Pro-competitive rationing in multi-unit auctions*

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Abstract

In multi-unit auctions, it is necessary to specify rationing rules to break ties between multiple marginal bids. The standard approach in the literature and in pratice is to ration marginal bids proportionally. This paper shows how bidding can be made more competitive if the rationing rule instead gives increasing priority to marginal bids with small quantities at price levels closer to the reservation price. In comparison to standard rationing, such a rule can have almost the same effect on the competitiveness of bids as a doubling of the number of bidders.

Key words: Divisible-good auctions, multi-unit auctions, rationing rules, bidding format

JEL Classification C72, D44, D45

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1 Introduction

A wide range of products, commodities and assets are traded in divisible-good or multi-unit auctions. For instance, auctions of electricity, treasury bills and emission permits as well as financial exchanges, all allow bids for more than one unit of the traded items. In multi-unit auctions, each bidder submits a stack of bids, where each bid specifies a bid price and a bid quantity, such that the bidder is willing to trade the specified bid quantity at the specified bid price or better. Unless by coincidence, it would normally not be possible to clear such auctions by either fully accepting or fully rejecting all bid quantities at any price level. For multi-unit auctions it is therefore necessary to specify rationing rules. Rationing rules are of particular importance for the outcome in auctions where bid prices accumulate at a few price levels, as usually happens in financial exchanges and treasury auctions, where bidders have similar valuations of the traded items. The purpose of this paper is to highlight how rationing rules can be designed in order to increase the competition among a set of bidders, to the benefit of the auctioneer.

In practice, the normal procedure is to only ration marginal bids, which have a bid price exactly at the clearing price. In auctions where all bids are cleared simultaneously, it is standard practice to ration marginal bids *pro-rata*, so that each bidder gets the same percentage of its marginal bid quantities accepted. In exchanges with continuous trading, it is also common to give priority to marginal bids that arrived early to the exchange; this is referred to as *price-time priority*. Field and Large (2012) empirically observe that, in comparison to price-time priority, pro-rata rationing significantly increases bid quantities in the order book of financial exchanges, but also the cancellation rate of bids. This verifies that the design of the rationing rule influences bidding behaviour in auctions.

This paper shows that an auctioneer can increase its surplus by rationing marginal bids non-proportionally. I focus on the procurement auction, where the auctioneer buys items, but results are analogous for sales auctions as well as for double auctions and exchanges, where bidders are both buying and selling items. Obviously a procurer benefits if bidders offer many items at low prices. Thus a procurer would like to encourage bids that specify large bid quantities at low bid prices. I consider a one-shot game, so it will be optimal for bidders to submit bids for all of their items with a marginal cost below the reservation price of the auctioneer. Thus, bids that specify large bid quantities at high bid prices should be discouraged by the auctioneer, as they will lead to less quantity being offered at low bid prices. In line with this argument, the paper shows that bidding gets closer to the competitive outcome when an auction gives disproportionate priority to marginal bids with large bid quantities at low clearing prices and disproportionate priority to marginal bids with small bid quantities at high clearing prices.

I evaluate rationing rules in uniform-price auctions, where all accepted bids are transacted at the clearing price. Uniform-price auctions are for example used in most wholesale electricity markets and in the U.S. treasury sales auctions. Assume that each bidder submits a stack of $v+1$ sell bids with different bid prices and that the auctioneer wants to maintain the same pro-competitive effect at each bid price.

In this case, I show that optimal use of disproportionate rationing on the margin in an auction with N symmetric bidders, gives the auctioneer approximately the same procurement cost as an auction with pro rata on the margin rationing and $\left(1+\frac{1}{v}\right)(N-1)+1>N$ symmetric bidders with the same aggregate production cost. Thus changing to the optimal rationing rule from pro-rata on the margin rationing, almost corresponds to a doubling of the number of bidders when each bidder submits a stack with two bid prices. The effect is smaller, the larger the number of bids each bidder makes. However, if the auctioneer is mostly concerned with competitiveness in a narrow price interval, perhaps because it has some prior knowledge of where the auction is going to clear, then the auctioneer can use disproportionate rationing to significantly boost competition in that short price interval, even if each bidder submits a stack with many bids.

The optimal rationing rule depends on the clearing price. Still, a disproportionate rationing rule can be pro-competitive even if the rule does not depend on the clearing price. Intuitively, assume that bidders are more concerned with bids at a low price, perhaps because the auction is more likely to clear at a low price or perhaps because bidders have significantly higher mark-ups at low prices. In this case, the auctioneer could also focus on encouraging large bid quantities at low clearing prices, so a rationing rule that gives priority to marginal bids with large bid quantities at all clearing prices would boost competition. Alternatively, if bidders are instead more concerned with bids at a high price, competition would be intensified if the rationing rule gives priority to marginal bids with small bid quantities at all clearing prices.

My model uses Nash equilibria of a static game to predict bidding behaviour for different rationing rules. A stepped supply function is used to represent the bid stack of each bidder. Similar to Holmberg et al. (2013), I use a discrete version of Klemperer and Meyer's (1989) Supply Function Equilibrium (SFE) concept to analyse Nash equilibria of stepped supply functions. But I generalize Holmberg et al.'s (2013) model to allow for disproportionate rationing on the margin and non-constant tick-sizes. Production costs of bidders are common knowledge and the auctioneer's demand is uncertain as in the standard SFE model. The underlying assumptions of the SFE model are particularly realistic for wholesale electricity markets,[1] but sales auction versions of the supply function equilibrium have also been used to analyse bidding in uniform-price treasury auctions in U.S. (Wang and Zender, 2002).[2]

Previously, Kremer and Nyborg (2004) have also shown that rationing rules

[1] In this market, technology characteristics and fuel prices are transparent and producers make offers before demand of electricity has been realized (Anderson and Hu, 2008; Green and Newbery, 1992; Holmberg and Newbery, 2009). Observed offers match the first-order condition of a stepped SFE model so well that the theory cannot be rejected (Wolak, 2007). The continuous SFE model is less precise in practice; it can only make accurate predictions of bids from large firms, whose submitted supply functions have many steps (Hortacsu and Puller, 2008; Sioshansi and Oren, 2007).

[2] In U.S. treasury auctions, an uncertain amount of non-competitive bids from small investors are given priority before regular bids. Thus there is an uncertain supply of securities that is available to large investors. Some treasury auctions also up-date the number of sold securities with respect to the latest market news, after buyers have submitted their bids.

can be used to improve competition. However, they analyze a rationing rule, where also infra-marginal bids (sell bids below and buy bids above the clearing price) would be rationed proportionally. But partly rationing infra-marginal bids would be inefficient in a market with increasing marginal costs. The spread rationing rule (SRR) and the concentrate rationing rule (CRR) examined by Saez et al. (2007) may also result in rationing of infra-marginal bids and similar inefficiency problems. Gresik (2001) proposes a new rule, ζ-rationing, where marginal bids (when possible) are rationed in proportion to the total amount that a bidder wants to trade at the marginal price. McAdams (2000) and Kweik and Schenone (2000) explore the extent to which rationing rules may provide the auctioneer with a tool for deterring collusive bidding. In order to ensure existence of Nash equilibria in theoretical models of auctions, such as in papers by Fabra et al. (2006), Simon and Zame (1990), and Jackson and Swinkels (1999), it is sometimes convenient to consider type dependent rationing rules, for example where priority is given to the most efficient marginal bids, e.g. sell bids with lowest cost. However, such rationing rules are difficult to apply in practice, where bidders' true costs are normally not observed by the auctioneer. The present paper is the first to use a rationing rule that depends on the clearing price. In this way competition can be improved in an almost mechanical way. Thus it is my belief that the pro-competitive effect would be robust to assumptions made on bidders' values/costs and uncertainties in the auctioneer's demand or supply.

Section 2 describes the setting of the game. The analysis is carried out in Section 3. Section 4 discusses some extensions that may be of practical relevance. Section 5 concludes. Technical lemmas are derived in Appendix A and proofs of the propositions and non-technical lemmas are provided in Appendix B.

2 Model

Consider a uniform-price procurement auction, so that all accepted bids are paid the Market Clearing Price (MCP). A stepped supply function is used to represent the bid stack of each bidder. As illustrated by Figure 1, the market is cleared at the lowest price where aggregated supply is larger than the auctioneer's demand. Any excess supply at the MCP is rationed on the margin. I calculate a pure strategy Nash equilibrium of a one-shot game, in which each risk-neutral supplier chooses a step supply function to maximize its expected profit.

Similar to Holmberg et al. (2013) there are M permissible price levels, P_j, $j \in \{1, \ldots, M\}$, with the price tick $\Delta P_j = P_j - P_{j-1} \geq 0$. The minimum quantity increment is zero, i.e. quantities can be continuously varied. The difference to Holmberg et al. (2013) is that I now allow for non-constant tick-sizes and non-pro rata rationing. I let $r = \frac{\Delta P_j}{\Delta P_{j+1}}$, where it is assumed that r is a bounded constant.

Producer $i \in \{1, \ldots, N\}$ submits a supply vector $\boldsymbol{S}^i = \left\{S_j^i\right\}_{j=1}^{M}$ consisting of non-negative maximum quantities it is willing to produce at each permissible price level. The quantity increment $\Delta S_j^i = S_j^i - S_{j-1}^i$ is non-negative (supply must be non-decreasing in price). Let $\boldsymbol{S} = \left\{\boldsymbol{S}^i\right\}_{i=1}^{N}$ and denote competitors' collective

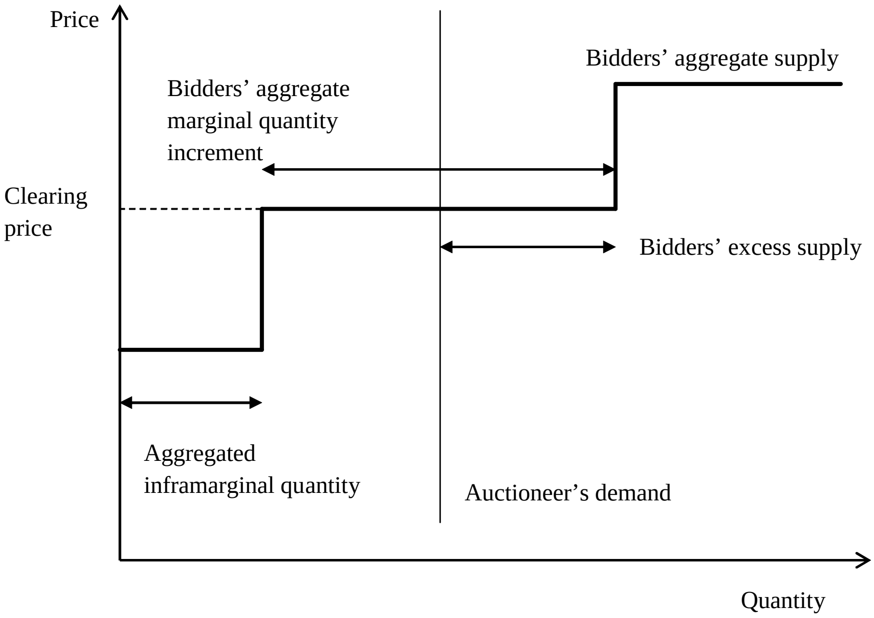

Figure 1: Clearing of and excess supply in the procurement auction.

offered quantity at price P_j as S_j^{-i} and the total market supply at P_j as S_j. The cost function of supplier i, $C_i\left(S^i\right)$, is a smooth, increasing and convex function up to the capacity constraint k_i. Let k be the total production capacity in the market. Costs are common knowledge. Klemperer and Meyer's (1989) continuous model is used as a benchmark. The set of individual smooth supply functions in the continuous model is given by $\{q_i\left(p\right)\}_{i=1}^N$.

The auctioneer's demand is perfectly inelastic up to the reservation price P_M. Demand is uncertain and given by the shock ε. The shock has a continuous probability density, $g(\varepsilon)$, with $\overline{g} \geq g(\varepsilon) \geq \underline{g} > 0$ on the support $[\underline{\varepsilon}, \overline{\varepsilon}]$. MCP is the lowest price at which the offered supply is (strictly) larger than the stochastic demand shock. Thus the equilibrium price as a function of the demand shock, $P\left(\varepsilon\right)$, is right continuous, and the MCP equals P_j if $\varepsilon \in [S_{j-1}, S_j)$. Given chosen step supply functions, the market clearing price can be calculated for each demand shock in the interval $[\underline{\varepsilon}, \overline{\varepsilon}]$. The lowest and highest prices that are realized are denoted by P_L and P_H, respectively, where $1 \leq L < H \leq M$. I let $s\left(\varepsilon\right)$ and $s_i\left(\varepsilon\right)$ be the total accepted supply and supplier i's accepted supply at ε, respectively.

2.1 The rationing rule

I consider a new class of rules that ration disproportionately on the margin. The rules are such that any bid accepted for some demand shock ε_0 is also accepted for any $\varepsilon > \varepsilon_0$, i.e. a bidder's acceptance is monotonic with respect to the demand shock. For a given set of supply schedules, the outcome of the auction is the same (irrespective of the sharing rule) when there is no excess supply at MCP. In this case we have:

$$s_i\left(S_{j-1}\right) \equiv S^i_{j-1}. \tag{1}$$

The rationing rule determines how to accept bids when $S_{j-1} < \varepsilon < S_j$. For the class of rationing rules that I consider, the increment of producer i's accepted supply Δs_i for a shock increment $\Delta\varepsilon$ is determined by the differential equation

$$\frac{ds_i\left(\varepsilon\right)}{d\varepsilon} = \frac{\left(S^i_j - s_i\left(\varepsilon\right)\right)^{\mu_j}}{\sum_{k=1}^N \left(S^k_j - s_k\left(\varepsilon\right)\right)^{\mu_j}} \text{ if } \varepsilon \in \left(S_{j-1}, S_j\right), \tag{2}$$

where the rationing parameter μ_j determines the non-linearity of the sharing rule at the price P_j, i.e. the extent to which large quantity increments at this clearing price are given priority to small increments. I consider $\mu_j \geq 0$, so that the rationing rule results in monotonic acceptance (in absolute terms) in the sense that a larger quantity increment at the marginal price will (weakly) increase the accepted volume from marginal bids of the supplier. Similarly the rationing rule gives monotonic rejection (in absolute terms), i.e. a larger quantity increment at the marginal price will also (weakly) increase the rejected volume from marginal bids of a supplier. For $\mu_j = 1$ we get pro rata on the margin rationing, where any additional demand $\Delta\varepsilon$ is allocated in proportion to a supplier's unmet supply at the clearing price, $S^i_j - s_i\left(\varepsilon\right)$.[3] It follows from (2) that with $\mu_j > 1$, disproportionate priority is given to producers with large unmet supply at the clearing price. When $\mu_j \to \infty$, $\Delta\varepsilon$ is shared equally among suppliers with the largest unmet supply at the clearing price, while suppliers with less unmet supply at P_j get no share of $\Delta\varepsilon$. We say that this rule gives maximum priority to large quantity increments at P_j (subject to that rejection is monotonic for the rationing rule). The case $0 \leq \mu_j < 1$ gives more priority to small quantity increments. In particular, $\mu_j = 0$ gives maximum priority to small quantity increments at P_j (subject to that acceptance is monotonic for the rationing rule). In this case, all suppliers with unmet supply at the clearing price get the same share of any additional marginal demand increment $\Delta\varepsilon$. Note that

$$\sum_{i=1}^N \frac{ds_i\left(\varepsilon\right)}{d\varepsilon} \equiv 1, \tag{3}$$

i.e. the marginal increase in total accepted supply always equals the marginal shock increment.

Together with the initial condition in (1) a system of differential equations of the type in (2), with one equation per bidder, can be used to calculate the accepted quantity for each supplier as a function of the demand shock for any given set of monotonic step supply functions.[4] From the supply $s_i\left(\varepsilon\right)$ allocated to each supplier, it is straightforward to calculate the supplier's expected profit:

$$E\left(\pi_i\right) = \int_{\underline{\varepsilon}}^{\overline{\varepsilon}} \left[P\left(\varepsilon\right) s_i\left(\varepsilon\right) - C_i(s_i\left(\varepsilon\right))\right] g\left(\varepsilon\right) d\varepsilon. \tag{4}$$

[3] Lemma 4 in Appendix A formally establishes that this corresponds to pro-rata on the margin rationing.

[4] Lemma 5 in Appendix A formally establishes that there exists a unique allocation for any given set of non-decreasing supply schedules.

3 Analysis

In the following subsection, I derive a first-order condition for optimal bids when rationing is disproportionate on the margin. Then I will analyse a case with two permissible price levels. The third subsection of the analysis section analyses cases with many permissible price levels.

3.1 The first-order condition

Optimal bids of a supplier can be determined from the following first-order condition.

Lemma 1 *The first-order condition for a uniform-price auction with N symmetric suppliers is given by:*

$$\begin{aligned} \left.\frac{\partial E(\pi_i)}{\partial S_j^i}\right|_{S_j^i=S_j^k} &= -\Delta P_{j+1} S_j^i g\left(S_j\right) \\ +\frac{(N-1)\Delta S_j}{N} &\int_0^1 \left[P_j - C_i'\left(\overline{S}_j\left(u\right)/N\right)\right]\left(1-u^{\mu_j}\right) g\left(\overline{S}_j\left(u\right)\right) du \\ +\frac{(N-1)\Delta S_{j+1}}{N} &\int_0^1 \left[P_{j+1} - C_i'(\overline{S}_{j+1}\left(u\right)/N)\right] u^{\mu_{j+1}} g\left(\overline{S}_{j+1}\left(u\right)\right) du = 0, \end{aligned} \tag{5}$$

where $k \neq i$ and $\overline{S}_j\left(u\right) := uS_{j-1} + \left(1-u\right)S_j$.

The first-order condition can be intuitively interpreted as follows. When calculating $\partial E\left(\pi_i\right)/\partial S_j^i$, supply is increased at P_j while holding the supply at all other price levels constant. This implies that the bid price of one (infinitesimally small) unit of quantity is decreased from P_{j+1} to P_j. This decreases the MCP for the event when the unit is price-setting, i.e. when $\varepsilon = S_j$. This event brings a negative contribution to the expected profit, which corresponds to the first term in the first-order condition (5). This term corresponds to the price effect; the term is negative as a bid price was decreased. Due to the rationing mechanism, decreasing the price of one unit (weakly) increases the accepted supply for demand outcomes $\varepsilon \in \left[S_{j-1}, S_{j+1}\right)$. This brings a positive contribution to the expected profit; the two integrals in (5). The first integral covers $\varepsilon \in \left[S_{j-1}, S_j\right)$ when the MCP is P_j, and the second for $\varepsilon \in \left[S_{j-1}, S_j\right)$ when the MCP is P_{j+1}. The first integral corresponds to the loss associated with the quantity effect at the price P_j and the second integral corresponds to the loss associated with the quantity effect at the price P_{j+1}. The two integral terms are positive as a bid price was decreased.

By means of Lemma 1 we can identify two reasons why supplier i's loss associated with the quantity effect at P_j dominates the loss associated with the quantity effect at P_{j+1}. First, if the market is more likely to clear at P_j than at P_{j+1}. The other reason is that supplier i has higher average mark-ups at P_j than at P_{j+1}. We also note the following from Lemma 1:

Remark 1 *For given supply schedules* $\mathbf{S}$*, the loss associated with supplier* i*'s quantity effect when increasing the bid price for some units of output from* P_j *to* P_{j+1} *becomes larger if*

1. the rationing rule gives increased priority to large quantity increments at P_j compared to P_{j+1}, i.e. μ_j increases and/or μ_{j+1} decreases.
2. supplier i's loss associated with the quantity effect at P_j dominates the loss associated with the quantity effect at P_{j+1}, the same rationing rule is used at P_j and P_{j+1}, and the rationing rule gives increased priority to large quantity increments, i.e. $\mu_j = \mu_{j+1}$ increases.
3. supplier i's loss associated with the quantity effect at P_{j+1} dominates the loss associated with the quantity effect at P_j, the same rationing rule is used at P_j and P_{j+1}, and the rationing rule gives increased priority to small quantity increments, i.e. $\mu_j = \mu_{j+1}$ decreases.

3.2 Two price-levels

To illustrate the effect that disproportionate rationing has on equilibrium bids, we first analyse a simple case with only two admissible price levels, P_1 and P_2. We make the following assumption.

Assumption 1. The uniform-price auction has two price levels, P_1 and P_2. Suppliers are symmetric, each supplier has capacity k_i and constant marginal cost $c \leq P_1 < P_2$, such that $(N-1)(P_2 - c) \leq N\Delta P_2$. Demand is uniformly distributed on $[0, k]$. We set $S_0^i = 0$.

We can deduce the following inequality from Assumption 1:

$$(N-1)(P_1 - c) \leq \Delta P_2. \tag{6}$$

$P_2 > c$ is the highest possible price, so irrespective of competitors' bids, it is the best response for each supplier to offer its entire capacity k_i at P_2. Thus market performance is determined by S_1^i. A higher S_1^i means that bids are more competitive, i.e. average mark-ups are lower. We get the following result.

Lemma 2 *Under Assumption 1, the solution to the first-order condition in Lemma 1 is:*

$$S_1^i = \frac{(N-1)\, k_i\, (P_2 - c)}{\left((\mu_2 + 1)\, \Delta P_2 + (N-1)(P_2 - c) - \frac{(N-1)(P_1 - c)(1+\mu_2)\mu_1}{(1+\mu_1)} \right)}. \tag{7}$$

As expected from Remark 1, we have from Lemma 2 and the inequality in (6) that S_1^i increases when μ_2 decreases and/or when μ_1 increases. We note that the inequality in (6), which follows from Assumption 1, ensures that the optimal supply at P_1 is never constrained by the capacity constraint k_i. Increasing μ_1

and decreasing μ_2 will weakly improve, but to a lower extent, market competitiveness also for circumstances when $(N-1)(P_2-c) > N\Delta P_2$, so that supply at P_1 is constrained by the capacity constraint k_i for the most high powered rationing parameters. We can verify that the following first-order solutions are Nash equilibria.

Proposition 1 *Under Assumption 1, we can establish Nash equilibria for the following cases*

1. A rationing rule that gives maximum priority to large quantity increments at P_1 ($\mu_1 = \infty$) and maximum priority to small quantity increments at P_2 ($\mu_2 = 0$) results in the most competitive first-order solution. The symmetric Nash equilibrium for this case is:
$$S_1^i = \frac{(N-1)\,k_i\,(P_2-c)}{N\Delta P_2}. \tag{8}$$

2. Auction competitiveness is also improved, but to a lower extent, when maximum priority is given to small quantity increments at both P_1 and P_2 ($\mu_2 = \mu_1 = 0$). The Nash equilibrium for this case is:
$$S_1^i = \frac{(N-1)\,k_i\,(P_2-c)}{\Delta P_2 + (N-1)\,(P_2-c)}. \tag{9}$$

3. The Nash equilibrium for pro rata on the margin rationing ($\mu_2 = \mu_1 = 1$) is:
$$S_1^i = \frac{(N-1)\,k_i(P_2-c)}{(N+1)\,\Delta P_2}. \tag{10}$$

   In this case, supplier i's loss associated with the quantity effect at P_2 dominates the loss associated with the quantity effect at P_1.

The second result, that competitiveness is improved (relative to standard rationing) by giving maximum priority to small quantity increments at both P_1 and P_2 can be explained by that supplier i's loss associated with the quantity effect at P_2 dominates the loss associated with the quantity effect at P_1 for pro rata on the margin rationing (the third result). In the special case when $P_1 = c$, the loss associated with the quantity effect at P_1 is zero, so that it is only P_2 that contributes to this loss. In this special case, giving maximum priority to small quantity increments at both P_1 and P_2 ($\mu_2 = \mu_1 = 0$) will have the same effect as the optimal rationing rule (our first result in Proposition 1).

We can multiply the first and third result in Proposition 1 by N to get expressions for the total market supply at P_1. By using that $k = Nk_i$, we can deduce the following:

Corollary 1 *Under Assumption 1, a uniform-price auction with N symmetric suppliers and optimal rationing on the margin gives the auctioneer the same total procurement cost as a uniform-price auction with pro rata on the margin rationing and $2N-1$ symmetric suppliers with the same total production cost (the same marginal cost c and total production capacity k).*

3.3 Many price levels

In this section we analyse the case where supply functions have many steps, so that the difference equation in Lemma 1 can be approximated by a differential equation. A difference equation is said to be consistent with a differential equation, if the difference equation converges to said differential equation as the number of steps in the supply schedules increase towards infinity (Holmberg et al., 2013).

Lemma 3 *For N symmetric suppliers, the discrete first-order condition in Lemma 1 is consistent with the continuous differential equation*

$$-q_i(P_j) + [P_j - C_i'(q_i(P_j))]\left(\frac{1}{(\mu_{j+1}+1)} + \frac{\mu_j r}{(\mu_j+1)}\right)(N-1)\,q_i'(P_j) = 0 \quad (11)$$

if $P_j > C_i'(q_i(P_j))$ *and* $\mu_j > 0$.

In the special case when tick-sizes are constant, i.e. $r = 1$, and rationing is proportionate on the margin, i.e. $\mu_j = 1$, (11) can be simplified to

$$-q_i(P_j) + [P_j - C_i'(q_i(P_j))](N-1)\,q_i'(P_j) = 0, \quad (12)$$

which is the differential equation of continuous supply function equilibria for symmetric suppliers with inelastic demand (Rudkevich, 1998; Anderson and Philpott, 2002; Holmberg, 2008). This confirms a previous consistency result in Holmberg et al. (2013) for pro rata on the margin rationing. A comparison of (11) and (12) implies that for constant tick-sizes and disproportionate rationing on the margin, competitiveness (the number of competitors, $N-1$) is approximately boosted by the factor

$$\lambda = \frac{1}{(\mu_{j+1}+1)} + \frac{\mu_j}{(\mu_j+1)} \quad (13)$$

relative to the case with pro rata on the margin rationing. As in the case with two price levels we note that it is beneficial for competition to use rationing parameters such that $\mu_j > \mu_{j+1}$. However, with more price levels there will be smaller changes in μ_j from one price level to the next, and a lower pro-competitive effect, if one wants to maintain the same effect on competition at each price level. We can write (13) in the following form:

$$\mu_j = \frac{1}{1 + \frac{1}{\mu_{j+1}+1} - \lambda} - 1.$$

By setting the competition boosting factor λ to a constant and $\mu_H = 0$ (the rationing parameter at the highest realized price), we can iteratively solve for μ_j for sequentially smaller j, until a non-negative solution of μ_j no longer exists. In this way we can approximately determine for how many steps in a supply function we can maintain λ at the desired level. The results are summarized in Table 1.

We can multiply the differential equation in (11) by N, so that we get an equation for the total supply, and then note the following from Table 1.

Table 1: The competition boosting factor λ and the number of steps in a supply function, for which the factor can be maintained.

No of steps	λ
1	2
2	1.4
3	1.3
4	1.2
6	1.15
9	1.1
19	1.05
49	1.02
99	1.01
199	1.005
499	1.002
999	1.001

Remark 2 *A uniform-price auction with optimal rationing on the margin and N symmetric suppliers with v steps in each supply function has approximately the same total procurement cost as a uniform-price auction with pro rata on the margin rationing and $(1 + 1/v)\ (N - 1) + 1$ symmetric suppliers with the same total production costs and v steps in each supply function.*

Even if supply functions have many steps, the auctioneer can still boost competition substantially locally by introducing large changes in μ_j in an interval with a few price levels, where the auctioneer expects the auction to clear or where the auctioneer is mostly concerned with market competitiveness. We also note the following from Lemma 3:

Remark 3 *If the rationing rule is the same for each price level, $\mu_j = \mu_{j+1} = \mu$, but tick-sizes are non-constant, then*

1. If tick-sizes decrease towards the reservation price ($r > 1$), then the competition boosting factor $\lambda = \frac{1}{(\mu+1)} + \frac{\mu r}{(\mu+1)}$ increases when the rationing rule gives increased disproportionate priority to large quantity increments at all prices ($\mu \uparrow$).

2. If tick-sizes increase towards the reservation price ($r < 1$), then the competition boosting factor $\lambda = \frac{1}{(\mu+1)} + \frac{\mu r}{(\mu+1)}$ increases when the rationing rule gives increased disproportionate priority to small quantity increments at all prices ($\mu \downarrow$).

The intuition behind this result is that smaller tick-sizes towards the reservation price tends to also decrease quantity increments, so that supplier i's loss associated with the quantity effect at P_j tends to dominate the loss associated with the quantity effect at P_{j+1}. The opposite is true if tick-sizes are instead larger towards the reservation price.

4 Extensions of the auction design

In the analysed model, each rationing parameter has been tied to a price level, but this may not be optimal in practice. In practice, the bidding format often restricts the number of steps in supply schedules and/or bidders do not always use all allowed steps, because the additional effort required of a supplier to submit another step may not be negligible (Kastl, 2011). In such cases, it should be sufficient to boost competition at bid prices that are used by the supplier, so that a higher boosting factor can be maintained at those fewer prices. In practice it may therefore be beneficial to have individual rationing parameters for suppliers, μ_j^i, where a supplier's parameter could for example depend on the step number in its supply function. The auctioneer may also want to weight supplier's unmet supply, in order to avoid that the disproportionate rationing rule favours small or large suppliers, or to optimize rationing for asymmetric bidders. As an example, a supplier's weight ω_i could be inversely proportional to its production capacity or maximum offered supply S_H^i. Thus (2) could be generalized as follows

$$\frac{ds_i(\varepsilon)}{d\varepsilon} = \frac{\left(\omega_i \left(S_j^i - s_i(\varepsilon)\right)\right)^{\mu_j^i}}{\sum_{k=1}^{N} \left(\omega_k \left(S_j^k - s_k(\varepsilon)\right)\right)^{\mu_j^k}}.$$

In a more advanced auction, the individual rationing parameters of a supplier may depend on its supply schedule. The auctioneer may for example want to set high μ_j^i values in price intervals where quantity increments of supplier i are decreasing and low μ_j^i values in price intervals where quantity increments of supplier i are increasing.

It has been shown that tick-sizes can be combined with the rationing rule in order to boost competition. It should be possible to get similar effects with other aspects of the bidding format, such as lot sizes, the distance between permissible quantity levels.

5 Conclusions

For an auctioneer it is beneficial if bidders increase quantity increments at prices far from the reservation price and if bidders decrease their quantity increments near the reservation price. It is shown that such a pro-competitive effect on bids can be achieved with rationing rules that prioritize large marginal quantity increments at clearing prices far from the reservation price and then gives increased priority to small marginal quantity increments at price levels closer to the reservation price. For supply schedules with one step, I show that optimal use of disproportionate rationing on the margin for a uniform-price auction with N symmetric suppliers, gives the auctioneer the same procurement cost as a uniform-price auction with pro rata on the margin rationing and $2N - 1$ symmetric suppliers with the same total production cost. The effect is smaller for supply schedules with more steps. For supply functions with v steps, a uniform-price auction with N symmetric suppliers and optimal use of disproportionate rationing on the margin at each step roughly gives the auctioneer the same procurement cost as a uniform-price auction with pro rata on the margin rationing and $\left(1+\frac{1}{v}\right)(N-1)+1$ symmetric suppliers with the same total production cost. However, even if supply functions have many steps, the auctioneer can still boost competition substantially locally by using disproportionate rationing on the margin at a few price levels, where the auctioneer expects the auction to clear or where the auctioneer is mostly concerned with market competitiveness. Forward prices or clearing prices of previous auctions can be used to predict the clearing price.

The paper also identifies situations where competitiveness of the auction can be improved if the same rationing rule is used at all price levels. It is also shown how the bidding format, such tick-sizes, can be tailored to create such situations.

The pro-competitive mechanism is almost mechanical, so although my results are derived for costs that are common knowledge, they should qualitatively hold for other standard models of divisible-good auctions.[5] I consider a uniform-price auction, where all accepted bids are paid the marginal price. However, intuitively similar results should hold for all or most multi-item auctions with non-truthtelling mechanisms[6], including pay-as-bid auctions[7]. Similarly, the pro-competitive mechanism should work also when then there is a finite set of permissible quantities, as in practice, so that quantities cannot be varied continuously as in the model. Rationing rules with normalizations of quantity increments with respect to the size of a bidder may improve performance in auctions with asymmetric bidders. Finally, although results are derived for a procurement auction with supply-side bidding, analogous results would hold for a sales auction with demand-side bidding as well as for double auctions and exchanges that have both demand-side and supply-side bidding.

The bidding format and parts of the auction software that receives and manages bids can be kept unchanged when implementing a pro-competitive rationing rule, so it should be straightforward to implement it in practice.

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[5] In the general case, costs would be asymmetric information. Costs (values in sales auctions) can for example be private (Reny, 1999) or affiliated information (Ausubel and Cramton, 1996; Vives, 2011). One extreme case of affiliated costs is when costs are common but uncertain (Hortacsu and McAdams, 2010; Wang and Zender, 2002; Wilson, 1979).

[6] Vickrey (1961), Clarke (1971), Groves (1973) and Ausubel (2004) introduce auctions that give bidders incentives to bid their true cost, so a rationing rule would not be able to improve competition in such auctions.

[7] Pay-as-bid (or discriminatory) auctions are used in many treasury auctions and some electricity markets around the world. They have been analysed by Anderson et al. (2013), Ausubel et al. (2014), Fabra et al. (2006), Holmberg (2009), Hortacsu and McAdams (2010), Kastl (2012) and Wang and Zender, (2002).

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Appendix A: Technical results

First we verify that the special case when $\mu_j = 1$ corresponds to pro rata on the margin rationing at the price level P_j.

Lemma 4 *The auction has pro rata on the margin rationing at the price level P_j when $\mu_j = 1$.*

Proof. We can use the identities $\sum_{k=1}^{N} s_k(\varepsilon) \equiv \varepsilon$ and $\sum_{k=1}^{N} S_j^k \equiv S_j$ to simplify and then solve (2) when $\mu_j = 1$:

$$\frac{ds_i(\varepsilon)}{d\varepsilon} = \frac{S_j^i - s_i(\varepsilon)}{S_j - \varepsilon}$$
$$\frac{\frac{ds_i(\varepsilon)}{d\varepsilon}}{S_j - \varepsilon} + \frac{s_i(\varepsilon)}{(S_j - \varepsilon)^2} = \frac{S_j^i}{(S_j - \varepsilon)^2}.$$

It now follows from the product rule and integration that:

$$\frac{d}{d\varepsilon}\left(\frac{s_i(\varepsilon)}{S_j-\varepsilon}\right) = \frac{S_j^i}{(S_j-\varepsilon)^2}$$
$$\frac{s_i(\varepsilon)}{S_j-\varepsilon} - \frac{s_i(S_{j-1})}{S_j-S_{j-1}} = \frac{S_j^i}{S_j-\varepsilon} - \frac{S_j^i}{S_j-S_{j-1}}.$$

It now follows from (1) that:

$$s_i(\varepsilon) = S_j^i - \frac{\Delta S_j^i (S_j - \varepsilon)}{\Delta S_j} = S_{j-1}^i + \frac{\Delta S_j^i (\varepsilon - S_{j-1})}{\Delta S_j},$$

which is identical to the accepted supply of a supplier in a uniform-price auction with pro rata on the margin rationing (Holmberg et al., 2013), when demand is inelastic. ■

The following statement ensures that there is a unique allocation under disproportionate rationing. Note that rationing is never required at price levels where no supplier has a quantity increment.

Lemma 5 *For a given set of non-decreasing stepped supply functions* $\mathbf{S}$*, such that* $S_j^k > S_{j-1}^k$ *for at least one supplier* $k \in \{1,\ldots,N\}$*, there exists a unique rationing allocation at price* P_j*, defined by the initial value problem (1) and (2). This unique solution satisfies* $s_i(\varepsilon) \leq S_j^i = s_i(S_j)$ *and* $s_i'(\varepsilon) \geq 0$ *for* $\varepsilon \in [S_{j-1}, S_j)$ *and* $\forall i \in \{1,\ldots,N\}$*.*

Proof. We have $S_j^i \geq S_{j-1}^i = s_i(S_{j-1})$. Thus it follows from (2) that $s_i'(\varepsilon) \geq 0$ when $s_i(\varepsilon) < S_j^i$ and that $s_i'(\varepsilon) = 0$ when $s_i(\varepsilon) = S_j^i$, as long as there is some supplier $k \in \{1,\ldots,N\}$ with $s_k(\varepsilon) < S_j^k$. There must be at least one such supplier for $\varepsilon \in [S_{j-1}, S_j)$, otherwise we would get the contradiction that $S_j \leq s(\varepsilon) = \varepsilon$ for some $\varepsilon \in [S_{j-1}, S_j)$. We also note that the right-hand side of (2) is Lipschitz continuous in the interval $[S_{j-1}, \varepsilon^*]$ for any $\varepsilon^* \in [S_{j-1}, S_j)$, so it follows from the Picard–Lindelöf theorem that the initial value problem has a unique solution in the interval $[S_{j-1}, S_j)$. ■

From the properties of the sharing rule, it is now possible to derive a first-order condition for the optimal supply schedule of a supplier.

Lemma 6 *The first-order condition for supplier i's optimal output at price* P_j *is:*

$$\begin{aligned}\frac{\partial E(\pi_i)}{\partial S_j^i} &= -\Delta P_{j+1} S_j^i g(S_j) \\ &+ \int_{S_{j-1}}^{S_j} \left[P_j - C_i'(s_i(\varepsilon))\right] \frac{\partial s_i(\varepsilon)}{\partial S_j^i} g(\varepsilon)\, d\varepsilon + \int_{S_j}^{S_{j+1}} \left[P_{j+1} - C_i'(s_i(\varepsilon))\right] \frac{\partial s_i(\varepsilon)}{\partial S_j^i} g(\varepsilon)\, d\varepsilon = 0.\end{aligned} \tag{14}$$

Proof. The accepted supply of supplier i only depends on S_j^i for $\varepsilon \in [S_{j-1}, S_j)$ when the clearing price is P_j and for outcomes $\varepsilon \in [S_j, S_{j+1})$ when the clearing price is P_{j+1}. The contribution to the expected profit from the outcomes $\varepsilon \in [S_{j-1}, S_j)$ is given by:

$$E_j^i = \int_{S_{j-1}}^{S_j} [P_j s_i(\varepsilon) - C_i(s_i(\varepsilon))] g(\varepsilon)\, d\varepsilon,$$

so

$$\frac{\partial E_j^i}{\partial S_j^i} = [P_j s_i(S_j) - C_i(s_i(S_j))] g(S_j) + \int_{S_{j-1}}^{S_j} [P_j - C_i'(s_i(\varepsilon))] \frac{\partial s_i(\varepsilon)}{\partial S_j^i} g(\varepsilon)\, d\varepsilon. \quad (15)$$

The contribution to the expected profit from outcomes $\varepsilon \in [S_j, S_{j+1})$ is given by:

$$E_{j+1}^i = \int_{S_j}^{S_{j+1}} [P_{j+1} s_i(\varepsilon) - C_i(s_i(\varepsilon))] g(\varepsilon)\, d\varepsilon,$$

so

$$\frac{\partial E_{j+1}^i}{\partial S_j^i} = -[P_{j+1} s_i(S_j) - C_i(s_i(S_j))] g(S_j) + \int_{S_j}^{S_{j+1}} [P_{j+1} - C_i'(s_i(\varepsilon))] \frac{\partial s_i(\varepsilon)}{\partial S_j^i} g(\varepsilon)\, d\varepsilon. \quad (16)$$

Summing the contributions from (15) and (16) establishes the result in (14). ■

In this paper, I will focus on characterizing symmetric Nash equilibria. Thus I want to find the optimal response of a supplier i when its $N-1$ competitors submit identical bids. It follows from (14) that the optimal stepped supply function is to a large extent determined by how supplier i's accepted supply $s_i(\varepsilon)$ depends on its supply function. The following Lemma specifies this dependence when the supplier's $N-1$ competitors submit identical bids.

Lemma 7 *For N symmetric producers we have that :*

$$\left.\frac{\partial s_i(\varepsilon)}{\partial S_j^i}\right|_{S_j^i = \frac{S_j}{N}} = \begin{cases} \frac{(N-1)}{N}\left(1 - \frac{(S_j-\varepsilon)^{\mu_j}}{(\Delta S_j)^{\mu_j}}\right) & \text{if } \varepsilon \in [S_{j-1}, S_j) \\ \frac{(N-1)(S_{j+1}-\varepsilon)^{\mu_{j+1}}}{N(\Delta S_{j+1})^{\mu_{j+1}}} & \text{if } \varepsilon \in [S_j, S_{j+1}) \\ 0 & \text{otherwise} \end{cases}$$

Proof. For fixed S_k^i $\forall k \neq j$, increasing S_j^i will increase producer i's quantity increment at the price p_j and decrease its quantity increment at the price p_{j+1}. The quantity increments and offered supply at all other price levels will remain unchanged. Thus a change in S_j^i will only influence the accepted supply for outcomes $\varepsilon \in [S_{j-1}, S_j)$ when the clearing price is p_j and outcomes $\varepsilon \in [S_j, S_{j+1})$

when the clearing price is p_{j+1}. Let $u_{ki}(\varepsilon) = \frac{\partial s_k(\varepsilon)}{\partial S_j^i}$ and first consider $\varepsilon \in (S_{j-1}, S_j)$. It follows from (2) that

$$\begin{aligned}
u_{ii}'(\varepsilon) = & \frac{\mu_j (1 - u_{ii}(\varepsilon)) \left(S_j^i - s_i(\varepsilon)\right)^{\mu_j - 1}}{\sum_{k=1}^N \left(S_j^k - s_k(\varepsilon)\right)^{\mu_j}} \\
& - \frac{\mu_j \left(S_j^i - s_i(\varepsilon)\right)^{\mu_j} (1 - u_{ii}(\varepsilon)) \left(S_j^i - s_i(\varepsilon)\right)^{\mu_j - 1}}{\left[\sum_{k=1}^N \left(S_j^k - s_k(\varepsilon)\right)^{\mu_j}\right]^2} \\
& - \frac{\mu_j \left(S_j^i - s_i(\varepsilon)\right)^{\mu_j} \sum_{k \neq i}^N (-u_{ki}(\varepsilon)) \left(S_j^k - s_k(\varepsilon)\right)^{\mu_j - 1}}{\left[\sum_{k=1}^N \left(S_j^k - s_k(\varepsilon)\right)^{\mu_j}\right]^2}.
\end{aligned}$$

Symmetry, i.e. $S_j^i = S_j^k$, yields

$$u_{ii}'(\varepsilon) = \frac{\mu_j (1 - u_{ii}(\varepsilon))}{N \left(S_j^i - s_i(\varepsilon)\right)} - \frac{\mu_j (1 - u_{ii}(\varepsilon))}{N^2 \left(S_j^i - s_i(\varepsilon)\right)} + \frac{\mu_j \sum_{k \neq i}^N u_{ki}(\varepsilon)}{N^2 \left(S_j^i - s_i(\varepsilon)\right)}. \tag{17}$$

Notice that $\sum_{k=1}^N s_k(\varepsilon) \equiv \varepsilon$ and accordingly $\sum_{k=1}^N u_{ki}(\varepsilon) \equiv 0$. Thus we can write (17) as follows:

$$\begin{aligned}
u_{ii}'(\varepsilon) &= \frac{\mu_j (1 - u_{ii}(\varepsilon))}{N \left(S_j^i - s_i(\varepsilon)\right)} - \frac{\mu_j}{N^2 \left(S_j^i - s_i(\varepsilon)\right)} \\
&= \frac{\mu_j ((N-1)/N - u_{ii}(\varepsilon))}{S_j - \varepsilon},
\end{aligned}$$

where $S_j = N S_j^i$. Hence,

$$(S_j - \varepsilon) u_{ii}'(\varepsilon) + \mu_j u_{ii}(\varepsilon) = \mu_j (N-1)/N.$$

We solve this differential equation by means of an integrating factor. Multiplying all terms by $\frac{1}{(S_j - \varepsilon)^{\mu_j + 1}}$ yields:

$$\frac{u_{ii}'(\varepsilon)}{(S_j - \varepsilon)^{\mu_j}} + \frac{\mu_j u_{ii}(\varepsilon)}{(S_j - \varepsilon)^{\mu_j + 1}} = \frac{\mu_j (N-1)/N}{(S_j - \varepsilon)^{\mu_j + 1}}.$$

By means of the product rule we get

$$\frac{d}{d\varepsilon} \frac{u_{ii}(\varepsilon)}{(S_j - \varepsilon)^{\mu_j}} = \frac{d}{d\varepsilon} \frac{(N-1)/N}{(S_j - \varepsilon)^{\mu_j}},$$

so that

$$\frac{u_{ii}(\varepsilon)}{(S_j - \varepsilon)^{\mu_j}} - \frac{u_{ii}(S_{j-1})}{(S_j - S_{j-1})^{\mu_j}} = \frac{(N-1)/N}{(S_j - \varepsilon)^{\mu_j}} - \frac{(N-1)/N}{(S_j - S_{j-1})^{\mu_j}}.$$

We have $u_{ii}(S_{j-1}) = 0$, so

$$\frac{\partial s_i(\varepsilon)}{\partial S_j^i} = u_{ii}(\varepsilon) = \frac{(N-1)}{N}\left(1 - \frac{(S_j - \varepsilon)^{\mu_j}}{(\Delta S_j)^{\mu_j}}\right) \text{ if } \varepsilon \in (S_{j-1}, S_j).$$

Now, we will repeat the same procedure for the interval $\varepsilon \in (S_j, S_{j+1})$ when the price is p_{j+1}. Again, let $u_{ki}(\varepsilon) = \frac{\partial s_k(\varepsilon)}{\partial S_j^i}$. In this interval, we have (compare with (2))

$$s_i'(\varepsilon) = \frac{\left(S_{j+1}^i - s_i(\varepsilon)\right)^{\mu_{j+1}}}{\sum_{k=1}^N \left(S_{j+1}^k - s_k(\varepsilon)\right)^{\mu_{j+1}}}.$$

Thus

$$\begin{aligned} u_{ii}'(\varepsilon) = &- \frac{\mu_{j+1} u_{ii}(\varepsilon)\left(S_{j+1}^i - s_i(\varepsilon)\right)^{\mu_{j+1}-1}}{\sum_{k=1}^N \left(S_{j+1}^k - s_k(\varepsilon)\right)^{\mu_{j+1}}} \\ &+ \frac{\left(S_{j+1}^i - s_i(\varepsilon)\right)^{\mu_{j+1}} \sum_{k=1}^N u_{ki}(\varepsilon)\,\mu_{j+1}\left(S_{j+1}^k - s_k(\varepsilon)\right)^{\mu_{j+1}-1}}{\left[\sum_{k=1}^N \left(S_{j+1}^k - s_k(\varepsilon)\right)^{\mu_{j+1}}\right]^2}. \end{aligned}$$

Symmetry implies that

$$u_{ii}'(\varepsilon) = -\frac{\mu_{j+1} u_{ii}(\varepsilon)}{N\left(S_{j+1}^i - s_i(\varepsilon)\right)} + \frac{\mu_{j+1}\sum_{k=1}^N u_{ki}(\varepsilon)}{N^2\left(S_{j+1}^i - s_i(\varepsilon)\right)}.$$

As before $\sum_{k=1}^N s_k(\varepsilon) \equiv \varepsilon$ implies that $\sum_{k=1}^N u_{ki}(\varepsilon) \equiv 0$, so

$$u_{ii}'(\varepsilon) = \frac{-\mu_{j+1} u_{ii}(\varepsilon)}{S_{j+1} - \varepsilon},$$

where $S_{j+1} = N S_{j+1}^i$. Hence,

$$(S_{j+1} - \varepsilon)\, u_{ii}'(\varepsilon) + \mu_{j+1} u_{ii}(\varepsilon) = 0.$$

As above, we solve this differential equation by means of an integrating factor. Multiplying all terms by $\frac{1}{(S_{j+1}-\varepsilon)^{\mu_{j+1}+1}}$ yields:

$$\frac{u_{ii}'(\varepsilon)}{(S_{j+1} - \varepsilon)^{\mu_{j+1}}} + \frac{\mu_{j+1} u_{ii}(\varepsilon)}{(S_{j+1} - \varepsilon)^{\mu_{j+1}+1}} = 0.$$

Thus it follows from the product rule that

$$\frac{d}{d\varepsilon}\frac{u_{ii}(\varepsilon)}{(S_{j+1} - \varepsilon)^{\mu_{j+1}}} = 0,$$

so that

$$\frac{u_{ii}(\varepsilon)}{(S_{j+1} - \varepsilon)^{\mu_{j+1}}} = \frac{u_{ii}(S_j)}{(S_{j+1} - S_j)^{\mu_{j+1}}}, \tag{18}$$

where $u_{ii}(S_j)$ can be determined from the relation

$$1 = \frac{dS_j^i}{dS_j^i} = \frac{ds_i(S_j)}{dS_j^i} = u_{ii}(S_j) + s_i'(S_j)\frac{dS_j}{dS_j^i}.$$

We have $s_i'(S_j) = s_i'(\varepsilon) = \frac{1}{N}$ due to symmetry and $\frac{dS_j}{dS_j^i} = 1$, so

$$u_{ii}(S_j) = 1 - \frac{1}{N} = \frac{N-1}{N}.$$

Now, it follows from (18) that

$$\frac{\partial s_i(\varepsilon)}{\partial S_j^i} = u_{ii}(\varepsilon) = \frac{(N-1)(S_{j+1}-\varepsilon)^{\mu_{j+1}}}{N(\Delta S_{j+1})^{\mu_{j+1}}} \text{ if } \varepsilon \in (S_j, S_{j+1}).$$

Finally, we note that $\frac{\partial s_i(\varepsilon)}{\partial S_j^i}$ is continuous at the points $\varepsilon = S_j$ and $\varepsilon = S_{j+1}$. ■

We can now conclude the following from Lemma 6 and Lemma 7 in Appendix A.

Corollary 2 *The first-order condition of a market with N symmetric suppliers is given by:*

$$\begin{aligned} \tfrac{\partial E(\pi_i)}{\partial S_j^i} &= -\Delta P_{j+1} S_j^i g(S_j) \\ +\tfrac{(N-1)}{N} &\int_{S_{j-1}}^{S_j} [P_j - C_i'(s_i(\varepsilon))]\left(1 - \tfrac{(S_j-\varepsilon)^{\mu_j}}{(\Delta S_j)^{\mu_j}}\right) g(\varepsilon)\, d\varepsilon \\ +\tfrac{(N-1)}{N(\Delta S_{j+1})^{\mu_{j+1}}} &\int_{S_j}^{S_{j+1}} [P_{j+1} - C_i'(s_i(\varepsilon))](S_{j+1}-\varepsilon)^{\mu_{j+1}} g(\varepsilon)\, d\varepsilon = 0. \end{aligned} \tag{19}$$

For extreme cases when $\mu_j = 0$ or $\mu_j = \infty$, the acceptance sensitivity with respect to quantity increments, i.e. $\frac{\partial s_i(\varepsilon)}{\partial S_j^i}$, can also be determined at asymmetric points, where $S_j^i \neq S_j^k$.

Lemma 8 *If $\mu_j = 0$ and competitors have identical supply functions, S_j^k, then:*

$$\frac{\partial s_i(\varepsilon)}{\partial S_j^i} = \begin{cases} 0 & \text{if } \Delta S_j^i > \Delta S_j^k \text{ and } \varepsilon \in (S_{j-1}, S_j) \\ 0 & \text{if } \Delta S_j^i < \Delta S_j^k \text{ and } \varepsilon \in (S_{j-1}, S_{j-1} + N\Delta S_j^i) \\ 1 & \text{if } \Delta S_j^i < \Delta S_j^k \text{ and } \varepsilon \in (S_{j-1} + N\Delta S_j^i, S_j) \end{cases}$$

and

$$\frac{\partial s_i(\varepsilon)}{\partial S_{j-1}^i} = \begin{cases} \frac{N-1}{N} & \text{if } \Delta S_j^i > \Delta S_j^k \text{ and } \varepsilon \in (S_{j-1}, S_{j-1} + N\Delta S_j^k) \\ 0 & \text{if } \Delta S_j^i > \Delta S_j^k \text{ and } \varepsilon \in (S_{j-1} + N\Delta S_j^k, S_j) \\ \frac{N-1}{N} & \text{if } \Delta S_j^i < \Delta S_j^k \text{ and } \varepsilon \in (S_{j-1}, S_{j-1} + N\Delta S_j^i) \\ 0 & \text{if } \Delta S_j^i < \Delta S_j^k \text{ and } \varepsilon \in (S_{j-1} + N\Delta S_j^i, S_j) \end{cases}$$

Proof. It follows from (2) that for $\mu_j = 0$ and $\Delta S_j^i > \Delta S_j^k$, all producers get the same accepted quantity from marginal bids for $\varepsilon \in \left(S_{j-1}, S_{j-1} + N\Delta S_j^k\right)$, while competitors' accepted quantity of marginal bids is constant in the interval $\left(S_{j-1} + N\Delta S_j^k, S_j\right)$. Thus

$$s_i(\varepsilon) = \begin{cases} S_{j-1}^i + \frac{\varepsilon - S_{j-1}}{N} & \text{if } \varepsilon \in \left(S_{j-1}, S_{j-1} + N\Delta S_j^k\right) \\ S_{j-1}^i + \Delta S_j^k + \varepsilon - S_{j-1} - N\Delta S_j^k & \text{if } \varepsilon \in \left(S_{j-1} + N\Delta S_j^k, S_j\right). \end{cases}$$

For $\mu_j = 0$ and $\Delta S_j^i < \Delta S_j^k$, all producers get the same accepted quantity of marginal bids for $\varepsilon \in \left(S_{j-1}, S_{j-1} + N\Delta S_j^i\right)$, while supplier i's accepted quantity from marginal bids is constant in the interval $\left(S_{j-1} + N\Delta S_j^i, S_j\right)$. Thus

$$s_i(\varepsilon) = \begin{cases} S_{j-1}^i + \frac{\varepsilon - S_{j-1}}{N} & \text{if } \varepsilon \in \left(S_{j-1}, S_{j-1} + N\Delta S_j^i\right) \\ S_j^i & \text{if } \varepsilon \in \left(S_{j-1} + N\Delta S_j^i, S_j\right). \end{cases}$$

The statement follows from differentiation of the expressions above with respect to S_{j-1}^i and S_j^i. ■

Lemma 9 *If $\mu_j = \infty$ and competitors have identical supply functions, S_j^k, then:*

$$\frac{\partial s_i(\varepsilon)}{\partial S_j^i} = \begin{cases} 0 & \text{if } \Delta S_j^i > \Delta S_j^k \text{ and } \varepsilon \in \left(S_{j-1}, S_{j-1} + \Delta S_j^i - \Delta S_j^k\right) \\ \frac{N-1}{N} & \text{if } \Delta S_j^i > \Delta S_j^k \text{ and } \varepsilon \in \left(S_{j-1} + \Delta S_j^i - \Delta S_j^k, S_j\right) \\ 0 & \text{if } \Delta S_j^i < \Delta S_j^k \text{ and } \varepsilon \in \left(S_{j-1}, S_{j-1} + (N-1)\left(\Delta S_j^k - \Delta S_j^i\right)\right) \\ \frac{N-1}{N} & \text{if } \Delta S_j^i < \Delta S_j^k \text{ and } \varepsilon \in \left(S_{j-1} + (N-1)\left(\Delta S_j^k - \Delta S_j^i\right), S_j\right) \end{cases}$$

and

$$\frac{\partial s_i(\varepsilon)}{\partial S_{j-1}^i} = \begin{cases} 0 & \text{if } \Delta S_j^i > \Delta S_j^k \text{ and } \varepsilon \in \left(S_{j-1}, S_{j-1} + \Delta S_j^i - \Delta S_j^k\right) \\ 0 & \text{if } \Delta S_j^i > \Delta S_j^k \text{ and } \varepsilon \in \left(S_{j-1} + \Delta S_j^i - \Delta S_j^k, S_j\right) \\ 1 & \text{if } \Delta S_j^i < \Delta S_j^k \text{ and } \varepsilon \in \left(S_{j-1}, S_{j-1} + (N-1)\left(\Delta S_j^k - \Delta S_j^i\right)\right) \\ 0 & \text{if } \Delta S_j^i < \Delta S_j^k \text{ and } \varepsilon \in \left(S_{j-1} + (N-1)\left(\Delta S_j^k - \Delta S_j^i\right), S_j\right). \end{cases}$$

Proof. It follows from (2) that for $\mu_j = \infty$ and $\Delta S_j^i > \Delta S_j^k$ marginal bids are only accepted from supplier i, as long as its unmet supply at P_j, $S_j^i - s_i(\varepsilon)$, is larger than for each other supplier. Thus

$$s_i(\varepsilon) = \begin{cases} S_{j-1}^i + \varepsilon - S_{j-1} & \text{if } \varepsilon \in \left(S_{j-1}, S_{j-1} + \Delta S_j^i - \Delta S_j^k\right) \\ S_j^i - \Delta S_j^k + \frac{\varepsilon - S_{j-1} - \Delta S_j^i + \Delta S_j^k}{N} & \text{if } \varepsilon \in \left(S_{j-1} + \Delta S_j^i - \Delta S_j^k, S_j\right). \end{cases}$$

If instead $\mu_j = \infty$ and $\Delta S_j^i < \Delta S_j^k$, then marginal bids are only accepted from competitors of supplier i, as long as each competitor's unmet supply at P_j, $S_j^k - s_k(\varepsilon)$, is larger than for supplier i.

$$s_i(\varepsilon) = \begin{cases} S_{j-1}^i & \text{if } \varepsilon \in \left(S_{j-1}, S_{j-1} + (N-1)\left(\Delta S_j^k - \Delta S_j^i\right)\right) \\ S_{j-1}^i + \frac{\varepsilon - S_{j-1} - (N-1)\left(\Delta S_j^k - \Delta S_j^i\right)}{N} & \text{if } \varepsilon \in \left(S_{j-1} + (N-1)\left(\Delta S_j^k - \Delta S_j^i\right), S_j\right). \end{cases}$$

The statement follows from differentiation of the expressions above with respect to S_{j-1}^i and S_j^i. ■

The following lemma is useful when we want to analyse the convergence properties of the first-order condition as the number of steps per supply function increases.

Lemma 10 *We can make the following statements for the first-order condition in Corollary 2 when* $P_j - C_i'\left(S_j^i\right) > 0$ *and* $\mu_j > 0$ *for all price levels :*

1. *The difference* $S_{j+1}^i - S_j^i$ *is of the order* ΔP_{j+1}*.*
2. *The discrete first-order condition in the first-order condition in Corollary 2 can be approximated by:*

$$\frac{\partial E(\pi_i)}{\partial S_j^i} = -\Delta P_{j+1} S_j^i g(S_j) + \frac{(N-1)}{N}\left[P_j - C_i'(S_j^i)\right] g(S_j)\left(\frac{\mu_j \Delta S_j}{(\mu_j+1)} + \frac{\Delta S_{j+1}}{(\mu_{j+1}+1)}\right) \tag{20}$$

$$+ O\left((\Delta P_{j+1})^2\right). \tag{21}$$

Proof. The sum

$$\begin{aligned} I :=& \tfrac{(N-1)}{N} \int_{S_{j-1}}^{S_j} \left[P_j - C_i'(s_i(\varepsilon))\right]\left(1 - \tfrac{(S_j-\varepsilon)^{\mu_j}}{(\Delta S_j)^{\mu_j}}\right) g(\varepsilon)\, d\varepsilon \\ &+ \tfrac{(N-1)}{N(\Delta S_{j+1})^{\mu_{j+1}}} \int_{S_j}^{S_{j+1}} \left[P_{j+1} - C_i'(s_i(\varepsilon))\right](S_{j+1}-\varepsilon)^{\mu_{j+1}} g(\varepsilon)\, d\varepsilon \end{aligned} \tag{22}$$

must be of the order ΔP_{j+1}, otherwise the first-order condition in Corollary 2 in Appendix A cannot be satisfied for small ΔP_{j+1}. Supply schedules are symmetric and non-decreasing. Moreover, $P_{j+1} - C_i'\left(S_{j+1}^i\right) > 0$, $\mu_j > 0$, $N \geq 2$, and $g(\varepsilon) > 0$, so it follows that we must have:

$$I \geq \frac{(N-1)\left[P_j - C_i'(S_j^i)\right]\underline{g}}{N} \int_{S_{j-1}}^{S_j} \left(1 - \frac{(S_j-\varepsilon)^{\mu_j}}{(\Delta S_j)^{\mu_j}}\right) d\varepsilon \geq 0. \tag{23}$$

We have that I is of the order ΔP_{j+1} and $\Delta S_j \geq \Delta S_j^i \geq 0$, so the inequality above implies that $\Delta S_j$and ΔS_j^i must both be of the order ΔP_{j+1}, or equivalently of the order ΔP_j, as $r = \frac{\Delta P_j}{\Delta P_{j+1}}$ is bounded.

In the next step we want to derive the Taylor expansions of the first-order conditions. Using Taylor expansions and the result above, the first-order condition in Corollary 2 can be written:

$$\begin{aligned} \tfrac{\partial E(\pi_i)}{\partial S_j^i} =& -\Delta P_{j+1} S_j^i g(S_j) \\ &+ \tfrac{(N-1)}{N} \int_{S_{j-1}}^{S_j} \left[P_j - C_i'(S_j^i) + O(\Delta P_j)\right]\left(1 - \tfrac{(S_j-\varepsilon)^{\mu_j}}{(\Delta S_j)^{\mu_j}}\right)\left[g(S_j) + O(\Delta P_j)\right] d\varepsilon \\ &+ \tfrac{(N-1)}{N} \int_{S_j}^{S_{j+1}} \left[P_{j+1} - C_i'(S_{j+1}^i) + O(\Delta P_{j+1})\right]\left(\tfrac{S_{j+1}-\varepsilon}{\Delta S_{j+1}}\right)^{\mu_{j+1}}\left[g(S_{j+1}) + O(\Delta P_{j+1})\right] d\varepsilon \end{aligned}$$

Hence, as ΔS_j and ΔS_j^i are of the order ΔP_{j+1}:

$$
\begin{array}{c}
\frac{\partial E(\pi_i)}{\partial S_j^i} = -\Delta P_{j+1} S_j^i g\left(S_j\right) \\
+\frac{(N-1)}{N}\left[P_j - C_i'(S_j^i)\right] g\left(S_j\right) \int\limits_{S_{j-1}}^{S_j} \left(1 - \frac{(S_j-\varepsilon)^{\mu_j}}{(\Delta S_j)^{\mu_j}}\right) d\varepsilon \\
+\frac{(N-1)}{N}\left[P_{j+1} - C_i'(S_{j+1}^i)\right] g\left(S_{j+1}\right) \int\limits_{S_j}^{S_{j+1}} \frac{(S_{j+1}-\varepsilon)^{\mu_j+1}}{(\Delta S_{j+1})^{\mu_j+1}} d\varepsilon \\
+O\left((\Delta P_{j+1})^2\right).
\end{array}
\tag{24}
$$

It can be shown that

$$
\begin{array}{c}
\int_{S_{j-1}}^{S_j} \left(1 - \frac{(S_j-\varepsilon)^{\mu_j}}{(\Delta S_j)^{\mu_j}}\right) d\varepsilon = \frac{\mu_j \Delta S_j}{(\mu_j+1)} \\
\int_{S_j}^{S_{j+1}} \frac{(S_{j+1}-\varepsilon)^{\mu_j+1}}{(\Delta S_{j+1})^{\mu_j+1}} d\varepsilon = \frac{\Delta S_{j+1}}{(\mu_{j+1}+1)}.
\end{array}
$$

Using these results and that ΔS_j and ΔS_j^i are of the order ΔP_{j+1}, the Taylor expansion in (24) can be simplified to:

$$
\begin{aligned}
\frac{\partial E\left(\pi_i\right)}{\partial S_j^i} &= -\Delta P_{j+1} S_j^i g\left(S_j\right) + \frac{(N-1)}{N}\left[P_j - C_i'(S_j^i)\right] g\left(S_j\right)\left(\frac{\mu_j \Delta S_j}{(\mu_j+1)} + \frac{\Delta S_{j+1}}{(\mu_{j+1}+1)}\right) \\
&\quad + O\left((\Delta P_{j+1})^2\right).
\end{aligned}
\tag{25}
$$

■

Appendix B: Proofs of lemmas and propositions in the main text

Proof. (Lemma 1) This follows from Corollary 2 in Appendix A and the substitutions $u = \frac{S_j-\varepsilon}{\Delta S_j}$ and $u = \frac{S_{j+1}-\varepsilon}{\Delta S_{j+1}}$, respectively. ■

Proof. (Lemma 2) We have

$$
\int\limits_0^1 \left(1 - u^{\mu_j}\right) du = \left[u - \frac{u^{\mu_j+1}}{\mu_j+1}\right]_0^1 = \frac{\mu_j}{\mu_j+1}
$$

and

$$
\int\limits_0^1 u^{\mu_{j+1}} du = \left[\frac{u^{\mu_{j+1}+1}}{\mu_{j+1}+1}\right]_0^1 = \frac{1}{\mu_{j+1}+1},
$$

so it follows from Lemma 1 and Assumption 1 that:

$$
\begin{aligned}
\Delta P_2 S_1^i &= \frac{(N-1)\left(P_1 - c\right)\mu_1 \Delta S_1^i}{(\mu_1+1)} + \frac{(N-1)\left(P_2 - c\right)\Delta S_2^i}{(\mu_2+1)} \\
\Delta P_2 S_1^i &= \frac{(N-1)\left(P_1 - c\right)\mu_1 S_1^i}{(\mu_1+1)} + \frac{(N-1)\left(P_2 - c\right)\left(k_i - S_1^i\right)}{(\mu_2+1)} \\
S_1^i &= \frac{(N-1)\left(P_2 - c\right)k_i}{(\mu_2+1)\left(\Delta P_2 - \frac{(N-1)(P_1-c)\mu_1}{(\mu_1+1)} + \frac{(N-1)(P_2-c)}{(\mu_2+1)}\right)}.
\end{aligned}
$$

■

Proof. (Proposition 1) It follows from (6) and Lemma 2 that the first-order solution of S_1^i increases when μ_2 decreases and that S_1^i increases when μ_1 increases. Thus competitiveness is maximized when $\mu_1 = \infty$ and $\mu_2 = 0$. (8) - (10) follows from Lemma 2.

In the next step we want to prove that the first-order solution in (8) constitutes an NE. It follows from Lemma 6, Lemma 8 and Lemma 9 in Appendix A that if $\mu_1 = \infty$ and $\mu_2 = 0$, and competitors have identical supply, $S_1^k = \frac{(N-1)k_i(P_2-c)}{N\Delta P_2}$, then:

$$\begin{array}{c} \frac{\partial E(\pi_i)}{\partial S_1^i} = -\Delta P_2 S_1^i g \\ +\frac{N-1}{N}(P_1-c)\, g \min\left(\Delta S_1 - \Delta S_1^i + \Delta S_1^k, \Delta S_1 - (N-1)\left(\Delta S_1^k - \Delta S_1^i\right)\right) \\ +\frac{N-1}{N}\min\left(N\Delta S_2^i, N\Delta S_2^k\right)(P_2-c)\, g \\ = -\Delta P_2 S_1^i g + (N-1)(P_1-c)\, g \min\left(S_1^k, S_1^i\right) \\ +(N-1)\min\left(k_i - S_1^i, k_i - S_1^k\right)(P_2-c)\, g. \end{array} \tag{26}$$

We note that $\frac{\partial E(\pi_i)}{\partial S_1^i}$ is piece-wise linear in S_1^i with a break point at $S_1^i = S_1^k$, where $\frac{\partial E(\pi_i)}{\partial S_1^i} = 0$. Moreover, $\frac{\partial E(\pi_i)}{\partial S_1^i} = (N-1)\left(k_i - S_1^k\right)(P_2-c)\, g \geq 0$ for $S_1^i = 0$ and it follows from (6) that $\frac{\partial E(\pi_i)}{\partial S_1^i} = -\Delta P_2 k_i g + (N-1)(P_1-c)\, g S_1^k \leq 0$ for $S_1^i = k_i$. Hence, we can conclude that $\frac{\partial^2 E(\pi_i)}{\partial\left(S_1^i\right)^2} \leq 0$. Thus $S_1^i = S_1^k$ is the best response to $S_1^k = \frac{(N-1)k_i(P_2-c)}{N\Delta P_2}$, which verifies that (8) constitutes a Nash equilibrium if $\mu_1 = \infty$ and $\mu_2 = 0$.

In the next step we want to prove that the first-order solution in (9) constitutes an NE. It follows from Lemma 6 and Lemma 8 in Appendix A that if $\mu_1 = \mu_2 = 0$, and competitors have identical supply , $S_1^k = \frac{(N-1)(P_2-c)\ k_i}{\Delta P_2 + (N-1)(P_2-c)}$, then:

$$\begin{array}{c} \frac{\partial E(\pi_i)}{\partial S_1^i} = -\Delta P_2 S_1^i g \\ +(N-1)\max\left(0, \Delta S_1^k - \Delta S_1^i\right)(P_1-c)\, g \\ +(N-1)\min\left(\Delta S_2^i, \Delta S_2^k\right)(P_2-c)\, g \\ = -\Delta P_2 S_1^i g \\ +(N-1)\max\left(0, S_1^k - S_1^i\right)(P_1-c)\, g \\ +(N-1)\min\left(k_i - S_1^i, k_i - S_1^k\right)(P_2-c)\, g. \end{array} \tag{27}$$

We have $\frac{\partial E(\pi_i)}{\partial S_1^i} = (N-1)\, S_1^k (P_1-c)\, g + (N-1)\left(k_i - S_1^k\right)(P_2-c)\, g \geq 0$ for $S_1^i = 0$ and $\frac{\partial E(\pi_i)}{\partial S_1^i} = -\Delta P_2 k_i g \leq 0$ for $S_1^i = k_i$. $\frac{\partial E(\pi_i)}{\partial S_1^i}$ is piece-wise linear in S_1^i with a break point at $S_1^i = S_1^k$, where $\frac{\partial E(\pi_i)}{\partial S_1^i} = 0$, so we can now conclude that $\frac{\partial^2 E(\pi_i)}{\partial\left(S_1^i\right)^2} \leq 0$. Thus $S_1^i = S_1^k$ is the best response to $S_1^k = \frac{(N-1)(P_2-c)\ k_i}{\Delta P_2 + (N-1)(P_2-c)}$, which verifies that (9) constitutes an Nash equilibrium for $\mu_1 = \mu_2 = 0$.

It follows from Holmberg et al. (2013) that (10) constitutes a Nash equilibrium. Finally, the following argument shows that supplier i's loss associated with the quantity effect at P_2 dominates the loss associated with the quantity effect at P_1

for pro rata on the margin rationing. It follows from Assumption 1 and (6) that

$$\begin{aligned}
\Delta S_1^i (P_1 - c) = S_1^i (P_1 - c) &= \frac{(N-1)(P_2 - c) k_i}{(N+1)\Delta P_2} (P_1 - c) \\
&= \frac{(N-1)(P_1 - c)}{(N+1)\Delta P_2} (P_2 - c) k_i \\
&\leq \frac{\Delta P_2}{(N+1)\Delta P_2} (P_2 - c) k_i \\
&\leq \frac{(N+1)\Delta P_2 - (N-1)(P_2 - c)}{(N+1)\Delta P_2} (P_2 - c) k_i \\
&= \left(k_i - S_1^i\right)(P_2 - c) = \Delta S_2^i (P_2 - c),
\end{aligned}$$

when $S_1^i = \frac{(N-1)k_i(P_2-c)}{(N+1)\Delta P_2}$. ■

Proof. (Lemma 3) We use the Taylor approximation in Lemma 10 to approximate the difference equation in Lemma 1:

$$-\Delta P_{j+1} S_j^i g(S_j) + \left(P_j - C_i'\left(S_j^i\right)\right) g(S_j) \frac{(N-1)}{N} \left[\frac{\Delta S_{j+1}}{\mu_{j+1}+1} + \frac{\mu_j \Delta S_j}{\mu_j + 1}\right] + O\left((\Delta P_{j+1})^2\right) = 0.$$

We have assumed that g is bounded away from zero. Thus

$$-\Delta P_{j+1} S_j^i + \left(P_j - C_i'\left(S_j^i\right)\right) \frac{(N-1)}{N} \left[\frac{\Delta S_{j+1}}{\mu_{j+1}+1} + \frac{\mu_j \Delta S_j}{\mu_j + 1}\right] + O\left((\Delta P_{j+1})^2\right) = 0. \tag{28}$$

Symmetry implies that

$$-\Delta P_{j+1} S_j^i + \left(P_j - C_i'\left(S_j^i\right)\right) (N-1) \left[\frac{\Delta S_{j+1}^i}{\mu_{j+1}+1} + \frac{\mu_j \Delta S_j^i}{\mu_j + 1}\right] + O\left((\Delta P_{j+1})^2\right) = 0.$$

Thus

$$-S_j^i + \left(P_j - C_i'\left(S_j^i\right)\right)(N-1) \frac{\left[\frac{\Delta S_{j+1}^i}{\mu_{j+1}+1} + \frac{\mu_j \Delta S_j^i}{\mu_j+1}\right]}{\Delta P_{j+1}} + O(\Delta P_{j+1}) = 0,$$

so with $\Delta P_j = r \Delta P_{j+1}$

$$-S_j^i + \left(P_j - C_i'\left(S_j^i\right)\right)(N-1)\left(\frac{\Delta S_{j+1}^i}{(\mu_{j+1}+1)\Delta P_{j+1}} + \frac{\mu_j r \Delta S_j^i}{(\mu_j+1)\Delta P_j}\right) + O(\Delta P_{j+1}) = 0.$$

Hence,

$$\begin{aligned}
&\frac{1}{\left(\frac{1}{(\mu_{j+1}+1)} + \frac{\mu_j r}{(\mu_j+1)}\right)} \left(\frac{\Delta S_{j+1}^i}{(\mu_{j+1}+1)\Delta P_{j+1}} + \frac{\mu_j r \Delta S_j^i}{(\mu_j+1)\Delta P_j}\right) \\
&= \frac{S_j^i}{\left(\frac{1}{(\mu_{j+1}+1)} + \frac{\mu_j r}{(\mu_j+1)}\right)(N-1)\left(P_j - C_i'\left(S_j^i\right)\right)} + O(\Delta P_{j+1}).
\end{aligned}$$

If S_j^i are replaced by samples of the continuous supply function $q_i(p)$ at price P_j, then the left-hand side becomes an estimate of $q_i'(P_j)$ and the right-hand side converges to:

$$\frac{q_i(P_j)}{(N-1)\left(\frac{1}{(\mu_{j+1}+1)}+\frac{\mu_j r}{(\mu_j+1)}\right)(P_j - C_i{}'(q_i(P_j)))}$$

when $q_i'(P_j)$ is bounded. Thus the first-order condition in Lemma 1 is consistent with the ordinary differential equation in (11) when $P_j > C_i'(q_i(P_j))$ and $\mu_j > 0$.

■