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LOCAL PAYOFF SECURITY AND THE EXISTENCE OF NASH EQUILIBRIUM IN DISCONTINUOUS GAMES

by

Philip J. Reny

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Postadress
Box 5501
114 85 Stockholm

Gatuadress
Industrihuset
Storgatan 19

Telefon
08-783 80 00
Telefax
08-661 79 69

Bankgiro
446-9995

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19 15 92-5

10.92 10.000

Local Payoff Security and the Existence of Nash Equilibrium in Discontinuous Games*

Philip J. Reny
Department of Economics, University of Pittsburgh
Pittsburgh, PA 15260
email: reny+@pitt.edu

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Abstract

Consider a game $G = < S_i, u_i >_{i=1}^N$ with bounded, measurable payoffs, and compact, metrizable strategy spaces. Fix a pure strategy, s_i, for player i, and a mixed strategy, μ_{-i}, for the others. The function u_i (extended to mixed strategies) is *locally payoff secure* at (s_i, μ_{-i}) if player i has a mixed strategy μ_i guaranteeing a payoff virtually no worse than $u_i(s_i, \mu_{-i})$ whenever the others play a strategy arbitrarily near μ_{-i}. The function u_i is locally payoff secure if it is locally payoff secure at each point (s_i, μ_{-i}). If the sum of the players' payoffs is upper-semicontinuous and each player's payoff function is locally payoff secure, then G possesses a mixed strategy Nash equilibrium.

1. Introduction

The minmax theorem due to von Neumann (1928) and Nash's (1950) equilibrium existence theorem are fundamental results of noncooperative game theory. Of the

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numerous generalizations that have since been obtained, those pertaining to games with infinite strategy spaces (for instance, Debreu (1952) or Glicksberg (1952)) are of particular importance to economists. This is because it is often natural to model strategic settings arising in economics as games with strategy spaces equal to, say, an interval of real numbers. For example, situations of quantity and price competition between firms were modelled precisely this way in the classic treatments of Cournot (1838) and Bertrand (1883). Both of their games however exhibit discontinuities in the payoffs. Consequently, neither Debreu's nor Glicksberg's theorem applies. Other economic examples involving discontinuous payoffs include models of auctions (Milgrom and Weber(1982)), patent races (Fudenberg et. al. (1983)), spatial competition (Hotelling (1929)), etc. There is as well a long history of the study of discontinuous games in the game theory literature including the comprehensive analysis of duels (see for example Dresher (1961)) and games of timing more generally.

The issue of existence of Nash equilibrium in games with discontinuities has certainly not been ignored. Existence results have been obtained by Dasgupta and Maskin (1986), Mertens (1986), Robson (1994) and Simon (1987).[1] An early result on the existence of approximate equilibria can be found in Blackwell and Girshick (1954).

The approach to existence followed here is closely related to that of Dasgupta and Maskin (1986) and Simon (1987). The idea is to approximate the original game by a sequence of finite games and to show that the limit of equilibria of the finite games (each of which has an equilibrium by Nash's (1950) theorem) is an equilibrium of the original game. Dasgupta and Maskin's (1986) conditions ensure that *any* sequence of finite approximations to the original strategy sets that is eventually dense in them will produce an equilibrium of the original game in this way. Simon's (1987) conditions require only the existence of an *appropriate* sequence of finite approximations. In both cases, the sets of *pure* strategies are approximated. In the present approach, on the other hand, it is the sets of *mixed* strategies that are approximated. In addition, as in Simon (1987), we only require that there exist a single appropriate sequence of approximating games. However, unlike Simon (1987) the existence of such an approximating sequence is not a hypothesis of the theorem, rather it is a consequence of our hypothesis that the players' payoffs are *locally secure*. Consequently, the hypotheses of the present

[1] A related result is due to Simon and Zame (1990) who consider games in which payoffs are correspondences rather than functions. They provide conditions under which such correspondences admit a selection such that the resulting game has an equilibrium.

theorem are easier to check in practical applications and as we mention below have a natural interpretation; the latter property being important for theoretical applications. Finally we note that the present theorem strictly generalizes each of the (exact) equilibrium existence theorems mentioned in the previous paragraph.[2]

2. The Theorem

There are N players $i = 1, 2, ..., N$. Each player i has a pure strategy set, S_i, a compact subset of some metric space, and a bounded, measurable payoff function $u_i : S \to \mathbf{R}$, where $S = \times S_i$. Under these conditions, $G = < S_i, u_i >_{i=1}^{N}$ is called a *compact game.* It is in addition called *upper semicontinuous (u.s.c.)-sum*, if the sum of the N players' payoff functions is u.s.c. as a function from S into $\mathbf{R}$. It has been noted by Dasgupta and Maskin (1986) and Simon (1987) that many games of interest to economists involving discontinuities are compact u.s.c.-sum games. This is also obviously true of compact zero-sum games (and hence of games of timing etc.).

Let M_i denote the set of Borel probability measures on S_i. Consequently, M_i is compact and metrizable (see Billingsley (1968)). Thus, whenever we write $\mu' \to \mu$ for some sequence of measures, convergence is understood to be weak convergence of measures. Extend each u_i to $M = \times_{i=1}^{N} M_i$ in the usual manner, and note that Billingsley (1968) shows that the upper-semicontinuity of $\sum u_i$ on S implies that on M as well.

Definition. The function u_i is *locally payoff secure* (for i) *at* $(s_i, \mu_{-i}) \in S_i \times M_{-i}$ if for every $\epsilon > 0$, player i has a mixed strategy μ_i satisfying:[3]

$$\inf_{\mu'_{-i} \to \mu_{-i}} u_i(\mu_i, \mu'_{-i}) \geq u_i(s_i, \mu_{-i}) - \epsilon \, .$$

The function u_i is *locally payoff secure* (for i) if it is locally payoff secure at every $(s_i, \mu_{-i}) \in S_i \times M_{-i}$.

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[2]Mertens' (1986) is an excpetion. However, if we restrict attention to strategy spaces that are compact and metrizable, then Mertens' result as well as Glicksberg's (1952) become special cases of the present theorem.

[3]The symbol $\inf_{x' \to x} f(x')$ denotes $\inf_{\{x'\}: x' \to x} (\liminf_{x' \to x} f(x'))$.

Thus, a player has locally secure payoffs when he can always ensure himself of a payoff which is virtually no worse than he expects, even if the others play slightly differently than expected. Clearly, continuous functions are locally payoff secure. In the classic Bertrand price competition game, it is trivial to check that payoffs are locally secure. For consider any price charged by one firm against a mixed strategy of the others. A slight reduction in the one's price either increases substantially or decreases only slightly his payoff so long as the others continue to play near their original mixed strategies. The case of Cournot competition with fixed costs also yields locally secure payoffs when, for example, these are the only discontinuities since in this case each firm's payoffs are continuous in the strategies of the others (see the Corollary below for a generalization). We now state our main result, the proof of which can be found in Section 3.

Theorem. *Suppose that* $G =< S_i, u_i >_{i=1}^{N}$ *is a compact u.s.c.-sum game. If each* u_i *is locally payoff secure, then* G *possesses a mixed strategy Nash equilibrium.*

Remark: The condition that the game be u.s.c.-sum was first introduced by Dasgupta and Maskin (1986) and it continues to be an essential ingredient. Mertens (1986) and Simon (1987) noted the importance of the robustness of one's payoff to perturbations in the others' strategies, and our condition of local payoff security is a natural continuation of their ideas.

The Theorem strictly generalizes the main results of Dasgupta and Maskin (1986), as well as the results of Robson (1994) and Simon (1987).[4] In addition, the Theorem yields the following corollary which, in the domain of compact games both generalizes Glicksberg's (1952) theorem and extends to the $N-$person non zero-sum case a theorem due to Mertens (1986).

Corollary. *Suppose that* $G =< S_i, u_i >_{i=1}^{N}$ *is a compact, u.s.c.-sum game. If each player's payoff is lower-semicontinuous in the pure strategies of the others, then* G *possesses a mixed strategy Nash equilibrium.*

Proof of the Corollary. By Billingsley (1968) each player's payoff is consequently lower-semicontinuous in the others' mixed strategies. Therefore for each

[4]It should be noted that Robson's (1994) focus was not on the existence of equilibrium, however he did obtain such a result in passing.

i, $\inf_{\mu'_{-i} \to \mu_{-i}} u_i(s_i, \mu'_{-i}) = u_i(s_i, \mu_{-i})$ for every $(s_i, \mu_{-i}) \in S_i \times M_{-i}$, so that u_i is locally payoff secure. ■

If the game is symmetric, then the conditions stated in the Theorem guarantee the existence of a symmetric mixed strategy Nash equilibrium.[5] The proof of the following Proposition is contained in Section 3.

Proposition. *Suppose that $G = < S_i, u_i >_{i=1}^{N}$ is a symmetric compact u.s.c.-sum game. If each u_i is locally payoff secure, then G possesses a symmetric mixed strategy Nash equilibrium.*

3. Proofs

The following "countable best response" Lemma eliminates the need to explicitly construct an appropriate sequence of approximating games. As shall become apparent in the proof of the Theorem, the Lemma shows that under local payoff security, such a sequence is guaranteed to exist. The idea behind the proof of the Lemma is based on a construction in Mertens (1986).

Lemma. *Suppose that X is an arbitrary set, Y is a compact subset of a metric space, and for each $x \in X$, $f_x : Y \to \mathbf{R}$ is lower semicontinuous. Then there is a countable subset, X^∞, of X such that for all $y \in Y$,*

$$\sup_{x \in X^\infty} f_x(y) = \sup_{x \in X} f_x(y).$$

Call such a set X^∞ a *countable best response set* for $f_x(y)$ against Y.

Proof. Given the assumptions on Y, $C(Y)$, the metric space of continuous real-valued functions on Y (endowed with the supnorm metric) is separable. Let then $g_1, g_2, ...$ denote a countable dense subset of $C(Y)$. For each $n = 1, 2, ...$, if possible choose $x_n \in X$ such that $f_{x_n}(y) \geq g_n(y)$ for all $y \in Y$. Note that this must be

[5] The game $G = < S_i, u_i >_{i=1}^{N}$ is symmetric if $S_i = S_j$ for all players i and j, and for any permutation π of $(1, 2, ..., N)$, and every $s \in S$, $u_i(s) = u_{\pi(i)}(s_\pi)$, where s_π is the element of S whose ith coordinate is $s_{\pi(i)}$. A Nash equilibrium $\mu \in M$ is symmetric if $\mu_i = \mu_j$ for all players i and j.

possible for infinitely many n since for each $x \in X$, $f_x(\cdot)$, being l.s.c., is bounded below on the compact set Y. Let X^∞ be the (countable) set consisting of each of these x_n's. We now show that X^∞ has the desired property.

Fix $x \in X$. Since $f_x(\cdot)$ is l.s.c., it is the pointwise limit from below of a sequence of functions among $g_1, g_2, \ldots$. That is, for some subsequence $n_1, n_2, \ldots$, and for all $y \in Y$, $g_{n_k}(y) \leq f_x(y)$ for all k, and $\lim_k g_{n_k}(y) \to f_x(y)$. Consequently, for each n_k, there is an $x_{n_k} \in X^\infty$ such that $g_{n_k}(y) \leq f_{x_{n_k}}(y)$ for all $y \in Y$. But this implies that $\limsup_k f_{x_{n_k}}(y) \geq f_x(y)$, for all $y \in Y$. Since each $x_{n_k} \in X^\infty$, it follows that $\sup_{x \in X^\infty} f_x(y) \geq f_x(y)$ for all $y \in Y$. The desired conclusion is obtained by noting that x was arbitrary. ■

Proof of the Theorem. For each i, and every $\mu \in M$, let $\underline{u}_i(\mu_i, \mu_{-i}) \equiv \inf_{\mu'_{-i} \to \mu_{-i}} u_i(\mu_i, \mu'_{-i})$. Then $\underline{u}_i(\mu_i, \cdot)$ is l.s.c. on M_{-i} for every $\mu_i \in M_i$. By the Lemma, we may then choose for each i a (countable) subset M_i^∞ of M_i that constitutes a countable best response set for $\underline{u}_i$ against M_{-i}. For each i, let $M_i^1, M_i^2 \ldots$, be an increasing sequence of finite subsets of M_i^∞ whose union is M_i^∞, and let μ^k denote an equilibrium of the finite game $< M_i^k, u_i >_{i=1}^N$, $k = 1, 2, \ldots$.[6] Since M is a compact metric space, we may assume without loss of generality that $\mu^k \to \mu^* \in M$. It suffices to show that μ^* is an equilibrium of G. Fix $\epsilon > 0$, and suppose that μ^* is not an equilibrium of G. Then for every player i,

$$\begin{aligned}
u_i(\mu^*) &\leq u_i(\hat{s}_i, \mu^*_{-i}), \text{ for some } \hat{s}_i \in S_i, \text{ with the inequality strict for at least one } i \\
&\leq \underline{u}_i(\mu_i, \mu^*_{-i}) + \epsilon/3, \text{ for some } \mu_i \in M_i \text{ , since } u_i \text{ is locally payoff secure} \\
&\leq \underline{u}_i(\mu_i^\infty, \mu^*_{-i}) + \epsilon/2, \text{ for some } \mu_i^\infty \in M_i^\infty, \text{ by definition of } M_i^\infty \\
&\leq \underline{u}_i(\mu_i^\infty, \mu^k_{-i}) + \epsilon, \text{ for } k \text{ large enough, since } \underline{u}_i \text{ is l.s.c.} \\
&\leq u_i(\mu_i^\infty, \mu^k_{-i}) + \epsilon, \text{ by definition of } \underline{u}_i \\
&\leq u_i(\mu^k) + \epsilon, \text{ by definition of } \mu^k, \text{ and since for } k \text{ large enough, } \mu_i^\infty \in M_i^k.
\end{aligned}$$

The above inequalities imply that $\liminf_{k \to \infty} u_i(\mu^k) \geq u_i(\mu^*)$ for every player $i = 1, 2, \ldots, N$, with the inequality strict for at least one player i. But this contradicts G being a u.s.c.-sum game. ■

[6] Note that each finite game has as its set of pure strategies finitely many *mixed* strategies of the original game. Thus μ^k is a mixture over finitely many *mixed* strategies of the original game and can therefore also be viewed as an element of M.

Proof of the Proposition. Follow precisely the proof of the Theorem and note that by symmetry we may choose the countable best response sets M_i^∞ such that they are identical for each player. Conequently, the finite games $< M_i^k, u_i >_{i=1}^N$ can be chosen to be symmetric for every k. Thus, they each possess symmetric mixed strategy equilibria (by standard results for finite symmetric games). Therefore, we may choose each μ^k to be symmetric so that μ^* is symmetric as well. The argument then continues as before. ■

4. Related Literature

In this section we shall sketch how the Theorem yields the results of Dasgupta and Maskin (1986), Mertens (1986), Robson (1994), and Simon (1987).

4.1. Mertens (1986) and Robson (1994)

Robson (1994), in the course of studying the informational robustness of equilibria, proves that in a compact game, if each player's payoff is u.s.c. in all players' strategies, and continuous in the other players' strategies, then the game possesses a mixed strategy Nash equilibrium. Mertens (1986) shows that quite generally, two-person zero-sum games possess a value whenever player **I**'s payoff is u.s.c. in his own strategy and l.s.c. in player **II**'s strategy. Both of these results follow directly from the Corollary to the Theorem. (In the case of Mertens' theorem, we must restrict attention to the compact game setting. Consequently, Mertens' (1986) full result is not covered by our Theorem.)

4.2. Dasgupta and Maskin (1986)

Dasgupta and Maskin (1986) impose the following conditions in order to guarantee the existence of a mixed strategy Nash equilibrium. (i) G is a compact u.s.c.-sum game; (ii) each S_i is a convex subset of $\mathbf{R}^m$; (iii) the discontinuities of each u_i lie along a finite number of "diagonal" sets; (iv) each u_i is "weakly" lower-semicontinuous in i's strategy choice.

A number of comments are in order. Condition (i) seems to be essential.[7] Con-

[7]Simon (1987) has noted that the import of this condition is that it leads to what he calls "complementary discontinuities" in the sense that whenever some player's payoff jumps down at a particular joint *mixed* strategy, then some other player's payoff jumps up there. Clearly, the proof of our theorem goes through when the u.s.c.-sum game condition is replaced by the complementary discontinuities assumption. Our preference for the u.s.c.-sum game condition

dition (ii) is somewhat unusual in that it requires convexity of the players' sets of *pure* strategies in order to guarantee the existence of a *mixed* strategy Nash equilibrium. Condition (iii) is only informally stated here, but it essentially requires discontinuities in payoffs to occur at particular coincidences of strategies. For instance, this condition is satisfied in Bertrand price competition or duels where discontinuities arise when players choose the same strategy. On the other hand, in a three firm Bertrand setting in which firms A and B produce intermediate goods which can be assembled at no cost by consumers to yield a final consumption good, while firm C directly produces the final consumption good, discontinuities in profits occur along the hyperplane of prices $p_A + p_B = p_C$. Such discontinuities, although rather well-behaved, do not satisfy Dasgupta and Maskin's "diagonal discontinuity" requirement (iii) above. Condition (iv) rules out, for example, the possibility that the payoff from some strategy of player i exceeds that from any other by some fixed positive amount. Thus, the Cournot model with fixed costs of production does not satisfy this condition.

For our purposes here, it is sufficient to note the following implications of Dasgupta and Maskin's conditions (ii), (iii) and (iv). Their diagonal discontinuities assumption implies that for each joint pure strategy of the others, there are finitely many pure strategies of player i at which u_i is discontinuous. Consequently, whenever player i chooses an atomless mixed strategy, his payoff is then continuous in the strategy choices of the others. Second, together with (ii), their weak lower-semicontinuity assumption implies (in fact, it essentially asserts) that for every $s_i \in S_i$, player i has a sequence of atomless mixed strategies $\{\mu_i^n\}_{n=1}^{\infty}$, such that $\liminf_n u_i(\mu_i^n, s_{-i}) \geq u_i(s_i, s_{-i})$, for all $s_{-i} \in S_{-i}$.[8] Consequently, by Fatou's lemma, $\liminf_n u_i(\mu_i^n, \mu_{-i}) \geq u_i(s_i, \mu_{-i})$, for all $\mu_{-i} \in M_{-i}$. Finally, since the continuity of $u_i(\mu_i^n, \mu_{-i})$ in μ_{-i} implies that $\inf_{\mu'_{-i} \to \mu_{-i}} u_i(\mu_i^n, \mu'_{-i}) = u_i(\mu_i^n, \mu_{-i})$, it follows that u_i satisfies local payoff security. Thus, Dasgupta and Maskin's (1986) conditions (i)-(iv) imply the hypotheses of our Theorem.

4.3. Simon (1987)

Simon (1987) strictly generalizes the result of Dasgupta and Maskin (1986). Simon makes the following assumptions: (i) G is a compact u.s.c.-sum game; (ii) there

stems from the fact that it is stated in terms of the players' pure strategies, rather than their mixed strategies.

[8] We make the rather harmless assumption here that $\#S_i \geq 2$. Thus, by (ii), the existence of an atomless measure on S_i is assured.

exists a sequence $\{\times_i S_i^n\}_{n\geq 1}$ of finite approximations to the original strategy sets satisfying, where M_i^n denotes i's mixed strategies in the nth finite approximation:

(∗) For each player i, for every $(\bar{s}_i, \bar{\mu}_{-i}) \in S_i \times M_{-i}$, there exists a sequence of pairs (μ_i^n, Y^n), with $\mu_i^n \in M_i^n$, and $Y^n \subseteq S_{-i}$, $n = 1, 2, \ldots$,such that for all $n : \bar{\mu}_{-i}(Y^n) > 1 - 1/n$, and
$\sum_{s_i \in S_i^n} \mu_i^n(s_i)[\inf_{s'_{-i}\to s_{-i}} u_i(s_i, s'_{-i})] > u_i(\bar{s}_i, s_{-i}) - 1/n$, for all $s_{-i} \in Y^n$.

Consequently, since by (i) u_i is bounded, for every $\epsilon > 0$, there is an n large enough such that $\inf_{\mu'_{-i}\to\bar{\mu}_{-i}} u_i(\mu_i^n, \mu'_{-i}) \geq \int_{S_{-i}}[\inf_{s'_{-i}\to s_{-i}} u_i(\mu_i^n, s'_{-i})]d\bar{\mu}_{-i}(s_{-i}) \geq \int_{S_{-i}}\{\sum_{s_i\in S_i^n} \mu_i^n(s_i)[\inf_{s'_{-i}\to s_{-i}} u_i(s_i, s'_{-i})]\}d\bar{\mu}_{-i}(s_{-i}) > u_i(\bar{s}_i, \bar{\mu}_{-i}) - \epsilon$.[9] Thus, u_i is locally payoff secure. Hence, Simon's (1987) conditions (i) and (ii) also imply the hypotheses of our Theorem.

4.4. An Example

To demonstrate that our Theorem is strictly more general than those mentioned above, we provide an example satisfying the conditions of the Theorem, but failing to satisfy Simon's (1987) conditions. Since Simon's result generalizes Dasgupta and Maskin's (1986) their conditions will not be satisfied either. Finally, it will be obvious that the example violates the conditions of Mertens (1986) and Robson (1994).

Consider the following two-person zero-sum game on the unit square. Player I chooses x and II chooses y, each from $[0,1]$. Player I's payoff, $u(x,y)$, is defined as follows. Consider the four quadrants in the unit square, A, B, C, and D around the point $(1/2, 1/2)$ with various boundaries excluded. Specifically, $A = \{(x,y) : x < 1/2,\ y > 1/2,\ y \neq 1\}$, $B = \{(x,y) : x,\ y > 1/2,\ y \neq 1\}$, $C = \{(x,y) : x,\ y < 1/2\}$, and $D = \{(x,y) : x > 1/2,\ y \leq 1/2\}$. If (x,y) is an element of:

(*i*) $A \cup D$, then $u(x,y) = 1$
(*ii*) B, then $u(x,y) = 1$ if $x \leq y$; -1, if $x > y$
(*iii*) C, then $u(x,y) = -1$ if $x < y$; 1 if $x \geq y$

As for the remaining boundaries of the quadrants, for all $y \neq 1/2$, and all x, $u(x,1) = u(1/2,y) = -1$, and $u(x,1/2) = 0$.

[9]The first inequaity follows since the function of s_{-i} in square brackets is l.s.c. and less than or equal to $u_i(\mu_i^n, s_{-i})$ for every s_{-i}; and the third by (ii) together with the facts that n is large enough and u_i is bounded (by (i)).

Consider the pure strategy pair $(1/2, 1/2)$. At this point, $u = 0$. However, it is easy to check that for all $x \in X$, $\inf_{y \to 1/2} u(x, y) = -1$. Consequently, $(*)$ above, and therefore Simon's (1987) condition (ii), fails for player **I** at $(1/2, 1/2)$. On the other hand, both players' payoffs are locally secure as we now argue. Player **II**'s payoffs are locally secure since $u(x, 1) = -1 \le u(x, y)$ for all (x, y). To see that player **I**'s payoffs are as well, consider the mixed strategy for **I**, $\mu_1^\delta, \delta < 1/2$, which is uniform on $[1/2 - \delta, 1/2 + \delta]$. Player **I**'s payoff $u(\mu_1^\delta, y)$ is then lower-semicontinuous in y.[10] Consequently, $\inf_{\mu_2' \to \mu_2} u(\mu_1^\delta, \mu_2') = u(\mu_1^\delta, \mu_2)$ for all mixed strategies, μ_2, of player **II**. Finally, it is easy to check that $\lim_{\delta \to 0} u(\mu_1^\delta, \mu_2) = 1 - [\mu_2(1/2) + 2\mu_2(1)] \ge u(x, \mu_2)$ for all pure strategies, x, of player **I** and all mixed strategies, μ_2, of player **II**. The desired conclusion follows. Since the game is clearly compact and u.s.c.-sum, the hypotheses of our Theorem are satisfied and guarantee the existence of a mixed strategy Nash equilibrium. Indeed, since $y = -1$ always yields **II** his highest possible payoff, the pure strategy pair $(1, 1)$ is one of many equilibria of this game.

[10]It is continuous at every $y < 1$, and jumps down from 1 to -1 at $y = 1$.

References

Bertrand, J. (1883): "Theorie Mathematique de la Richesse Sociale," *Journal des Savants*, 499-508.

Billingsley, P. (1968): *Convergence of Probability Measures.* John Wiley and Sons, NY.

Blackwell, D. and Girshick (1954): *The Theory of Games and Statistical Decisions.* Dover.

Cournot, A. (1838): "Recherches sur les Principes Mathematiques de la Theoriedes Richesses," English Translation: "Researches into the Mathematical Principles of the Theory of Wealth," ed. N. Bacon, Macmillan, 1897.

Dasgupta, P. and E. Maskin (1986): "The Existence of Equilibrium in Discontinuous Economic Games, I: Theory," *Review of Economic* Studies 53, 1-26.

Debreu, G. (1952): "A Social Equilibrium Existence Theorem," *Proceedings of the National Academy of Sciences,* 38, 886-893.

Dresher, M. (1961): *The Mathematics of Games of Strategy,* New York: Dover.

Fudenberg, D., R. Gilbert, J. Stiglitz, and J. Tirole (1983): "Preemption, Leapfrogging, and Competition in Patent Races," *European Economic Review* 22, 3-31.

Glicksberg, I.L. (1952): "A Further Generalization of the Kakutani Fixed Point Theorem," *Proceedings of the American Mathematical Society,* 170-174.

Hotelling, H. (1929): "The Stability of Competition," *Economic Journal* 39, 41-57.

Mertens, J. F. (1986): "The Minmax Theorem for U.S.C.-U.S.C. Payoff Functions," *International Journal of Game Theory* 15, 237-250.

Milgrom, P. and R. Weber (1982): "A Theory of Auctions and Competitive Bidding," *Econometrica* 50, 1089-1122.

Nash, J. F. (1950): "Equilibrium Points in n-Person Games," *Proc. Nat. Acad. Sci.* 36, 48-49.

Neumann, J. von (1928): "Zur Teorie der Gesellschaftsspiele," *Math. Annalen* 100, 295-320.

Robson, A. J. (1994): "An 'Informationally Robust' Equilibrium in Two-Person Nonzero-Sum Games," *Games and Economic Behavior*, 2, 233-245.

Simon, L. (1987): "Games with Discontinuous Payoffs," *Review of Economic Studies,"* 54, 569-597.

Simon, L. and W. Zame (1990): "Discontinuous Games and Endogenous Sharing Rules," *Econometrica* 58, 861-872.