ON RAY-HOMOTHETIC PRODUCTION FUNCTIONS*

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1. Homogeneous scalar valued production functions have played an important role in econometric studies of production for estimating returns to scale. But since this class of functions models very simple technologies, others have been developed. Shephard [1953], [1970] introduced the class of homothetic functions, in which returns to scale can vary with output but not with the input mix. Eichhorn [1969], [1970] derived the class of ray-homogeneous functions by solving a multiplicative Cauchy functional equation. For such a class returns to scale can vary with the input mix, but not with output. The homothetic and ray-homogeneous classes were combined by Färe [1973], who solved a translation functional equation to obtain the class of ray-homothetic functions. Such functions are homothetic along each ray in input space, but possibly in different ways for different rays. As a result, returns to scale can vary both with output and with the input mix. It naturally follows that technically optimal (i.e., cost minimizing) output can vary both with output and with the input mix when the production function is ray-homothetic.

Homothetic production functions have been estimated by Zellner and Revankar [1966] among many others, but to the best of our knowledge neither ray-homogeneous nor ray-homothetic production functions have ever been estimated. The present paper represents an attempt to fill that gap by specifying and providing estimates of a ray-homothetic production function. We also demonstrate that the implications of ray-homotheticity for returns to scale, and

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hence for technically optimal output, differ substantially from those of homotheticity and ray-homogeneity.

2. Let $\phi: R_+^n \to R_+$ be a production function with properties:[1]

- $\phi.1$ $\phi(0) = 0$, and $\phi(x) > 0$ for some $x \geq 0$.[2]
- $\phi.2$ ϕ is bounded for bounded input vectors x.
- $\phi.3$ $\phi(\lambda \cdot x) \geqq \phi(x)$ for $\lambda \geqq 1$.
- $\phi.4$ For any $x \geq 0$ such that $\phi(\lambda \cdot x) > 0$ for some scalar $\lambda > 0$, $\phi(\lambda \cdot x) \to +\infty$ as $\lambda \to +\infty$.
- $\phi.5$ ϕ is upper semi-continuous.

Also, consider the functions $F: R_+ \to R_+$ and $H: \{x/|x| \,|\, x \geq 0\} \to R_+$ with the properties

- F.1 $F(0) = 0$.
- F.2 $F(v)$ is bounded for $|v| < +\infty$.
- F.3 F is strictly increasing.
- F.4 $F(v) \to +\infty$ as $v \to +\infty$.
- F.5 F is continuous.
- H. $H(x/|x|) > 0$ and bounded.

A production function ϕ is ray-homothetic if

$$(1) \qquad \phi(\lambda \cdot x) = F(\lambda^{H(x/|x|)} \cdot G(x)), \quad \lambda > 0,$$

where $G(x) = F^{-1}(\phi(x))$. If F is the identity function, then

$$(2) \qquad \phi(\lambda \cdot x) = \lambda^{H(x/|x|)} \cdot \phi(x), \quad \lambda > 0,$$

and ϕ is ray-homogeneous. On the other hand, if $H(x/|x|)$ is a positive constant α, then

$$(3) \qquad \phi(\lambda \cdot x) = F(\lambda^{\alpha} \cdot G(x)), \quad \lambda > 0,$$

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[1] These properties are adapted from Shephard [1974], who also assumes that the efficient subsets are bounded.

[2] $x \geq 0$ means $x \geqq 0$ but $x \neq 0$.

and ϕ is homothetic. Thus the ray-homothetic function (1) provides a straightforward generalization of the functions of Eichhorn and Shephard. Finally, if F is the identity function and $H(x/|x|)$ is a positive constant, then

(4) $\quad \phi(\lambda\cdot x) = \lambda^{\alpha}\cdot\phi(x), \quad \lambda > 0,$

and ϕ is homogeneous.

Goldman and Shephard [1972] have proved that the ray-homogeneous function (2) satisfies (global) strong disposability of inputs $(x' \geqq x \Rightarrow \phi(x') \geqq \phi(x))$ or (global) quasiconcavity if and only if $H(x/|x|)$ is a positive constant, in which case it is homogeneous. Färe [1975] has proved a similar theorem stating that the ray-homothetic function (1) satisfies the same two (global) properties if and only if $H(x/|x|)$ is a positive constant, in which case it is homothetic. Although neither of these two strong properties is imposed globally by the ray-homothetic function, they may be satisfied locally (i.e., for some neighborhood of a point $x \in R^n_+$) even if $H(x/|x|)$ is not a positive constant.

Defining the elasticity of scale ε as

$$\varepsilon = \lim_{\lambda\to 1}\left[\frac{\partial\phi(\lambda\cdot x)}{\partial\lambda}\;\frac{\lambda}{\phi(x)}\right],$$

one can easily calculate this elasticity, assuming sufficient regularity, for the above functions. Clearly $\varepsilon = \varepsilon_1(x/|x|, \phi(x))$ for the ray-homothetic function, $\varepsilon = \varepsilon_2(x/|x|)$ for the ray-homogeneous function, $\varepsilon = \varepsilon_3(\phi(x))$ for the homothetic function, and $\varepsilon = \varepsilon_4 = \alpha$, a constant, for the homogeneous function. Technically optimal output is obtained for the ray-homothetic and homothetic functions by setting $\varepsilon_i = 1$, $i = 1,3$. Technically optimal output is zero, indeterminate or infinite for the ray-homogeneous and homogeneous functions.

3. In their article on generalized production functions, Zellner and Revankar [1966] discuss various properties of homothetic production functions. They also provide an econometric example showing how a

parametric homothetic production function can be estimated. However a simple inspection of a plot of their data, along lines suggested by Hanoch and Rothschild [1972], led us to conclude that the data need not necessarily have been generated from a technology satisfying (global) strong disposability of inputs or (global) quasi concavity. For that reason we demonstrate how a parametric ray-homothetic production function can be estimated and interpreted. We borrow the data, and a portion of the parametric specification, from Zellner and Revankar. The functional specification is

$$(5) \quad Ve^{\theta V} = AK^{\alpha+\gamma(K/L+\delta L/K)^{-1}} L^{\beta+\gamma(K/L+\delta L/K)^{-1}},$$

with $\theta, \gamma \varepsilon R$, $A, \alpha, \beta, \delta \varepsilon R_+$, and $[\alpha + \gamma(K/L + \delta L/K)^{-1}] > 0$, $[\beta + \gamma(K/L + \delta L/K)^{-1}] > 0$, for all K/L. If $\gamma = 0$ then (5) is the homothetic Cobb-Douglas function used by Zellner and Revankar. If $\theta = 0$ then (5) is ray-homogeneous, and if $\theta = \gamma = 0$ then (5) is a homogeneous Cobb-Douglas function.

For the statistical model we follow the methodology of Zellner, Kmenta and Drèze [1969] by assuming that the data were generated by a process consistent with the maximization of the mathematical expectation of profits.

Introducing a multiplicative random error term in (5) and taking natural logarithms gives the estimating equation

$$(6) \quad \ln V_i + \theta V_i = \ln A + \left[\alpha + \gamma((K_i/L_i) + \delta(L_i/K_i))^{-1}\right] \ln K_i + \left[\beta + \gamma((K_i/L_i) + \delta(L_i/K_i))^{-1}\right] \ln L_i + \mu_i ,$$

where i = 1,...,25 indexes observations. The variables, V, K, L refer to per-establishment means of value added, capital and labor for each of 25 states in the U.S. Transportation Equipment Industry in 1957, and are described in greater detail by Zellner and Revankar. It is assumed that $\mu_i \sim NID(0,\sigma^2)$, and that $E(\mu_i \mu_j) = 0$, $i \neq j$. Under these assumptions the parameters of (6) may be estimated by maximum likelihood methods. The results are presented in Table 1;

column (1) contains estimates of the ray-homothetic function (5), while columns (2) and (3) contain estimates of the ray-homogeneous and homothetic versions of (5) respectively.

All three specifications provide excellent fits to the data, although the least restrictive of the three, the ray-homothetic function, is clearly to be preferred. Estimates of all parameters of the ray-homothetic function are highly significant. The estimated value of θ is significantly greater than zero, suggesting that technology is not ray-homogeneous; and the estimated value of γ is significantly less than zero, suggesting that technology is not homothetic either. The estimated ray-homothetic function is depicted by a series of isoquants in Figure 1. The single dashed isoquant belongs to the estimated homothetic function.

4. The empirical estimates obtained above can be used to draw some inferences for returns to scale and technically optimal output. Applying the definition of ε to the parametric ray-homothetic production function (5) gives

$$(7) \quad \varepsilon\left(x/|x|,\, \theta(x)\right) = \frac{\alpha + \beta}{1 + \theta V} + \frac{2\gamma}{(1+\theta V)(K/L+\delta L/K)}.$$

At technically optimal output, $\varepsilon_1(x/|x|, \theta(x)) = 1$, and thus

$$(8) \quad V_1^o = \frac{\alpha + \beta - 1}{\theta} + \frac{2\gamma}{\theta(K/L+\delta L/K)}.$$

Both $\varepsilon_1(x/|x|,\, \theta(x))$ and V_1^o can be computed for each observation, using parameter estimates given in Table 1. Computed values of $\varepsilon_1(x/|x|,\, \theta(x))$ measure returns to scale at each observation, while computed values of V_1^o can be compared with actual values of V for each observation to determine the magnitude of the resulting deviation of actual from technically optimal output. These results are given in Table 2, along with analogous results for the ray-homgeneous and homothetic versions of (5). Table 2 emphasizes shortcomings of the latter two functions that are not otherwise apparent. For the homothetic function returns to scale is a (monotonically decreasing) function of output only, and so technically optimal output is the same constant for all observations. For the ray-homogeneous function returns to scale is a U-shaped function of the input mix only, reaching a minimum at $K/L = \delta^{1/2} = 0.844$. Since this minimum value exceeds unity, technically optimal output is infinite for all observations. Neither of these scenarios is plausible.

For the ray-homothetic function, however, returns to scale is a monotonically decreasing function of output and a U-shaped function of the input mix, reaching a minimum with respect to the latter at $K/L = \delta^{1/2} = 0.635$. As a result, technically optimal output varies across observations, as one would expect. Despite this variation the majority of production is carried out in the region of increasing returns to scale and so actual output is on average only 65.5 % of technically optimal output.

5. The ray-homothetic function includes ray-homogeneous and homothetic functions as special cases, and is considerably more flexible than either. We have constructed and estimated a parametric version of a ray-homothetic function, using a Cobb-Douglas function as a base. Undoubtedly more complex bases can be used (e.g., the CES function), but there seems to be no reason to do so. Our specification is relatively easy to estimate, and it is sufficiently flexible to permit returns to scale to attain a different value at every point in input space. This flexibility of the elasticity of scale in turn permits technically optimal output to vary with the input mix, a desirable property that is absent in both the homothetic and the ray-homogeneous functions.

Table 1. Estimated Production Function Parameters

Parameter	Ray-Homothetic (1)	Ray-Homogeneous (2)	Homothetic (3)
Θ	0.098 (0.009)	-	0.114 (0.015)
A	14.941 (1.007)	7.989 (1.075)	19.298 (1.891)
α	0.330 (0.026)	0.221 (0.083)	0.355 (0.026)
β	1.440 (0.046)	1.319 (0.186)	1.104 (0.380)
γ	-0.259 (0.047)	-0.403 (0.351)	-
δ	0.403 (0.096)	0.712 (0.774)	-
$\bar{R}^2$	0.969	0.957	0.919
$\ln\alpha$	6.526	6.323	5.633

Figures in parentheses are asymptotic standard errors.

Table 2. Implied Values of Returns to Scale and Technically Optimal Output

State	V	K/L	Ray-Homothetic $\varepsilon_1(x/\|x\|,\phi(x))$	Ray-Homothetic V_1^o	Ray-Homogeneous $\varepsilon_2(x/\|x\|)$	Ray-Homogeneous V_2^o	Homothetic $\varepsilon_3(\phi(x))$	Homothetic V_3^o
Florida	0.193	0.341	1.403	4.387	1.208	$+\infty$	1.428	4.026
Maine	0.364	0.304	1.402	4.617	1.215	"	1.401	"
Iowa	0.477	0.337	1.368	4.415	1.211	"	1.383	"
Louisiana	0.638	0.237	1.414	5.135	1.291	"	1.340	"
Massachusetts	1.404	0.389	1.236	4.149	1.176	"	1.258	"
West Virginia	1.513	0.380	1.228	4.192	1.182	"	1.244	"
Texas	1.712	0.207	1.310	5.405	1.318	"	1.221	"
Alabama	1.855	0.121	1.371	6.335	1.405	"	1.204	"
New York	2.040	0.384	1.174	4.173	1.179	"	1.184	"
Virginia	2.052	0.229	1.257	5.202	1.298	"	1.182	"
California	2.333	0.410	1.138	4.067	1.164	"	1.152	"
Wisconsin	2.463	0.417	1.124	4.041	1.160	"	1.139	"
Illinois	2.629	0.667	1.084	3.702	1.075	"	1.122	"
Pennsylvania	2.651	0.350	1.131	4.341	1.201	"	1.121	"
New Jersey	2.701	0.401	1.108	4.101	1.169	"	1.115	"
Maryland	3.219	0.253	1.132	4.997	1.277	"	1.067	"
Washington	3.558	0.350	1.057	4.339	1.201	"	1.038	"
Indiana	3.816	0.760	0.996	3.764	1.065	"	1.017	"
Kentucky	4.031	0.434	0.996	3.979	1.151	"	1.000	"
Georgia	4.289	0.253	1.049	4.995	1.277	"	0.980	"
Ohio	4.440	0.608	0.950	3.701	1.086	"	0.969	"
Connecticut	4.485	0.320	1.002	4.513	1.223	"	0.966	"
Missouri	5.217	0.387	0.931	4.159	1.178	"	0.915	"
Kansas	6.507	0.120	0.990	6.339	1.406	"	0.838	"
Michigan	7.182	0.887	0.813	3.920	1.063	"	0.802	"

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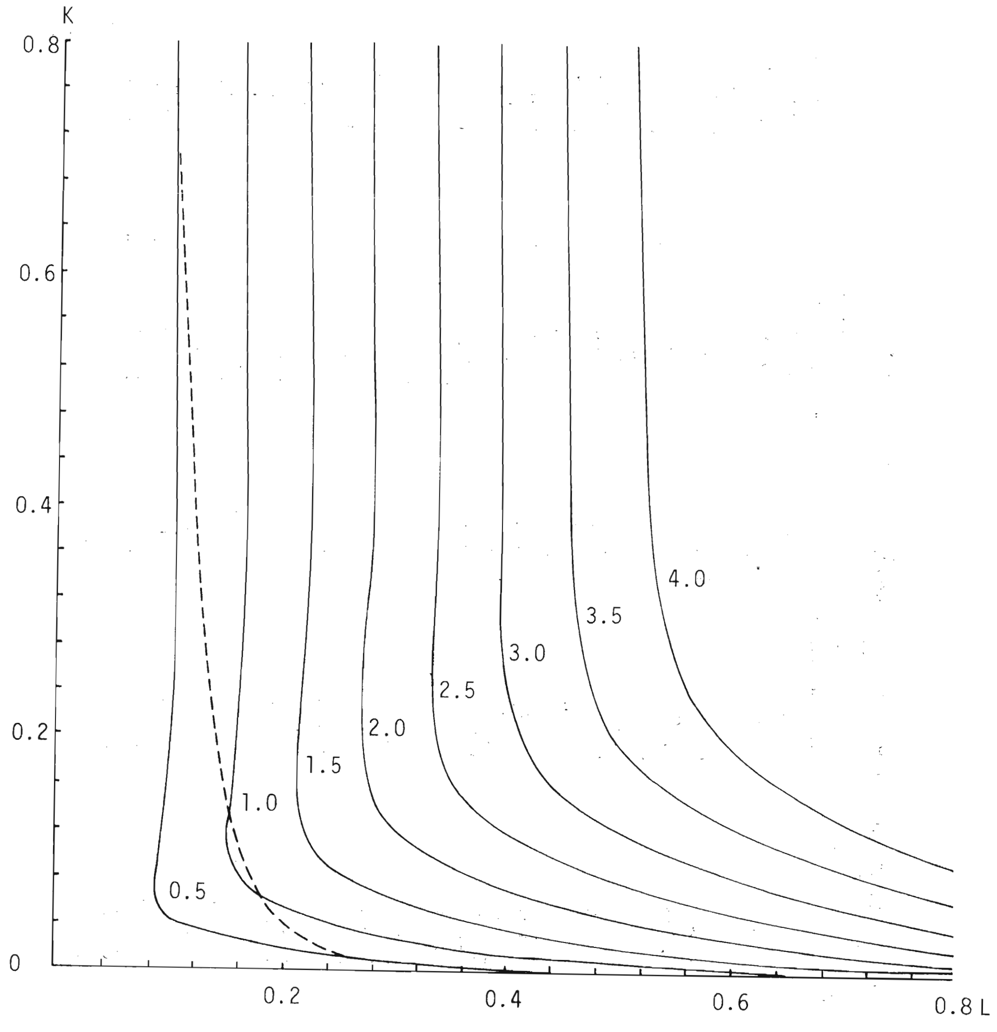
Figure 1



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