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SEARCH THEORY, DOWNWARD MONEY WAGE RIGIDITY AND THE MICRO FOUNDATIONS OF THE PHILLIPS CURVE

by

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This is a preliminary paper. Comments are welcome.

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Abstract

The present paper has two aims. The first one concerns primarily an issue of method. I set up and analyse an explicitly stochastic model of the optimal behaviour of a firm, which recruits from a search labour market. The second aim of my paper concerns very much an issue of substance in economics. I show that when the firm is not allowed to decrease its money wage, its optimal response to lower unemployment is to increase its wage, if a plausible (and testable) condition with regard to its expected horizon is met. Hence search theory predicts the existence of a micro Phillips relation under plausible assumptions.

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Search Theory, Downward Money Wage Rigidity and the Microfoundations of the Phillips Curve

The present paper, which is largely based on my dissertation [Schager (1987)], has two aims. The first one concerns primarily an issue of method. I set up and analyse an explicitly stochastic model of the optimal behaviour of a firm, which recruits from a search labour market. The second aim of my paper concerns very much an issue of substance in economics. I show that when the firm is not allowed to decrease its money wage, its optimal response to lower unemployment is to increase its wage, if a plausible (and testable) condition with regard to its expected horizon is met. Hence search theory predicts the existence of a micro Phillips relation under plausible assumptions.

Introduction and Basic Assumptions

My basic model is that of search in the labour market under incomplete information. The job applicants know the distribution of wage offers over vacant jobs and the firms know the distribution of reservation wages over job applicants, but no one can tell where a specific wage offer or job applicant is to be found.

Mortensen (1970) gives the first stringent analysis of the optimal wage policy of a firm in such a labour market environment. Its crucial feature is the ability of the firm to control the (expected) speed of its hiring process by its wage. Hence the firm faces the dynamic optimization problem of balancing higher wage costs against reaching more rapidly more profitable employment states. Mortensen studied this problem by using deterministic control methods, assuming that expected expansion is always realized, and his analysis has been extended by subsequent authors [Salop (1973), Pissarides (1976), Siven (1979), Virén (1979), Leban (1982a,b)].

No extension of the model of firm behaviour under labour market search has explicitly taken into consideration the stochastic nature of the hiring flow to the firm (at least not as far as analytical models are concerned; Eaton and Watts (1977) simulated numerical results on the basis of a stochastic model).

From a methodological point of view the novel feature of my contribution is the treatment of the firm's dynamic optimization problem as explicitly stochastic. Consequently I apply the methods of stochastic dynamic programming to find the solution. More precisely, I analyse the firm's hiring policy as a Markov decision process.

In order to be able to use the tools of Markov decision processes I assume that the firm faces a flow of job applicants, that forms a stationary Markov process in continuous time. This is in analogy to the way in which the flow of customers to a service station is modelled in operations research. I choose two examples of simple birth-and-death processes to represent the flow: the Poisson process and the linear-death process, respectively.

The interaction between the firm and the labour market works as follows. When the firm opens up vacancies, it is contacted by job applicants according to a stationary Markov process with intensity $\gamma \cdot \delta(v)$, $v \geq 0$ being the number of vacancies announced by the firm. In the Poisson case $\delta(v) = 1$ as long as $v > 0$; $\delta(0) = 0$. In the linear–death case $\delta(v) = v$. (The latter case is equivalent to any single vacancy generating a Poisson flow of job applicants with intensity γ.) The sojourn time between successive contacts is an exponentially distributed random variable with expectation $[\delta(v) \cdot \gamma]^{-1}$.

According to results from labour market search literature, each job applicant should under fairly general conditions conduct his search by calculating a reservation wage such that a job offer should be accepted if and only if the corresponding wage offer exceeds the reservation wage [see Zuckerman (1988) for the continuous time case]. Let $F(\cdot)$ denote the distribution function of reservation wages over all job applicants. The corresponding density function $f(\cdot)$ is supposed to be continuous and differentiable with a range $(\underline{s}, \bar{s})$, $\underline{s} \geq 0$; $\bar{s} \leq \infty$. Any randomly selected job applicant would accept a job offer from the firm with probability $F(w)$, if the firm offers the wage w. Consequently at the wage offer w the process of contacts is transformed into a corresponding process of hires with intensity $\gamma \cdot \delta(v) \cdot F(w) = \delta(v) \cdot \lambda(w)$.

The firm is modelled as a one-product production unit. Labour is the only variable factor of production. I further assume that production takes place at constant returns to scale up to a fixed capacity limit. The analysis can be extended to cover diminishing returns to scale [see Schager (1987)], but here constant returns are assumed in order to simplify the presentation. The product price is exogenously given.

Consequently the reward or profit rate of the firm per unit of time can be written

$$r(i,w) = (pa-w)\cdot i, \qquad i \leq N,$$

where i denotes the number of employees, w the wage, p the product price, a the physical productivity per worker and N denotes the capacity limit (in terms of employees).

Let us first consider the decision problem of the firm under the standard assumption in search theory that wages are completely flexible. The state variable of the firm is $i \leq N$ and its decision variables are w and $\delta(v)$. The firm moves from employment state i to the adjacent state i+1 with an instantaneous transition probability $\delta_i(v)\cdot\lambda(w_i)$, if w_i and $\delta_i(v)$ are chosen by the firm in state i.

It is a property of Markov decision processes that the optimal policy is dependent on the state of the process only. Hence we need to consider only stationary policies $[w_i, \delta_i(v)]$, when searching for the optimal policy, although w and $\delta(v)$ are free to vary at any point of time.

Basic theory of Markov decision processes tells us that the solution to the firm's decision problem is the solution to the following functional equation of dynamic programming

$$L(i) = \max_{w_i, \delta_i} \left[\frac{r(i, w_i)}{\delta_i \cdot \lambda(w_i) + \alpha} + \frac{\delta_i \cdot \lambda(w_i)}{\delta_i \cdot \lambda(w_i) + \alpha} \cdot L(i+1) \right],$$

where L(i) is the maximum expected discounted total profits in state i, α denotes the (instantaneous) discount rate and w_i, δ_i and i are subject to appropriate restrictions. This functional equation can be easily derived from first principles.

Following the approach in Lippman (1980) the functional equation above is extensively analysed in Schager (1987) for the case where $\delta(v) = 1$ for $v > 0$, $\delta(0) = 0$. The results on the optimal policy accord well with those given by deterministic control applications to search models, in which wages are flexible. The optimal wage sequence $(w_j^*)_i^N$ is decreasing in j. Sensitivity analysis establishes that w_j^* is increasing in the contact intensity γ. As γ must be increasing in the number of unemployed, the counterintuitive result obtains, first established in Mortensen (1970), that the optimal wage is increasing in unemployment.

The remainder of this paper, however, is devoted to analysing the case where the wage of the firm is downward rigid. The existence of downward money wage rigidity is as acknowledged by economists as they find it hard to derive from the optimizing behaviour of job applicants and firms. The theoretical issue is not settled, though some recent approaches such as the notion of efficiency wages appear fruitful [see Yellen (1984), Lindbeck and Snower (1986)]. My position is simply that as firms are obviously constrained in their choice on feasible wage policies, an analysis of the optimal policy must take such a constraint into account.

A condition such that only non-decreasing wage policies are feasible is clearly important, as the optimal flexible wage policy is decreasing in the employment state and hence over time during employment expansion. Like myself, other European contributors to labour market search theory as Virén (1979) and Leban (1982a,b) have found this feature troublesome and counterfactual.

These authors have analysed deterministic optimal wage policy solutions under the assumption of downward money wage rigidity. They have not, however, considered sensitivity analysis results.

Before proceeding with the analysis, let me make a remark about the absence of quits in my hiring model. The reason for neglecting quits is purely technical. A Markov decision process, in which the firm moves to both higher and lower employment states, is far less tractable than a process in which the movement is one-way. In Schager (1987) some conjectures are presented as to the properties of the solution to a hiring-cum-quit Markov decision process. It seems that the qualitative results which are of interest in the present context are not altered by such an extension. Let me also add that formally a stochastic control of hires and a deterministic control of hires and quits give rise to functional equations with similar structure. This is demonstrated in some detail in Schager (1987). The reason is that the deterministic representation of the hiring and quit processes consolidates these processes into one net flow, which moves one-way with certainty at any wage level. The problem of technical tractability arises when this condition is not met.

The Optimal Policy and the Value Function

In the Technical Appendix it is shown that under downward wage rigidity the optimal wage policy of the firm is to choose one wage level at the initial employment state n and to keep that wage level fixed during employment expansion up to the capacity limit N. The number of vacancies at the firm is $v = N-n$. It is also shown in the Appendix that the optimal wage level w^* is the solution to

(i) $$\max_{w \geq w_{n-1}} [W_P(n,w)],$$

$$\alpha \cdot W_P(n,w) = (pa-w)\left\{n+\frac{\lambda(w)}{\alpha}\left[1-\left(\frac{\lambda(w)}{\lambda(w)+\alpha}\right)^v\right]\right\} =$$

$$= (pa-w)\left\{n+\sum_{i=1}^{v}\left[\frac{\lambda(w)}{\lambda(w)+\alpha}\right]^i\right\}$$

when the contact process is a Poisson process

or to

(ii) $\max\limits_{w \geq w_{n-1}} [W_D(n,w)]$,

$$\alpha \cdot W_D(n,w) = (pa-w)\left[n+\frac{v \cdot \lambda(w)}{\lambda(w)+\alpha}\right]$$

when the contact process is a linear-death process.

w_{n-1} is the inherited wage level at employment state n.

W_P and W_D are the expected discounted total profits and their closed form expressions are easy to interpret once the optimality of a constant wage level policy is established.

Two remarks should be made about the consequences of the character of the optimal wage policy as compared to the case when wages are fully flexible.

The first consequence is not apparent in the presentation of this paper, because of the simplifying assumption of a constant marginal productivity up to the capacity limit N. The optimality of a constant wage level policy holds, however, also when the marginal productivity is strictly decreasing in the number of employees. In such a case the decision on the optimal wage level implies a decision on the highest employment level, to which it is optimal to expand. The decisions on the wage level and on the number of vacancies are interdependent and must be simultaneously considered by the firm in order for it to arrive at an optimal combination. By contrast, when wages are flexible the decision on the optimal number of vacancies is a trivial matter and is taken independently of the wage policy [see further Schager (1987)].

The second consequence is that the firm will be in a kind of wage disequilibrium as employment expands. It is a fundamental property of

$W_P(n,w)$ and of $W_D(n,w)$ that their derivatives with respect to w ($W_P'(n,w)$ and $W_D'(n,w)$, respectively) are strictly decreasing in n. As $W_P'(n,w^*) \le 0$; $W_D'(n,w^*) \le 0$, $W_P'(i,w^*)$ and $W_D'(i,w^*)$ are strictly smaller than zero for $i > n$, and they become increasingly negative as i increases. Sensitivity analysis of parameter changes must clearly take into account, that the expanding firm is very likely to be in a situation, in which W_P' or W_D' is negative. Hence to speak of non-decreasing rather than increasing effects on the optimal wage of a parameter change is not only a matter of prudent formalism but reflects an essential property of the optimal wage, when downward money wage rigidity prevails.

Sensitivity Analysis

In this section we consider the effect on the optimal wage of changes in the marginal value productivity pa, in the capacity limit N and hence in $v = N-n$ and in the contact intensity γ.

Straightforward differentiation with respect to w yields

$$\text{(i)} \qquad \alpha \cdot W_P'(n,w) = (pa-w)\frac{\alpha \cdot \lambda'}{(\lambda+\alpha)^2} \cdot \sum_{i=1}^{v} i\left(\frac{\lambda}{\lambda+\alpha}\right)^{i-1} - n - \sum_{i=1}^{v}\left(\frac{\lambda}{\lambda+\alpha}\right)^{i} =$$

$$= (pa-w) \cdot \frac{\lambda'}{\alpha}\left[1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{v}\left(1+\frac{\alpha \cdot v}{\lambda+\alpha}\right)\right] - n - \frac{\lambda}{\alpha}\left[1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{v}\right]$$

and

$$\text{(ii)} \qquad \alpha \cdot W_D'(n,w) = (pa-w) \cdot \frac{v \cdot \alpha \cdot \lambda'}{(\lambda+\alpha)^2} - n - \frac{v \cdot \lambda}{\lambda+\alpha}$$

It is immediately obvious that <u>the optimal wage is non-decreasing in pa</u>.

Next we consider changes in v. For case (i) we can write

$$\alpha\cdot W_P'(n,w|v)+n = \sum_{i=1}^{v}\left(\frac{\lambda}{\lambda+\alpha}\right)^{i-1}\left[(pa-w)\frac{\alpha\cdot\lambda'}{(\lambda+\alpha)^2}\cdot i-\frac{\lambda}{\lambda+\alpha}\right] =$$

$$= \sum_{i=1}^{v-1}\left(\frac{\lambda}{\lambda+\alpha}\right)^{i-1}\left[(pa-w)\frac{\alpha\cdot\lambda'}{(\lambda+\alpha)^2}\cdot i-\frac{\lambda}{\lambda+\alpha}\right] +$$

$$+\left(\frac{\lambda}{\lambda+\alpha}\right)^{v-1}\left[(pa-w)\frac{\alpha\cdot\lambda'}{(\lambda+\alpha)^2}\cdot v-\frac{\lambda}{\lambda+\alpha}\right] =$$

$$= \alpha\cdot W_P'(n,w|v-1)+n+\left(\frac{\lambda}{\lambda+\alpha}\right)^{v-1}\left[(pa-w)\frac{\alpha\cdot\lambda'}{(\lambda+\alpha)^2}\cdot v-\frac{\lambda}{\lambda+\alpha}\right]$$

From this structure it can be seen that

$$W_P'(n,w|v-1)+n \geq 0 \quad \Rightarrow W_P'(n,w|v) > W_P'(n,w|v-1)$$

$$W_P'(n,w|v)+n \leq 0 \quad \Rightarrow W_P'(n,w|v-1)+n \leq 0$$

When $\left(\frac{\lambda}{\lambda+\alpha}\right)^{v-1}\left[(pa-w)\frac{\alpha\cdot\lambda'}{(\lambda+\alpha)^2}\cdot v-\frac{\lambda}{\lambda+\alpha}\right] < 0$, it holds that $W_P'(n,w|v) <$ $< W_P'(n,w|v-1) < 0$, but as $\left(\frac{\lambda}{\lambda+\alpha}\right)^{i}$ is increasing in w it must hold that $W_P'(n,w^*|v-1) = 0$ and $\int_{w_{n-1}}^{w^*} W_P'(n,w|v-1)dw \geq 0$ implies

$$\int_{w_{n-1}}^{w^*} W_P'(n,w|v)dw > 0.$$

Together these conditions establish that w^* is non–decreasing in v for case (i).

For case (ii) the same result follows from the easily established relation

$$\alpha\cdot W_D'(n,w|v) = \frac{v}{v-1}\cdot\alpha\cdot W_D'(n,w|v-1)+\frac{n}{v-1}$$

We conclude that <u>the optimal wage is non–decreasing in v</u>.

As to changes in γ there are no unambiguous effects on the optimal wage. The following interesting relations hold, however.

(i)

$$\frac{dW'_P(n,w)}{\frac{d\gamma}{\gamma}} =$$

$$= (pa-w)\cdot\frac{\lambda'}{\alpha}\cdot\frac{\alpha^2}{(\lambda+\alpha)^2}\cdot\sum_{i=1}^{v} i\cdot\left(\frac{\lambda}{\lambda+\alpha}\right)^{i-1}\cdot\frac{\alpha\cdot i-\lambda}{\lambda+\alpha} - \frac{\lambda}{\alpha}\left[1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{v}\left(1+\frac{\alpha\cdot v}{\lambda+\alpha}\right)\right] =$$

$$= (pa-w)\frac{\lambda'}{\alpha}\cdot\frac{\alpha^2}{(\lambda+\alpha)^2}\cdot\sum_{i=1}^{v}\left[i\cdot\left(\frac{\lambda}{\lambda+\alpha}\right)^{i-1} - (v+1)\left(\frac{\lambda}{\lambda+\alpha}\right)^{v}\right] - \frac{\lambda}{\alpha}\left[1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{v}\left(1+\frac{\alpha\cdot v}{\lambda+\alpha}\right)\right] =$$

$$= (pa-w)\frac{\lambda'}{\alpha}\left[1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{v}\left(1+\frac{\alpha\cdot v}{\lambda+\alpha}+v(v+1)\cdot\frac{\alpha^2}{(\lambda+\alpha)^2}\right)\right] - \frac{\lambda}{\alpha}\left[1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{v}\left(1+\frac{\alpha\cdot v}{\lambda+\alpha}\right)\right]$$

and

(ii)

$$\frac{dW'_D(n,w)}{\frac{d\gamma}{\gamma}} = \frac{v}{(\lambda+\alpha)^2}\left[(pa-w)\frac{\lambda'}{\lambda+\alpha}(\alpha-\lambda)-\lambda\right]$$

It is clear that increases in γ may have a decreasing impact on $W'_P(n,w)$ as well as on $W'_D(n,w)$. As $\left[1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{v}\left(1+\frac{\alpha\cdot v}{\lambda+\alpha}\right)\right] > 0$, a sufficient condition for this to happen in case (i) is that

$$\sum_{i=1}^{v} i\cdot\left(\frac{\lambda}{\lambda+\alpha}\right)^{i-1}\cdot\frac{\alpha\cdot i-\lambda}{\lambda+\alpha} \leq 0,$$

a condition which must at least be satisfied for $\lambda \geq \alpha\cdot v$.

The corresponding condition for case (ii) is simply

$$\lambda \geq \alpha$$

Such sufficient conditions cannot, however, be derived from the necessary properties of the optimal solution (regardless of whether it is an interior one or not). It is also obvious that if w_{n-1} is close to $\underline{s}$, $\lambda(w) \geq \alpha$ cannot hold for all feasible values of w. If, on the other hand, the conditions hold for $\lambda(w_{n-1})$, they must hold for every feasible w.

We are, in fact, not restricted just to draw the interesting theoretical conclusion, that the optimal wage may be non–increasing in γ. Both λ and α are easily interpreted entities, which can be empirically identified.

α^{-1} is the expected horizon of the firm. $v \cdot \lambda^{-1}$ and λ^{-1} are the expected waiting times until all currently announced vacancies are filled in case (i) and case (ii), respectively; in case (ii) λ^{-1} is also the expected duration of any announced vacancy.

Hence we can formulate our result such that <u>the optimal wage is non-increasing in γ, if the firm at the initial wage expects to fill its vacancies before reaching its horizon</u>.

Conclusion

A search theoretical analysis of the optimal behaviour of a firm, which cannot lower its money wage, yields a new result on the effect of changes in the supply of job applicants to the firm. An increased flow of unemployed (or employed) applicants has a non-increasing effect on the optimal wage of the firm, if the job vacancies of the firm are in expectation filled before the horizon is reached. This result is not obtained in search theoretical models with full wage flexibility, in which the wage is always non–decreasing in the flow of job applicants. Otherwise, changes in profit flow parameters have the same intuitively reasonable effects on the optimal wage under both wage regimes [see further Schager (1987)].

The relation between vacancy durations and horizons of business operations is an empirical issue, which can and should be investigated at different levels of aggregation. In Schager (1988) the model of case (ii) of this paper is extended as to serve as a model for aggregate wage dynamics. Applied to data for Swedish manufacturing the estimation results are not only supporting the specification as such. They also give a point estimate of the expected horizon of the average firm in Swedish manufacturing of 54 months, a quite reasonable figure. Measured average durations of vacancies in manufacturing do not exceed 6 weeks. For Sweden, at least, theory and empirical evidence seem to reveal the existence of a very pronounced Phillips–relation type of behaviour at the micro level.

Technical Appendix: The Structure of the Optimal Wage Policy, when Wages are Downward Rigid

In this Appendix we derive formally the wage policy solution to the hiring Markov decision process under the condition that the feasible wage policy is non-decreasing. Morover, the solution to the functional equation of the decision process, which is the maximum expected discounted total profits, is obtained in closed form.

The Appendix is organised as follows. First we reformulate the decision process so as to apply to a situation, in which only non-negative wage changes are feasible. Theorem 1 then shows the existence and properties of a hiring stopping state. Given such a stopping state, a Lemma establishes the character of a solution to the decision process, if such a solution belongs to the class of constant wage level policies. At last, Theorem 2 – which yields the principal result of the analysis – proves that the solution to the decision process indeed belongs to the class of constant wage level policies.

With the notation given in the first section of the main text we recall

Profit rate: $r(i,w_i)=(pa-w_i)\cdot i, \quad i=0...N$

$$r(i, w_i) = pa\cdot N-i\cdot w_i, \quad i=N+1...$$

Intensity of hiring process: $\delta(v_i) \cdot \lambda(w_i) = \delta(v_i) \cdot \gamma \cdot F(w_i)$,

case (i): $\delta(v_i) = 1$ for $v_i > 0$, $\delta(0) = 0$;

case (ii): $\delta(v_i) = v_i$.

We introduce the notation

Initial employment level: n

Initial wage level: w_{n-1}

To avoid the trivial case with no employment expansion, we postulate that $w_{n-1}<pa$ and $n<N$.

The existence of downward wage rigidity affects the formal structure of the firm's decision problem. We must now consider a decision process in two state variables, the number of employees i and the wage level w_i, and two control variables, the vacancy control $\delta(v_i)$ and the non–negative wage change, which we denote x_i; x_i is the wage increase to be chosen at the point of time state (i, w_{i-1}) is entered.

The functional equation of the decision problem reads:

$$\text{(1)}\quad \left|\begin{aligned} &H(i,w_{i-1}) = \max_{\substack{x_i \geq 0 \\ v_i = 0,1\ldots}} [H(i,w_{i-1},x_i,\delta(v_i)] \\ &H[i,w_{i-1},x_i,\delta(v_i)] = \frac{r(i,w_{i-1}+x_i)}{\delta(v_i)\cdot\lambda(w_{i-1}+x_i)+\alpha} + \\ &\qquad + \frac{\delta(v_i)\cdot\lambda(w_{i-1}+x_i)}{\delta(v_i)\cdot\lambda(w_{i-1}+x_i)+\alpha}\cdot H(i+1,w_i), \\ &w_i = w_{i-1}+x_i; \quad i = n,n+1\ldots; \quad 0 \leq n < N \end{aligned}\right.$$

The existence of a solution to (1) is not immediately guaranteed. In the case of flexible wages the existence of a solution is easily verified, as the state space is countable and the wage control influences continuously the profit rate and the transition intensities in such a way that a maximum must obtain at a bounded wage level. In the present case the wage state variable is not countable and its value is directly changed by the decision variable x (such a decision variable is sometimes called not a control but an intervention).

Nevertheless, a solution to (1) exists. Suppose that H(i+1,w) is realvalued, continuous and non–increasing in w. Then $H[i,w_{i-1},x_i,\delta(v_i)]$ is continuous in x_i and strictly decreasing in x_i for $x_i \geq \bar{s}-w_{i-1}$ and $\delta(v_i)$ bounded. Consequently, a realvalued function H(i,w) exists and it is continuous in w according to the Maximum Theorem. Furthermore, an increase in w must have a non–increasing impact on H(i,w). According to Theorem 1 below, (N,w_N) is stopping state such that $H(N,w_N)$ has the properties that were supposed to hold for H(i+1,w). Induction with respect to i establishes the existence of a solution to (1) for every i=n...N.

The existence and character of an optimal stopping state is given by

Theorem 1. There exists an optimal stopping state (S^*,w_{S^*}) as part of the solution to (1) such that $S^*=N$, $w_{n-1}\leq w_{S^*}=w_{N-1}=w_N<pa$ and $H(N,w_{N-1})=H(N,w_N)=\frac{(pa-w_N)N}{\alpha}$.

Proof. An optimal stopping employment state $S^*\leq N$ exists, as employment expansion beyond N yields reductions $-(i-N)w_N$ to all future profit rates and hence reduces total expected discounted profits.

To show that S<N is inoptimal, let us suppose that the firm stops at (S,w_S), S<N. Stopping at S implies $v_S=\delta(v_S)=0$, so the value of such a policy is $(pa-w_S)\cdot S/\alpha$. Choosing to expand employment with one unit at an unchanged wage and then stop, which is a feasible policy, implies $v_S=\delta(v_S)=1$, $v_{S+1}=\delta(v_{S+1})=0$ and the value of that policy is

$$\frac{\alpha}{\lambda(w_S)+\alpha}\cdot\frac{(pa-w_S)\cdot S}{\alpha}+\frac{\lambda(w_S)}{\lambda(w_S)+\alpha}\cdot\frac{(pa-w_S)(S+1)}{\alpha}>\frac{(pa-w_S)\cdot S}{\alpha}.$$

Consequently, stopping at S cannot be an optimal policy and as the inequality holds for every S<N, N must be the optimal employment stopping state S*.

It is obvious that an increase of the wage level w_i up to pa cannot be optimal in any employment state $i\leq N$, as it implies zero value of that state. There exist feasible policies that ensure a positive value of every state, as long as $w_{n-1}<pa$. Hence the hiring process stops at state (N,w_N), $w_{n-1}\le w_N<pa$, in which $H(N,w_N)=\frac{(pa-w_N)N}{\alpha}$. It is immediately clear that stopping cannot be optimally conbined with a strictly positive wage increase, so $x^*_N=0$ and $w_N=w_{N-1}$.

Q.E.D.

We consider next the relation between the optimal stopping policy and the number of vacancies v_i in state i, $n\le i\le N$. Clearly we should interpret $v_i=N-i$, so $\delta(v_i) = \delta(N-i)$. Stopping occurs at i, when $\delta(v_i)=0$ and $\delta(v_j)>0$ for $j<i$, so in both case (i) and (ii) $v_i=N-i$ yields the optimal stopping rule.

In case (i), where $\delta(0)=0$, $\delta(v)=1$ for $v>0$, the creation of vacancies is just a stopping policy. $v > 0$ means "hiring process on", regardless of the value of v, while $v=0$ means "hiring process off". It is of no consequence, whether the firm announces all its $(N-i)$ vacancies or just one vacancy in employment state $i<N$.

In case (ii), where $\delta(v)=v$, the creation of vacancies affects in addition the intensity of the hiring process. It is easy to show that the value of every employment state $i\le N-2$ increases, if the firm announces all its $(N-i)$ vacancies instead of a lower number. On the other hand, we must explicitly rule out the possibility of "false" vacancies, by which the firm increases artificially the hiring intensity. Hence we require "state–consistency" of the optimal policy in the sense that, if it is optimal to announce zero vacancies in employment state i, the number of vacancies in a lower employment state j must be $(i-j)$.

Having identified the optimal stopping state, we can reformulate the decision problem (1). Introducing the simplifying notation $\lambda_i(w_i)=\gamma\cdot\delta(N-i)\cdot F(w_i)$, where $\delta(N-i)$ is either equal to 1 or equal to $N-i$, we get

$$(2)\quad \left|\begin{array}{l} H(i,w_{i-1}) = \max\limits_{x_i \geq 0} [H(i,w_{i-1},x_i)] \\[2ex] H(i,w_{i-1},x_i) = \dfrac{r(i, w_{i-1}+x_i)}{\lambda_i(w_{i-1}+x_i)+\alpha} + \dfrac{\lambda_i(w_{i-1}+x_i)}{\lambda_i(w_{i-1}+x_i)+\alpha} \cdot H(i+1,w_i) \\ \qquad w_i=w_{i-1}+x_i; \qquad i=n\ldots N-1; \qquad 0\leq n<N \\[2ex] H(N,w_{N-1}) = \dfrac{r(N, w_N)}{\alpha}, \qquad w_{n-1}\leq w_N=w_{N-1}<pa \end{array}\right.$$

In order to find the solution to the decision process (2), we will first consider the case where the firm applies a very special wage (increase) policy: to make an increase at the initial state (n,w_{n-1}) and to keep the established wage level w_n constant during expansion up to employment state N.

Selecting the best future constant wage level in initial state (n,w_{n-1}) corresponds to solving the functional equation system

$$(3)\quad \left|\begin{array}{l} V(n,w_{n-1}) = \max\limits_{x_n \geq 0} [V(n,w_{n-1},x_n)] = \max\limits_{w_n \geq w_{n-1}} [V(n,w_n,0)] \\[2ex] V(i,w_n,0) = \dfrac{r(i, w_n)}{\lambda_i(w_n)+\alpha} + \dfrac{\lambda_i(w_n)}{\lambda_i(w_n)+\alpha} \cdot V(i+1,w_n,0) \\ \qquad w_n = w_{n-1} + x_n < pa; \qquad i=n,\ldots N-1 \\[2ex] V(N,w_n,0) = \dfrac{r(N, w_n)}{\alpha} \end{array}\right.$$

Note that the maximization in (3) is only carried out in state (n, w_{n-1}); the states (i,w_n), $i=n+1\ldots N$, follow automatically without any new decisions on the wage level. Note also the additive property of the second and the third argument in $V(i,w,x)$, such that $V(i,w,x) = V(i,w+x,0)$.

We intend to show that the easily found solution to (3) is the solution to the decision problem (2). To do so we need the following

Lemma

$$\alpha \cdot V(i,w_n,0) = (pa-w_n) \cdot \left[i + \frac{1}{\Delta\lambda+\alpha} \cdot \left(\lambda_i - \lambda_N \cdot \prod_{j=i}^{N-1} \frac{\lambda_j}{\lambda_j+\alpha}\right)\right]$$

$$i=n\ldots N; \qquad \lambda_i=\lambda_i(w_n); \qquad \Delta\lambda=\lambda_i-\lambda_{i+1}\geq 0.$$

Furthermore,

$$\frac{dV(i,w_n,0)}{dw_n} = V'(i,w_n,0) \text{ is strictly decreasing in i.}$$

Proof

The profit rate $r(i,w_n)=(pa-w_n)i$. Taking the total discounted profits $(pa-w_n)/\alpha$, contributed by each of the i employees already in place, and adding the same total discounted profits, contributed by each of the (N–i) employees to be hired, evaluated with respect to the expected discounted waiting time until the hire, establishes

$$\alpha \cdot V(i,w_n,0) = (pa-w_n)i + (pa-w_n) \cdot \sum_{j=i}^{N-1} \prod_{k=i}^{j} \frac{\lambda_k(w_n)}{\lambda_k(w_n)+\alpha}, \qquad i=n,\ldots N$$

We denote $\varphi(i,w_n) = \sum_{j=i}^{N-1} \prod_{k=i}^{j} \frac{\lambda_k(w_n)}{\lambda_k(w_n)+\alpha}$; $\varphi(N,w_n)=0$.

It holds that

$$\varphi(i,w_n) = \frac{\lambda_i(w_n)}{\lambda_i(w_n)+\alpha} \cdot [\varphi(i+1,w_n)+1], \qquad i=n,\ldots N-1$$

We denote by $\Delta\lambda=\lambda_i-\lambda_{i+1}$, which is independent of i. In case (i) $\Delta\lambda=0$, in case (ii) $\Delta\lambda=\gamma\cdot F(w_n)$.

Some manipulation gives the useful recursive formula

$$\lambda_i(w_n)-(\Delta\lambda+\alpha)\cdot\varphi(i,w_n)=\frac{\lambda_i(w_n)}{\lambda_i(w_n)+\alpha}\cdot[\lambda_{i+1}(w_n)-(\Delta\lambda+\alpha)\cdot\varphi(i+1,w_n)].$$

By induction from N we obtain

$$\varphi(i,w_n)=\frac{1}{\Delta\lambda+\alpha}\cdot\left[\lambda_i(w_n)-\lambda_N(w_n)\cdot\prod_{j=i}^{N-1}\frac{\lambda_j(w_n)}{\lambda_j(w_n)+\alpha}\right]$$

Substitution into the expression for $\alpha\cdot V(i,w_n,0)$ yields the first part of the Lemma.

Note that in both case (i) and case (ii) $\varphi(i,w_n)$ is simplified.
In case (i), $\lambda_n(w_n)=...=\lambda_N(w_n)=\gamma\cdot F(w_n)=\lambda(w_n)$ and $\Delta\lambda=0$, so

$$\varphi(i,w_n)=\frac{\lambda(w_n)}{\alpha}\cdot\left\{1-\left[\frac{\lambda(w_n)}{\lambda(w_n)+\alpha}\right]^{N-i}\right\}$$

In case (ii), $\lambda_i(w_n)=(N-i)\cdot\gamma\cdot F(w_n)=(N-i)\cdot\lambda(w_n)$, $\lambda_N(w_n)=0$ and $\Delta\lambda=\lambda(w_n)$, so

$$\varphi(i,w_n)=\frac{(N-i)\cdot\lambda(w_n)}{\lambda(w_n)+\alpha}.$$

In general it holds that

$$0<\varphi(i,w_n)-\varphi(i+1,w_n)=1-\frac{\alpha}{\Delta\lambda+\alpha}\cdot\left[1-\frac{\lambda_N(w_n)}{\lambda_i(w_n)}\cdot\prod_{j=i}^{N-1}\frac{\lambda_j(w_n)}{\lambda_j(w_n)+\alpha}\right]<1,$$

as $\lambda_N(w_n)\leq\lambda_i(w_n)$, i=n,...N–1.

Straightforward differentiation of $\varphi(i+1,w_n)-\varphi(i,w_n)$ with respect to w_n yields a cumbersome expression which nevertheless establishes that $\varphi'(i+1,w_n)-\varphi'(i,w_n)<0$. Treating the cases (i) and (ii) separately and using the simpler formulas gives the result immediately.

We form

$$\alpha\cdot[V(i+1,w_n,0)-V(i,w_n,0)]=(pa-w_n)\cdot[1+\varphi(i+1,w_n)-\varphi(i,w_n)]$$

Differentiating with respect to w_n gives

$$\alpha[V'(i+1,w_n,0) - V'(i,w_n,0)] =$$
$$= \varphi(i,w_n) - \varphi(i+1,w_n) - 1 + (pa-w_n) \cdot [\varphi'(i+1,w_n) - \varphi'(i,w_n)] < 0$$

according to the properties of $\varphi(i,w_n)$ and $\varphi'(i,w_n)$ as established above. Hence the second part of the Lemma is proved.

Q.E.D.

We are now in the position to state the important

Theorem 2. Let the firm be in initial state (n, w_{n-1}) and face the decision problem (2). The optimal wage (increase) policy is to make one wage increase x^*_n and to keep the established wage level $w^*=w^*_n=w_{n-1}+x^*_n$ fixed, as higher employment states i are realised, $i=n+1,\ldots N$, i.e. the optimal wage increases $x^*_i=0$ for $i=n+1,\ldots N$. Thus $H(n,w_{n-1})=V(n,w_{n-1})=V(n,w^*,0)$ and $H(i,w_{i-1})=V(i,w^*,0)$, $i=n+1,\ldots N$.

Proof. As established in the formulation of (2) $x^*_N=0$. When the firm enters state $(N-1,w_{N-2})$ it makes a wage increase x_{N-1} and establishes a wage $w_{N-1}, w_{N-2} \leq w_{N-1} < pa$ to hold in all subsequent states. Consequently

$$H(N-1,w_{N-2},x_{N-1}) = V(N-1,w_{N-2},x_{N-1})$$

and x_{N-1} is to be chosen such that

$$H(N-1,w_{N-2}) = V(N-1,w_{N-2}) = V(N-1,w_{N-2},x^*_{N-1}) = V(N-1,w^*_{N-1},0)$$

so the Theorem is true for $n=N-1$.

Let us assume that the firm has entered state (i,w_{i-1}), $n \leq i \leq N-2$, and that it has been established that the optimal policy, when state $(i+1,w_i)$ is reached, is to make one wage increase x^*_{i+1} and to keep the wage $w_i+x^*_{i+1}=w^*_{i+1}$ constant in all subsequent states. This changes (2) into

$$\left| \begin{array}{l} H(i,w_{i-1}) = \max\limits_{x_i \geq 0}[H(i,w_{i-1},x_i)] \\ \\ H(i,w_{i-1},x_i) = \dfrac{r(i,w_{i-1}+x_i)}{\lambda_i(w_{i-1}+x_i)+\alpha} + \dfrac{\lambda_i(w_{i-1}+x_i)}{\lambda_i(w_{i-1}+x_i)+\alpha} \cdot V(i+1,w_i,x^*_{i+1}) \\ \\ \qquad w_i=w_{i-1}+x_i \end{array} \right.$$

From (3) we have

$$V(i,w_{i-1},x_i) = \frac{r(i,w_{i-1}+x_i)}{\lambda_i(w_{i-1}+x_i)+\alpha} + \frac{\lambda_i(w_{i-1}+x_i)}{\lambda_i(w_{i-1}+x_i)+\alpha} \cdot V(i+1,w_i,0)$$

and as it holds that

$$V(i+1,w_i,x^*_{i+1}) = V(i+1,w_i,0) + \int_0^{x^*_{i+1}} V'(i+1,w_i,x)dx$$

substitution yields

$$H(i,w_{i-1},x_i) = V(i,w_{i-1},x_i) + \frac{\lambda_i(w_{i-1}+x_i)}{\lambda_i(w_{i-1}+x_i)+\alpha} \cdot \int_0^{x^*_{i+1}} V'(i+1,w_i,x)dx.$$

Let $\hat{x}_i \geq 0$ be any choice of x_i that establishes a $\hat{w}_i=w_{i-1}+\hat{x}_i$ such that the corresponding $x^*_{i+1}=\hat{x}^*_{i+1}>0$. The value of applying such a policy is

$$H(i,w_{i-1},\hat{x}_i) = V(i,w_{i-1},\hat{x}_i) + \frac{\lambda_i(\hat{w}_i)}{\lambda_i(\hat{w}_i)+\alpha} \int_0^{\hat{x}^*_{i+1}} V'(i+1,\hat{w}_i,x)\, dx.$$

Consider now the strictly different policy $x_i=\hat{x}_i+\hat{x}^*_{i+1}$, $x_{i+1}=0$. It yields the value

$$V(i,w_{i-1},\hat{x}_i+\hat{x}^*_{i+1}) = V(i,w_{i-1},\hat{x}_i) + \int_0^{\hat{x}^*_{i+1}} V'(i,\hat{w}_i,x)\, dx.$$

The difference in value between the first and the second policy is

$$H(i,w_{i-1},\hat{x}_i) - V(i,w_{i-1},\hat{x}_i+\hat{x}^*_{i+1}) =$$

$$= \frac{\lambda_i(\hat{w}_i)}{\lambda_i(\hat{w}_i)+\alpha} \cdot \int_0^{\hat{x}^*_{i+1}} V'(i+1,\hat{w}_i,x)dx - \int_0^{\hat{x}^*_{i+1}} V'(i,\hat{w}_i,x)dx < 0,$$

because

$\int_0^{\hat{x}^*_{i+1}} V'(i+1,\hat{w}_i,x)dx \geq 0$, as $\hat{x}^*_{i+1}$ is by assumption a strictly positive, optimal wage increase in state $(i+1,\hat{w}_i)$,

$$0 < \frac{\lambda_i(\hat{w}_i)}{\lambda_i(\hat{w}_i)+\alpha} < 1,$$

and

$\int_0^{\hat{x}^*_{i+1}} V'(i+1,\hat{w}_i,x)dx < \int_0^{\hat{x}^*_{i+1}} V'(i,\hat{w}_i,x)dx$, as $V'(i,w,x)$ is strictly decreasing in i according to the Lemma.

Thus $\hat{x}_i$ cannot be an optimal wage increase policy. $\hat{x}_i$, however, is any policy in state (i,w_{i-1}), which establishes a wage level $\hat{w}_i$ such that the optimal increase $\hat{x}^*_{i+1}$ in state $(i+1,\hat{w}_i)$ is strictly positive. Consequently, an optimal increase x^*_i in state (i,w_{i-1}) implies $x^*_{i+1}=0$, provided that it is an optimal policy to keep the wage $w_i+x^*_{i+1}=w^*_{i+1}$ fixed in all subsequent states.

It has already been shown that it is optimal to keep the wage $w_{N-1}=w_{N-2}+x_{N-1}$ fixed in subsequent states. Consequently, $x^*_{N-1}=0$ if x_{N-2} is optimally chosen as a part of the solution to the decision problem. This implies that it is an optimal policy to keep w_{N-2} fixed in subsequent states. Induction establishes the sequence of optimal wage increases as

$(x^*_N=)x^*_{N-1}=\ldots=x^*_{n+1}=0$, where (n,w_{n-1}) is the initial state of the decision process. As w_{n-1} is by definition not determined as a part of the optimal policy, $x^*_n \geq 0$.

The decision problem (2) has consequently been reduced to (3), i.e. the choice of the best constant wage level to hold in all subsequent employment states i=n,...N. Hence

$$H(n,w_{n-1}) = \max_{x \geq 0}[V(n,w_{n-1},x)] = V(n,w^*,0)$$

and

$$H(i,w_{i-1}) = V(i,w^*,0)$$

for all i=n,...N.

Q.E.D.

References

Eaton, B.C. and Watts, M. (1977), "Wage Dispersion, Job Vacancies and Job Search in Equilibrium", Economica, **44**, 23-35.

Leban, R. (1982a), "Employment and Wage Strategies of the Firm over a Business Cycle", Journal of Economic Dynamics and Control, **4**, 371-394.

Leban, R. (1982b), "Wage Rigidity and Employment Policy of the Firm in a Dynamic Environment", in G. Feichtinger (ed.), Optimal Control Theory and Economic Applications (Amsterdam: North Holland).

Lindbeck, A. and Snower, B.J. (1986), "Explanations of Unemployment", Oxford Review of Economic Policy, **1**, 34-59.

Lippman, S.A. (1980), "Optimal Pricing to Retard Entry", Review of Economic Studies, **47**, 723-731.

Mortensen, D.T. (1970), "A Theory of Wage and Employment Dynamics", in E.S. Phelps (ed.), Microeconomic Foundations of Employment and Inflation Theory (London: Macmillan).

Pissarides, C.A. (1976), Labour Market Adjustment: Microeconomic Foundations of Short–run Neoclassical and Keynesian Dynamics (Cambridge: Cambridge University Press).

Salop, S.C. (1973), "Wage Differentials in a Dynamic Theory of the Firm", Journal of Economic Theory, **6**, 321-344.

Schager, N.H. (1987), Unemployment, Vacancy Durations and Wage Increases: Applications of Markov Processes to Labour Market Dynamics (Stockholm: Industrial Institute for Economic and Social Research).

Schager, N.H. (1988), "Causes of Wage Increases in Swedish Manufacturing: A Remarkable Case of Regular Behaviour", Working Paper No. 195, Industrial Institute for Economic and Social Research, Stockholm.

Siven, C.H. (1979), A Study in the Theory of Inflation and Unemployment (Amsterdam: North Holland).

Virén, M. (1979), Human Capital and Wage Differentials in a Dynamic Theory of the Firm (Helsinki: Societas Scientiarum Fennica).

Yellen, J.L. (1984), "Efficiency Wage Models of Unemployment", American Economic Review (Papers and Proceedings), **74**, 200-205.

Zuckerman, D. (1988), "Job Search with General Stochastic Offer Arrival Rates, Journal of Economic Dynamics and Control, **12**, 679-684