A list of Working Papers on the last pages

No. 173, 1986

COMPARATIVE STATICS IN DYNAMIC PROGRAMMING MODELS OF ECONOMICS

by

Harald Lang

Dec., 1986

COMPARATIVE STATICS IN DYNAMIC PROGRAMMING MODELS OF ECONOMICS

by

Harald Lang*

Industrial Institute for Economic and Social Research, S-11453 Stockholm, Sweden.

1 INTRODUCTION

A dynamic programming problem of economics typically has the following formal form:

$$v(x) = \max_{s} T(s,\Theta,v)(x) \tag{1}$$

where v is the "value function", s some policy parameter, Θ some parameter of the problem and $T(s,\Theta,\cdot)$ is a mapping taking functions of x into new functions of x. For a comparative statics analysis, the differentiability properties of (1) are of interest.

Our main result (theorem 4) essentially states that as long as the optimal policy parameter s is unique, if one can formally differentiate (1) with respect to Θ, treating the policy s as being fixed and v as being a priori differentiable, then this differentiation is a posteriori justified.

We also give a version of the "envelope theorem" which sometimes can be used to differentiate (1) with respect to the variable x. (The formal distinction in this context between variables and parameters of v is that the right-hand side of (1)

---

* Most of this work was done while I was visiting researcher at the Institute for Economic Studies, University of Stockholm.

depends on Θ and $v(\theta,\cdot)$ for the current value of Θ only, whereas it depends on $v(\Theta,x')$ also for x' different from the current value of x.) A result in this direction has also been presented by Benveniste and Scheinkman (1979).

2 SOME MATHEMATICAL RESULTS

Lemma 1. Let D be an open subset of R^n, $f(x)$ a continuous function on D, $g(x,z)$ a function on DxD, and assume that

$$f(x) \geqq g(x,z) \text{ for x close enough to z, and}$$

$$f(x) = g(x,x) \text{ for all } x \in D.$$

Assume further that $g(x,z)$ is differentiable w.r.t. x and that $g_1(x,x)$ (the subscript denotes differentiation w.r.t. the first n variables) is continuous on D.

Then $f(x)$ is continuously differentiable, and

$$f_x(x) = g_1(x,x).$$

Proof. Since we differentiate w.r.t. one variable at a time, we may without loss of generality set n=1. Now, for any $x_0 \in D$

$$\liminf_{0<h\to 0} [f(x_0+h) - h(x_0)]/h \geqq$$

$$\liminf_{0<h\to 0} [g(x_0+h,x_0)$$

$$-g(x_0,x_0)]/h = g_1(x_0,x_0). \tag{2}$$

Now take h>0 arbitrarily and define

$$m(h) \equiv h + \max\{g_1(x,x) \mid x_0 \leq x \leq x_0 + h\}.$$

For h fixed, define

$$F(x) \equiv f(x) - m(h)(x-x_0), \quad x \in [x_0,x_0+h].$$

F(x) is continuous, and its maximum in $[x_0,x_0+h]$ must be attained at x_0. Indeed, for any $x \in [x_0,x_0+h]$

$$\limsup_{0<k\to 0} [F(x-k) - F(x)]/(-k) =$$

$$\limsup_{0<k\to 0} [f(x-k)-f(x)]/(-k) - m(h) \leq$$

$$\limsup_{0<k\to 0} [g(x-k,x) - g(x,x)]/(-k) - m(h) =$$

$$g_1(x,x) - m(h) \leq - h < 0,$$

which is impossible at a maximum point. Hence $F(x_0+h) \leq F(x_0)$, i.e. $f(x_0+h) - f(x_0) \leq m(h)h$, thus

$$\limsup_{0<h\to 0} [f(x_0+h) - f(x_0)]/h \leq \limsup_{0<h\to 0} m(h) =$$

$$g_1(x_0,x_0). \tag{3}$$

Of course, (2) and (3) show that the right-hand derivative of f at x_0 exists and equals $g_1(x_0,x_0)$. The left-hand derivative is treated similarly. Q.E.D.

<u>Theorem 1</u> ("envelope theorem"). Let D be an open subset of R^n, $A:D\to R^m$ a continuous, convex- and compact-valued correspondence, h(x,y) a continuous function on DxR^m and define

$$f(x) = \max_{y\in A(x)} h(x,y), \quad x \in D.$$

Then f(x) is continuous. If we further assume that the maximizer $y^* = y^*(x)$ is uniquely determined by x, then $y^*(x)$ is continuous. If further $h_1(x,y)$ exists and is continuous and $y^*(x)$ is an interior point of A(x) then f(x) is differentiable and

$$f_x(x) = h_1(x,y^*(x)).$$

It is important to note that h(x,y) is not assumed be differentiable w.r.t. y.

Proof. The continuity of f(x) and of $y^*(x)$ is well known, and we do not repeat the arguments here. Now define

$$g(x,z) \equiv h(x,y^*(z))$$

The differentiability conclusion of f(x) now follows from Lemma 1 applied to the pair f and g. Q.E.D.

Theorem 2 (Blackwell, 1965). Let D be some subset of R^n, and B(D) the Banach space of bounded, real-valued continuous functions on D, normed by the supremum norm.

Let $T:B(D)\to B(D)$ be a mapping with the following two properties (Blackwell's conditions, abbreviated B.C. in the sequel):

(monotonicity) $f \geqq g$ implies $T(f) \geqq T(g)$

(discounting) there is a number $\beta<1$ (the modulus of T) such for all $f\in B(D)$ and all constants $c > 0$, $T(f+c) \leqq T(f) + \beta c$.

Then T is a contraction mapping with modulus β. In particular, the equation

$$f = T(f) \tag{4}$$

has a unique solution $f \in B(D)$, and if S is a closed subset of B(D) such that T maps S into S, then the solution f lies in S.

Theorem 3 ("Bellman's principle"). With the notation of Theorem 2, let $T = T(s;\cdot)$ depend continuously on some parameter $s \in K \subset R^m$ and assume that the modulus β can be choosen independently of s. Let $A: D \to K$ be a compact-valued correspondence, and consider the two equations

$$f(x) = \max_{s \in A(x)} T(s;f)(x) \equiv T^*(f)(x)$$

$$g(x) = T(s_0(x);f)(x)$$

where $s_0(x): D \to A(x)$ is any continuous function (it is easy to see that both right-hand sides define mappings satisfying B.C., so both equations have unique solutions). Then

$$g(x) \leqq f(x) \text{ for all } x \in D.$$

Proof. Let S be the closed subset of B(D) consisting of functions $\geqq g$. Then for any $h \in S$ we have

$$T^*(h)(x) \geqq T(s_0(x);h)(x) \geqq T(s_0(x);g)(x) = g(x)$$

where in the second relation we used the monotonicity of T. Hence, by theorem 2, $f \in S$. Q.E.D.

For the rest of this section we will adopt notions from differential calculus for mappings between Banach spaces. We refer to Dieudonné (1960) as a general reference.

Lemma 2. With the notation of Theorem 2, let $T=T(\Theta;\cdot)$ depend continuously on some real parameter $\Theta\epsilon(\Theta_0,\Theta_1)$, and assume that the modulus β of T can be choosen independently of Θ. Assume further that $T(\cdot;\cdot):(\Theta_0,\Theta_1)xB(D) \to B(D)$ is differentiable. Then the solution $f=f(\Theta;\cdot)$ to

$$f = T(\Theta;f) \tag{5}$$

is differentiable w.r.t. Θ, and f_Θ is the unique solution to

$$f_\Theta = T_\Theta(\Theta;f) + T_f(\Theta,f;f_\Theta) = T^*(f_\Theta). \tag{6}$$

Here the mapping T* satisfies B.C.

Proof. Equation (4) may be written

$$f - T(\Theta;f) = 0$$

so the conclusion that f is differentiable, as well as formula (6), follows from the implicit function theorem. The only non-trivial thing to check is that the derivative of $f-T(\Theta;f)$ w.r.t. f is invertible. But this derivative is $I-T_f(\Theta,\cdot)$, where I is the identity mapping. We show below that the linear mapping T_f satisfies B.C., so its norm is at most $\beta<1$, and the invertibility of $I-T_f$ follows.

To prove monotonicity of T*, take $g \geqq 0$ in B(D). For any $h\epsilon B(D)$ and $\epsilon > 0$

$$T_f(\Theta,h;\epsilon g) \geqq T(\Theta;h+\epsilon g) - T(\Theta;h) -$$
$$\| T(\Theta;h+\epsilon g) - T(\Theta;h) - T_f(\Theta,h;\epsilon g) \| \geqq 0+o(\epsilon)$$

by the monotonicity of $T(\Theta;\cdot)$ and the definition of differentiality. But $T_f(\Theta,h;\cdot)$ is linear, so dividing by ϵ and letting $\epsilon\to 0$ gives $T_f(\Theta,h;g) \geq 0$. Since $T_f(\Theta,h;\cdot)$ is linear, this provides monotonicity of T_f, and hence of T^*.

Discounting is proved similarly. Q.E.D.

Theorem 4. With the notation of Theorem 2, let $T=T(s,\Theta;\cdot)$ depend continuously on $s\in K\subset R^m$ and $\Theta\in(\Theta_0,\Theta_1)$ with a modulus of T being independent of s and of Θ. Let $D(\Theta)$ denote $(\Theta_0,\Theta_1)\times D$ and let $A:D(\Theta)\to K$ be a continuous, compact-valued correspondence and consider the equation

$$f(x) = \max_{s\in A(\Theta,x)} T(s,\Theta;f)(x) \tag{7}$$

Assume that the maximizer $s=s^*(\Theta;x)$ is uniquely determined by Θ and x and that $s^*(\Theta,x)\in A(\Theta_1,x)$ for all $x\in D$ if Θ_1 is close to Θ[1]

Then f is differentiable w.r.t. Θ, and f_Θ is the unique solution to

$$f_\Theta = T_\Theta[s^*(x),\Theta;f] + T_f[s^*(x),\Theta,f;f_\Theta] \tag{8}$$

where the right-hand side defines a mapping in f_Θ satisfying B.C.

Proof. First we must prove that f is jointly continuous in Θ and x. To this end, we may temporarily think of the right-hand side of (7) being a mapping $B[D(\Theta)]\to B[D(\Theta)]$. Indeed, for any $g\in B[D(\Theta)]$

$$\max_{s\in A(x,\Theta)} T(s,\Theta;g)(x)$$

----

[1] E.g., if $A(\Theta,x) = A(x)$ is independent of Θ.

is continuous on $D(\Theta)$ by Theorem 1 (although the maximizer need not be unique for arbitrary g). Now, by Theorem 2, $f \in B(D(\Theta))$, which proves the joint continuity of f in Θ and x.

Now define $g(\Theta,\Theta';x)$ on $(\Theta_0,\Theta_1)x(\Theta_0,\Theta_1)xD$ by the equation

$$g(\Theta,\Theta';x) = T(s^*(\Theta';x),\Theta;g)(x)$$

where $s^*(\cdot;\cdot)$ is continuous by Theorem 1. By Theorem 3,

$$f(\Theta;x) \geqq g(\Theta,\Theta';x), \quad \Theta \text{ close to } \Theta'$$

and by definition

$$f(\Theta;x) = g(\Theta,\Theta;x).$$

By Lemma 2, $g(\Theta,\Theta';x)$ is differentiable w.r.t. Θ, so by Lemma 1, f is differentiable w.r.t. Θ, $f_\Theta(\Theta;x) = g_1(\Theta,\Theta;x)$ and substituting f_Θ for g_1 in the equation for g_1 given by (6), gives (8). Q.E.D.

3 AN EXAMPLE

To illustrate the results of Section 2, let us consider the following simple dynamic problem;

$$v(x) = \max_{0 \leqq y \leqq f(x)} \{u(f(x)-y) + \beta v(y)\}. \tag{9}$$

Here u is utility flow of consumption, x capital stock and f(x) a production function. The discount factor is $\beta<1$. This period's production f(x) is split into consumption c and next period's capital y; $c+y=f(x)$.

Assume that u and f are continuously differentiable, increasing and concave, and denote the right-hand side of (9) by T(v). Obviously T satisfies B.C., and the value function v is increasing and concave by Theorem 2, since T maps the set of (non-strictly) increasing, concave functions into itself.

The correspondence $x \to [0,f(x)]$ is continuous and by concavity, optimal $y=y^*$ is unique, so if y^* is not a corner solution, v(x) is differentiable by Theorem 1 and

$$v'(x) = u'[f(x)-y^*(x)]f'(x).$$

(Observe that in order to apply Theorem 1 we need only a priori know that v(y) is continuous.)

Now we may write the first order condition for y (assuming away corner solutions, for simplicity)

$$-u'[f(x)-y] + \beta v'(y) = 0.$$

Using this equality it is possible to show that both $y^*(x)$ and $f(x)-y^*(x)$ are increasing in x (we omit the details), which we will exploit below.

Now assume that $f=f(\Theta;x)$ is parametrized by Θ such that $f_\Theta > 0$ and $f_{\Theta x} \leqq 0$. We may now use Theorem 4 to differentiate (9) w.r.t. Θ to get

$$v_\Theta(x) = u'[f(x)-y^*(x)]f_\Theta(x) + \beta v_\Theta[y^*(x)]. \quad (10)$$

As anticipated by Theorem 4, the right-hand side of (10) is a mapping in v_Θ satisfying B.C.; we denote it by $T^*(v_\Theta)$. Hence we can use Theorem 2 to derive properties of v_Θ; for instance v_Θ is decreasing in x. Indeed, using u' decreasing, $f_{x\Theta} \leqq 0$, the increasingness of $f(x)-y^*(x)$ and of $y^*(x)$, we see that T^* maps decreasing functions into decreasing functions. Using this fact one can show (we omit the details) that optimal consumption c^* increases with increasing θ; i.e., an improvement in production according to the specification $f_\theta > 0$, $f_{\theta x} \leq 0$ will (not surprisingly) increase consumption during the first consumption period for any initial capital stock x.

REFERENCES

Benveniste, L.M. and J.A. Scheinkman, 1979, "On the Differentiability of the Value Function in Dynamic Models of Economics, Econometrica, Vol.47, No.3.

Blackwell, D., 1965, Discounted Dynamic Programming; Annals of Mathematical Statistics, 36.

Dieudonné, J., 1960, Foundations of Modern Analysis; Academic Press, New York and London.