Social Interactions and Spillovers

Antoni Calvó-Armengol, Antonio Cabrales
& Yves Zenou

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Introduction

- Social interactions are important.
- social interactions have a crucial importance in the determination of an individual's outcomes
- Need a model with both network formation and effort choices

Introduction

- Network formation, where do we stand?
  - Non-cooperative games of network formation with nominal lists of intended links are plagued by coordination problems.
  - Cooperative-like stability concepts solve them partially, but heavy combinatorial costs still jeopardize a full characterization.
  - Bringing predictions down to stylized facts about the topology of networks calls for heuristic (non-equilibrium) dynamic models.

Non-cooperative models of network formation

- Each player picks $(s_{i1}, ..., s_{in}) \in \{0, 1\}^{n-1}$, one among the 2^{n-1} possible link announcements
- Strategy profiles are then mapped into a network adjacency matrix:

$$\begin{bmatrix} 0 & g_{ij}(s) \\ g_{ji}(s) & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & s_{ij} \times s_{ji} \\ s_{ij} \times s_{ji} & \end{bmatrix}}_{un-directed-network} \text{ or } \underbrace{\begin{bmatrix} 0 & s_{ij} \\ s_{ji} & \end{bmatrix}}_{directed-network}$$

- We have a combinatorial (coordination) equilibrium multiplicity

One possible way out

- Players devote a *generic* amount of resources to socializing, a scalar $s_i \geq 0$, rather than deciding whether to interact with every other agent
- Then, socialization efforts (n values) are mapped into link intensities $g_{ij}(\mathbf{s}) : \mathbb{R}_+^n \to [0, 1]$ (n^2 values)
- We still get a rich network with heterogeneous link intensities, but abandon the paradigm of a discrete graph.

Aim of this paper

- We want a simple static model of network formation that:
  - allows for a joint analysis of network formation and the optimal economic use of this collectively created device
  - off-the-shelf Nash equilibrium analysis leads to sharp predictions
  - permits a full-fledged welfare analysis of individual decisions and to derive unambiguous comparative statics
  - can explain some (if not all) facts about network topology

What do we do?

- Agents take two decisions: production and synergy efforts.
- Payoffs depend on own and others' production efforts –spillovers.
- Spillovers vary with the synergistic strength between pairs of agents.
- The innovation: agents devote a joint (generic) effort to networking, rather than deciding how much to interact with every other agent.
- We still get a rich network with heterogeneous ($n(n-1)/2$) link intensities, but abandon the paradigm of a discrete graph.

What do we get?

- We characterize all the (approximate) Nash equilibria:
  - one (strict but unstable) no-synergies equilibrium
  - two (stable) interior equilibria, both actions- and Pareto-ranked
- We document the comparative statics w.r.t. the spillover size and the individual traits: the synergy effort is more responsive
- We relate the equilibrium synergies to a standard random graph model, and draw welfare implications of the network topology

Synergistic effort is *generic* within a community —a scalar decision.

This is realistic in many applications, particularly when networks are so large that keeping track of every participant becomes a burdensome task, or when the individuals do not yet know one another.

Examples: businessmen go to fairs, and researchers to conferences and workshops to present their ideas or products, to listen to those of other people's, and to meet peers in general.

Concrete example:

Decision makers: parents.

Each parent exerts two types of costly effort:

Productive effort with the child (i.e. doing homework with the child, doing sport activities together, driving him to different activities, and so on)

Socialization effort related to education (going to parental evenings, birthday parties, or any activity that involves other parents).

Related literature

- Games on networks:
  - Ballester, C.-A. & Zenou *Eca* 06
- Network formation:
  - Jackson *WCES* 06 (Myerson 1991, Jackson & Wolinsky *JET* 96, Bala & Goyal *Eca* 00, C.-A. & Ilkiliç 06)
- Networks with link intensities:
  - Bloch and Dutta (IJGT 2009)
- Topology of random graphs:
  - Chang & Lu *PNAS* 02; Jackson & Rogers *AER* 07; Köning, Tessone and Zenou (2009)

The game

The replica game

$N = \{1, ..., n\}$ is a finite set of players

$T = \{1, \ldots, t\}$ finite set of types for these players.

We let n be a multiple of t, that is, $n = mt$ for some integer $m \geq 1$, so that there is the same number of players of each type.

Case $n = t$ as *the baseline game*,

Case $n = mt$ as *the* $m-$*replica* of this baseline game.

In an $m-$replica game, there are exactly m players of each type $\tau \in T$.

This replica game allows us to take limits as the population becomes large without having to specify the types of the new individuals that are added.

For each player $i \in N$, we denote by $\tau(i) \in T$ his type.

Simultaneous move game of network formation (or social interactions) and investment.

Network formation Consider some $m-$replica game, $m \geq 1$. Let $n = mt$.

Mapping socialization efforts to a network adjacency matrix

Lemma

Suppose that, for all $\mathbf{s} \neq \mathbf{0}$*, the link intensity satisfies:*

(A1) *symmetry:* $g_{ij}(\mathbf{s}) = g_{ji}(\mathbf{s})$*, for all* i, j*;*

(A2) *constant returns to scale:* $\sum_{j=1}^{n} g_{ij}(\mathbf{s}) = s_i$*;*

(A3) *homogeneous mixing:* $g_{ij}(\mathbf{s}) / s_i = g_{kj}(\mathbf{s}) / s_k$*, for all* i, j, k*;*

then, the link intensity is:

$$g_{ij}(\mathbf{s}) = \frac{s_i s_j}{\sum_{k=1}^{n} s_k}, \text{ if } \mathbf{s} \neq \mathbf{0}, \text{ and } 0 \text{ otherwise.}$$

- With $\max_i s_i^2 < \sum_{k=1}^{n} s_k$, a random graph with expected degree sequence $\mathbf{s} = (s_1, ..., s_n)$.

t players with individual traits $0 < b_1 \leq ... \leq b_t$

Let $c > 0$. Player i chooses a productive $k_i \geq 0$ and a synergy effort $s_i \geq 0$:

$$u_i(\mathbf{s}, \mathbf{k}) = b_{\tau(i)}\, k_i + a \sum_{j=1, j \neq i}^{n} g_{ij}\left(\mathbf{s}\right) k_j k_i - \frac{1}{2} c k_i^2 - \frac{1}{2} s_i^2 \tag{1}$$

$$\frac{\partial^2 u_i(\mathbf{s}, \mathbf{k})}{\partial k_i \partial k_j} = a g_{ij}(\mathbf{s}), \text{ for all } i \neq j, \tag{2}$$

Strategic complementarities in productive investments.

$a g_{ij}(\mathbf{s}) \geq 0$: Size of these complementarities depends on the profile of socialization efforts, and varies across different pairs of players.

Payoffs also display strategic complementarities in socialization efforts, $\partial^2 u_i(\mathbf{s}, \mathbf{k}) / \partial s_i \partial s_j \geq 0$.

Education example.

$b_{\tau(i)}\, k_i + a \sum_{j=1, j\neq i}^{n} g_{ij}\left(\mathbf{s}\right) k_j k_i$: outcome of the child (future wage or current grade)

$-\frac{1}{2} c k_i^2 - \frac{1}{2} s_i^2$ the cost of both efforts for parent i.

$$u_i(\mathbf{s}, \mathbf{k}) = \underbrace{b_{\tau(i)}\, k_i + a \sum_{j=1, j\neq i}^{n} g_{ij}\left(\mathbf{s}\right) k_j k_i}_{\text{Reward from activity}} \underbrace{- \frac{1}{2} c k_i^2 - \frac{1}{2} s_i^2}_{\text{Cost from activity}}$$

Equilibrium characterization

Lemma

There always exists a pure strategy Nash equilibrium with no player investing in synergies. For high enough m, this partially corner equilibrium is unstable.

- An increasing function of population heterogeneity:

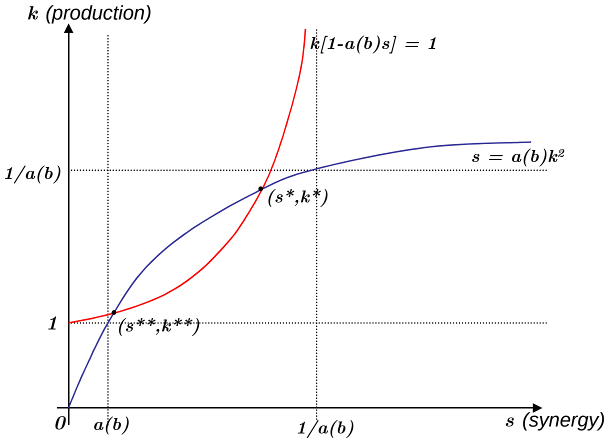
$$a(\mathbf{b}) = a\frac{\sum_{\tau=1}^{t} b_\tau^2}{\sum_{\tau=1}^{t} b_\tau}$$

Theorem 1 *Suppose that* $2\,(c/3)^{3/2} > a(\mathbf{b}) > 0$. *For m large enough, there are exactly two interior pure strategy Nash equilibria with strategies converging to* $b_{\tau(i)}(k,s)$, *where the baseline* (k,s) *are the positive solutions to:*

$$\left\{ \begin{array}{l} s = a(\mathbf{b})k^2 \\ k\,[c - a(\mathbf{b})s] = 1 \end{array} \right. . \tag{3}$$

Proposition 2 *For m sufficiently large, the two interior equilibria are stable while the equilibrium with* $\left(s_i^*, k_i^*\right) = \left(0, b_{\tau(i)}/c\right)$ *for all* $i = 1, ..., mt$ *is not stable.*

Equilibrium networks and actions



Homogeneous population with common trait b

Table 1: Simulations with $a = 2, c = 1, t = 1$ and $b = 0.1$.

m	2	5	10	20	50	100	500	∞
Low equilibrium								
k^*	1,898	1,195	1,101	1,065	1,049	1,046	1,046	1,046
s^*	2,366	815	458	303	234	222	218	219
High equilibrium								
k^*	3,346	4,643	4,591	4,508	4,444	4,420	4,400	4,394
s^*	3,506	3,923	3,911	3,891	3,875	3,869	3,864	3,862

Sketch of the proof

- Fix **s** and let **G**(**s**)=[g_{ij}(**s**)], a matrix.
- The game in **k** has a pure strategy Nash equilibrium (then interior) if and only if a(**s**)<1 (Ballester *et al.* 2006, Ballester & C.-A. 2006)
- Then $\left[\mathbf{I} - a\mathbf{G}(\mathbf{s})\right]^{-1} = \mathbf{I} + \frac{a\overline{\mathbf{s}}}{\overline{\mathbf{s}} - a\overline{\mathbf{s}^2}}\mathbf{G}(\mathbf{s})$
- Let {m^h} increasing, and {$\mathbf{s}^h$} such that a($\mathbf{s}^h$)<1. Then, $s_i^h \in O(1)$ and thus $k_i^h \in O(1)$ as well.
- Then the FOC converge to the simple marginal cost/benefit equations, and all positive (**k**,**s**) solutions are such that a(**s**)<1.

The approximated equilibria $(\mathbf{s}^*, \mathbf{k}^*)$ display three important features.

1) The level of socialization per unit of productive investment is the same for all players, that is, $s_i^*/k_i^* = s_j^*/k_j^*$, for all i, j.

2) Differences in productive investments reflect differences in idiosyncratic traits, i.e. $k_i^*/k_j^* = b_{\tau(i)}/b_{\tau(j)}$, for all i, j.

3) In the presence of synergies, productive investments are all scaled up (compared to the case without synergies) by a *synergistic multiplier*, which is $\frac{1}{1-a(\mathbf{b})s^*/c}$.

Welfare

Approximate equilibrium $(\mathbf{s}^*, \mathbf{k}^*)$:

$$\mathbf{u}(\mathbf{s}^*, \mathbf{k}^*) = (u_1(\mathbf{s}^*, \mathbf{k}^*), ..., u_m(\mathbf{s}^*, \mathbf{k}^*))$$

corresponding equilibrium payoffs.

$\left(\mathbf{s}^E, \mathbf{k}^E\right)$ the (approximate) efficient outcome, i.e., the one (almost) maximizing the sum of payoffs for all players in a large $m-$replica game.

Proposition 3 *Assume* $0 < a(\mathbf{b}) < 2(c/3)^{3/2}$ *and let* $(\mathbf{s}^*, \mathbf{k}^*)$ *and* $(\mathbf{s}^{**}, \mathbf{k}^{**})$ *be the two different approximate equilibria of an* $m-$*replica game. Then, without loss of generality,* $(\mathbf{s}^*, \mathbf{k}^*) \geq \left(\mathbf{s}^E, \mathbf{k}^E\right) \geq (\mathbf{s}^{**}, \mathbf{k}^{**})$ *and* $\mathbf{u}\left(\mathbf{s}^E, \mathbf{k}^E\right) \geq \mathbf{u}(\mathbf{s}^*, \mathbf{k}^*) \geq \mathbf{u}(\mathbf{s}^{**}, \mathbf{k}^{**})$*, where* $\geq$ *is the component-wise ordering.*

Pareto-superior and Pareto-inferior approximate equilibrium as high and low equilibrium.

Socially efficient outcome lies in between the two equilibria.

Education: Parents put too much effort in socializing and educating their children (hign equilibrium)

Parents exert too little effort (low-equilibrium).

Comparative statics: actions

Theorem

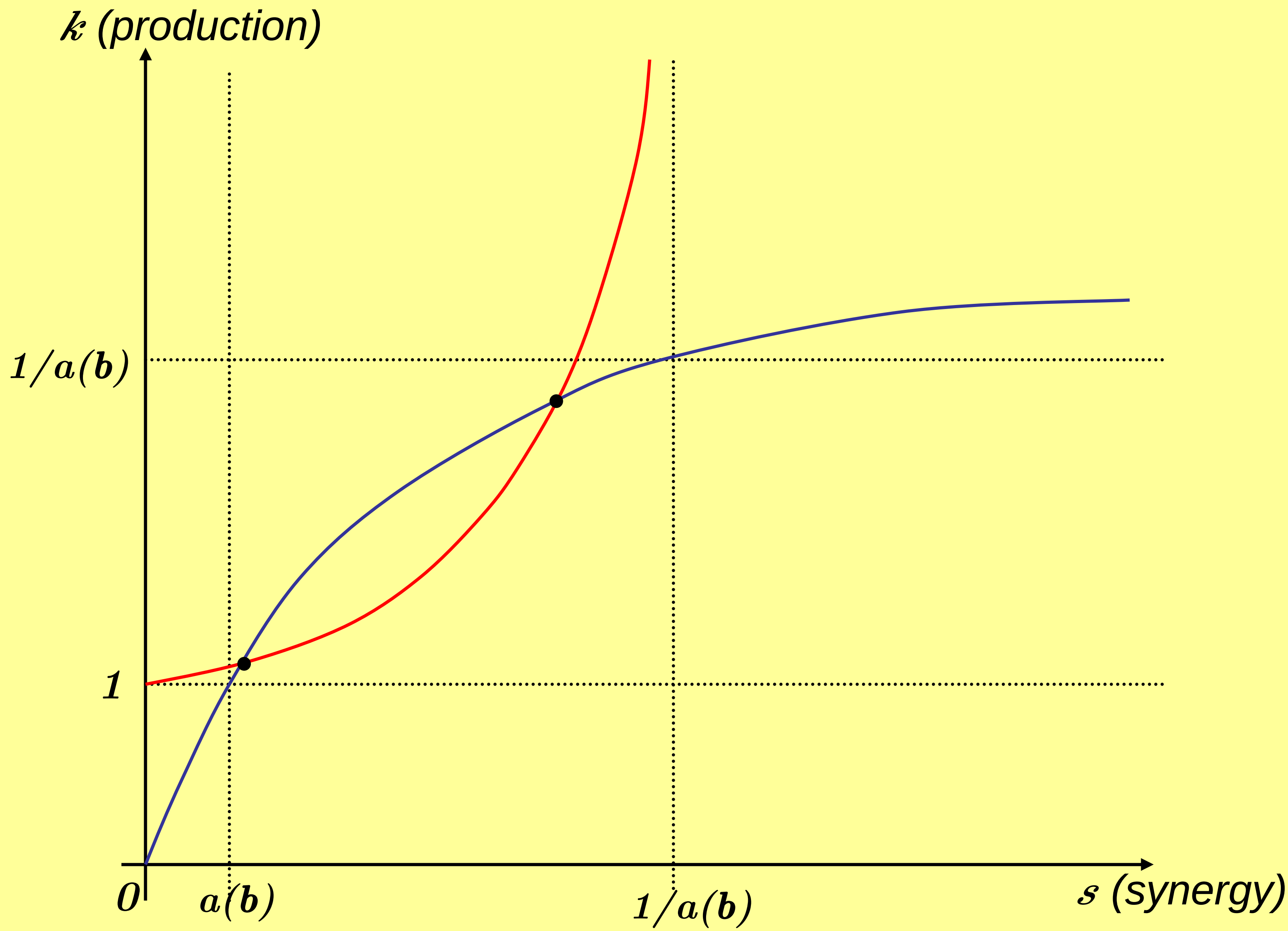
Let $a(\mathbf{b})$ increase. Then, at the Pareto-inferior [superior] approximate equilibrium, all actions increase [decrease]. The percentage change of the synergy effort is higher than that of the production effort.

- At the low [high] equilibrium, there is under-provision [over-provision]
- Socialization is more responsive that production

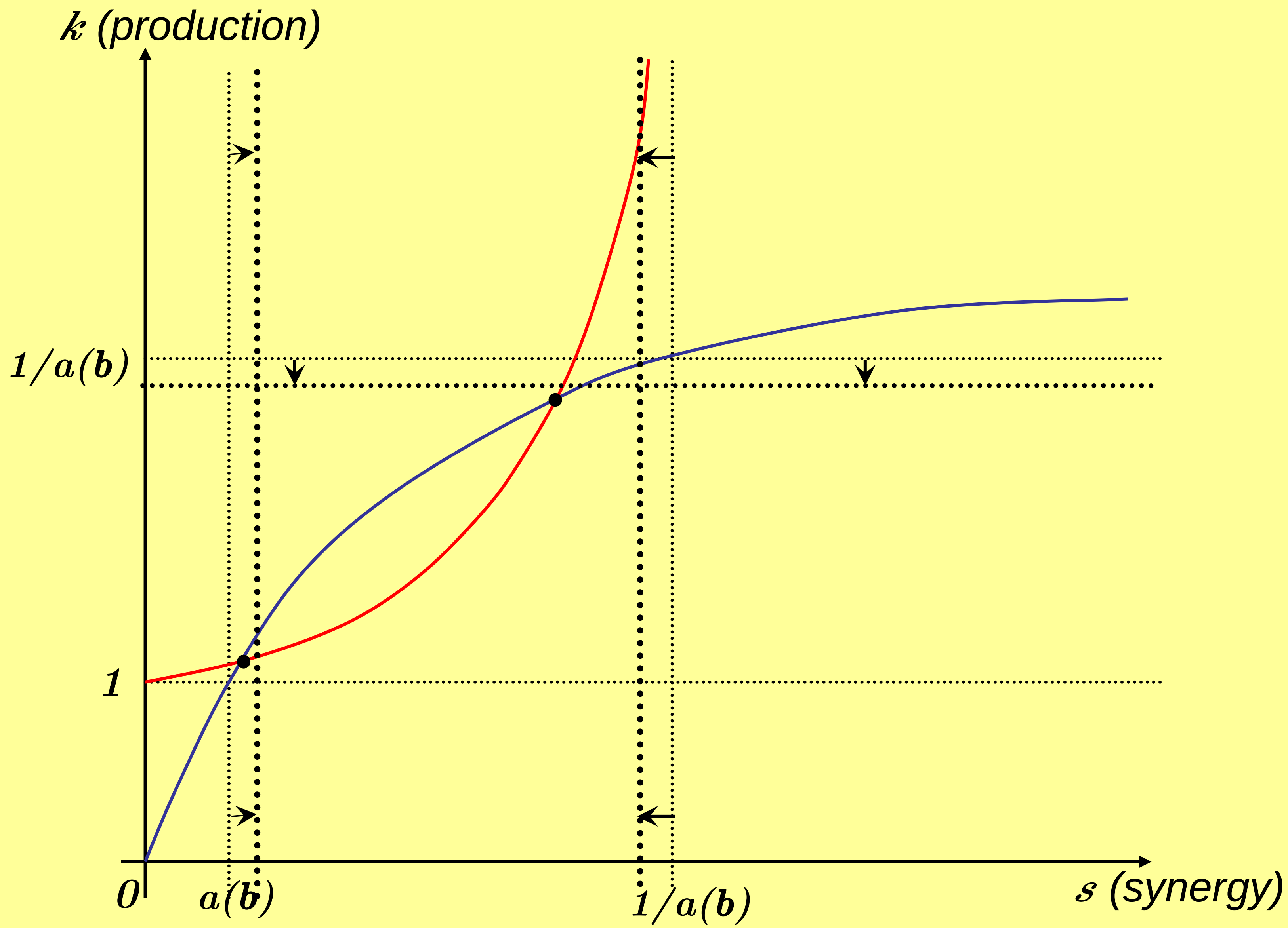
Proposition 4 *Suppose that $a(\mathbf{b})$ increases. Then, in both approximate equilibria of the replica game, the percentage change in socialization effort is higher than that of productive investment for all agents. Besides, the baseline equilibrium actions that solve (3) both increase at the low equilibrium and decrease at the high equilibrium.*

Elasticity of socialization with respect to productive investment is smaller than one at all equilibria.

$$\frac{1}{s}\frac{\partial s}{\partial a(\mathbf{b})} = \frac{k^2}{s} + 2\frac{1}{k}\frac{\partial k}{\partial a(\mathbf{b})} > \frac{1}{k}\frac{\partial k}{\partial a(\mathbf{b})}$$



k (production)
$1/a(\boldsymbol{b})$
1
0
$a(\boldsymbol{b})$
$1/a(\boldsymbol{b})$
s (synergy)



k (production)
1/a(b)
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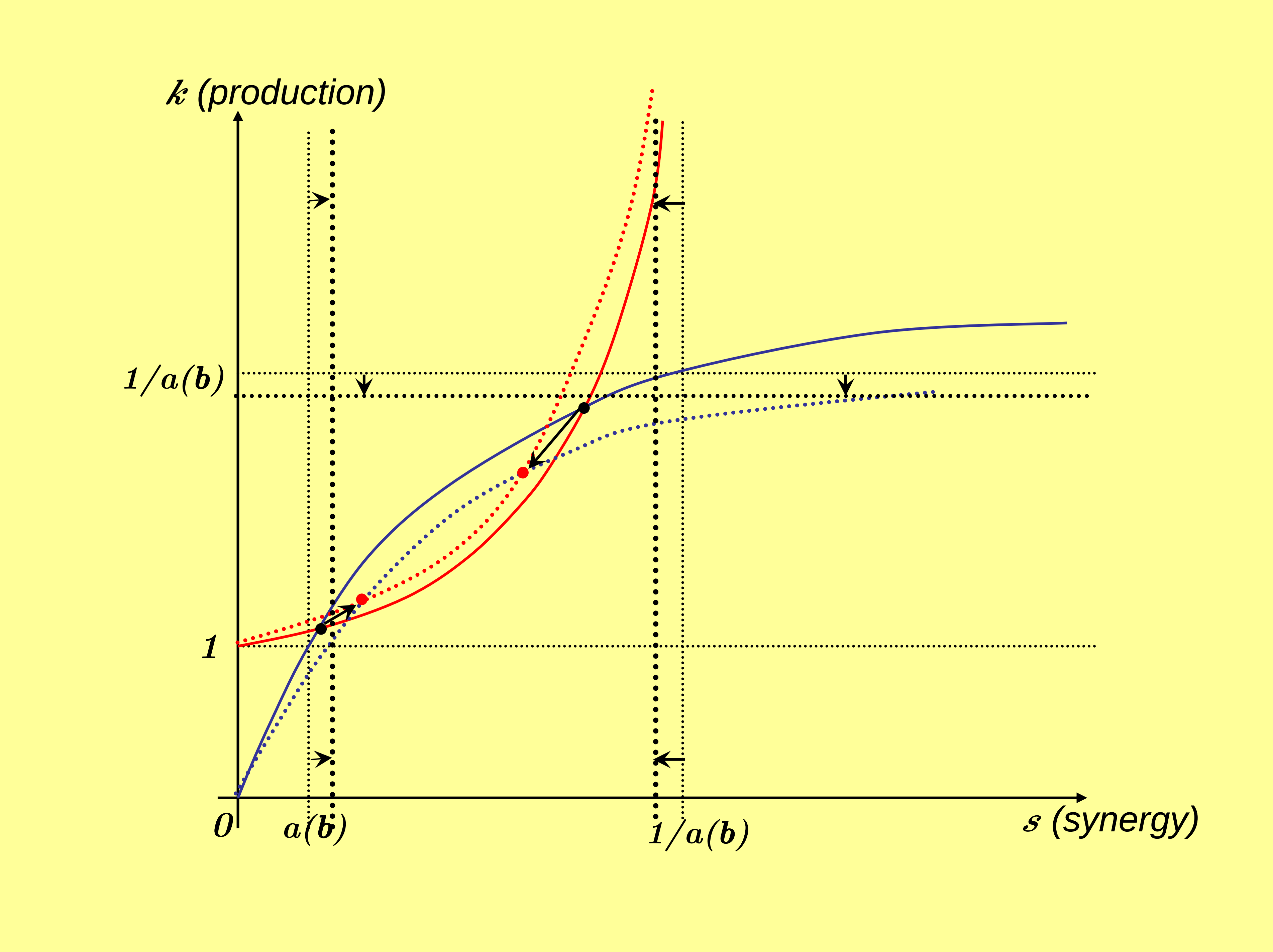
a(b)
1/a(b)
s (synergy)



k (production)
1/a(b)
1
0
a(b)
1/a(b)
s (synergy)

Comparative statics: payoffs

- For m high enough, approximate equilibrium payoffs are:

$$u_i^* \simeq \frac{b_{\tau(i)}^2}{2a(\mathbf{b})} \frac{s^*}{k^*} = \frac{1}{2} b_{\tau(i)}^2 k^*$$

Theorem

An increase in $a(\mathbf{b})$ external to i increases [decreases] i's payoffs at the low [high] equilibrium.

- An increase in a, or a multiplicative increase in $(a, b_1, \ldots, b_t)$, increases [decreases] everyone's payoffs at the low [high] equilibrium
- a mean-preserving spread in individual traits increases [decreases] total payoffs at the low [high] equilibrium, etc.

Empirical implications in terms of education

Education: Empirical predictions of our model.

(i) Better educated parents exert more productive effort educating their kids than low-educated parents, i.e. k_i and b_i are positively related (Patacchini and Zenou, 2009).

(ii) Better educated parents are more prone to socialize with other parents than low-educated parents, i.e. s_i and b_i are positively related (Putman, 1995).

(iii) Multiple equilibria. Explain why, in different locales, children whose parents have similar characteristics (e.g. income, education level) or are similarly talented as other children (say, measured by I.Q.) end up having very different educational outcomes or different levels of parental educational efforts.

(iv) Different levels of parental education affect (positively or negatively) proportionally more the socialization effort of those parents s_i than their direct effort k_i with children.

Robustness Analysis

Ours relies on specific assumptions due to the functional form of payoffs.

Three main characteristics:

(a) Linear-quadratic returns to productive investment,

(b) Aggregate constant returns to scale in socialization effort (condition $(A2)$ in lemma),

(c) Generic socialization effort (condition $(A3)$ in Lemma).

Relax some of these assumptions to see how robust are our results in terms of characterization of equilibria, multiple equilibria, welfare and comparative statics results.

Lemma 5 *Suppose that, for all* $\mathbf{s} \neq \mathbf{0}$*, the link intensity satisfies:*

(A1) symmetry: $g_{ij}(\mathbf{s}) = g_{ji}(\mathbf{s})$*, for all* i, j*;*

(A2) aggregate constant returns to scale: $\sum_{j=1}^{n} g_{ij}(\mathbf{s}) = s_i$*;*

(A3) anonymous socialization: $g_{ji}(\mathbf{s})/s_j = g_{ki}(\mathbf{s})/s_k$*, for all* i, j, k*;*

then, the link intensity is given by

$$g_{ij}(\mathbf{s}) = \frac{s_i s_j}{\sum_{j=1}^{n} s_j}$$

Heterogeneous populations

Modify assumptions $(A2)$ and $(A3)$ in Lemma 5 to identify the likely changes in the analysis.

Productivity based socialization

Let us consider a link-formation process between i and j, which not only depends on efforts s_i and s_j but also on individual characteristics b_i and b_j.

Lemma 6 *Assume* $0 < \alpha \leq 1$ *and suppose that, for all* $\mathbf{s} \neq \mathbf{0}$*, the link intensity satisfies:*

(A1) Symmetry: $g_{ij}(\mathbf{s}) = g_{ji}(\mathbf{s})$*, for all* i, j*;*

(A2) Aggregate constant returns to scale (in b_i *and* s_i*):*

$$\sum_{j=1}^{n} g_{ij}(\mathbf{s}) = b_i^{1-\alpha} s_i^{\alpha}$$

(A3) Productivity based socialization:

$$\frac{g_{ji}(\mathbf{s})}{g_{ki}(\mathbf{s})} = \frac{b_j^{1-\alpha} s_j^{\alpha}}{b_k^{1-\alpha} s_k^{\alpha}}$$

for all i, j, k*;*

then, the link intensity is given by

$$g_{ij}(\mathbf{s}) = \begin{cases} \frac{(b_i b_j)^{1-\alpha}(s_i s_j)^{\alpha}}{\sum_{k=1}^{n} b_k^{1-\alpha} s_k^{\alpha}}, \text{ if } \mathbf{s} \neq \mathbf{0} \\ 0, \text{ if } \mathbf{s} = \mathbf{0} \end{cases}$$

Assumptions $(A2)$ and $(A3)$ have now be relaxed.

Individual i controls his total number of friends but this depends on both his own characteristic b_i as well as his socialization effort s_i (assumption $(A2)$).

Each player *does not* devote the same share of his total socialization effort to interacting with i.

This share also depends on each intrinsic characteristic (assumption $(A3)$).

Solving these two conditions as we did before so that $s_i = b_i s$ and $k_i = b_i k$, we easily obtain the following first-order conditions:

$$\begin{cases} s = \left[a(\mathbf{b})k^2\right]^{1/(2-\alpha)} \\ k\left[c - a(\mathbf{b})s^\alpha\right] = 1 \end{cases} \tag{4}$$

Proposition 7 *Consider the link-formation process described in Lemma 6 and the utility function (1). If*

$$a(\mathbf{b}) > \left[\frac{2-\alpha}{(2\alpha)^\alpha (2+\alpha)^{(1-\alpha)} c^{(1-\alpha)\left(4\alpha-\alpha^2+1\right)}}\right]^{(2-\alpha)/2}$$

then the system of equations (4) has exactly two solutions $(s, k) \in \mathbb{R}^2_+$ *such that* $c > a(\mathbf{b})s^\alpha$*.*

The graph of equilibrium conditions looks very much the same as in the original model.

The condition is different than that in Theorem 1 because it is a sufficient condition.

It is also easily verified that the elasticity result, i.e.

$$\frac{1}{s}\frac{\partial s}{\partial a(\mathbf{b})} > \frac{1}{k}\frac{\partial k}{\partial a(\mathbf{b})}$$

as well as the results on comparative statics, welfare and Pareto-ranking are preserved under this new formulation.

Socialization with non-constant returns to scale

Lemma 8 *Suppose that, for all* $\mathbf{s} \neq \mathbf{0}$*, the link intensity satisfies:*

(A1) Symmetry: $g_{ij}(\mathbf{s}) = g_{ji}(\mathbf{s})$*, for all* i, j*;*

(A2) Aggregate returns to scale: $\sum_{j=1}^{n} g_{ij}(\mathbf{s}) = s_i^{\alpha}$*, which are constant if* $\alpha = 1$*, increasing if* $\alpha > 1$ *and decreasing if* $\alpha < 1$*.*

(A3) Anonymous socialization: $g_{ji}(\mathbf{s}) / g_{ki}(\mathbf{s}) = s_j^{\alpha} / s_k^{\alpha}$*, for all* i, j, k*;*

then, the link intensity is given by

$$g_{ij}(\mathbf{s}) = \begin{cases} \frac{(s_i s_j)^{\alpha}}{\sum_{k=1}^{n} s_k^{\alpha}}, \text{ if } \mathbf{s} \neq \mathbf{0} \\ 0, \text{ if } \mathbf{s} = \mathbf{0} \end{cases}$$

Let us also adjust the synergistic returns so that

$$u_i(\mathbf{s},\mathbf{k}) = b_i k_i + a \sum_{j=1, j\neq i}^{n} g_{ij}(\mathbf{s}) \left(k_i k_j\right)^{(2-\alpha)} - \frac{1}{2} k_i^2 - \frac{1}{2} s_i^2 \tag{5}$$

For this problem to make sense, we assume that $0 < \alpha < 2$.

This adjustment in the utility function is necessary to obtain closed-form solutions and to express the equilibrium values as $s_i = b_i s$ and $k_i = b_i k$.

Denote

$$a_\alpha(\mathbf{b}) = a\frac{\sum_{j=1}^n b_j^2}{\sum_{j=1}^n b_j^\alpha}$$

Then, in equilibrium we have that $s_i = b_i s$ and $k_i = b_i k$, so that:

$$\begin{cases} s = k^2 \left[\alpha a_\alpha(\mathbf{b})\right]^{1/(2-\alpha)} \\ ck\left[c - s^\alpha k^{2-2\alpha} a_\alpha(\mathbf{b})\right] = c + s^\alpha k^{2-2\alpha} a_\alpha(\mathbf{b})\,(1-\alpha) \end{cases} \tag{6}$$

We then have the following results:

Proposition 9 *Consider the link-formation process described in Lemma 8 and the utility function (5). If*

$$a_\alpha(\mathbf{b}) < \left(\frac{32(2-\alpha)c^3}{2\alpha^{\alpha/(2-\alpha)}\left[6+\alpha+\sqrt{(2+\alpha)^2+16(1-\alpha)}\right]^3}\right)^{(2-\alpha)/2},$$

then the system of equations (6) has exactly two solutions $(s,k) \in \mathbb{R}_+^2$ *such that* $c > s^\alpha k^{2-2\alpha} a_\alpha(\mathbf{b})$.

We provide again a sufficient condition for the existence of two equilibria.

The result on elasticity holds in this case.

The results on comparative statics and welfare can also be shown to be robust to this formulation for values of α bigger than 0.55.

Socialization without congestion

No congestion in the meeting process so each pairwise interaction strength/probability depended only on the socialization efforts of the pair.

Lemma 10 *Suppose that, for all* $\mathbf{s} \neq \mathbf{0}$*, the link intensity satisfies:*

(A1) symmetry: $g_{ij}(\mathbf{s}) = g_{ji}(\mathbf{s})$*, for all* i, j*;*

(A2) aggregate returns to scale: $\sum_{j=1}^{n} g_{ij}(\mathbf{s}) = s_i \sum_{j=1}^{n} s_j$*.*

(A3) anonymous socialization: $g_{ji}(\mathbf{s}) / s_j = g_{ki}(\mathbf{s}) / s_k$*, for all* i, j, k*;*

then, the link intensity is given by

$$g_{ij}(\mathbf{s}) = s_i s_j$$

A necessary condition for invertibility of the socialization matrix $\mathbf{M}$ (which also ensures that $k_i > 0$) is now $c > a\, n\overline{\mathbf{s^2}}$. We now show that this condition cannot be fulfilled for large n.

Denote $a(\mathbf{b}) = a \sum_{j=1}^{j=n} b_j^2$ and let $s_i = b_i s$ and $k_i = b_i k$. Then, when n is large (and thus $a(\mathbf{b})$ is large)

$$s \approx \sqrt{\frac{c}{(n-1)a(\mathbf{b})}}$$

The condition $c > a\, n\overline{\mathbf{s^2}}$ can now be written as:

$$c > s\, na(\mathbf{b}) \approx \sqrt{cna(\mathbf{b})} \Leftrightarrow c > na(\mathbf{b}).$$

Thus clearly as n get large this condition will eventually be impossible to satisfy.

When there is no congestion in the socialization process (like here), then for large population n, the condition on invertibility of the socialization matrix $\mathbf{M}$ will not be satisfied.

This is because, when there is no congestion, benefits from socialization explode when n is large. There are, in a sense, “too many” synergies from friends.

The case of homogeneous populations

Homogeneous population of players with a single type b, i.e. $b_1 = ... = b_n = b$.

Generalizing the cost function

Non-linear marginal costs of both socialization and investment.

$$u_i(\mathbf{s}, \mathbf{k}) = bk_i + a \sum_{j=1, j \neq i}^{n} g_{ij}(\mathbf{s}) k_i k_j - \frac{1}{d+1} k_i^{d+1} - \frac{1}{d+1} s_i^{d+1}, \quad (7)$$

where $a, b \geq 0$ and $d \geq 1$.

Symmetric equilibria, where all (homogeneous) players choose the same socialization effort and productive investment.

For all $\alpha < \beta$, define:

$$\mathbf{1}_{(\alpha,\beta]}(x) = \begin{cases} 1, \text{ if } \alpha < x < \beta \\ 0, \text{ otherwise} \end{cases}.$$

Function $\phi : \mathbb{R} \to \mathbb{R}$ given by:

$$\phi(x) = d^{\frac{1}{d+1}} \left[x^{d+1} - b^{1+\frac{1}{d}} \right]^{\frac{1}{d+1}}.$$

Proposition 11 *Consider an homogeneous population with the link-formation process described in Lemma 5 and the utility function (7). Then, if* $\frac{d^2}{2+d^2}\left(\frac{2}{2+d^2}\right)^{\frac{2}{d^2}} > a^{1+\frac{1}{d}} b^{\frac{2}{d^2}}$, *the system of equations*

$$\left\{ \begin{array}{l} s^d = \mathbf{1}_{(-\infty,\phi(k)]}(s)\, ak^2 \\ k^d \left[1 - as\right] = b \end{array} \right. . \tag{8}$$

has exactly two solutions $(s, k) \in \mathbb{R}^2_+$.

In this formulation, all results from the benchmark model (multiple equilibria, elasticity, comparative statics and welfare results) are also preserved.

Socialization with non-constant returns to scale

Link intensity:

$$g_{ij}(\mathbf{s}) = \begin{cases} \frac{\left(s_i s_j\right)^{\alpha}}{\sum_{k=1}^{n} s_k^{\alpha}}, \text{ if } \mathbf{s} \neq \mathbf{0} \\ 0, \text{ if } \mathbf{s} = \mathbf{0} \end{cases}$$

Change the utility function as follows:

$$u_i(\mathbf{s}, \mathbf{k}) = b\ k_i + a \sum_{j=1, j \neq i}^{n} g_{ij}(\mathbf{s}) \left(k_i k_j\right)^{\beta} - \frac{1}{2} k_i^2 - \frac{1}{2} s_i^2 \tag{9}$$

In equilibrium we have that $s_i = bs$ and $k_i = bk$ so that:

$$\begin{cases} s = k^{2\beta/(2-\alpha)} \left[\alpha b^{\alpha+2\beta-2} a\right]^{1/(2-\alpha)} \\ ck = 1 + \frac{\beta \frac{a}{c} b^{2\beta+\alpha-2} s^{\alpha} k^{2\beta-2}}{1-\frac{a}{c} b^{2\beta+\alpha-2} s^{\alpha} k^{2\beta-2}} \end{cases} \tag{10}$$

Let

$$g\left(a, b, \alpha, \beta\right) = \alpha^{\alpha/(2-\alpha)} b^{(2\alpha+4\beta-4)/(2-\alpha)} a^{2/(2-\alpha)}$$

$$h_1\left(\alpha, \beta, c\right) = \frac{4^{\frac{4\beta+\alpha-2}{2-\alpha}} (2-\alpha)\beta c^{\frac{2-\alpha+4\beta}{2-\alpha}}}{(4\beta+2\alpha-4)\left(\frac{6\beta+5\alpha\beta+2\alpha^2-8}{(4\beta+2\alpha-4)} + \sqrt{\left(\frac{6\beta+5\alpha\beta+2\alpha^2-8}{(4\beta+2\alpha-4)}\right)^2 + 16}\right.}$$

and

$$h_2\left(\alpha, \beta, c\right) = \frac{(2-\alpha)\beta}{(4\beta+2\alpha-4)\left(\max\left\{\begin{array}{c} \frac{\frac{6\beta+5\alpha\beta+2\alpha^2-8}{(4\beta+2\alpha-4)} + \sqrt{\left(\frac{6\beta+5\alpha\beta-}{(4\beta+2}\right.}}{4c} \\ \left(\frac{\frac{6\beta+5\alpha\beta+2\alpha^2-8}{(4\beta+2\alpha-4)} + \sqrt{\left(\frac{6\beta+5\alpha\beta+2\alpha^2}{(4\beta+2\alpha-4)}\right.}}{4c}\right. \end{array}\right.\right.}$$

Proposition 12 *Consider an homogeneous population with the link-formation process described in Lemma 8 and the utility function (9). If* $\alpha > 2$ *or* $\alpha < 2(1-\beta)$*, then there exists a unique interior equilibrium. Suppose* $2(1-\beta) < \alpha < 2$*. Then when* $\beta < 1$ *and* $g(a, b, \alpha, \beta) < h_1(\alpha, \beta, c)$*, or when* $\beta \geq 1$ *and* $g(a, b, \alpha, \beta) < h_1(\alpha, \beta, c)$ *there are two interior equilibria.*

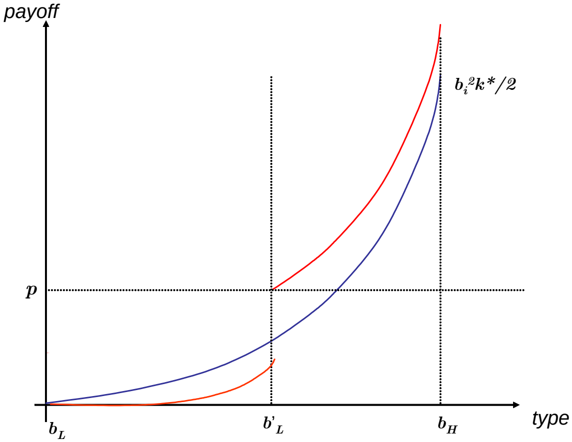
Clearly, the conditions for existence of multiple equilibria are satisfied if a or b are low enough. In this case, one can also easily show that the result on elasticity and welfare are preserved.

Individual preferences over group composition

- At the low equilibrium, player *i* prefers a higher $a(b_{\tau(i)}, \mathbf{b}_{-\tau(i)})$
  - a mean-preserving spread in $\mathbf{b}_{-\tau(i)}$ improves welfare. Holding the average trait constant, a more diverse group is better.
  - however, the highest type prefers full segregation to diversity.
  - note also $a(b_1) \le a(b_1, b_2) \le ...a(\mathbf{b}) \le ...a(b_{t-1}, b_t) \le a(b_t).$
- So, whenever viable, unraveling segregation of high types expected

(reciprocally for the high equilibrium)

Group composition and segregation



Network topology

- Large social networks display key empirical regularities (connectivity distribution, average distance, clustering, giant component, etc.)
- With $0 \leq g_i(\mathbf{s}^*) \leq 1$ and *mt* high, equilibrium defines a random graph with independent link probabilities and expected connectivities *s****b***:*
  - any connectivity distribution matched with adequate **b**
  - the average distance is roughly $(1 + o(1))\frac{\log(mt)}{\log\left(s^*\bar{\mathbf{b}}\right)}$
  - clustering is low (for large populations), roughly $\frac{1}{mt}\frac{s^*}{\bar{\mathbf{b}}}\left(1 + \frac{v(\mathbf{b})}{\bar{\mathbf{b}}}\right)^2$
- Endogenous changes in welfare and topology available

Low equilibrium: welfare and topology

	Av(**k**)	Var(**k**)	Clust.	Dist.	u^*_i
a up	+	+	+	-	+
(a,**b**) scaled up	+	+	+	-	+
b spread up	+	+	+	-	+ (total)

High equilibrium: welfare and topology

	Av(**k**)	Var(**k**)	Clust.	Dist.	u^*_i
a up	-	-	-	+	-
(a,**b**) scaled up	?	?	?	?	-
b spread up	-	?	?	+	- (total)

Homogeneous population

- A single (common) trait, but non-linear marginal synergy nor production costs:

$$u_i(\mathbf{k}, \mathbf{s}) = bk_i + a \sum_{j=1, j \neq i}^{n} g_{ij}(\mathbf{s})\, k_j k_i - \frac{1}{c+1} k_i^{c+1} - \frac{1}{c+1} s_i^{c+1}$$

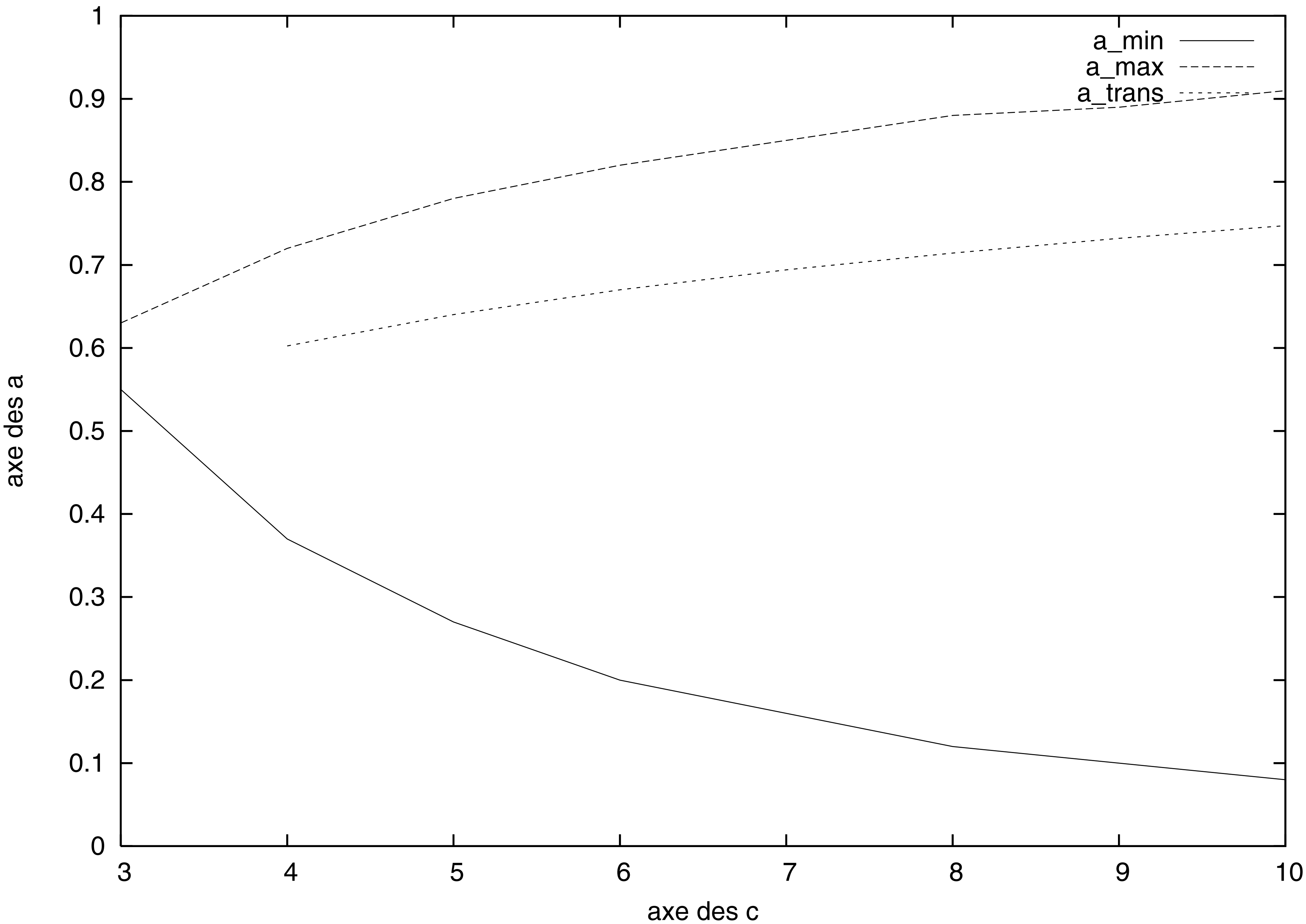
- We still have a no-synergy equilibrium, and one or two interior equilibria with ranked actions. Same comparative statics in actions:

$$\begin{aligned} s^c &= \mathbf{1}_{(-\infty, \phi(k)]}(s)\, ak^2 \\ k^c [1 - as] &= b \end{aligned}$$

Poisson equilibrium network and giant component

Theorem

Let $a < 1$. Then, only the high-actions equilibrium has a giant component if and only if $ab^{2/c} < (1-a)^{2/c}$. If, instead, $ab^{2/c} > (\frac{c}{a+c})^2$, then both equilibrium networks have a giant component.



a_min
a_max
a_trans
axe des a
axe des c
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
3
4
5
6
7
8
9
10

Conclusions

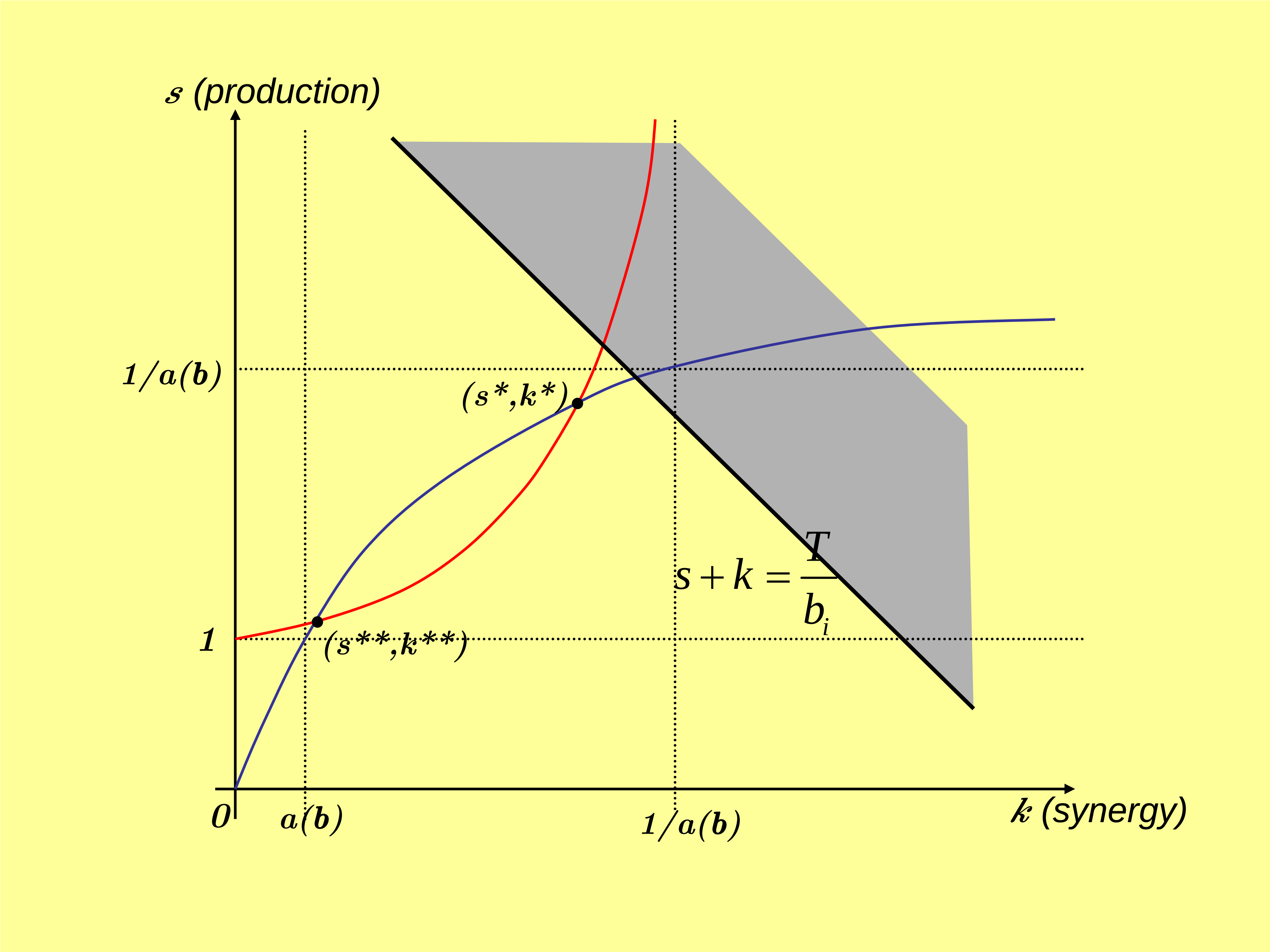
A simple operational model of network formation with welfare and topology predictions, and comparative statics:

- a "too cold" and a "too hot" equilibrium
- socialization is more responsive than production
- preferences over group composition hint towards assortative segregation

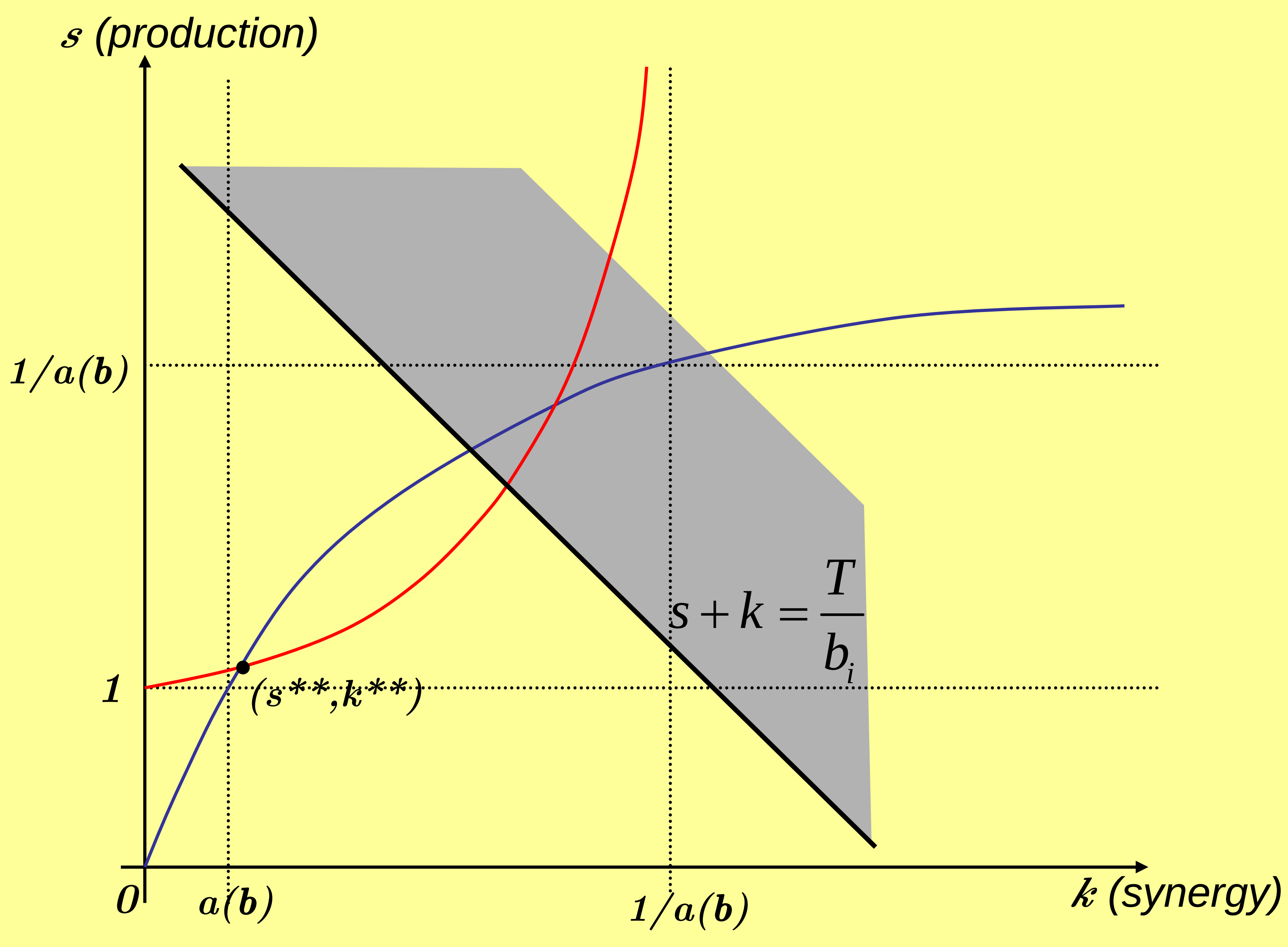
A variation of the model with several groups allows for partially directed socialization within groups −clustering, communities.

Equilibrium characterization (with bounds)

- Let $T \geq 3\sqrt{3}b_t$. There exists an $0 < \varepsilon_T < 2b_t / T$ such that:
  - when $\varepsilon_T \leq a(\mathbf{b}) < 2/3\sqrt{3}$, we have the two equilibria
  - when $0 < a(\mathbf{b}) < \varepsilon_T$, only the "low-actions" equilibrium
- When $a(\mathbf{b}) \downarrow 0$, the high $\uparrow (1/a(\mathbf{b}), 1/a(\mathbf{b}))$ and the low $\downarrow (1, a(\mathbf{b}))$



s (production)
1/a(b)
(s*,k*)
$s + k = \frac{T}{b_i}$
1
(s**,k**)
0
a(b)
1/a(b)
k (synergy)



s (production)
1/a(b)
1
0
a(b)
1/a(b)
k (synergy)
(s**,k**)
$s + k = \frac{T}{b_i}$