I work out the $Prob(b_i < b_j)$. This probability is determined by the integral of the joint distribution in the grey area in figure 8.

$$
\begin{aligned}
prob(b_i < b_j) &= \left(\int_{\underline{b}}^{P} f_j(b_j)\left(\int_{0}^{b_j} f_i(b_i)\partial b_i\right)\partial b_j\right) + F_i(P) - F_j(P) = \\
&= \int_{\underline{b}}^{P} f_j(b_j)F_i(b_j)\partial b_j + F_i(P) - F_j(P) = \\
&= \int_{\underline{b}}^{P} \frac{min\left\{\theta_i+\theta_j,\theta_i+T,k\right\}}{min\left\{\theta_i+\theta_j,\theta_i+T,k\right\} - max\left\{0,\theta_i-T,\theta_i+\theta_j-k\right\}}\frac{\underline{b}}{b^2} \\
&\quad \frac{min\left\{\theta_i+\theta_j,\theta_j+T,k\right\}}{min\left\{\theta_i+\theta_j,\theta_j+T,k\right\} - max\left\{0,\theta_j-T,\theta_i+\theta_j-k\right\}}\frac{b-\underline{b}}{b}\partial b + \\
&\quad 1 - F_j(P) \\
&= \frac{min\left\{\theta_i+\theta_j,\theta_i+T,k\right\}}{min\left\{\theta_i+\theta_j,\theta_i+T,k\right\} - max\left\{0,\theta_i-T,\theta_i+\theta_j-k\right\}} \\
&\quad \frac{min\left\{\theta_i+\theta_j,\theta_j+T,k\right\}}{min\left\{\theta_i+\theta_j,\theta_j+T,k\right\} - max\left\{0,\theta_j-T,\theta_i+\theta_j-k\right\}}\underline{b} \\
&\quad \left[\int_{\underline{b}}^{P}\frac{\partial b}{b^2} - \int_{\underline{b}}^{P}\frac{\underline{b}}{b^3}\partial b\right] + 1 - F_j(P) = \\
&= \frac{min\left\{\theta_i+\theta_j,\theta_i+T,k\right\}}{min\left\{\theta_i+\theta_j,\theta_i+T,k\right\} - max\left\{0,\theta_i-T,\theta_i+\theta_j-k\right\}} \\
&\quad \frac{min\left\{\theta_i+\theta_j,\theta_j+T,k\right\}}{min\left\{\theta_i+\theta_j,\theta_j+T,k\right\} - max\left\{0,\theta_j-T,\theta_i+\theta_j-k\right\}}\underline{b} \\
&\quad \left[\frac{\underline{b}}{4b^4} - \frac{1}{3b^3}\right]_{\underline{b}}^{P} + 1 - F_j(P) = \\
&= \frac{min\left\{\theta_i+\theta_j,\theta_i+T,k\right\}}{min\left\{\theta_i+\theta_j,\theta_i+T,k\right\} - max\left\{0,\theta_i-T,\theta_i+\theta_j-k\right\}} \\
&\quad \frac{min\left\{\theta_i+\theta_j,\theta_j+T,k\right\}}{min\left\{\theta_i+\theta_j,\theta_j+T,k\right\} - max\left\{0,\theta_j-T,\theta_i+\theta_j-k\right\}}\underline{b} \\
&\quad \left[\frac{3\underline{b}-4b}{12b^4}\right]_{\underline{b}}^{P} + 1 - F_j(P) = \\
&= \frac{min\left\{\theta_i+\theta_j,\theta_i+T,k\right\}}{min\left\{\theta_i+\theta_j,\theta_i+T,k\right\} - max\left\{0,\theta_i-T,\theta_i+\theta_j-k\right\}} \\
&\quad \frac{min\left\{\theta_i+\theta_j,\theta_j+T,k\right\}}{min\left\{\theta_i+\theta_j,\theta_j+T,k\right\} - max\left\{0,\theta_j-T,\theta_i+\theta_j-k\right\}}\underline{b} \\
&\quad \left[\frac{3\underline{b}-4P}{12P^4} + \frac{1}{12\underline{b}^3}\right]_{\underline{b}}^{P} + 1 - F_j(P) = \\
&= \frac{min\left\{\theta_i+\theta_j,\theta_i+T,k\right\}}{min\left\{\theta_i+\theta_j,\theta_i+T,k\right\} - max\left\{0,\theta_i-T,\theta_i+\theta_j-k\right\}} \\
&\quad \frac{min\left\{\theta_i+\theta_j,\theta_j+T,k\right\}}{min\left\{\theta_i+\theta_j,\theta_j+T,k\right\} - max\left\{0,\theta_j-T,\theta_i+\theta_j-k\right\}}\underline{b} \\
&\quad \frac{(3\underline{b}-4P)\underline{b}^3 + P^4}{12P^4\underline{b}^3} + 1 - F_j(P) \qquad (12)
\end{aligned}
$$

Using 12, the $prob(b_s < b_n)$ in Area $A1$ is equal to,

Figure 8: $(b_i, b_j) \mid b_i < b_j$



$$prob(b_s < b_n) = \frac{(\theta_s + \theta_n)}{(k + T - \theta_n)}\underline{b}\frac{(3\underline{b} - 4P)\underline{b}^3 + P^4}{12P^4\underline{b}^3} + 1 - F_n(P)$$

Using 12, the $prob(b_s < b_n)$ in Area $B1$ is equal to,

$$prob(b_s < b_n) = \frac{(\theta_s + T)k}{(k + T - \theta_n)^2}\underline{b}\frac{(3\underline{b} - 4P)\underline{b}^3 + P^4}{12P^4\underline{b}^3} + 1 - F_n(P)$$

Finally, due to the symmetry in the strategies in area B, the probability that firm i submit the lower bid is $prob(b_i < b_j) = \frac{1}{2}$.

Corollary 1. Area $A1$

$$\frac{\partial \underline{b}}{\partial \theta_n} = \frac{P(\theta_s + T)}{(\theta_n + \theta_s)^2} > 0$$

$$\begin{aligned}\frac{\partial F_n(P)}{\partial \theta_n} &= \frac{-\left[(k + T - \theta_n) - (\theta_n + \theta_s)\right]}{\left[(\theta_n + \theta_s)(k + T - \theta_n)\right]^2} \\ &= \frac{2\theta_n + \theta_s - k - T}{\left[(\theta_n + \theta_s)(k + T - \theta_n)\right]^2} > 0 \iff \theta_n > \frac{k + T - \theta_s}{2}\end{aligned}$$

$$\begin{aligned}\frac{\partial E(b_n)}{\partial \theta_n} = \; & \frac{\theta_s + T}{(k + T - \theta_n)^2}\underline{b}ln\left(\frac{P}{\underline{b}}\right) + \\ & \frac{\theta_s + T}{k + T - \theta_n}\frac{\partial \underline{b}}{\partial \theta_n}ln\left(\frac{P}{\underline{b}}\right) + \\ & \frac{\theta_s + T}{k + T - \theta_n}\underline{b}\frac{\underline{b}}{P}\frac{\partial \underline{b}}{\partial \theta_n} - \frac{\partial F_n(P)}{\partial \theta_n} > 0\end{aligned}$$

Where all the elements are positive and $-\frac{\partial F_n(P)}{\partial \theta_n} > 0$ when $\theta_n < \frac{k+T-\theta_s}{2}$.

$$\begin{aligned}\frac{\partial E(b_s)}{\partial \theta_n} &= \frac{1}{(T+\theta_s)}\underline{b}ln\left(\frac{P}{\underline{b}}\right)+\\ &\frac{1}{(T+\theta_s)}\frac{\partial \underline{b}}{\partial \theta_n}ln\left(\frac{P}{\underline{b}}\right)+\\ &\frac{1}{(T+\theta_s)}\underline{b}\frac{\underline{b}}{P}\frac{\partial \underline{b}}{\partial \theta_n} > 0\end{aligned}$$

$$\frac{\partial \overline{\pi}_n}{\partial \theta_n} = P > 0$$

$$\frac{\partial \overline{\pi}_s}{\partial \theta_n} = \frac{P}{k}(\theta_s + T) > 0$$

Area $B1$

$$\frac{\partial \underline{b}}{\partial \theta_n} = \frac{P}{k} > 0$$

$$\frac{\partial F_n(P)}{\partial \theta_n} = 0$$

$$\begin{aligned}\frac{\partial E(b_n)}{\partial \theta_n} &= \frac{\theta_s+T}{(k+T-\theta_n)^2}\underline{b}ln\left(\frac{P}{\underline{b}}\right)+\\ &\frac{\theta_s+T}{k+T-\theta_n}\frac{\partial \underline{b}}{\partial \theta_n}ln\left(\frac{P}{\underline{b}}\right)+\\ &\frac{\theta_s+T}{k+T-\theta_n}\underline{b}\frac{\underline{b}}{P}\frac{\partial \underline{b}}{\partial \theta_n} > 0\end{aligned}$$

$$\begin{aligned}\frac{\partial E(b_s)}{\partial \theta_n} &= \frac{k}{(k+T-\theta_n)^2}\underline{b}ln\left(\frac{P}{\underline{b}}\right)+\\ &\frac{k}{k+T-\theta_n}\frac{\partial \underline{b}}{\partial \theta_n}ln\left(\frac{P}{\underline{b}}\right)+\\ &\frac{k}{k+T-\theta_n}\underline{b}\frac{\underline{b}}{P}\frac{\partial \underline{b}}{\partial \theta_n} > 0\end{aligned}$$

$$\frac{\partial \overline{\pi}_n}{\partial \theta_n} = P > 0$$

$$\frac{\partial \overline{\pi}_s}{\partial \theta_n} = \frac{P}{k}(\theta_s + T) > 0$$

Corollary 2. Area $A1$.

$$\frac{\partial \underline{b}}{\partial \theta_s} = \frac{-P(\theta_n - T)}{(\theta_n + \theta_s)^2} < 0$$

$$\begin{aligned}\frac{\partial F_n(P)}{\partial \theta_s} &= \frac{2(\theta_s + T)\left[(\theta_n + \theta_s)(k + T - \theta_n)\right] - (\theta_s + T)^2\left[k + T - \theta_n\right]}{\left[(\theta_n + \theta_s)(k + T - \theta_n)\right]^2} \\ &= \frac{(\theta_s + T)(k + T - \theta_n)(2\theta_n + \theta_s - T)}{\left[(\theta_n + \theta_s)(k + T - \theta_n)\right]^2} > 0\end{aligned}$$

$$\begin{aligned}\frac{\partial E(b_n)}{\partial \theta_s} = & \frac{1}{k + T - \theta_n}\underline{b}ln\left(\frac{P}{\underline{b}}\right) + \\ & \frac{\theta_s + T}{k + T - \theta_n}\frac{\partial \underline{b}}{\partial \theta_s}ln\left(\frac{P}{\underline{b}}\right) \\ & \frac{\theta_s + T}{k + T - \theta_n}\underline{b}\frac{\underline{b}}{P}\frac{\partial \underline{b}}{\partial \theta_s} - \frac{\partial F_n(P)}{\partial \theta_s} < 0\end{aligned}$$

Where, $\frac{\theta_s + T}{k + T - \theta_n}\frac{\partial \underline{b}}{\partial \theta_s}ln\left(\frac{P}{\underline{b}}\right)$, $\frac{\theta_s + T}{k + T - \theta_n}\underline{b}\frac{\underline{b}}{P}\frac{\partial \underline{b}}{\partial \theta_s}$ and $\frac{\partial F_n(P)}{\partial \theta_s}$ are negative and $\frac{1}{k + T - \theta_n}\underline{b}ln\left(\frac{P}{\underline{b}}\right)$ is positive.

$$\begin{aligned}\frac{\partial E(b_s)}{\partial \theta_s} = & \frac{T - \theta_n}{(T + \theta_s)^2}\underline{b}ln\left(\frac{P}{\underline{b}}\right) + \\ & \frac{\theta_s + \theta_n}{T + \theta_s}\frac{\partial \underline{b}}{\partial \theta_s}ln\left(\frac{P}{\underline{b}}\right) \\ & \frac{\theta_s + \theta_n}{T + \theta_s}\underline{b}\frac{\underline{b}}{P}\frac{\partial \underline{b}}{\partial \theta_s} < 0\end{aligned}$$

$$\frac{\partial \overline{\pi}_n}{\partial \theta_s} = 0$$

$$\frac{\partial \overline{\pi}_s}{\partial \theta_s} = \frac{-P(\theta_n - T)}{(\theta_n + \theta_s)^2}(\theta_s + T) + \frac{-P(\theta_n - T)}{(\theta_n + \theta_s)} > 0$$

Area $B1$.

$$\frac{\partial \underline{b}}{\partial \theta_s} = 0$$

$$\frac{\partial F_n(P)}{\partial \theta_s} = \frac{1}{k} > 0$$

$$
\begin{aligned}
\frac{\partial E(b_n)}{\partial \theta_s} &= \frac{1}{k+T-\theta_n}\frac{P(\theta_n-T)}{k}ln\left(\frac{P}{\underline{b}}\right)-\frac{P}{k}>0 \Leftrightarrow \\
& \frac{(\theta_n-T)}{k+T-\theta_n}ln\left(\frac{P}{\underline{b}}\right)>1
\end{aligned}
$$

$$
\frac{\partial E(b_s)}{\partial \theta_s} = 0
$$

$$
\frac{\partial \overline{\pi}_n}{\partial \theta_s} = 0
$$

$$
\frac{\partial \overline{\pi}_s}{\partial \theta_s} = \underline{b} > 0
$$

Proposition 2. Area $A1$

$$
\frac{\partial \underline{b}}{\partial T} = \frac{-P}{(\theta_n+\theta_s)} < 0
$$

$$
\begin{aligned}
\frac{\partial F_n(P)}{\partial T} &= \frac{2(\theta_s+T)(\theta_n+\theta_s)(k+T-\theta_n)-(\theta_s+T)^2(\theta_n+\theta_s)}{(\theta_n+\theta_s)^2(k+T-\theta_n)^2} = \\
&= \frac{2(\theta_s+T)(\theta_n+\theta_s)\left[2(k+T-\theta_n)-(\theta_s+T)\right]}{(\theta_n+\theta_s)^2(k+T-\theta_n)^2} > 0
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial E(b_n)}{\partial T} &= \frac{(k+T-\theta_n)-(\theta_s+T)}{(k+T-\theta_n)^2}\underline{b}ln\left(\frac{P}{\underline{b}}\right) \\
& \frac{(\theta_s+T)}{k+T-\theta_n}\frac{\partial \underline{b}}{\partial T}ln\left(\frac{P}{\underline{b}}\right)+ \\
& \frac{(\theta_s+T)}{k+T-\theta_n}\underline{b}\frac{\underline{b}}{P}\frac{\partial \underline{b}}{\partial T}-\frac{\partial F_n(P)}{\partial T} ?0
\end{aligned}
$$

Where, $\frac{(\theta_s+T)}{k+T-\theta_n}\frac{\partial \underline{b}}{\partial T}ln\left(\frac{P}{\underline{b}}\right)$ and $\frac{(\theta_s+T)}{k+T-\theta_n}\underline{b}\frac{\underline{b}}{P}\frac{\partial \underline{b}}{\partial T}-\frac{\partial F_n(P)}{\partial T}$ are negative and $\frac{(k+T-\theta_n)-(\theta_s+T)}{(k+T-\theta_n)^2}\underline{b}ln\left(\frac{P}{\underline{b}}\right)$ is positive.

$$
\begin{aligned}
\frac{\partial E(b_s)}{\partial T} &= \frac{-(\theta_s+\theta_n)}{(k+T-\theta_n)^2}\underline{b}ln\left(\frac{P}{\underline{b}}\right) \\
& \frac{-(\theta_s+\theta_n)}{(k+T-\theta_n)}\frac{\partial \underline{b}}{\partial T}ln\left(\frac{P}{\underline{b}}\right)+ \\
& \frac{-(\theta_s+\theta_n)}{(k+T-\theta_n)}\underline{b}\frac{\underline{b}}{P}\frac{\partial \underline{b}}{\partial T} < 0
\end{aligned}
$$

$$\frac{\partial \overline{\pi}_n}{\partial T} = -P < 0$$

$$\frac{\partial \overline{\pi}_s}{\partial T} = \frac{-P}{k}(\theta_s + T) + \frac{P(\theta_n - T)}{k} = \frac{P(\theta_n - 2T - \theta_s)}{k} > 0 \Leftrightarrow \theta_n > 2T + \theta_s$$

Area $B1$

$$\frac{\partial \underline{b}}{\partial T} = \frac{-P}{k} < 0$$

$$\frac{\partial F_n(P)}{\partial T} = \frac{1}{k} > 0$$

$$\begin{aligned}\frac{\partial E(b_n)}{\partial T} = & \frac{(k + T - \theta_n) - (\theta_s + T)}{(k + T - \theta_n)^2}\underline{b}ln\left(\frac{P}{\underline{b}}\right) \\ & \frac{(\theta_s + T)}{k + T - \theta_n}\frac{\partial \underline{b}}{\partial T}ln\left(\frac{P}{\underline{b}}\right) + \\ & \frac{(\theta_s + T)}{k + T - \theta_n}\underline{b}\frac{\underline{b}}{P}\frac{\partial \underline{b}}{\partial T} - \frac{\partial F_n(P)}{\partial T} < 0\end{aligned}$$

$$\begin{aligned}\frac{\partial E(b_s)}{\partial T} = & \frac{-k}{(k + T - \theta_n)^2}\underline{b}ln\left(\frac{P}{\underline{b}}\right) \\ & \frac{-k}{(k + T - \theta_n)}\frac{\partial \underline{b}}{\partial T}ln\left(\frac{P}{\underline{b}}\right) + \\ & \frac{-k}{(k + T - \theta_n)}\underline{b}\frac{\underline{b}}{P}\frac{\partial \underline{b}}{\partial T} < 0\end{aligned}$$

$$\frac{\partial \overline{\pi}_n}{\partial T} = -P < 0$$

$$\frac{\partial \overline{\pi}_s}{\partial T} = \frac{-P}{k}(\theta_s + T) + \frac{P(\theta_n - T)}{k} = \frac{P(\theta_n - 2T - \theta_s)}{k} > 0 \Leftrightarrow \theta_n > 2T + \theta_s$$

Proposition 3. Uniform Price Auction (pure strategies equilibrium). Using lemma one, the proof is straight forward.

Uniform Price Auction and Discriminatory Price Auction (mixed strategies equilibrium).

Proof:

Using the same steps that I have used in Proposition one, I will work out the equilibrium strategies.

First step, the payoff function for any firm is:

$$\begin{aligned}\pi_i^{II}(b) &= b\left[F_j^{II}(b)max\left\{0,\theta_i-T,\theta_i+\theta_j-k\right\}\right] \\ &\quad +(1-F_j^{II}(b))\left[bmin\left\{\theta_i,k\right\}+E(b_j\mid b_j\geq b)max\left\{0,min\left\{T,k-\theta_i\right\}\right\}\right]= \\ &= -F_j^{II}(b)\left[bmin\left\{\theta_i,k\right\}+E(b_j\mid b_j\geq b)max\left\{0,min\left\{T,k-\theta_i\right\}\right\}\right] \\ &\quad -F_j^{II}(b)\left[-bmax\left\{0,\theta_i-T,\theta_i+\theta_j-k\right\}\right]+ \\ &\quad bmin\left\{\theta_i,k\right\}+E(b_j\mid b_j\geq b)max\left\{0,min\left\{T,k-\theta_i\right\}\right\}\end{aligned} \tag{13}$$

For further reference, I define:
$L_i(b)^{II}=bmin\left\{\theta_i,k\right\}+E(b_j\mid b_j\geq b)max\left\{0,min\left\{T,k-\theta_i\right\}\right\}$ and
$H_i(b)^{II}=bmax\left\{0,\theta_i-T,\theta_i+\theta_j-k\right\}$

Second step, $\pi_i^{II}(b)=\overline{\pi}_i^{II}\forall b\in S_i, i=n,s$, where S_i is the support of the mixed strategies. Then,

$$\begin{aligned}\overline{\pi}_i^{II} &= -F_j^{II}(b)\left[bmin\left\{\theta_i,k\right\}+E(b_j\mid b_j\geq b)max\left\{0,min\left\{T,k-\theta_i\right\}\right\}\right] \\ &\quad -F_j^{II}(b)\left[-bmax\left\{0,\theta_i-T,\theta_i+\theta_j-k\right\}\right]+ \\ &\quad bmin\left\{\theta_i,k\right\}+E(b_j\mid b_j\geq b)max\left\{0,min\left\{T,k-\theta_i\right\}\right\}\Rightarrow \\ F_j^{II}(b) &= \frac{bmin\left\{\theta_i,k\right\}+E(b_j\mid b_j\geq b)max\left\{0,min\left\{T,k-\theta_i\right\}\right\}-\overline{\pi}_i^{II}}{L_i(b)-H_i(b)}\end{aligned} \tag{14}$$

Third step, at $\underline{b}^{II}$, $F_i^{II}(\underline{b}^{II})=0\forall i=n,s$. Then,

$$\overline{\pi}_i^{II} = \underline{b}^{II}min\left\{\theta_i,k\right\}+E(b_j\mid b_j\geq\underline{b}^{II})max\left\{0,min\left\{T,k-\theta_i\right\}\right\} \tag{15}$$

Fourth step, Plug in 15 into 14, we obtain the mixed strategies for both firms.

$$F_j^{II}(b) = \frac{L_i^{II}(b)-L_i^{II}(\underline{b}^{II})}{L_i^{II}(b)-H_i^{II}(b)}\forall i=n,s \tag{16}$$

It is easy to verify that equation $F_j^{II}(b)\forall i,j$ is indeed a cumulative distribution function. First, in the third step, I have established that $F_j^{II}(\underline{b}^{II})=0$. Second, $F_j^{II}(b)$ is an increasing function in b. At $\underline{b}^{II}$, $L_i^{II}(\underline{b}^{II})=H_i^{II}(b)$, for any $b>\underline{b}^{II}$, $L_i^{II}(\underline{b}^{II})<H_i^{II}(b)$; moreover, $\dfrac{\partial L_i^{II}(b)}{\partial b}>0$, $\dfrac{\partial L_i^{II}(\underline{b}^{II})}{\partial b}=0$ and $\dfrac{\partial H_i^{II}(b)}{\partial b}>0$, therefore, $\dfrac{\partial\left(L_i^{II}(b)-L_i^{II}(\underline{b}^{II})\right)}{\partial b}>\dfrac{\partial\left(L_i^{II}(b)-H_i^{II}(b)\right)}{\partial b}$. Third, $F_j^{II}(b)\leq 1\forall b\in S_i$. Finally, $F_j^{II}(b)$ is continuous in the support because $L_i^{II}(b)-L_i^{II}(\underline{b}^{II})$ and $L_i^{II}(b)-H_i^{II}(b)$ are continuous functions in the support.

In area $A1$, using equation 16, the cumulative distribution function for both firms is defined by:

$$
F_s^{II}(b) = \begin{cases} 0 & \text{if } b < \underline{b}^{II} \\ \dfrac{\theta_n + \theta_s}{\theta_s + T} \dfrac{b - \underline{b}^{II}}{b} & \text{if } \underline{b}^{II} \leq b < P \\ 1 & \text{if } P \leq b \end{cases}
$$

$$
F_n^{II}(b) = \begin{cases} 0 & \text{if } b < \underline{b}^{II} \\ \dfrac{\theta_s \left(b - \underline{b}^{II}\right) + T\left(E(b_n \mid b_n \geq b) - E\left(b_n \mid b_n \geq \underline{b}^{II}\right)\right)}{b\theta_s + E\left(b_n \mid b_n \geq b\right) T} & \text{if } \underline{b}^{II} \leq b < P \\ 1 & \text{if } P \leq b \end{cases}
$$

Moreover,

$$
\begin{aligned}
F_s^{II}(P) &= \frac{\theta_n + \theta_s}{\theta_s + T} \frac{P - \dfrac{P(\theta_n - T)}{\theta_n + \theta_s}}{P} = 1 \\
F_n^{II}(P) &= \frac{\theta_s \left(P - \underline{b}^{II}\right) + T\left(P - E\left(b_n \mid b_n \geq \underline{b}^{II}\right)\right)}{P(\theta_s + T)} < 1
\end{aligned}
$$

In area $B1$, using equation 16, the cumulative distribution function for both firms is defined by:

$$
F_s^{II}(b) = \begin{cases} 0 & \text{if } b < \underline{b}^{II} \\ \dfrac{\theta_n \left(b - \underline{b}^{II}\right) + (k - \theta_n)\left(E(b_s \mid b_s \geq b) - E\left(b_s \mid b_s \geq \underline{b}^{II}\right)\right)}{bT + E\left(b_s \mid b_s \geq b\right)(k - \theta_n)} & \text{if } \underline{b}^{II} \leq b < P \\ 1 & \text{if } P \leq b \end{cases}
$$

$$
F_n^{II}(b) = \begin{cases} 0 & \text{if } b < \underline{b}^{II} \\ \dfrac{\theta_s \left(b - \underline{b}^{II}\right) + T\left(E(b_n \mid b_n \geq b) - E\left(b_n \mid b_n \geq \underline{b}^{II}\right)\right)}{b(k - \theta_n) + E\left(b_n \mid b_n \geq b\right) T} & \text{if } \underline{b}^{II} \leq b < P \\ 1 & \text{if } P \leq b \end{cases}
$$

Moreover,

$$
\begin{aligned}
F_s^{II}(P) &= 1 \\
F_n^{II}(P) &= \frac{\theta_s \left(P - \underline{b}^{II}\right) + T\left(P - E\left(b_n \mid b_n \geq \underline{b}^{II}\right)\right)}{P(T + k - \theta_n)} < 1
\end{aligned}
$$

In area $B2$, using equation 16, the cumulative distribution function for both firms is defined by:

$$
F_s^{II}(b) = \begin{cases} 0 & \text{if } b < \underline{b}^{II} \\ \dfrac{k}{T+k-\theta_n}\dfrac{b-\underline{b}^{II}}{b} & \text{if } \underline{b}^{II} \leq b < P \\ 1 & \text{if } P \leq b \end{cases}
$$

$$
F_n^{II}(b) = \begin{cases} 0 & \text{if } b < \underline{b}^{II} \\ \dfrac{\theta_s\left(b-\underline{b}^{II}\right)+T\left(E(b_n \mid b_n \geq b)-E\left(b_n \mid b_n \geq \underline{b}^{II}\right)\right)}{b(k-\theta_n)+E\left(b_n \mid b_n \geq b\right)T} & \text{if } \underline{b}^{II} \leq b < P \\ 1 & \text{if } P \leq b \end{cases}
$$

Moreover

$$
\begin{aligned}
F_s^{II}(P) &= 1 \\
F_n^{II}(P) &= \frac{\theta_s\left(P-\underline{b}^{II}\right)+T\left(P-E\left(b_n \mid b_n \geq \underline{b}^{II}\right)\right)}{P(T+k-\theta_n)} < 1
\end{aligned}
$$

In area B, using equation 16, the cumulative distribution function for both firms is defined by:

$$
F_i(b) = \begin{cases} 0 & \text{if } b < \underline{b} \\ \dfrac{k}{2k-\theta_i-\theta_j}\dfrac{b-\underline{b}}{b} & \text{if } b \in (\underline{b}, P) \quad \forall i = s, n \\ 1 & \text{if } b = P \end{cases}
$$

Second, I work out the support of the mixed strategy equilibrium.

In the diagonal, the payoff function for both firms is equal, therefore $\underline{b}_n^{II} = \underline{b}_s^{II}$.

In area $A1$, $\underline{b}_n^{II} \geq \underline{b}_s^{II}$. The residual demand in area $A1$ for firm s is zero, therefore $\underline{b}_s^{II} = 0$, moreover $\underline{b}_n^{II} = \dfrac{P(\theta_n - T)}{k}$. Therefore $S^{II} = \left[max\left\{\underline{b}_n^{II}, \underline{b}_s^{II}\right\}, P\right] = \left[\underline{b}_n^{II}, P\right] = \left[\dfrac{P(\theta_n - T)}{k}, P\right]$.

In area $B2$, $\underline{b}_n^{II} = \underline{b}_s^{II}$. In area $B2$, $\theta_n > k > \theta_s$. Therefore, using lemma three, $b_n^{II} = b_n$ and $b_s^{II} \leq b_s$. Using proposition two, I know that in areas $B2$, $b_n > b_s$. Hence $b_s^{II} < b_s < b_n < b_n^{II}$. Then, in area $B2$, $S^{II} = \left[max\left\{\underline{b}_n^{II}, \underline{b}_s^{II}\right\}, P\right] = \left[\underline{b}_n^{II}, P\right] = \left[\dfrac{P(\theta_n - T)}{k}, P\right]$.

In area B, $\underline{b}_n^{II} \geq \underline{b}_s^{II}$. According with lemma three, in area B, $\underline{b}_n^{II}$ solves $\underline{b}_n^{II}\theta_n + E\left(b_s \mid b_s \geq \underline{b}_n^{II}\right)(k-\theta_n) = P(\theta_n+\theta_s-k)$ and $\underline{b}_s^{II}$ solves $\underline{b}_s^{II}\theta_s + E\left(b_n \mid b_n \geq \underline{b}_s^{II}\right)(k-\theta_s) = P(\theta_n+\theta_s-k)$. The right hand side of both expressions is equal. Therefore $\underline{b}_n^{II}\theta_n + E\left(b_s \mid b_s \geq \underline{b}_n^{II}\right)(k-\theta_n) = \underline{b}_s^{II}\theta_s + E\left(b_n \mid b_n \geq \underline{b}_s^{II}\right)(k-\theta_s)$. When the realization of the demand is an ϵ over the diagonal $(\theta_n, \theta_s) = (\theta_n, \theta_n - \epsilon)$. Therefore, $\underline{b}_n^{II}\theta_n + E\left(b_s \mid b_s \geq \underline{b}_n^{II}\right)(k-\theta_n) = \underline{b}_s^{II}(\theta_n - \epsilon) + E\left(b_n \mid b_n \geq \underline{b}_s^{II}\right)(k-\theta_n+\epsilon)$. My claim is that in area B, $b_n^{II} > b_s^{II}$. In order to prove it, I will start assuming that when we move an ϵ to the left of the diagonal $E\left(b_s \mid b_s \geq \underline{b}_n^{II}\right) = E\left(b_n \mid b_n \geq \underline{b}_s^{II}\right)$.[14] Under this assumption, $\underline{b}_n^{II}\theta_n + E\left(b_s \mid b_s \geq \underline{b}_s^{II}\right)(k-\theta_n) = \underline{b}_s^{II}(\theta_n - \epsilon) + E\left(b_s \mid b_s \geq \underline{b}_s^{II}\right)(k - \theta_n + \epsilon)$. After simple algebra, I obtain $\underline{b}_n^{II}\theta_n = \underline{b}_s^{II}\theta_n + \epsilon\left[E\left(b_s \mid b_s \geq \underline{b}_s^{II}\right) - \underline{b}_s^{II}\right]$, where $\epsilon\left[E\left(b_s \mid b_s \geq \underline{b}_s^{II}\right) - \underline{b}_s^{II}\right] > 0$. Therefore $\underline{b}_n^{II} > \underline{b}_s^{II}$. Therefore, using lemma three,

$$S^{II} = \left[max\left\{\underline{b}_n^{II}, \underline{b}_s^{II}\right\}, P\right] = \left[\frac{P(\theta_n + \theta_s - k) - E\left(b_s \mid b_s \geq \underline{b}_n^{II}\right)(k-\theta_n)}{\theta_n}, P\right]$$

Finally, in area $B1$, $b_n^{II} \geq b_s^{II}$. In areas $A1$, $B2$ and B (all the areas that border $B1$), $b_n^{II} \geq b_s^{II}$ and $b_i^{II} \forall i = n, s$ is a continuous and monotone function. Therefore, $b_n^{II} \geq b_s^{II}$. Hence, using lemma 2,

$$S^{II} = \left[max\left\{\underline{b}_n^{II}, \underline{b}_s^{II}\right\}, P\right] = \left[\frac{P(\theta_n - T) - E\left(b_s \mid b_s \geq \underline{b}_n^{II}\right)(k-\theta_n)}{\theta_n}, P\right]$$

Third, I work out the payoff function.

Using equation 15. The payoff function in area $A1$ is:

$$\begin{aligned}\overline{\pi}_n^{II} &= \underline{b}^{II}(\theta_n + \theta_s) \\ \overline{\pi}_s^{II} &= \underline{b}^{II}\theta_s + E\left(b_n \mid b_n \geq \underline{b}^{II}\right)T\end{aligned}$$

Using equation 15. The payoff function in area $B1$ is:

$$\begin{aligned}\overline{\pi}_n^{II} &= \underline{b}^{II}\theta_n + E\left(b_s \mid b_s \geq \underline{b}^{II}\right)(k-\theta_n) \\ \overline{\pi}_s^{II} &= \underline{b}^{II}\theta_s + E\left(b_n \mid b_n \geq \underline{b}^{II}\right)T\end{aligned}$$

Using equation 15. The payoff function in area $B2$ is:

$$\begin{aligned}\overline{\pi}_n^{II} &= \underline{b}^{II}k \\ \overline{\pi}_s^{II} &= \underline{b}^{II}\theta_s + E\left(b_n \mid b_n \geq \underline{b}^{II}\right)T\end{aligned}$$

Using equation 15. The payoff function in area B is:

$$\overline{\pi}_i^{II} = \underline{b}^{II}k \forall i = s, n$$

Finally, the expected bid and $Prob(b_i \leq b_j)$ can not be worked out analytically because the cumulative distribution function does not present a close form solution. However, below in the annex, I will describe an algorithm to work out the cumulative distribution

[14] First, in the diagonal $\theta_n = \theta_s$, moreover the firms are symmetric in capacity and costs, therefore in the diagonal the mixed strategies and the support are symmetric. Hence, assume $E\left(b_s \mid b_s \geq \underline{b}_n^{II}\right) = E\left(b_n \mid b_n \geq \underline{b}_s^{II}\right)$ when the realization of demand is an ϵ over the diagonal is reasonable. Second, if under the assumption $E\left(b_s \mid b_s \geq \underline{b}_n^{II}\right) = E\left(b_n \mid b_n \geq \underline{b}_s^{II}\right)$, I obtain that $b_n^{II} > b_s^{II}$, then $F_s(b)^{II}$ stochastic dominate $F_n(b)^{II}$ and $E\left(b_s \mid b_s \geq \underline{b}_n^{II}\right) \leq E\left(b_n \mid b_n \geq \underline{b}_s^{II}\right)$, but if I assume $E\left(b_s \mid b_s \geq \underline{b}_n^{II}\right) \leq E\left(b_n \mid b_n \geq \underline{b}_s^{II}\right)$, the proof shows even stronger evidence in favour of $b_n^{II} > b_s^{II}$.

function and the expected bid.

Algorithm to work out the cumulative distribution function.

The payoff function of the model in which the transmission rights are assigned to the firm that submits the lower bid in the spot electricity auction satisfies the properties that guarantee that a mixed equilibrium exists, however the cumulative distribution function defined by equation 16 is a function of its own expected value, therefore it does not exist a close form solution for it. In the next lines, I present an algorithm that gives me the opportunity to work out an approximation of the cumulative distribution function.

To made the exposition easier, I will focus only in the equilibrium in Area $B2$ in figure 6. The support and mixed strategies equilibrium in area $B2$ are defined by the next equations:

$$S^{II} = \left[max\left\{\underline{b}_n^{II}, \underline{b}_s^{II}\right\}, P\right] = \left[\underline{b}_n^{II}, P\right] = \left[\frac{P(\theta_s + \theta_n - k)}{k}, P\right] \tag{17}$$

$$F_s^{II}(b) = \frac{k(b - \underline{b}^{II})}{(2k - \theta_n - \theta_s)b} \tag{18}$$

$$F_n^{II}(b) = \frac{\theta_s(b - \underline{b}^{II}) + (k - \theta_s)\left[E\left(b_n \mid b_n \geq b\right) - E\left(b_n \mid b_n \geq \underline{b}^{II}\right)\right]}{b(k - \theta_n) + E\left(b_n \mid b_n \geq b\right)(k - \theta_s)} \tag{19}$$

Equations 17 and 18 do not depend of any expectation. Therefore, they can be easily computed. Nevertheless, equation 19, depends on its own expectation. Therefore, it does not exist a close form solution for it. To work out the cumulative distribution function defined by 19, I have developed the algorithm that I describe below. The key point in the algorithm is guarantee that the prior expected value that I use to work out 19 and the posterior expected value calculated using 19 coincide, i.e., a fix point exist.

The algorithm to work out the cumulative distribution function defined in 19 consists in the next sequence of iterations. In the first iteration, I have taken the cumulative distribution function for area $B2$ when the transmission rights are assigned to the grid operator defined by equation 11. In the rest of iterations, I use the cumulative distribution function generated in the previous iteration to work out the cumulative distribution function defined in 19. In each iteration, I work out the difference in the expected value between two consecutive iterations, (figure 10 summarizes this information). The iteration process concludes when the difference in means between two consecutive iterations is zero.

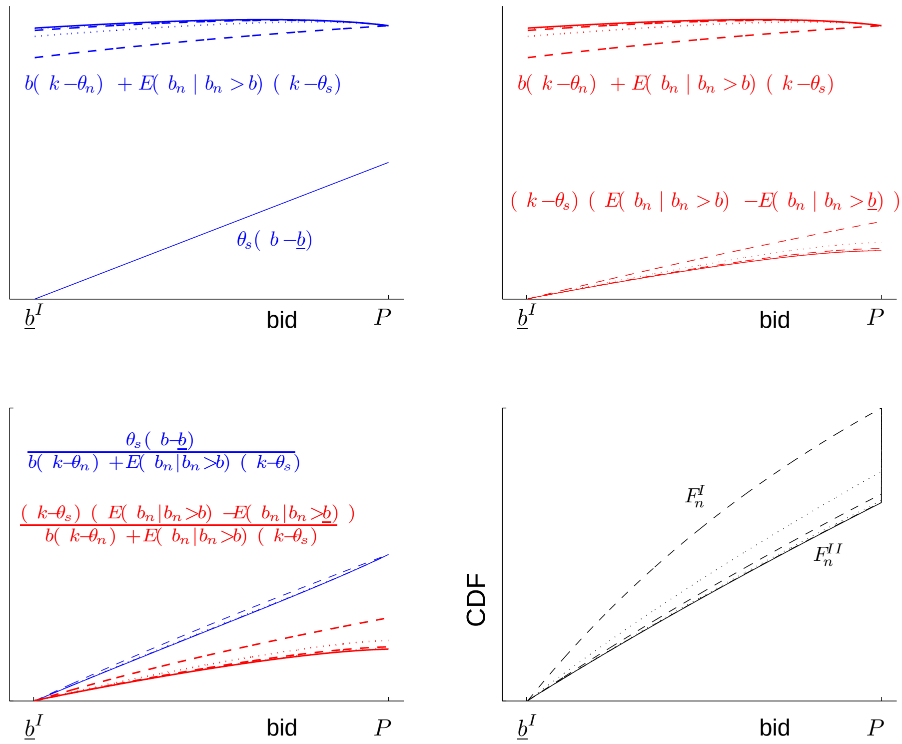
In the next lines, I provide evidence that the algorithm that I have described above converges to a fix point.

Equation 19 can be split in two equations:

$$E_1(b) = \frac{\theta_s(b - \underline{b}^{II})}{b(k - \theta_n) + E\left(b_n \mid b_n \geq b\right)(k - \theta_s)} \tag{20}$$

$$E_2(b) = \frac{(k - \theta_s)\left[E\left(b_n \mid b_n \geq b\right) - E\left(b_n \mid b_n \geq \underline{b}^{II}\right)\right]}{b(k - \theta_n) + E\left(b_n \mid b_n \geq b\right)(k - \theta_s)} \tag{21}$$

Figure 9: Existence and Uniqueness of the CDF



In figure 9, I have plotted the first six iterations of the algorithm that I have described above. In the first panel (starting from the top left), I have plotted in blue the iterations for the numerator and the denominator of equation 20. As can be observed the numerator does not change and the denominator converges quickly. In the second panel, I have plotted in red the six first iterations for the numerator and denominator of equation 21. As can be observed, both converge quickly. In the third panel, I have plotted in blue the six first iterations for equation 20 and in red the six first iteration for equation 21. As can be observed both equations converges quickly. Finally, in the last panel, I have plotted in black the six first iterations for equation 19, as can be observed the iteration process converges quickly.

In figure 10 I have plotted the difference in means between two consecutive iterations. As can be observed, in the second iteration the difference in means is big and in latter iterations the difference decreases smoothly. The difference in mean becomes zero between iteration six and iteration seven. This means that there exists a cumulative distribution function between the one obtained in iteration six and the one obtained in iteration seven for which the prior mean used to work out the cumulative distribution function and the posterior mean derived from the cumulative distribution function coincide, i.e., a fix point exists.

Above, I have described the algorithm to work out the equilibrium and I have shown that the algorithm converges. In the next lines, I provide evidence that the algorithm do not have internal mistakes, i.e., generates the results predicted by the model. Lemma two stablish $\underline{b}^{II}\theta_i + E(b_j \mid b_j \geq \underline{b}^{II})(k_i - \theta_i) = P(\theta_i - T)$. Column four in table

Figure 10: Existence of a fix point

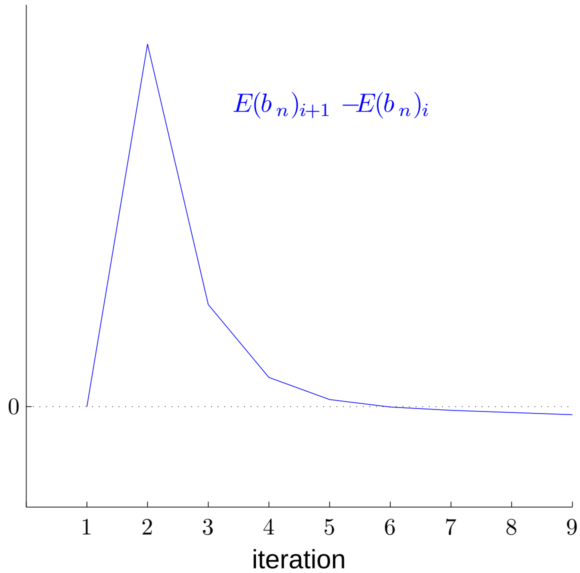


Table 7: $\theta_n = 55, \theta_s = 5, k_n = k_s = 60, c_n = c_s = 0, P = 7$

	$\underline{b}^{II}$	$E(b_s)$	$\overline{\pi}_n = \underline{b}^{II}\theta_n + E(b_s)(k - \theta_n)$	$\overline{\pi}_n = P(\theta_n - T)$
$T = 60$	0	0	0	0
$T = 50$	0.50	1.49	$(0.5 \cdot 55) + (1.49 \cdot 5) = 34.95$	35
$T = 40$	1.62	3.14	$(1.62 \cdot 55) + (3.14 \cdot 5) = 104.8$	105
$T = 30$	2.79	4.31	$(2.79 \cdot 55) + (4.31 \cdot 5) = 175$	175
$T = 20$	3.97	5.23	$(3.97 \cdot 55) + (5.23 \cdot 5) = 244.5$	245
$T = 10$	5.18	6.02	$(5.18 \cdot 55) + (6.02 \cdot 5) = 315$	315
$T = 0$	7	7	$(7 \cdot 55) + (7 \cdot 5) = 385$	385

7 presents the expected profit for firm n using the values generated by the algorithm $(\underline{b}^{II}\theta_i + E(b_j \mid b_j \geq \underline{b}^{II})(k_i - \theta_i))$. Column five presents the expected profit predicted by the theory $(P(\theta_i - T)\ )$. As can be observed both values coincide.

Proposition 4.

Area $B2$. Using proposition two and four it is straight forward to check that the lower bound of the support, the expected value of the bids of the firm located in the South and the expected payoff of the firm located in the North are equal in both models.

In area $B2$, $F_n(\underline{b}) = F_n^{II}(\underline{b}^{II}) = 0$.
$F_n(P) = \dfrac{(\theta_s + T)(P - \underline{b})}{P(T + k - \theta_n)}$ and $F_n^{II}(P) = \dfrac{\theta_s(P - \underline{b}^{II}) + T(P - E(b_n \mid b_n \geq \underline{b}^{II})}{P(T + k - \theta_n)}$. As I have shown before, $\underline{b} = \underline{b}^{II}$, moreover $E(b_n \mid b_n \geq \underline{b}^{II}) \geq \underline{b})$, therefore, $F_n^{II}(P) \leq F_n(P)$. Furthermore, $F_n^{II}(b)$ and $F_n(b)$ are monotone increasing and continue in the support. Therefore, $F_n(b) \geq F_n^{II}(b) \forall b \in [\underline{b}, P]$. Hence, $F_n(b)$ stochastic dominate $F_n^{II}(b)$. Then, $E(b_n^{II}) \geq E(b_n)$. Moreover, the expected payoff of the firm located in the South $\overline{\pi}_s^{II} \geq \overline{\pi}_s$.

Areas $B1$. Using proposition one and three it is straight forward to check that the lower bound of the support when the transmission rights are assigned to the firm that submits the lower bid in the spot electricity auction is lower than the lower bound of the support when the transmission rights are assigned to the grid operator. Moreover, the expected payoff of the firm located in the North is equal in both models, $\overline{\pi}_n = \overline{\pi}_n^{II}$.

In area $B1$, $F_n(\underline{b}) = 0 \leq F_n^{II}(\underline{b}) = 0$. $F_n(P) = \dfrac{(\theta_s + T)(P - \underline{b})}{P(T + k - \theta_n)}$ and $F_n^{II}(P) = \dfrac{\theta_s(P - \underline{b}^{II}) + T(P - E(b_n \mid b_n \geq \underline{b}^{II})}{P(T + k - \theta_n)}$. As I have shown before, $\underline{b}^{II} \leq \underline{b}$, $E(b_n \mid b_n \geq \underline{b}^{II}) \geq \underline{b}^{II})$. However, with this information, $F_n^{II}(P)$ and $F_n(P)$ can not be ranked. Therefore, it can not be determined the stochastic dominate relation between $F_n^{II}(b)$ and $F_n(b)$. And so, the expected value of bids for the firm located in the North.

In area $B1$, $F_s(\underline{b}) = 0 \leq F_s^{II}(\underline{b})$. $F_s(P) = F_s^{II}(P) = 1$. Moreover, $F_s^{II}(b)$ and $F_s(b)$ are monotone increasing and continue in the support. Therefore, $F_s \leq F_s^{II} \forall b \in [\underline{b}^{II}, P]$. Hence, $F_s^{II}(b)$ stochastic dominate $F_s(b)$. Then, $E(b_s) \geq E(b_s^{II})$.

Finally, in areas $B1$, $E(b_s) \geq E(b_s^{II})$, but the relation between $E(b_n)$ and $E(b_s^{II})$ can not be determined. Hence, the relation between the expected payoff of the firm located in the South can not be ranked in both models.

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