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NASH EQUILIBRIUM AND EVOLUTION BY IMITATION*

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Abstract

Nash's [16] "mass action" interpretation of his equilibrium concept does not presume that the players know the game or are capable of sophisticated calculations. Instead, players are repeatedly and randomly drawn from large populations to play the game, one population for each player position, and base their strategy choice on observed payoffs. The present paper examines in some detail such an interpretation in a class of population dynamics based on adaptation by way of imitation of successful behaviors. Drawing from results in evolutionary game theory, implications of dynamic stability for aggregate Nash equilibrium play are discussed.

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1 Introduction

The Nash equilibrium criterion is usually justified on rationalistic grounds, in terms of the involved players' "rationality", and, in some way or other, shared knowledge or beliefs about each others rationality and/or strategies etc. (see e.g. Tan and Werlang [20] and Aumann and Brandenburger [2]).

1.1 Nash's rationalistic interpretation

In his unpublished Ph.D. dissertation, John Nash provided the following rationalistic interpretation of his equilibrium criterion:[1]

> "We proceed by investigating the question: what would be a 'rational' prediction of the behavior to be expected of rational playing the game in question? By using the principles that a rational prediction should be unique, that the players should be able to deduce and make use of it, and that such knowledge on the part of each player of what to expect the others to do should not lead him to act out of conformity with the prediction, one is led to the concept of a solution defined before.
>
> If $S_1, S_2, ..., S_n$ were the sets of equilibrium strategies of a solvable game, the 'rational' prediction should be. "the average behavior of rational men playing in position i would define a mixed strategy s_i in S_i if an experiment were carried out.
>
> In this interpretation we need to assume the players know the full structure of the game in order to be able to deduce the prediction for themselves. It is quite strongly a rationalistic and idealizing interpretation." ([16], p. 23)

Nash used the phrase *position i* as we today would use the phrase "player i". Hence, "playing in position i" here means "choosing from the i:th player's strategy set with accompanying payoffs".

[1]For a discussion of the context of Nash's work, see Leonard [13]. We are grateful to Harold Kuhn for showing that study and for providing a copy of Nash's dissertation.

Note the restriction to *solvable* games. Nash defines a game to be such if all its (Nash) equilibria are *interchangeable* in the sense that if s and s' are equilibria, then also the strategy profile s'' in which some player i plays according to s but all others according to s', is an equilibrium. All two-person constant-sum games are solvable in this sense. However, many relevant games for economics are not solvable, in which Nash suggested something in the spirit of set-wise refinement:

> "In an unsolvable game it sometimes happens that good heuristic reasons can be found for narrowing down the set of equilibrium points to those in a single sub-solution, which then play the role of a solution." (op. cit., p23)

Here a *sub-solution* is a maximal set of interchangeable equilibria ([16], p. 10). For instance, a strict Nash equilibrium, i.e., a strategy profile s in which every strategy s_i is the *unique* best reply to s, viewed as a singleton set, is a sub-solution in this sense.

1.2 Nash's "mass-action" interpretation

In fact, Nash also provided a quite distinct, "as if" interpretation, which he called the *mass-action* interpretation:

> "We shall now take up the 'mass-action' interpretation of equilibrium points. In this interpretation solutions have no great significance. It is unnecessary to assume that the participants have full knowledge of the total structure of the game, or the ability and inclination to go through any complex reasoning processes. But the participants are supposed to accumulate empirical information on the relative advantages of the various pure strategies at their disposal.
>
> To be more detailed, we assume that there is a population (in the sense of statistics) of participants for each position of the game. Let us also assume that the 'average playing' of the game involves n participants selected at random from the n populations, and that there is a stable average frequency with which each pure

strategy is employed by the 'average member' of the appropriate population.

Since there is to be no collaboration between individuals playing in different positions of the game, the probability that a particular n-tuple of pure strategies will be employed in a playing of the game should be the product of the probabilities indicating the chance of each of the n pure strategies to be employed in a random playing" ([16], pp. 21-22.)

Nash notes that if s_i is a population distribution over the pure strategies available to the i:th player position, then $s = (s_i)$ is formally identical with a mixed strategy profile, and the expected payoff to any pure strategy in a random matching between individuals, one from each player population, is identical with the expected payoff of the pure strategy when played against the mixed strategy profile s. With $p_{i\alpha}(s)$ denoting the i:th player's expected payoff when using pure strategy α against a mixed-strategy profile s, Nash continues:

> "Now let us consider what effects the experience of the participants will produce. To assume, as we did, that they accumulated empirical evidence on the pure strategies at their disposal is to assume that those playing in position i learn the numbers $p_{i\alpha}(s)$. But if they know these they will employ only optimal pure strategies, i.e., those pure strategies [...] such that $p_{i\alpha}(s) = \max_\beta p_{i\beta}(s)$. Consequently, since s_i expresses their behavior, s_i attaches positive coefficients only to optimal pure strategies, [...]. But this is simply a condition for s to be an equilibrium point.
>
> Thus the assumption we made in this 'mass-action' interpretation lead to the conclusion that the mixed strategies representing the average behavior in each of the populations form an equilibrium point." (op cit., p. 22)
>
> "Actually, of course, we can only expect some sort of approximate equilibrium, since the information, its utilization, and the stability of the average frequencies will be imperfect." (op. cit., p 23)

Hence, in the "mass-action" interpretation, Nash argues that stationarity in population frequencies over pure strategies implies that the corresponding frequency distribution constitutes a Nash equilibrium. Although Nash did not discuss the matter, this approach presumes that there is no issue of strategically influencing the future behavior of other individuals. One could imagine that the populations are so large that unilateral deviations cannot be detected by others.

1.3 Difficulties with the "mass action" interpretation

There are a few problems with this interpretation, though.

First, if the frequency distribution is indeed stationary but not completely mixed, i.e., does not involve *all* pure strategies in the game, then how can an individual know the payoffs of unused pure strategies? One way out would be to make the stronger assumption that all individuals know their whole payoff *function,* as well as the current population state, and can *deduce* the payoff associated with any deviation. But this would seem to run against the very spirit of the "mass action" interpretation which emphasizes that the participants need not know much about the game or be able to make complex calculations, but instead base their strategy choice on *empirical* information about the "relative advantages" of different pure strategies. Alternatively, one could assume that every now and then individuals "experiment" by momentarily using some unused strategy in order to learn the payoffs of unused strategies. But in order to obtain precise information about the *expected* payoffs to such alternative strategies, a substantial amount of aggregate experimentation is needed, which would perturb the population frequency distribution and hence reduce the informational value of experimentation. It is also not clear what *incentives* individuals would have to experiment.

Secondly, the "mass action" interpretation does not say what would happen if the population frequency were not stationary but changes over time. The only suggestion in this direction is that we should only expect an "approximate equilibrium" if the "stability of the average frequency is imperfect" (see quote above). But this is not entirely convincing, since small fluctuations in frequencies (for example due to individuals' experimentation) may trigger a "mass movement" away from the current "approximate equilibrium". (Think, for example, of population frequencies near those of some weakly dominated Nash equilibrium.) In order to handle non-stationary population frequency distributions we need a model of population *dynamics* in which robustness properties of stationary distributions with respect to small frequency perturbations can be examined; in other words a dynamic stability analysis is called for.

Thirdly, the postulated behavior in the "mass action" interpretation, to play optimally against the observed population distribution, implicitly involves some form of inertia, even in the context of a stationary and completely mixed Nash equilibrium. For even if all frequencies and payoffs are common knowledge, and the frequency distribution constitutes a Nash equilibrium, "rationality" does not imply that all individuals will *continue* to play according to this equilibrium. First, such players are by definition completely indifferent between all pure strategies in the support of the Nash equilibrium in question, which in a completely mixed Nash equilibrium is the full pure-strategy space, and hence may without any payoff loss change pure strategy within this support. Hence, implicit in Nash's interpretation is a notion that indifferent individuals do not change strategy, at least in the aggregate. Secondly, and more profoundly, even if a certain stationary Nash-equilibrium frequency distribution has been observed for any amount of time, this does not imply that the only "rational" expectation is that this distribution will prevail. In fact, "rational" players may "rationally" believe that a certain equilibrium will be played up to a certain time, where after another equilibrium will be played, etc.[2] Nash's presumption, viz. that individuals who have observed a stationary frequency distribution will not use any strategy which is sub-optimal against this distribution, thus implicitly involves some form of inertia. Indeed, from a behavioral viewpoint, some form of inertia appears natural.

1.4 Towards a behavioral model of population dynamics

Rather than developing a learning model, the present paper examines in some detail the implications of the "mass-action" interpretation in the context of a class of population dynamics based on "evolution by imitation". Indeed, prominent social scientists and biologists argue that adaptation by way of

[2]There is also the further possibility that a certain non-stationary time pattern of frequency distributions is common knowledge.

imitation of successful behaviors is a fundamental driving force shaping human behavior and intelligence (see e.g. Ullman-Margalit [21] and Dawkins [8]).

As will be seen, if a population frequency distribution has full support over the pure strategies and is stationary in such an imitation dynamics then the distribution indeed constitutes a completely mixed Nash equilibrium, just as the above reasoning did suggest. Since there is no explicit "experimentation" in the dynamics considered here, also other population distributions than Nash equilibria can be stationary, viz. precisely those distributions which are such that all pure strategies in the support of each player-population distribution earn the same payoff. In a sense, such a population distribution constitutes a Nash equilibrium *relative* to the pure strategy subsets which constitute its support. However, one can show that any such non-Nash stationary distribution is dynamically unstable, since by definition some unused *better* pure strategy is then available to some player population.[3]

The above implication is true even for the relatively weak notion of *Lyapunov stability*, which, in essence requires that a small perturbation of the population distribution does not trigger a movement *away* from the distribution. For predictive purposes, however, the more stringent notion of *asymptotic stability* is more reliable; a population distribution has this stability property if it is Lyapunov stable and, moreover, attracts nearby population distributions *towards* itself. Only under asymptotic stability are predictions robust against occasional perturbations of the population state, e.g. due to "experimentation", "mistakes", "random mutations" etc. Unfortunately, however, many, if not most, relevant games possess *no* asymptotically stable population distribution *at all*.

For instance, a Nash equilibrium which does not reach all the information sets in an underlying extensive-form representation of the game is in general not asymptotically stable in any of the considered population dynamics. The reason is simply that local strategies at unreached information sets can be changed without affecting payoffs. In a generic extensive-form game, the Nash equilibrium in question will for this reason even belong to a non-trivial connected set of Nash equilibria, and, since all Nash equilibria are stationary

[3]There are games in which the unique Nash equilibrium is unstable but a whole *set* of strategies, containing the Nash equilibrium, constitute an attractor in the dynamics, see Swinkels [19] and Ritzberger and Weibull [17].

in the studied population dynamics, there is no dynamic force "pulling" the population distribution back towards the Nash equilibrium in question within the set to which it belongs, and hence the Nash equilibrium in question is *not* asymptotically stable in any such population dynamics.

Moreover, it has been shown that even an *isolated* Nash equilibrium, i.e., one which has a neighborhood without other Nash equilibria, is asymptotically stable in "bench-mark" populations dynamics (if and) *only* if it is *strict*. Many games of interest lack strict Nash equilibria, so again one cannot hope to find asymptotically stable population distributions.

These observations suggest that the "mass-action" interpretation may have quite limited validity when used to predict *individual* strategy profiles, and, indeed, that the very criterion of Nash equilibrium may not have so strong "as if" foundations as could be hoped. This has lead some researchers to instead consider set-valued predictions.

Here, we will reformulate Nash's "mass-action" interpretation in terms of asymptotically stable *sets* of strategy profiles. A set-valued prediction of this type simply means that once the population distribution has entered such a set, it will remain there, and, moreover, if the distribution is perturbed to fall slightly outside the set, then the dynamics will bring it back towards the set over time. Existence of asymptotically stable sets is no problem (just make the set sufficiently large), and, as will be seen, set-wise asymptotic stability does have a *set-valued* implication for Nash equilibrium play.

2 The Model

2.1 The game

Consider a finite n-player game in normal (or strategic) form. Let $I = \{1, ..n\}$ be the set of *player positions* in the game, A_i the pure-strategy set of player position i, S_i its mixed-strategy simplex, and $S = \times_{i \in I} S_i$ the polyhedron of mixed-strategy profiles. For any player position i, pure strategy $a \in A_i$ and mixed strategy $s_i \in S_i$, let s_{ia} denote the probability assigned to a. The *support* or *carrier* of a mixed strategy $s_i \in S_i$ is the subset $C_i(s_i)$ of pure strategies to which s_i assigns positive probability, i.e. $C_i(s_i) = \{a \in A_i : s_{ia} > 0\}$. A strategy profile s is called *completely mixed* or *interior* if *all* pure strategies are used with positive probability, i.e., if $C_i(s_i) = A_i$ for all player positions

$i \in I$. In contrast, a strategy profile is *pure* if only *one* pure strategy in each player position is assigned positive probability, i.e., if $C_i(s_i)$ is a singleton for each player position $i \in I$.

The expected payoff to player position i when a profile $s = (s_1, .., s_n) \in S$ is played will be denoted $u_i(s)$, and $u_{ia}(s)$ denotes the payoff to player position i when the individual in this position uses pure strategy $a \in A_i$ against the profile $s \in S$. The pure-strategy best-reply correspondence for player position i is denoted $\beta_i : S \rightarrow A_i$. Hence, a strategy profile $s \in S$ is a *Nash equilibrium* if and only if every pure strategy in its support is a best reply to s, i.e., if $C_i(s_i) \subset \beta_i(s)$, or, more explicitly, $s_{ia} > 0 \Rightarrow a \in \beta_i(s)$.

2.2 The transmission mechanism

A *population state* is formally identical with a mixed-strategy profile $s \in S$, but now each component $s_i \in S_i$ represents the distribution of pure strategies in player population i, i.e., s_{ia} is the probability that a randomly selected individuals in population i will use pure strategy $a \in A_i$.

At each play of the game, n individuals are randomly drawn, one from each player population. We assume that no individual ever randomizes but always uses some pure strategy in every play of the game. However, every now and then each individual reviews her pure-strategy choice. Let $r_{ia}(s)$ denote the average *time-rate* in population state $s \in S$ at which an individual in player-population i, who currently uses strategy is $a \in A_i$, reviews her strategy choice. Likewise, let $p^b_{ia}(s)$ denote the probability that such a reviewing individual will choose strategy $b \in A_i$. We write $p_{ia}(x)$ for the induced probability *distribution* over A_i; formally this is a mixed strategy for player position i: $p_{ia}(x) \in S_i$. Note that $p^a_{ia}(x)$ is the probability that the reviewing individual decides not to change strategy.[4]

In a finite population one may imagine that the review times of an a-strategist in population i constitute the arrival times of a Poisson process with average (across such individuals) arrival rate $r_{ia}(s)$, and that at each such arrival time the individual instantly selects a pure strategy according to the conditional probability distribution $p_{ia}(s)$ over S_i. Assuming that all individuals' Poisson processes are statistically independent, the probability

[4]Alternatively, one could re-interpret what we here call reviewing as the "exit" of one individual who is instantaneously replaced by a new "entrant".

that any two individuals happen to review simultaneously is zero, and the aggregate of reviewing times in the i:th player population among a-strategists is a Poisson process with arrival rate $s_{ia}r_{ia}(s)$ when the population is in state s. If strategy choices are statistically independent random variables, the aggregate arrival rate of the Poisson process of individuals who switch from one pure strategy $a \in A_i$ to another, $b \in A_i$, is $s_{ia}r_{ia}(s)p_{ia}^b(s)$.[5]

2.3 The induced dynamics

Since we consider large (technically speaking infinite) populations, we invoke the law of large numbers and model these aggregate stochastic processes as deterministic flows, each such flow being set equal to the expected rate of the corresponding Poisson arrival process.[6] Rearranging terms, one obtains the following population dynamics

$$\dot{s}_{ia} = \sum_{c \in S_i} r_{ic}(s)p_{ic}^a(s)\, s_{ic} - r_{ia}(s)\, s_{ia}\ . \qquad (1)$$

In order to guarantee that this system of differential equations has a unique solution through every initial population state $s^o \in S$, we henceforth assume that the involved review functions $r_{ia} : S \to R_+$ and choice-probability functions $p_{ia} : S \to S_i$ are Lipschitz continuous. Under this hypothesis, the equation system (1) induces a well-defined dynamics on the state space S (by the Picard-Lindelöf Theorem, see e.g. Hirsch and Smale [10]). In particular, a solution trajectory starting in S never leaves S, and a solution trajectory starting in the interior of S remains for ever in the interior (but may of course converge to a limit point on the boundary of S).

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[5]A *Poisson process* is a stochastic point process in continuous time, the points usually being called *arrival times*. The probability distribution of arrival times is given by a function $\lambda : R \to R$, called the *intensity* of the process, such that $\lambda(t)dt$ is the probability for an arrival in the infinitesimal time interval $(t, t + dt)$. Superposition of independent Poisson processes is again a Poisson process, and its intensity is the sum of the constituent intensities. Likewise, statistically independent decomposition, such as at the strategy switchings above, of a Poisson process also result in Poisson processes. See e.g. Çinlar [6] for details.

[6]Such determinstic approximations are not always innocent, see Boylan [5] for a critical analysis.

3 Imitation Dynamics

3.1 Definition

A population dynamics of the form (1) will be called *imitative* if more prevalent strategies are more likely to be adopted by reviewing individuals, ceteris paribus. More precisely, if two pure strategies $b, c \in A_i$ currently have the same expected payoff, but b is currently more popular than c in the sense that more individuals in player population i currently use b, then the choice probability for b, $p_{ia}^b(s)$, should exceed that of c, $p_{ia}^c(s)$. Technically this can be expressed as the requirement that each choice probability $p_{ia}^b(s)$ be strictly increasing in s_{ib}, the population share of the potential "target" strategy b.

Of course, such an imitative feature does not preclude that individuals also are sensitive to payoffs. For instance, the behaviorally plausible notion that individuals with less successful strategies are, on average, more inclined to review their strategy choice than individuals with more successful strategies can be formalized by having the average review rate $r_{ia}(s)$ non-increasing in the current expected payoff $u_{ia}(s)$. The likewise plausible notion that more successful strategies are, on average, more prone to being adopted than less successful ones can be formalized by letting each choice probability $p_{ia}^b(s)$ be non-decreasing in the expected payoff $u_{ib}(s)$ of the potential "target" strategy b. These possibilities will now be illustrated in a few examples.

3.2 Pure imitation by failing individuals

As a model of "pure", or "unbiased" imitation, assume that a reviewing individual adopts the pure strategy of "the first man in the street". Hence, independently of which strategy the reviewing individual has used so far, it is as if she were to draw an individual at random from her player population, according to a uniform probability distribution across individuals, and adopt the pure strategy of the individual she happened to sample. Formally:

$$p_{ia}^b(s) = s_{ib} \tag{2}$$

for all population states s and pure strategies $a, b \in A_i$. In a sense, one could say that a reviewing individual then decides to "just do what others are doing" in her player population. [7]

If, moreover, the average review rates within a player population are independent of the current strategy of the potential reviewer, i.e. $r_{ia}(s) = r_i(s)$ for some functions r_i, then this form of pure imitation leads to no change at all in the population state; all states are stationary. In particular, Nash's contention that a stationary frequency distribution in his "mass-action" interpretation necessarily constitutes a Nash equilibrium, is not valid. But this is no surprise, since by assumption payoffs are here completely irrelevant both for review propensities and choice probabilities.

More plausibly, suppose instead that individuals with less successful strategies review their strategy at a higher average rate than individuals with more successful strategies, i.e., let

$$r_{ia}(s) = \rho_i\left[u_{ia}(s), s\right] \tag{3}$$

for some positive function $\rho_i : R \times S \to R_+$ which is strictly decreasing in its first argument.

Note that this monotonicity assumption does not presumed that an a-strategist in population i necessarily *knows* the expected value $u_{ia}(s)$ of her current pure strategy, nor that she knows the current state s. For instance, some or all such individuals could have some noisy empirical data on their own current payoff and perhaps also on some alternative pure strategies or, say, the current average payoff $u_i(s)$ in their player population. The only informational assumption in (3) is that, on average, the review rate of a-strategists is higher if their expected payoff is lower, ceteris paribus.

Under assumptions (2) and (3), the population dynamics (1) becomes

$$\dot{s}_{ia} = \left(\sum_{c \in S_i} \rho_i\left[u_{ic}(s), s\right] s_{ic} - \rho_i\left[u_{ia}(s), s\right]\right) s_{ia} . \tag{4}$$

Note that the *growth rate* $\dot{s}_{ia}/s_{ia}$ of the population share of a-strategists by assumption is higher than that of the population share of b-strategists if and only if the current payoff $u_{ia}(s)$ to strategy a exceeds that of strategy b.

Despite this monotone connection between payoffs and growth rates, it is still not true that stationarity implies Nash equilibrium. For instance,

[7] This "pure" form of imitation may alternatively be thought of in terms of "naïve entrants", i.e., an "old" individual being replaced by an uninformed "newcomer".

any *pure* strategy profile is a stationary state in (4), for the simple technical reason that the dynamics does not allow any population share s_{ia} to increase from zero. The intuitive explanation is simply that all reviewing individuals imitate "the first man in the street", and in a population state with (at least) one pure strategy absent, *no* "man in the street" uses that pure strategy, and so no reviewing individual with adopt that strategy. In particular, any *pure* population distribution, whether it be a Nash equilibrium or not, is stationary.

However, stationarity in the *interior* of the state space S *does* imply Nash equilibrium. For in such a population state *all* pure strategies in the game are used by some individuals, and so all pure strategies available to each player position must earn the same payoff, by stationarity in (4), and hence the population state constitutes a Nash equilibrium. To see the intuition for this, suppose, on the contrary, that some pure strategy $a \in A_i$ earns more than another, $b \in A_i$. Then a-strategists would on average abandon their strategy at a lower time rate than b-strategists, while both types of strategist would be equally likely to be imitated (by (2)). Hence, the share of a-strategists would grow at a higher rate than that of b-strategists. In particular, both growth rates could not be zero, and so the population state would not be stationary.

In sum, this sort of imitation process does lend some support to completely mixed Nash equilibria. Unfortunately, though, such equilibria turn out to have poor stability properties, and hence are not likely to be observed (see Section 4 below).

Note, finally, that if the function ρ_i is *linear* (or, more exactly, *affine*) in the payoff argument, i.e., $\rho_i(z, s) = \alpha_i(s) - \beta_i(s)z$ for some positive functions α_i, β_i, then (4) boils down to the following simple expression

$$\dot{s}_{ia} = \beta_i(s)\,[u_{ia}(s) - u_i(s)]\,s_{ia}. \tag{5}$$

In other words, then the growth rate of the share s_{ia} of a-strategists in player population i is proportional to the difference between the payoff to a-strategists, $u_{ia}(s)$, and the average payoff in player population i, $u_i(s)$. But this is merely a player-specific rescaling of time in the so-called *replicator dynamics,* studied in evolutionary biology! That dynamics is derived from particular assumptions about of biological asexual reproduction, and takes the form (5) with $\beta_i(s) = 1$ for all $i \in I$ and $s \in S$.[8]

3.3 Imitation of successful individuals

While the transmission mechanism in the preceding sub-section is "purely" imitative in the sense of being independent of the payoffs of potential "target" strategies, it seems behaviorally more plausible to assume that more successful individuals are more likely to be imitated than less successful ones.

An example of such a "success-oriented" imitation dynamics will here be outlined. In order to isolate this "pull" effect from the above studied "push" effect away from less successful strategies, we now set all average review rates equal to one:

$$r_{ia}(s) = 1 \tag{6}$$

for all population states s, player positions i and pure strategies $a \in A_i$. Instead, now let the choice probabilities $p_{ia}^b(s)$ be increasing both in the target population share s_{ib} and in the target payoff $u_{ib}(s)$, as follows

$$p_{ia}^b(s) = \pi_{ia}\left[u_{ib}(s), s\right] s_{ib}, \tag{7}$$

where $\pi_{ia} : R \times S \to R_+$ is strictly increasing in its first argument. In other words, the "pull" towards a pure strategy $b \in A_i$ is proportional to its "popularity" s_{ib}, where the proportionality factor may depend on the reviewing individual's current pure strategy a and on the current population state s, and, most importantly, is an increasing function of the current expected payoff of the "target" strategy b. In order to have the choice probabilities sum up to one, we require $\sum_b \pi_{ia}\left[u_{ib}(s), s\right] s_{ib} = 1$ (in all population states s and for all player positions i and associated pure strategies a).

As in the earlier case of differentiated review rates, the informational assumption behind choice probabilities of the form (7) is *not* that an a-strategist in population i necessarily knows the current expected payoffs $u_{ib}(s)$ of all available pure strategies b, nor does she have to know the current population state s. It is sufficient that some individuals have some, perhaps noisy, empirical information about some available payoffs, and, on average, move more towards those with higher current expected payoffs than towards those

[8] In an alternative formulation, the replicator dynamics is written on the form (5) with $\beta_i(s) = 1/u_i(s)$, see e.g. Hofbauer and Sigmund [11].

with lower. In fact, some individuals may, due to observational noise, change to strategies which perform worse than the strategy they abandoned.

Conversely, (7) allows for the opposite possibility that virtually all reviewing individuals adopt one of the currently optimal (pure) strategies, by making the "attraction" function π_i sufficiently "payoff sensitive". For instance, suppose each function π_{ia} is exponential,

$$\pi_{ia}\left[u_{ib}(s), s\right] = \frac{\exp\left[\sigma_i u_{ib}(s)\right]}{\sum_c s_{ic} \exp\left[\sigma_i u_{ic}(s)\right]} \tag{8}$$

for some $\sigma_i > 0$.[9] The boundary case $\sigma_i = 0$ then corresponds to "pure" imitation as discussed above, and the limit case $\sigma_i \to \infty$ corresponds to "pure" best-reply behavior at interior states s, in the sense that all reviewing individuals then switch to currently optimal pure strategies.[10] Hence, the form (7) for choice probabilities spans a whole range of myopic choice behaviors from pure imitation to pure optimization. In particular, individuals with very high *individual* review rates (given the over-all unit average rate) can be made to move virtually instantaneously to a currently optimal strategy, and hence almost always play optimally against the current population strategy s even if this fluctuates over time.

Choice probabilities of the form (7), combined with the earlier made assumption (6) of unit review rates, result in the following population dynamics:

$$\dot{s}_{ia} = \left(\sum_{c \in S_i} \pi_{ic}\left[u_{ia}(s), s\right] s_{ic} - 1 \right) s_{ia} \, . \tag{9}$$

Just as in the above case (4) of pure imitation combined with payoff-dependent review rates, pure strategies a with higher expected payoffs $u_{ia}(s)$ have higher growth rates in (9) than pure strategies with low expected payoffs. Again, and essentially for the same reasons, it is still *not* true that stationarity implies Nash equilibrium, while stationarity in the *interior* does.

[9]Cf. the logit model of discrete choice, see e.g. McFadden [14].

[10]Some technical subtleties arise in the limit: the limit vector field is (i) Lipschitz continuous only on the interior of each of a class of geometrically well-behaved subsets which together partition the state space, and (ii) does not define a *monotonic* dynamics in the sense defined above.

Similarly as in the pure imitation dynamics (4), we note that if the "attraction" functions π_{ia} are *linear* (strictly speaking *affine*) in the "target" payoff, i.e., $\pi_{ia}(z,s) = \lambda_{ia}(s) + \mu_{ia}(s)z$ for some positive functions λ_{ia} and μ_{ia}, then (9) boils down to the following player-specific rescaling of time in the replicator dynamics:[11]

$$\dot{s}_{ia} = \left(\sum_{c\in A} s_{ic}\mu_{ic}(s)\right)\left[u_{ia}(s) - u_i(s)\right]s_{ia} \ . \tag{10}$$

It is not difficult (but somewhat tedious) to verify that a combination of affine review rates *and* affine attraction functions, i.e., a combination of the "linear" transmission mechanisms behind the dynamics (5) and (10), leads to a dynamics of the general form

$$\dot{s}_{ia} = \lambda_i(s)\left[u_{ia}(s) - u_i(s)\right]s_{ia} \ , \tag{11}$$

where the function λ_i is a polynomial which is positive on S and depends on the parameters of the review rates and attraction functions.

4 Dynamic Stability and Nash Equilibrium

All population dynamics in the preceding section can be written in the form

$$\dot{s}_{ia} = g_{ia}(s)s_{ia} \ , \tag{12}$$

for some growth-rate functions $g_{ia} : S \to R$ which are Lipschitz continuous and such that $\sum_a g_{ia}(s)s_{ia} = 0$ for every player position $i \in I$ and strategy profile $s \in S$. (This identity is equivalent to saying that the sum of population shares remains constant over time.) Such growth-rate functions will be called *regular.*

4.1 Monotonicity properties

As noted above, the studied imitation dynamics (4) and (9) also satisfy the following monotonicity condition with respect to payoffs:

[11]To see this, note that the requirement that choice probabilities sum to one implies $\lambda_{ia}(s) = 1 - \mu_{ia}(s)u_i(s)$, and hence $\sum_c \pi_{ic}\left[u_{ia}(s), s\right]s_{ic} = \left(\sum_c \mu_{ic}(s)\right)\left[u_{ia}(s) - u_i(s)\right] + 1$.

$$u_{ia}(s) > u_{ib}(s) \;\Leftrightarrow\; g_{ia}(s) > g_{ib}(s) \;. \tag{13}$$

Such growth rate functions, and their induced dynamics, are usually called *monotonic* in the evolutionary game-theory literature (see [15], [9], [18]).

Moreover, in some special cases, more precisely in (5), (10) and more generally (11), we found that each growth rate $g_{ia}(s)$ was *proportional* to the strategy's payoff excess $u_{ia}(s) - u_i(s)$. In the terminology of Samuelson and Zhang [18], such growth-rate functions, and the induced dynamics, are called *aggregate monotonic*:

$$g_{ia}(s) = \lambda_i(s)\,[u_{ia}(s) - u_i(s)] \tag{14}$$

for some positive function $\lambda_i : S \to R$ (such that g_{ia} is Lipschitz continuous).

A special case of aggregate monotonicity is evidently the *replicator dynamics* (see e.g. Hofbauer and Sigmund [11]):

$$\dot{s}_{ia} = [u_{ia}(s) - u_i(s)]\,s_{ia} \;. \tag{15}$$

4.2 Stability properties

Turning now to stability concepts, suppose some population dynamics is given in the form of a system of autonomous ordinary differential equations, such as, for example (12).

A population state $s^* \in S$ is called *Lyapunov stable* if small perturbations of the state do not lead it away, in the precise sense that every neighborhood Θ of s^* contains a neighborhood Θ^o of s^* such that all solution orbits starting in Θ^o remains in Θ forever.[12]

A more stringent stability notion is that of asymptotic stability, which essentially requires that the population also returns towards the state after any small perturbation. Formally, a population state $s^* \in S$ is called *asymptotically stable* if it is Lyapunov stable *and* has a neighborhood Θ' from which all solution orbits converge to s^* over time.[13]

[12]By a *neighborhood* of a point is here meant an *open* set containing the point.

[13]See e.g. Hirsch and Smale [10] for definitions and discussions of stability concepts.

Both these stability concepts are analogously defined for non-empty and closed *subset* $S^* \subset S$. Just let a *neighborhood* of such a set S^* mean an open set containing it. Then Lyapunov stability means that every neighborhood Θ of S^* contains a neighborhood Θ^o of S^* such that all solution orbits starting in Θ^o remains in Θ forever, and asymptotic stability means that, in addition, there is a neighborhood Θ' of S^* from which all solution orbits converge to S^* (in the sense that the distance to the set S^* shrinks towards zero over time).

4.3 Results for monotonic dynamics

We first note that *all* monotonic population dynamics (12) have the same set of stationary states, viz. those states in which all non-extinct pure strategies in each player position have the same expected payoff. Formally, this is the set $S^o = \times_{i \in I} S_i^o$, where

$$S_i^o = \{s_i \in S_i : u_{ia}(s) = u_i(s) \text{ for all } a \in A_i \text{ with } s_{ia} > 0\} .$$

It follows immediately from this observation that every Nash equilibrium s is a stationary state in all monotonic population dynamics, and that every interior stationary state s is a (completely mixed) Nash equilibrium. As noted earlier, a non-interior population state s may be stationary without being a Nash equilibrium. However, such states are not Lyapunov stable in any monotonic population dynamics, and are thus not likely to be observed in "practice":

Proposition 1 (Bomze [4], Nachbar [15], Friedman [9]): *If $s \in S$ is Lyapunov stable in some monotonic population dynamics (12), then s is a Nash equilibrium.*

It is easily established that every *strict* Nash equilibrium is asymptotically stable in any monotonic population dynamics: if $s \in S$ is a strict Nash equilibrium, then the population state s is asymptotically stable in every monotonic dynamics (12). However, as mentioned in the introduction, the "typical relevant" case is that the game has *no* strict Nash equilibrium at all. This is the motivation behind set-valued stability approaches. In particular, a set-valued connection has been established between, on the one hand, certain asymptotically stable *sets* of population states and, on the other hand, *sets* of Nash equilibria which meet the requirement of ***strategic stability*** in the sense of Kohlberg and Mertens [12]. A set $S^* \subset S$ of Nash equilibria is strategically stable if it is robust with respect to any sequence of small "trembles" or perturbations of the strategies. This robustness criterion was designed with the intention to meet certain "rationalistic" desiderata. However, it so happens that there is a connection between set-wise asymptotic stability in population dynamics of the type studied here (in fact in a larger class of dynamics) and strategic stability. The connection takes the form of set-wise inclusion:

Proposition 2 (Swinkels [19]): *Suppose $S^* \subset S$ is non-empty, closed and convex. If S^* is asymptotically stable in some monotone population dynamics (12), then S^* contains a strategically stable set of Nash equilibria.*[14]

Hence, *if* we (a) have a monotonic population dynamics defined on the mixed-strategy space S of some game, and (b) have found a (closed and convex) subset S^* which is asymptotically stable in this dynamics, then we are sure that S^* contains some subset S' of Nash equilibria which meets the stringent requirement of strategic stability. In some cases the set S' might be much smaller than S^* while in others these two sets may even coincide. Likewise, in some cases S^* will correspond to only one payoff outcome, while in others S^* may involve many different payoff outcomes. Hence, the *precision* of the evolutionary prediction may differ between games and between subsets within games.

4.4 Results for aggregate monotonic dynamics

From an operational viewpoint, the above result has the drawback that it may be hard to verify asymptotic stability in a given population dynamics. Moreover, the modeler may not be so sure of which exact specification of the dynamics is appropriate to the application at hand, and hence may want to focus on sets which are asymptotically stable in a fairly wide range of population dynamics.[15] In the special case of aggregate monotonic dynamics,

[14]Swinkels' result (his Theorem 1), is more general both with respect to the dynamics and with respect to the shape of the set S^*.

[15]Note, however, that asymptotic stability in one population dynamics implies asymptotic stability in all "nearby" population dynamics, so the predictions according to the above result are at least locally robust.

there is some headway in these directions - both towards operationality and towards robustness with respect to the dynamics.

We saw above that only for *interior* population states is it true that stationarity implies Nash equilibrium. However, as indicated above, such states have poor stability properties. For instance, Hofbauer and Sigmund [11] show that *no* interior population state is asymptotically stable in the replicator dynamics (15). The proof of this claim is based on an important observation by Amann and Hofbauer [1], viz. that the replicator dynamics induces the *same* solution orbits in the interior of the state space S as a certain volume-preserving dynamics.[16] Such a dynamics has no asymptotically stable state, and since the solution orbits are the same as for the replicator dynamics, the latter has no asymptotically stable state in the interior of the strategy space.

A slight generalization of this result leads to the conclusion no *interior* closed set S^* is asymptotically stable in the replicator dynamics. Moreover, since the restriction of the replicator dynamics to any sub-polyhedron of the polyhedron S of mixed strategy profiles is the replicator dynamics for the associated "subgame", no closed set S^* which is contained in the relative interior of S or any of its sub-polyhedra is asymptotically stable in the replicator dynamics (15). Hence, if we search for sets which are asymptotically stable in a class of population dynamics including the replicator dynamics, then we have to discard all such relatively interior subsets.

Formally, first note that the strategy space S is the Cartesian product of n simplexes, one for each player, so S is a polyhedron. More generally, the set of possible randomizations over any non-empty subset of pure strategies for some player i defines a *sub-simplex* of mixed strategies for that player, i.e., a subset $S_i' \subset S_i$ which also is a simplex of mixed strategies. The Cartesian product of such sub-simplexes constitutes a polyhedron of mixed strategy profiles, a *sub-polyhedron* $S' = \times S_i'$ of S. In particular, each *pure* strategy profile, viewed as a singleton set, is a minimal sub-polyhedron, and the full set S is the maximal sub-polyhedron.

A closed set $S^* \subset S$ will here be called *relatively interior* if it is contained in the interior of some sub-polyhedron S' of S. Clearly every sub-polyhedron

[16] The *divergence* of a dynamics $\dot{x} = f(x)$ at a state x is the trace of the Jacobian of f at x, i.e., $div[f(x)] = \sum_i \partial f_i(x)/\partial x_i$. The dynamics is called *volume preserving* if $div[f(x)] \equiv 0$. Such a dynamics behaves like water flowing under constant temperature and pressure, and can hence have no asymptotically stable state.

is a closed and convex set.

Having thus defined the relevant mathematical concepts, the negative result mentioned above, on dynamic stability, can be summarized as follows:[17]

Proposition 3 (Hofbauer and Sigmund [11]): *Suppose S^* is a non-empty and closed subset of S. If S^* is relatively interior, then it is not asymptotically stable in the replicator dynamics (15).*

It turns out that, for subsets $S^* \subset S$ which are themselves sub-polyhedra of S, the property of asymptotic stability in aggregate monotonic population dynamics can be concisely characterized in terms of a certain correspondence on S, called the *"better-reply"* correspondence (Ritzberger and Weibull [17]). This "new" correspondence γ assigns to each mixed-strategy profile $s \in S$ those pure strategies a for each player i which give him at least the same payoff as he obtains in s. Formally, the image $\gamma(s)$ of any profile $s \in S$ is the Cartesian product of the subsets

$$\gamma_i(s) = \{a \in S_i : u_{ia}(s) \geq u_i(s)\}, \tag{16}$$

one such set for each player $i \in I$. The pure strategies in $\gamma_i(s)$ are thus weakly *better* replies to s than s_i is.

Clearly all pure strategies which are *best* replies are *better* replies in this sense, so the image $\gamma(s)$ of any strategy profile s under the better-reply correspondence γ always contains the image $\beta(s)$ of the best-reply correspondence β. In particular, if $s \in S$ is a Nash equilibrium, then the set of weakly better pure replies coincides with the set of best pure replies, i.e., $\beta(s) = \gamma(s)$. Although $\gamma(s)$ by definition is a subset of pure strategy profiles, it will be notationally convenient to identify it with the associated subset of ("degenerate") mixed strategies (vertices of the polyhedron S).

Ritzberger and Weibull [17] call a sub-polyhedron S' *closed under the better-reply correspondence* if it contains all its weakly better replies, i.e., if $\gamma(s) \subset S'$ for all $s \in S'$, or, more concisely, $\gamma(S') \subset S'$. For instance, a singleton set $S' = \{s\}$ is closed under γ if and only if s is a *strict* Nash equilibrium, and the full polyhedron S is trivially closed under γ. Moreover,

[17] Hofbauer and Sigmund [11] state this result only in the special case of an interior singleton in a two-player game. For details concerning its present generalization, see Ritzberger and Weibull [17].

there always exists at least one *minimal* sub-polyhedron which is closed under γ (there are only finitely many sub-polyhedra).[18]

Closedness of sub-polyhedra $S' \subset S$ under the better-reply correspondence γ *characterizes* asymptotically stable in aggregate monotonic population dynamics, as follows:

Proposition 4 (Ritzberger and Weibull [17]): *If a sub-polyhedron $S' \subset S$ is closed under γ, then S' is asymptotically stable in all aggregate monotonic population dynamics. If a sub-polyhedron $S' \subset S$ is asymptotically stable in some aggregate monotonic population dynamics, then S' is closed under γ.*

It follows from Swinkels' [19] result above, Proposition 2, that any sub-polyhedron S' which is asymptotically stable in some monotonic population dynamics contains a set of Nash equilibria which is strategically stable in the sense of Kohlberg and Mertens [12]. However, for aggregate monotonic dynamics, the implication is even stronger: every sub-polyhedron S' which is asymptotically stable in some *aggregate* monotonic population dynamics contains an *essential component* of Nash equilibria.[19] Moreover, an essential component contains a strategically stable set (Kohlberg and Mertens [12]).

This slightly stronger implication of asymptotic stability follows from Proposition 3 combined with the following result:

Proposition 5 (Ritzberger and Weibull [17]): *If a sub-polyhedron $S' \subset S$ is closed under γ, then it contains an essential component of Nash equilibria, and hence also a strategically stable set.*

Just as in the case of Proposition 2 above, the cutting power of this result on the connection between evolutionary selection and Nash equilibrium play is game dependent.

The games under consideration being finite, also the set of sub-polyhedra $S' \subset S$ is finite. Hence, it is immediate that there exists at least one *minimal*

[18]C.f. *curb* sets, i.e., sets closed under the best-reply correspondence, a notion introduced in Basu and Weibull [3].

[19]The set of Nash equilibria is a finite union of connected sets, or *components*, and such a component is called *essential* if it is robust with repect to perturbations of payoffs, see van Damme [7].

such sub-polyhedron set S' which is closed under the better-reply correspondence γ, and every strategically stable set of Nash equilibria is contained in some such set. In particular, since a generic extensive form game has at least one strategically stable payoff *outcome* (Kohlberg and Mertens [12]), the associated strategy set is contained in a minimal sub-polyhedron which is closed under γ, and hence is asymptotically stable in all aggregate monotonic population dynamics.

5 Conclusions

The discussed dynamic models of evolution by imitation do lend some support to the "mass-action" interpretation of Nash equilibrium. However, the analysis also suggests that evolutionary predictions may be context dependent. The social, cultural, institutional etc. environment in which the interaction takes place presumably shapes the transmission mechanism by which behaviors spread in society. And different transmission mechanisms induce different population dynamics, and hence possibly different dynamically stable sets. However, the discussed set-valued approach suggests that some, perhaps less precise, predictions can be made with only some qualitative knowledge about the dynamics in question. More exactly, if one requires that predictions be valid for all aggregate monotonic population dynamics - which we saw appeared as "first order" approximations of the studied imitation dynamics - then one can identify the relevant sets directly by means of the better-reply correspondence, which relies only on the data of the game.

Many direction for further research in the broad area of behavioral foundations for non-cooperative solution concepts appear highly relevant. Rather than sketching here a few such possibilities we advise the interested reader to consult the recent special issues on evolutionary game theory and game dynamics in *Journal of Economic Theory* (August 1992) and *Games and Economic Behavior* (4 and 5, 1993).

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