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On the Measurement of the Degree of Progression

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1. Introduction

It is generally agreed upon that a progressive tax system should be defined as one where the average rate of tax increases with income before tax. The degree of progression, however, is often referred to by politicians and economists with no precise meaning attached to it.

The ambiguity of the latter concept was discussed by Musgrave and Tun Thin [1948] (M & T) in their well-known article "Income Tax Progression 1929-1948". They suggested the following four local measures of progression.

1. Average rate progression
   (The derivative of the tax rate with respect to income before tax.)
2. Marginal rate progression
   (The derivative of the marginal tax rate with respect to income before tax.
3. Liability progression
   (Elasticity of tax liability with respect to income before tax.)
4. Residual income progression
   (Elasticity of income after tax with respect to income before tax.)

These measures are all compatible with the basic definition of a progressive tax system. Any progessive tax is namely by each measure considered as "more progressive" than a proportional tax.

As could be expected the different measures all had different stories to tell about the development of the progression in the U.S. M. & T. could also contend that it were not possible, on the grounds of any of the sacrifice formulae to single out one measure of progression as the "correct" one. To day it seems natural to choose income redistribution instead of traditional equity theory as a framework for a discussion of the degree of progression. Recent work[1)] on the measurement of income inequality has provided strong justification for the use of the Lorenz-criterion when ranking income distributions with respect to income inequality. When the income distribution before tax is given this criterion could also be used to decide whether one tax system is more redistributed than another. Suppose namely

1) Atkinson [1970], Kolm [1969] and Rotchild & Stiglitz [1973].

that two tax schedules give rise to income distributions after tax with non-intersecting Lorenz-curves. Then the tax schedule relative to the dominated Lorenz-curve can be considered unambiguously more redistributive than the other.

The purpose of this note is to show that as soon as the context choosen is income redistribution judged by the Lorenz-criterion, there is just one logical measure of progression. The argument rests on the following reasonable requirement for a local measure of progression: If one tax system everywhere is more progressive than another, then it should also be unambigously more redistributive than the other.

In section 2 it is shown that the only measure to meet this test is the elasticity of income after tax, or with the M & T terminology, the residual progression. This relation between the global concept of redistribution and the proposed local measure of progression certainly is of analytical value, but it could also be useful in the practical work of e.g. assessing alternative tax systems.

Generally the redistributive effect of a tax system is affected by a change in the income distribution before tax. Of a special interest are the effects from a mere change of the scale of the income distribution before tax. For a tax schedule with constant progression redistribution is unaffected by any proportional change in income distribution before tax. In fact it is the only tax schedule with this property, which is shown in section 3. In the last section the effects of an increased progression on tax rates and marginal tax rates, at different income levels, are discussed.

2. Progression and Lorenz-domination

In order to decide whether a particular income distribution is more equal than another the concept of Lorenz-domination (LD) will be used.[1] According to this criterion an income distribution is more equal than another if and only if its Lorenz-curve lies completely inside the Lorenz-curve of the other The criterion has a long tradition in the study of income inequality. It can moreover be shown[1] that it is equivalent with the intuitively appealing "principle of transfers".[2] LD provides a partial ordering on the set of income distributions. Each one of the conventional measures of inequality

1) See Kolm [1969], Rotschild and Stiglitz [1973], Atkinson [1969].

2) Dalton [1920].

gives a particular total ordering on the same set. It can be shown[1)] that most of these orderings are in accordance with LD. A further justification for the use of LD in this context is provided by the corollary of this section.

In this section and the following one we will work with discrete representations of income distributions. An income distribution is thus given a vector $p = (p_1,\ldots,p_n)$; $p_1 \leq p_2,\ldots,\leq p_i,\ldots,\leq p_n$, where p_i is the income of the i:th person. The Lorenz-curve of a particular distribution is given by the curve[2)]

$$\left(\frac{\nu}{n} : \frac{\sum_{i=1}^{\nu} p_i}{\sum_{i=1}^{n} p_i}\right); \quad \nu = 1,\ldots,n$$

A formal definition of LD now is: The vector p is Lorenz-dominated by the vector q if and only if

$$\frac{\sum_{i=1}^{\nu} p_i}{\sum_{i=1}^{n} p_i} \leq \frac{\sum_{i=1}^{\nu} q_i}{\sum_{i=1}^{n} q_i}; \qquad \nu = 1,\ldots,n \tag{1}$$

with equality for $\nu = n$.

The following lemma is essential for the proposition in this section.

<u>Lemma</u>: Consider two vectors $p = (p_1,\ldots,p_n)$ and $q = (q_1,\ldots,q_n)$, where $0 \leq p_1 \leq p_2,\leq,\ldots,\leq p_n$, and $0 \leq q_1 \leq q_2 \leq \ldots \leq q_n$. If for $1<i\leq n$

$$\frac{q_i}{q_{i-1}} < \frac{p_i}{p_{i-1}} \tag{2}$$

then the vector p is Lorenz-dominated by the vector q.

<u>Proof:</u> The inequality (2) implies that $\sum_{i=1}^{n} \frac{q_i}{q_1} < \sum_{i=1}^{n} \frac{p_i}{p_1}$ and $\sum_{i=1}^{n} \frac{q_i}{q_n} > \sum_{i=1}^{n} \frac{p_i}{p_n}$.

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1) See i.e. Atkinson [1969].

2) See Kolm [1969].

Since both vectors are positive we then have $\frac{q_1}{\sum_{i=1}^{n} q_i} > \frac{p_i}{\sum_{i=1}^{n} p_i}$ and $\frac{q_n}{\sum_{i=1}^{n} q_i} < \frac{p_n}{\sum_{i=1}^{n} p_i}$.

Since both p and q are arranged in increasing order there now exists an intege ν' such that

$$\frac{q_i}{\sum_{i=1}^{n} q_i} \geqq \frac{p_i}{\sum_{i=1}^{n} p_i}, \text{ where } i \leqq \nu' \qquad (3)$$

and

$$\frac{q_i}{\sum_{i=1}^{n} q_i} < \frac{p_i}{\sum_{i=1}^{n} p_i}, \text{ where } i > \nu' \qquad (4)$$

From the identity $\frac{\sum_{i=1}^{n} q_i}{\sum_{i=1}^{n} q_i} \equiv \frac{\sum_{i=1}^{n} p_i}{\sum_{i=1}^{n} p_i}$ and (4) we have $\frac{\sum_{i=1}^{\nu} q_i}{\sum_{i=1}^{n} q_i} \geqq \frac{\sum_{i=1}^{\nu} p_i}{\sum_{i=1}^{n} p_i}$; $\nu > \nu'$

The rest of (1) follows directly from (3).

We now can prove

<u>Proposition 1</u>: Consider two tax schedules with elasticities in income after tax as a function of income before tax given by $a_1(y)$ and $a_2(y)$.[1)]

i) If $a_1(y) < a_2(y)$ everywhere then system 1 is unambigously more redistributive than system 2.

ii) A necessary condition for system 1 to be more redistributive than system 2, for any distribution of income before tax is that $a_1(y) < a_2(y)$ everywhere.

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1) For an individual we have
y = income before tax
s = amount of tax paid (s is a function of y, where the form of the function is specified in the tax laws)
x = (y-s) = income after tax
$t = (\frac{s}{y})$ = average tax rate
$\frac{ds}{dy}$ = marginal tax rate
$a(y) = (\frac{dx}{dy}\frac{y}{x})$ = elasticity of income after tax
$e(y) = (\frac{ds}{dy}\frac{y}{s})$ = tax elasticity.

Proof:

i) Income distribution before tax is given by the vector (y). The distribution of income after tax given by system one is represented by the vector x^1 and system 2 gives the vector x^2. It follows now by the definition of elasticity of income after tax that when $a_1(y) < a_2(y)$ everywhere then $\frac{x_i^1}{x_{i-1}^1} < \frac{x_i^2}{x_{i-1}^2}$. The first part of the proposition now follows from the Lemma.

ii) If $a_1(y) < a_2(y)$ except for one interval where $a_1(y) > a_2(y)$, we could always imagine an income distribution before tax that lies completely within the latter interval.

As a corollary to this proposition we get an interesting relation between LD and the basic definition of the progression.

Corollary: If and only if a specific tax system is progressive everywhere any income distribution before tax will be Lorenz-dominated by the resulting income distribution after tax.

3. Constant progression

The Lorenz-curve of an income distribution is not affected by the scale of incomes. Or in other words, a proportionate change of all incomes leaves the Lorenz-curve of the distribution intact.

It is easily verified that a tax schedule of constant progression has the property to preserve the Lorenz-curve for the distribution of incomes after tax, when the scale of all incomes is changed. In the following proposition it is also shown that in fact it is the only tax schedule with this property. That is, in any other tax system, there is a change in the redistributive effect resulting from e.g. an inflationary proportional increase of all incomes before tax.

Proposition 2: The redistributive effect of a particular tax schedule is unaffected by a proportionate change in all incomes before tax if and only if a(y) is constant (i.e. $x = by^a$, where y is income before tax and x is income after tax).

Proof:

i) The sufficiency part is trivial.

ii) To prove the necessity let the relation between income before tax (y) and income after tax (x) be given by $x = f(y)$. If the before tax distribution with a scale factor is given by $\alpha y = (\alpha y_1, \ldots, \alpha y_n)$ we than get the after tax distribution $x = \big(f(\alpha y_1),\ f(\alpha y_2), \ldots, f(\alpha y_n)\big)$. It is easily checked that the requirement that the Lorenz-curve of α is invariant for changes in α implies that for any pair of elements $y_i\ y_j$ the ratio $\frac{f(\alpha y_i)}{f(\alpha y_j)}$ is a function of y_i and y_j only. I.e.

$$\frac{f(\alpha y_i)}{f(\alpha y_j)} = G(y_i;\ y_j)$$

For $y_j = 1$ we get

$$f(\alpha y_i) = f(\alpha)\ g(y_i)$$

There is a well-known theorem first proved by Cauchy [1821] that the only function to satisfy this equation is $x = by^a$.

4. Tax rates, marginal tax rates and progression

The measure of progression (a) advocated here can be written as a function of the individual's tax rate (t) and marginal tax rate (M) in the following way

$$a(y) = \frac{1-M(y)}{1-t(y)}. \qquad (5)$$

So a could be lowered either by an increase in the marginal tax rate or a decrease in the average tax rate. If we for a given revenue constraint on the macro level consider a uniform increase in the rate of progression we would expect people with low incomes to get their tax rates reduced while they would be raised for high-income people. The latter category must consequently also get their marginal tax rates increased, while there would be a possibility for low income people to get marginal tax rates reduced as well.

For a log-normal income distribution, and a tax schedule with constant progression can prove.

Proposition 3. Suppose the income distribution before tax is log-normal (μ,σ) and consider tax schedules with constant progressivity. Then an increased progressivity with constant revenue, gives a majority of income earners a decreased tax rate and another majority an increased marginal tax rate (if $\sigma<1$).

Proof: The tax schedule is given by

$$x = by^{a}, \tag{6}$$

where b and a are public parameters.

Income before tax (y) is log-normally distributed with the parameters μ and σ. Therefore the mean income of the distribution is given by

$$E[y] = e^{\mu + \frac{1}{2}\sigma^2}. \tag{7}$$

The median is

$$M[y] = e^{\mu}. \tag{8}$$

Now by (1) and well-known properties of log-normal distributions[1] we have

$$E[x] = E[by^{a}] = be^{a\mu + \frac{1}{2}a^2\sigma^2}.$$

The aggregate tax rate (t) is given by

$$t = 1 - \frac{E[x]}{E[y]} = 1 - be^{\mu(a-1) + \frac{1}{2}\sigma^2(a^2-1)} \tag{9}$$

By differentiating in (9) we get

$$db = -b(\mu+\sigma^2 a)da \tag{10}$$

which is the relation between changes in the public parameters holding revenue constant.

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1) See i.e. Aitchison & Brown [1969].

Natural enough $\frac{db}{da}$ is negative. An increase in the progressivity (decrease in a) admits a compensating proportional increase in the disposable incomes.

To get the effect on the individual level of changing public parameters we differentiate in the tax function (6)

$$dx = y^a db + by^a \log y \, da \tag{11}$$

Substituting for db in (11) we get

$$dx = -y^a b(\mu+\sigma^2 a)da + by^a \log y \, da \tag{12}$$

From (12) it can be seen that an increase progressivity (da<0) gives an increased disposable income if and only if

$$y < e^{\mu+a\sigma^2}. \tag{13}$$

As the median income of the distribution is given by e^μ it is clear from (13) that a majority of income-earners always ($a \geq 0$) get their disposable income increased when the progression, and consequently the tax redistribution, increases.[1)]

Turning to the marginal tax rates (M) we have for an individual

$$M = 1 - aby^{a-1}. \tag{14}$$

The effect on M of a change in the public parameters is given by

$$dM = -ay^{a-1}db - (by^{a-1} + aby^{a-1}\log y)da. \tag{15}$$

The revenue constraint implies, by (10)

$$dM = ab\, y^{a-1}(\mu + \sigma^2 a)da - (by^{a-1} + aby^{a-1}\log y)da \tag{16}$$

1) One can also see that, the higher the value of a the larger the function that gets an increased disposable income from a marginal increase in the progressivity. (When a = 0.7 more than 75 percent of the income earners could be expected to get their disposable income increased by an increased progressivity.

It is clear from (16) that $\frac{dM}{da} < 0$ if and only if

$$y > e^{\mu + (\sigma^2 a - 1/a)} \tag{17}$$

so a majority gets an increased marginal tax rate from an increased progressivity if and only if $\sigma < \frac{1}{a}$.

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