IFN Working Paper No. 1537, 2025

# Multi-Product Supply Function Equilibria

Pär Holmberg, Keith Ruddell and Bert Willems

Research Institute of Industrial Economics
P.O. Box 55665
SE-102 15 Stockholm, Sweden
info@ifn.se
www.ifn.se

# Multi-Product Supply Function Equilibria*

Pär Holmberg† Keith Ruddell‡ Bert Willems§

September 24, 2025

**Abstract**

We characterize Nash equilibria in multi-product markets in which producers commit to vectors of supply functions contingent on all prices. The framework accommodates (dis)economies of scope in production, and goods may be substitutes or complements in demand. We show that equilibrium allocations of underlying goods and payoffs are invariant under bundling. With quadratic costs and linear demand, this invariance reduces the multi-product problem to an equivalent set of single-product markets that can be analyzed independently. We introduce Lerner and pass-through matrices to capture markups and welfare losses; their eigenvalues summarize fundamental market properties, remain invariant under bundling, and lend themselves to comparative statics analysis.

**Keywords:** Supply function equilibrium, multi-product pricing, divisible-good auction, bundling, pass-through, welfare

**JEL codes:** C62, C72, D43, D44, L94

*We are particularly grateful for comments from Paul Klemperer, Edward Anderson, Mark Armstrong, Nicolas Astier, Elizabeth Baldwin, Matteo Bizzarri, Andres Carvajal, Olof de Blanche, Simon Finster, Paul Milgrom, Shmuel Oren, Andrew Philpott, Marek Pycia, Sven Seuken, Thomas Tangerås, Milena Wittwer, Frank Wolak and Kyle Woodward. Pär Holmberg and Keith Ruddell have been financially supported by The Jan Wallander and Tom Hedelius Foundation, the Torsten Söderberg Foundation (E37/13) and the Swedish Energy Agency (40653-1,46227-1)

†Research Institute of Industrial Economics (IFN), Stockholm. Associate Researcher of the Energy Policy Research Group (EPRG), University of Cambridge. Affiliated with the Program on Energy and Sustainable Development (PESD), Stanford University.

‡Research Institute of Industrial Economics (IFN), Stockholm.

§LIDAM, UCLouvain; Department of Economics, TILEC & CentER, Tilburg University. Toulouse School of Economics

# 1 Introduction

In centrally organized multi-product markets, firms often trade several related goods simultaneously, such as electricity across hours of delivery or financial products with correlated payoff structures. Goods may be related because the marginal cost of one good may depend on the quantity of another good produced, or because consumers derive joint utility from bundles. Market designers have recognized those linkages by allowing bidders to condition trades in one good on prices or volumes in others. Many electricity markets use *complex intertemporal offers* to capture ramping costs (Reguant, 2014);[1] the Bank of England's *product-mix auction* enabled collateral-specific bidding (Klemperer, 2008; 2010); and financial exchanges clear *contingent multi-leg orders* to reduce execution risk (Rostek and Yoon, 2024b; Wittwer, 2021). Despite their practical importance, the theoretical analysis of competition under such interconnected bidding remains incomplete.

This paper develops a general framework for competition in multi-product supply function equilibrium and makes six contributions. First, we extend the classic supply function model of Klemperer and Meyer (1989, hereafter KM) from one product to many, allowing firms' bids in one good to depend on the prices of others. Second, we prove an invariance result: any invertible linear bundling of goods yields the same payoffs and allocations of the underlying goods. Third, we show how to construct bundles that locally decouple the markets, so that a firm's optimal supply of one bundle depends only on the bundle's own price. Fourth, we characterize equilibria under quadratic costs and linear demand, showing that with unbounded shocks the unique equilibrium is linear. Fifth, we introduce Lerner and pass-through matrices as tools for analyzing markups and welfare across multiple goods. Sixth, we also show how separation can simplify market design by reducing the complexity of bidding languages and mitigating the combinatorial explosion in package bidding.

Our first contribution is to extend the supply function equilibrium model of KM from a single product to a multi-product setting. In our framework, symmetric firms submit vector-valued supply schedules in which the output of each good is a function of all market prices, reflecting both economies of scope in production and strategic interactions across markets. As in KM, additive demand shocks generate ex-post optimality conditions, requiring that supply schedules be best responses for every realization of the shocks. The single-product Supply Function Equilibrium (SFE) has been widely applied in electricity markets, both in theoretical and simulation studies (Green

[1] From an engineering perspective, such intertemporal linkages have long been central to unit commitment and dispatch models (Wood and Wollenberg 1996).

and Newbery, 1992; Anderson and Philpott, 2002; Holmberg and Newbery, 2010) and in empirical work (Wolak, 2007; Hortaçsu and Puller, 2008; Sioshansi and Oren, 2007; Reguant, 2014). Our formulation provides the first general framework for analyzing strategic competition when firms submit multi-product supply schedules in the KM demand-shock tradition.

Our second contribution is to prove an invariance result: any invertible linear bundling of goods yields the same equilibrium payoffs and allocations of the underlying goods. Intuitively, just as a multi-dimensional problem in mathematics can be expressed in different coordinate systems, a market equilibrium can be expressed in terms of different bundles of goods without changing the underlying outcome. While similar invariance has been noted in consumer demand and monopoly models (Lancaster, 1966; Carvajal et al., 2010), our result establishes it in an oligopolistic setting with supply function competition.

Our third contribution is to establish local separation of demand and costs. By applying a linear transformation, we construct separating bundles for a given set of prices and production quantities, such that the cross-price effects of the demand slopes and the cross-quantity effects of the marginal cost slopes vanish. This removes local cross-effects in slopes, although firms may still create strategic linkages and cost shocks may remain correlated. In earlier work, Theil (1983) applied orthogonal rotations to diagonalize covariance matrices of shocks, while Johnson and Myatt (2003) introduced a demand transformation to simplify self-selection constraints.[2] In contrast, our approach applies results from matrix analysis to the joint structure of demand and cost slopes in a strategic equilibrium setting.[3]

We then prove that this decomposition also delivers local strategic separation. In separating bundles, the slope of a firm's supply function in one bundle depends only on the price of that bundle, not on the prices of other bundles. This means that small changes in the price of one bundle have no effect on the optimal supply of another bundle. As a result, at a given equilibrium point, a multi-product supply function equilibrium can be studied locally as a collection of single-product equilibria. These separation results can also be exploited for numerical calculations, making it possible to compute non-linear, multi-dimensional supply function equilibria.

Our fourth contribution is to characterize the full set of equilibria in the case of quadratic costs and linear demand. In this setting, we obtain a *global separation* result: markets can be represented in bundles that remain separated at every set of prices

[2]Related techniques are used in Johnson and Myatt (2006a, 2006b).

[3]Formally, this involves simultaneous diagonalization of two matrices (Magnus and Neudecker, 2019; Horn and Johnson, 2013; Theil, 1983); the cost Hessian and the demand Jacobian are jointly diagonalized.

and production quantities. Strategic separation then implies that each bundle can be analyzed independently, so that the multi-product problem reduces to a collection of single-bundle problems. Known results for single-good SFEs (KM), including the existence of non-linear equilibria, can then be applied bundle by bundle. Equilibria in the original goods are obtained by applying a linear change of coordinates to those in separated bundles, so that the entire set of equilibria can be recovered in the product space. A key implication is that when demand shocks are unbounded, the unique equilibrium is a linear supply function equilibrium, generalizing the classic KM result to multi-product markets.

Our fifth contribution is to develop a welfare analysis for multi-product firms in linear supply function equilibrium with non-constant marginal costs. We introduce a Lerner matrix, generalizing the scalar Lerner index, and a pass-through matrix, which describes how cost or demand shocks transmit into prices and quantities across goods. The eigenvalues of these matrices are invariant to the choice of product representation, and provide fundamental measures of distortions in multi-product markets. Markups depend on the number of firms and on the eigenstructure of the competitive pass-through matrix, which reflects both demand slopes and the marginal cost structure. A central result is that relative welfare losses are determined by the relative difference between the competitive and oligopoly pass-through matrices. We study how eigenvalues of the Lerner and the oligopoly pass-through matrices depend on the number of symmetric producers and eigenvalues of the competitive pass-through matrix. From these comparative statics results, we find that at least four symmetric firms are needed to ensure that relative welfare losses are always below 1%. [4]

Unlike the existing pass-through literature, which relies on elasticity-based representations and conduct parameters (Weyl and Fabinger, 2013; Adachi and Fabinger, 2022) or on conjectural variations (Ritz, 2024), our approach applies to multi-product supply function equilibria. In our setting, elasticities are not coordinate-invariant and distortions are more naturally captured by Lerner and pass-through matrices with invariant properties.

Our sixth contribution is to highlight how separation can improve market design. We show that firms' strategies are simple for separating bundles: they do not need to make supply in one bundle contingent on the price of another. This suggests that auctions could be organized around such bundles, reducing the complexity of bidding languages. In package bidding auctions, the number of possible bids grows exponen-

[4]For Cournot models, we show that at least 9 symmetric firms are needed to ensure that relative welfare losses are weakly below 1%. Corchón (2008) gets the same result for single-good markets with constant returns to scale.

tially with the number of products, creating the well-known problem of combinatorial explosion (Rothkopf et al., 1998; Pekeč and Rothkopf, 2003; Milgrom, 2009). By constructing bundles that separate markets, our approach mitigates the need for fully combinatorial bidding while still allowing firms to capture relevant cross-good linkages.

## Example: Two-period electricity market

To illustrate the role of bundling and motivate the general framework, consider a simple two-period electricity market in which offers for both periods are submitted simultaneously and cleared jointly. **A storage technology** such as a battery buys in the first period and resells in the second. Its profitability depends on the price difference across the two periods, so without bundling its bid would need to condition on both prices. **An inflexible generator** such as a nuclear plant must produce the same output in each period. Its profitability depends on the average price across the two periods, so again its bid would need to condition on both prices.

Bundling resolves this complexity. A **storage bundle** (buy one unit in period 1, sell one in period 2) lets the battery operator submit a simple one-dimensional supply schedule as a function of the bundle price (the difference).[5] Likewise, a **block bundle** (produce one unit in both periods) lets the nuclear plant submit a one-dimensional schedule as a function of the block-bundle price (the average).[6] Trading in bundles therefore simplifies strategies without changing allocations. This simple case illustrates the general mechanism: appropriate bundling removes cross-market dependencies, simplifies bidding languages, and leaves equilibrium allocations unchanged.

## Related literature

Another strand of the supply function literature emphasizes private information in treasury auctions and financial markets (see Wilson 1979; Back and Zender 1993; Hortaçsu and McAdams, 2010; Vives 2011; Kastl 2011; Rostek and Weretka, 2012). Rostek and Yoon (2021, 2024a, 2025) and Wittwer (2021) extend this literature to study bids that are contingent on several prices, compare such equilibria with bids that are only contingent on one price and evaluate the effect of synthetic products in the latter case.

[5]Note that the storage bundle is similar to the quality upgrade (buy a high-quality unit and sell a low quality unit) in Johnson and Myatt (2003).

[6]Block bids of this kind are widely used in EU electricity markets (see Ahlqvist et al. 2022; Herrero, Rodilla and Batlle 2020), where offers for all 24 hours of the next day are submitted simultaneously and cleared jointly.

Our analysis is also related to supply function competition in networked electricity markets. Graf et al. (2020), Holmberg and Philpott (2018), Ruddell (2018), and Wilson (2008) study bidding across locations linked by transmission constraints. In these models, locational goods may act as perfect substitutes when the network is uncongested, or as completely separate markets when congestion binds. By contrast, in our framework product interactions arise from economies of scope in costs, strategic bidding incentives, and demand linkages, rather than from physical network constraints. Moreover, our supply functions are contingent on all market prices. Bizzarri (2022) also considers multivariate supply functions, but focuses on vertical strategic interaction in input–output networks. In contrast, we study horizontal interaction across products.

A different body of work studies auction formats developed for practical applications, allowing discrete, package, or complex bids in which bidders can submit conditional offers across goods or directly report their underlying preferences or costs. Examples include the Product-Mix Auction introduced at the Bank of England (Klemperer 2010; Giese and Grace, 2023), its theoretical foundations (Baldwin and Klemperer, 2019; Finster, 2020), combinatorial auctions for spectrum allocation (Ausubel and Cramton, 2004; Ausubel, Cramton and Milgrom, 2006), and complex bids in electricity markets (O'Neill et al., 2005; Gribik, Hogan and Pope, 2007; Reguant, 2014; Herrero, Rodilla and Batlle, 2020). Like our analysis, these mechanisms are motivated by the need to capture interactions across products, but they also accommodate non-convex preferences or indivisibility constraints. The crucial distinction is that in our framework bundles are defined by invertible linear transformations of the good space, so each bundle remains perfectly divisible and is traded at a uniform price. Rebundling is therefore equivalent to a change of coordinates, leaving allocations and payoffs invariant to the representation. In contrast, package-bidding formats allow all-or-nothing offers, so effective prices depend on the composition of the package. This violates the 'one-good one-price' principle, making equilibrium outcomes dependent on the chosen bidding format rather than invariant across equivalent product representations. Nevertheless, our results should approximately apply to these practical auction formats when non-convexities and indivisibilities are minor.

In the Industrial Organization literature, bundling is typically studied as a tool for market segmentation and price discrimination (Armstrong, 2016). Carvajal et al. (2010) show that if bundles do not span the full product space, bundling choices can affect profits even under linear pricing. Our analysis is different: we restrict attention to bundles that span the entire product space, and centralized market clearing guarantees a single linear price for each good, so price discrimination is ruled out. Allocations and payoffs are invariant to the choice of bundles.

Armstrong and Vickers (2018) analyze multi-product Cournot competition and show that market power raises prices above marginal cost while also distorting relative prices. They find that markups are inversely proportional to the number of firms and are determined by the gradient of net utility. In our SFE framework, this relationship no longer holds: relative price distortions change with the number of firms due to strategic interactions of supply slopes, which lead to quadratic rather than linear matrix equations. In both frameworks, however, relative prices are undistorted either for a suitable combination of goods corresponding to a separating bundle or when the pass-through matrix converges to the identity. In an extension, we show that our approach can also be applied to Cournot models.

### Overview

The used notation is summarized in Table 1. Section 2 presents the model. Section 3 establishes invariance to bundling and identifies separating bundles. Section 4 characterizes multi-product SFE and non-linear equilibria. Section 5 focuses on quadratic costs and linear demand, where global separation can be obtained. Section 6 introduces Lerner and pass-through matrices and applies them to welfare and comparative statics, including a comparison with Cournot. Section 7 concludes. The Appendix provides proofs of lemmas and propositions, numerical and graphical illustrations of separating bundles and a pseudocode that shows how non-linear SFE can be computed.

## 2 Model description

We consider a sealed-bid procurement auction for $I$ heterogeneous divisible goods $i \in \mathcal{I} \equiv \{1, ..., I\}$. A set of $N$ suppliers $n \in \mathcal{N} \equiv \{1, \ldots, N\}$ simultaneously submit joint supply schedules for the provision of the goods. The demand for those goods is stochastic and is realized after supply schedules have been submitted. The auctioneer clears the markets jointly and determines a market price for each good. Suppliers produce according to their supply schedules. We refer to an equilibrium in this one-shot game as a supply function equilibrium (SFE).

Define $\mathcal{P}$ as the set of permissible price vectors $\mathbf{p} \in \mathcal{P} \subseteq \mathbb{R}^I$, and $\mathcal{Q}$ as the set of permissible quantity vectors $\mathbf{q} \in \mathcal{Q} \subseteq \mathbb{R}^I$. Supplier $n$'s cost for producing the quantities $\mathbf{q} \in \mathcal{Q}$ is given by a convex cost function $c_n(\mathbf{q})$. The cost functions are common knowledge among producers. Supplier $n$'s profit for selling the quantities $\mathbf{q}$ at prices $\mathbf{p}$ is

$$\pi_n(\mathbf{p}, \mathbf{q}) = \mathbf{p}^\top \mathbf{q} - c_n(\mathbf{q}), \tag{1}$$

**Table 1:** Summary of Notation

| Symbol | Description |
|---|---|
| *Model primitives* | |
| $I$ | Number of goods |
| $N$ | Number of symmetric suppliers (firms) |
| $\mathbf{p} \in \mathbb{R}^I$ | Vector of market prices |
| $\mathbf{q} \in \mathbb{R}^I$ | Vector of quantities |
| $c(\mathbf{q})$ | Convex cost function (symmetric across firms) |
| $\mathbf{d}(\mathbf{p})$ | Price-sensitive component of the demand function |
| $\boldsymbol{\varepsilon}$ | Additive demand shock vector |
| $f(\boldsymbol{\varepsilon})$ | Probability density of the demand shock |
| $\boldsymbol{\mu}, \Sigma$ | Mean and covariance of $\boldsymbol{\varepsilon}$ |
| $\mathbf{s}(\mathbf{p})$ | Symmetric (per firm) supply function |
| $\boldsymbol{C}(\mathbf{q}) = \partial^2 c(\mathbf{q})/\partial \mathbf{q}^2$ | Positive-definite cost curvature matrix (Hessian) |
| $\boldsymbol{D}(\mathbf{p}) = -\partial \mathbf{d}(\mathbf{p})/\partial \mathbf{p}$ | Positive-definite (sign flipped) demand-slope matrix (Jacobian) |
| $\boldsymbol{S}(\mathbf{p}) = \partial \mathbf{s}(\mathbf{p})/\partial \mathbf{p}$ | Positive-definite supply slope-matrix (Jacobian) |
| *Bundling* | |
| $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{I \times I}$ | Bundling matrices that transform between goods and bundles. Separating bundles are given by the eigenvectors of $\boldsymbol{DC}$ |
| We denote transformed vectors/matrices in bundle space with a tilde: e.g. $\widetilde{\mathbf{p}} = \boldsymbol{A}^T \mathbf{p}$ | |
| *Lerner, pass-through and welfare* | |
| $\boldsymbol{L} = \boldsymbol{I} - \boldsymbol{CS}$ | Lerner matrix |
| $\boldsymbol{\rho} = (\boldsymbol{I} + \frac{1}{N}\boldsymbol{S}^{-1}\boldsymbol{D})^{-1}$ | Oligopoly pass-through matrix |
| $\boldsymbol{\rho}_0 = (\boldsymbol{I} + \frac{1}{N}\boldsymbol{CD})^{-1}$ | Competitive pass-through matrix |
| $\boldsymbol{R} = \boldsymbol{\rho}\boldsymbol{\rho}_0^{-1}$ | Relative pass-through matrix |
| $\boldsymbol{W}_0 = \frac{1}{2}\boldsymbol{\rho}_0 \boldsymbol{D}^{-1}$ | Competitive welfare matrix |
| $\boldsymbol{M} = \boldsymbol{W}_0(\boldsymbol{\Sigma} + \boldsymbol{\mu}\boldsymbol{\mu}^\top)$ | Welfare-weighted shock importance matrix |
| $\bar{r} = \operatorname{tr}(\boldsymbol{MR})/\operatorname{tr}(\boldsymbol{M})$ | Welfare-weighted average relative pass-through |
| $\lambda \in 1, \ldots, I$ | Index of diagonal matrix elements in separating-bundles space. For $\boldsymbol{L}$, $\boldsymbol{\rho}$, $\boldsymbol{\rho}_0$ and $\boldsymbol{R}$, these elements are (invariant) eigenvalues |
| $r_\lambda = \rho_\lambda/\rho_{0,\lambda}$ | Relative pass-through ratio for eigenvalues indexed by $\lambda$ |
| $m_\lambda = (\widetilde{\boldsymbol{M}})_{\lambda\lambda}$ | Weight in diagonal element $\lambda$ of $\widetilde{\boldsymbol{M}}$ for separating bundles. |
| $\bar{r} = \sum_\lambda m_\lambda r_\lambda / \sum_\lambda m_\lambda$ | Welfare-weighted average relative pass-through |

where the first term represents revenue as the inner product of the price vector with the output vector.[7]

The demand in the auction is non-strategic, represented by the stochastic vector-valued demand function $\mathbf{d} : \mathcal{P} \times \mathcal{E} \to \mathcal{Q}$, which gives demand for the goods as func-

[7]For simplicity, we consider risk-neutral producers, but similar to Klemperer and Meyer (1989), risk aversion would not change our results, because the equilibria are ex-post optimal.

tion of the price vector $\mathbf{p}$ as well as an additive vector shock $\varepsilon$: [8]

$$\mathbf{d} : \mathcal{P} \times \mathcal{E} \to \mathcal{Q} : (\mathbf{p}, \varepsilon) \mapsto \mathbf{q} = \mathbf{d}(\mathbf{p}) + \varepsilon.$$

The demand shock vector $\varepsilon$ has a probability distribution with density $f(\varepsilon)$, on the support $\mathcal{E} \subseteq \mathbb{R}^I$, which is a convex set that contains the origin. Similar to KM, we assume that demand shocks are additive. Throughout the paper, we make the following assumptions regarding demand and cost functions.

**Assumption 1 (demand and cost functions).**

a. *The production cost functions $c_n$ are twice continuously differentiable, convex and common knowledge among the producers. The Hessian matrix of the cost function $\boldsymbol{C}_n$ is positive-definite everywhere.*

$$\boldsymbol{C}_n(\mathbf{q}) = \frac{\partial^2 c_n(\mathbf{q})}{\partial \mathbf{q}^2} > 0$$

b. *The Jacobian matrix of demand with respect to price is continuously differentiable and negative-definite everywhere.*

$$\boldsymbol{D}(\mathbf{p}) = -\frac{\partial \mathbf{d}(\mathbf{p})}{\partial \mathbf{p}} > 0$$

c. *The matrix product $\partial \mathbf{d}/\partial \mathbf{p}\ \partial^2 c_n/\partial \mathbf{q}^2$ has no repeated eigenvalues.*

Definiteness of the derivative matrices implies that they are symmetric. In particular this means that cross-elasticities in demand are equal $\partial d_i/\partial p_j = \partial d_j/\partial p_i$, for all $i, j$. Goods might be complements $\left(\partial d_i/\partial p_j < 0\right)$ or substitutes $\left(\partial d_i/\partial p_j > 0\right)$. Assumption 1.b. implies that own price effects are negative $(\partial d_i/\partial p_i < 0)$ and that own price effects dominate cross-price effects. Assumption 1.c. is a regularity assumption which rules out degenerate outcomes where demand and costs interactions offset each other. Without this assumption, proofs would need to include separate statements for those degenerate cases.

Each supplier $n \in \mathcal{N}$ submits a vector of supply functions $\mathbf{s}_n : \mathcal{P} \to \mathcal{Q}$ which specifies how much supplier $n$ is willing to produce of each good at every price vector $\mathbf{p} \in \mathcal{P}$.

[8] Weyl and Fabinger (2013) study shocks from exogenous (and inelastic) entry. Except for the sign, such shocks are equivalent to the additive demand shocks that we study.

**Assumption 2 (permissible bidding strategies).** *The auction only allows supply schedules for which the Jacobian matrix of the supply function vector of supplier $n \in \mathcal{N}$, is symmetric, continuously differentiable and positive-definite everywhere.*

$$S_n(\mathbf{p}) = \frac{\partial \mathbf{s}_n(\mathbf{p})}{\partial \mathbf{p}} > 0.$$

This assumption implies that the vector of supply functions $\mathbf{s}_n(\mathbf{p})$ by producer $n$ corresponds to a reported convex production cost function $k_n(\mathbf{q})$. That is, the reported marginal costs $\partial k_n / \partial \mathbf{q}$, which are strategic and may differ from the true marginal costs, are equal to the inverse of the supply functions:

$$\frac{\partial k_n(\mathbf{q})}{\partial \mathbf{q}} = \mathbf{s}_n^{-1}(\mathbf{q}) \qquad \text{and} \qquad \boldsymbol{K}_n(\mathbf{q}) = \frac{\partial^2 k_n(\mathbf{q})}{\partial \mathbf{q}^2} > 0,$$

where $\boldsymbol{K_n}$ is the Hessian of the reported cost function.

The auctioneer collects the bids of the suppliers and determines the market clearing price after the shock $\varepsilon$ has been realized. The clearing price vector $\mathbf{p}^c(\varepsilon)$ is set such that aggregate supply equals realized demand:

$$\mathbf{p}^c(\varepsilon) : \qquad \sum_{n \in \mathcal{N}} \mathbf{s}_n(\mathbf{p}) = \mathbf{d}(\mathbf{p}) + \varepsilon. \tag{2}$$

Assumptions 1 and 2 ensure that any vector of clearing prices, if it exists, is unique (Varian, 1975). Similar to KM and Vives (2011), we assume that there are no transactions (the payoff is zero) if supply functions are such that the market does not clear for some $\varepsilon$.

Each supplier chooses its vector of supply functions in order to maximize its expected profit. Given the clearing price $\mathbf{p}^c(\varepsilon)$, supplier $n$ produces $\mathbf{s}_n(\mathbf{p}^c(\varepsilon))$ units of goods and its expected profit is equal to

$$\Pi_n = \int_{\mathcal{E}} \pi_n\left(\mathbf{p}^c(\varepsilon), \mathbf{s}_n\left(\mathbf{p}^c(\varepsilon)\right)\right) f(\varepsilon)\, d\varepsilon. \tag{3}$$

The strategy profile $\{\mathbf{s}_n(\mathbf{p})\}_{n \in \mathcal{N}}$ is a Nash equilibrium in the procurement game if no player $n$ can increase its expected profit by selecting another permissible bidding strategy $\hat{\mathbf{s}}_n(\mathbf{p})$.

# 3 Bundling goods

The markets for goods might be interrelated because of: (dis-)economies of scope in production through the cost function $c_n$, goods that are complements or substitutes through the demand function $\mathbf{d}$, correlated demand shocks across markets through $\boldsymbol{\varepsilon}$, or suppliers who link their supply schedules $\mathbf{s}_n$ across markets for strategic reasons. This interaction between markets complicates the analysis of players' strategies. We can simplify the problem by making a coordinate transformation, i.e. to bundle the goods. However, first we need to show that such a transformation will not change the outcome of the game. Below we find that the auction outcome is invariant to bundling of goods. That is, in equilibrium firms receive the same allocation of the underlying goods and the same payoffs, irrespective of how the goods are bundled by the auctioneer. In the next step, we will rely on this property and choose specific bundles, the separating bundles, for which cost and demand interactions vanish. The two other interactions, correlated shocks and strategic linking, will be addressed in Section 4. Ex-post optimality will imply that the distribution and correlation of shocks do not influence the producers' decisions (the supply functions) and it will follow from our analysis that strategic linkage disappears for separating bundles.

Instead of procuring $I$ goods $i = 1, ..., I$, we now assume that the auctioneer procures $I$ bundles of goods $j = 1, ..., I$, where each bundle $j$ is divisible but consists of a fixed proportion of goods $i$. Formally, one unit of the bundle $j$ consists of $a_{1,j}$ units of good 1 and $a_{2,j}$ units of good 2, etc. Thus the amount of underlying goods $\mathbf{q}$ that are contained in $\tilde{\mathbf{q}}$ units bundles can be calculated as follows:

$$\mathbf{q} = A\tilde{\mathbf{q}}.$$

We refer to $A$ as the *bundling matrix*. We shall consider bundles which span the full set of goods and are linearly independent, so that the bundling matrix $A$ is of full rank.[9] The bundling transformation corresponds to a coordinate transformation. Note that typically defined bundles that span the space of goods would not be orthogonal. In such cases, the coordinate transformation is non-orthogonal.[10]

For each bundle $j$, it is possible to compute a price $\tilde{p}_j$ for one unit of the bundle $j$

[9]This means that $A$ has to be an element of the general linear group over $\mathbb{R}$: $A \in \mathrm{Gl}_I(\mathbb{R})$. Carvajal et al. (2010), who analyze the effects of bundling for a monopoly market, also consider 'partial' bundles that do not span the space of goods. In this case $A$ would not be of full rank and would not belong to $\mathrm{Gl}_I(\mathbb{R})$.

[10]Hence, bundling matrices are typically not unitary. This implies that distances are normally not preserved after a bundling transformation, i.e. the transformation is not isometric. For example, the price-cost margins will typically change after bundling, so that $\|\mathbf{p}_0 - \partial c(\mathbf{q}_0)/\partial \mathbf{q}\| \neq \|\tilde{\mathbf{p}}_0 - \partial \tilde{c}(\tilde{\mathbf{q}}_0)/\partial \tilde{\mathbf{q}}\|$.

from the prices of goods contained in that bundle, that is $\tilde{p}_j = \sum_j a_{i,j} p_i$. Hence, $\tilde{\mathbf{p}} = \boldsymbol{A}^\top \mathbf{p}$. From a vector of bundle prices $\tilde{\mathbf{p}}$, it is also possible to compute prices for the underlying goods $\mathbf{p}$ that would give those bundle prices:

$$\mathbf{p} = \boldsymbol{B}\tilde{\mathbf{p}},$$

where $\boldsymbol{B} = \left(\boldsymbol{A}^{-\mathbf{1}}\right)^{\boldsymbol{T}}$. From the transformations above, we can determine permissible bundle prices $\tilde{\mathcal{P}}$ and bundle quantities $\tilde{\mathcal{Q}}$ that correspond to permissible prices and quantities for the underlying goods.

$$\tilde{\mathcal{P}} = \{\tilde{\mathbf{p}} \in \mathbb{R}^I | \boldsymbol{B}\tilde{\mathbf{p}} \in \mathcal{P}\} \qquad \tilde{\mathcal{Q}} = \{\tilde{\mathbf{q}} \in \mathbb{R}^I | \boldsymbol{A}\tilde{\mathbf{q}} \in \mathcal{Q}\}.$$

Producer $n$'s cost of producing $\tilde{\mathbf{q}}$ bundles is equal to the cost of the goods $\boldsymbol{A}\tilde{\mathbf{q}}$. So,

$$\tilde{c}_n(\tilde{\mathbf{q}}) = c_n(\boldsymbol{A}\tilde{\mathbf{q}}). \tag{4}$$

The demand shock for bundles $\tilde{\boldsymbol{\varepsilon}} = \boldsymbol{A}^{-1}\boldsymbol{\varepsilon}$ has the support $\tilde{\mathcal{E}} = \{\tilde{\boldsymbol{\varepsilon}} \in \mathbb{R}^I | \boldsymbol{A}\tilde{\boldsymbol{\varepsilon}} \in \mathcal{E}\}$.The demand for bundles at bundle prices $\tilde{\mathbf{p}}$ and demand shock $\tilde{\boldsymbol{\varepsilon}}$ is determined by the demand for underlying goods and the transformations:

$$\tilde{\mathbf{d}}(\tilde{\mathbf{p}}) + \tilde{\boldsymbol{\varepsilon}} = \boldsymbol{A}^{-1}\mathbf{d}(\boldsymbol{B}\tilde{\mathbf{p}}) + \tilde{\boldsymbol{\varepsilon}}. \tag{5}$$

Similarly, a non-strategic supply function of some producer $n$, which corresponds to its true marginal cost, can be transformed as follows.

$$\tilde{\mathbf{s}}_n(\tilde{\mathbf{p}}) = \boldsymbol{A}^{-1}\mathbf{s}_n(\boldsymbol{B}\tilde{\mathbf{p}}).$$

The relationship between the supply functions $\mathbf{s}$ in goods and $\tilde{\mathbf{s}}$ in bundles is illustrated in Figure 1. The non-strategic supply of bundles $\tilde{q} = \tilde{s}(\tilde{p})$ can be found by three subsequent mappings: $\tilde{p} \xrightarrow{B} p \xrightarrow{s} q \xrightarrow{A^{-1}} \tilde{q}$: converting the bundle prices into good prices, identifying the non-strategic supply of underlying goods by firm $n$, and transforming goods supply into bundle supply.

Furthermore, we will use the properties of the lemma below to show that the properties of Assumption 1 (concavity of gross utility and convexity of cost) and Assumption 2 (permissible bidding format) are invariant to bundling.

**Lemma 1.** *The transformations of the cost Hessian, $\boldsymbol{C}(\mathbf{q})$ the demand Jacobian $-\boldsymbol{D}(\mathbf{p})$ and*

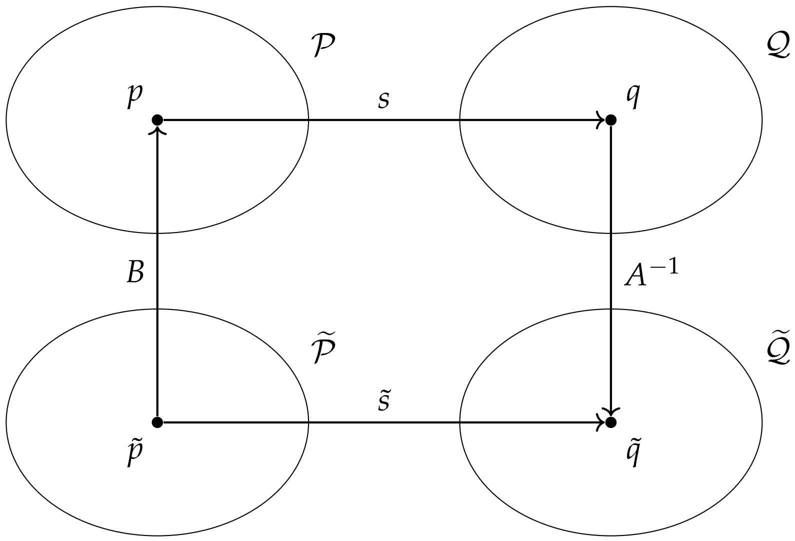



**Figure 1:** The mapping of supply functions in bundle space to good space.

*the supply Jacobian $\boldsymbol{S}(\mathbf{p})$ from good to bundle space are congruence transformations:*[11]

$$\tilde{\boldsymbol{C}} = \boldsymbol{A}^\top \boldsymbol{C}\boldsymbol{A}, \qquad -\tilde{\boldsymbol{D}} = -\boldsymbol{B}^\top \boldsymbol{D}\boldsymbol{B} \qquad \text{and} \qquad \tilde{\boldsymbol{S}} = \boldsymbol{B}^\top \boldsymbol{S}\boldsymbol{B}$$

*A congruence transformation is a mapping $\mathbb{R}^{I\times I} \to \mathbb{R}^{I\times I}$ which maps a matrix $\boldsymbol{X}$ on $\boldsymbol{A}^\top \boldsymbol{X}\boldsymbol{A}$ for some non-singular matrix $\boldsymbol{A} \in \mathrm{Gl}_I(\mathbb{R})$:*

With a congruence transformation, eigenvalues change, but the number of positive, negative and zero eigenvalues remains constant. This is Sylvester's law of inertia (Horn and Johnson, 2013). Moreover, symmetry of transformed matrices is preserved. Hence, positive definiteness of the cost Hessian and demand/supply Jacobians do not change when transforming between the good space and the bundle space even if the coordinate transformation would be non-orthogonal. We note that the product of the two matrices $\boldsymbol{D}\boldsymbol{C}$ transforms as a similarity transformation: $\tilde{\boldsymbol{D}}\tilde{\boldsymbol{C}} = \boldsymbol{A}^{-1}\boldsymbol{D}\boldsymbol{C}\boldsymbol{A}$. This transformation preserves eigenvalues, so that the assumed property of non-repeated eigenvalues is independent of the transformation (Horn and Johnson, 2013). Hence, Assumptions 1 and 2 are satisfied in bundle space if and only if they are satisfied in good space.

Cost and net-utility are invariant to bundling. We now formulate a proposition which shows that the Nash equilibrium is also invariant to the bundling transformation.

**Proposition 1.** *The strategy profile $\{\tilde{\mathbf{s}}_n(\tilde{\mathbf{p}})\}_{n\in\mathcal{N}}$ is a Nash equilibrium in the auction for bundles if and only if the strategy profile $\{\mathbf{s}_n(\mathbf{p})\}_{n\in\mathcal{N}} = \{\boldsymbol{A}\tilde{\mathbf{s}}_n(\boldsymbol{A}^\top\mathbf{p})\}_{n\in\mathcal{N}}$ is a Nash equilibrium*

[11] Recall that $-\boldsymbol{D}(\mathbf{p}) = \frac{\partial \mathbf{d}(\mathbf{p})}{\partial \mathbf{p}}$.

*in the auction for goods.*

Thus if we can solve for an SFE for a specific bundle of goods, then we can use the expressions above to transform the equilibrium to a market with bundles.

When operating the auction, or solving for the optimal offers or the supply function equilibrium, we have a choice in how to bundle the goods. In our case, we would like to find a bundling such that the markets for bundles are (locally) separate, with no cross-interactions in either cost or demand. That is, we would like to find a congruence transformation $\boldsymbol{A}$ such that Jacobian of demand and the Hessian of production cost are diagonal. The following Lemma formalizes this approach.

**Lemma 2** (Separating Bundles)**.** *Let $\boldsymbol{A}$ be a bundling that consists of the eigenvectors of the matrix product of the demand Jacobian and the cost Hessian, $\boldsymbol{D}(\mathbf{p})\boldsymbol{C}(\mathbf{q})$,*

$$\boldsymbol{A} = \text{eigenvector}(\boldsymbol{D}(\mathbf{p})\boldsymbol{C}(\mathbf{q}))$$

*then in bundle coordinates the cost Hessian $\tilde{\boldsymbol{C}}(\tilde{\mathbf{q}})$ and demand Jacobian $-\tilde{\boldsymbol{D}}(\tilde{\mathbf{p}})$ are diagonal:*

$$\begin{aligned} \tilde{\boldsymbol{D}} &= \boldsymbol{B}^\top \boldsymbol{D}\boldsymbol{B} \in diag(\mathbb{R}^{I\times I}) \\ \tilde{\boldsymbol{C}} &= \boldsymbol{A}^\top \boldsymbol{C}\boldsymbol{A} \in diag(\mathbb{R}^{I\times I}) \end{aligned}$$

*Moreover this is the unique bundling that simultaneously diagonalizes both matrices, up to scalar multiplication and permutation of columns.*[12]

The eigenvectors of $\boldsymbol{DC}$ are generalized eigenvectors (and conjugate vectors) of $\boldsymbol{D}$ and $\boldsymbol{C}$, which normally differ from the eigenvectors of $\boldsymbol{D}$ or $\boldsymbol{C}$. We discuss this further in Appendix B.2. Appendix B.1 has numerical examples that illustrate bundling and how separating bundles can be identified. In general, a bundling $\boldsymbol{A}$ will only separate markets locally, at a particular price and quantity. Section 5 focuses on a linear equilibrium, where there is a bundling $\boldsymbol{A}$ that will separate markets globally.

# 4 Characterization of bidding equilibrium

## 4.1 Best response

In a supply function equilibrium, a seller $n$ acts as a monopolist in response to its *residual demand*, defined as the market demand less the offers of all the other sellers:

[12]Assumption 1.c. rules this out, but if the matrix $\boldsymbol{DC}$ had repeated eigenvalues, then the separating bundles would still exist but would no longer be unique.

$$\mathbf{d}_n(\mathbf{p}) + \boldsymbol{\varepsilon} = \mathbf{d}(\mathbf{p}) + \boldsymbol{\varepsilon} - \sum_{m \neq n} \mathbf{s}_m(\mathbf{p}). \tag{6}$$

This inherits the properties of Assumption 1.b. — its price Jacobian $\partial \mathbf{d}_n / \partial \mathbf{p}$ is negative-definite and shocks are additive.

There will be a one-to-one correspondence between the shock vector and price vector, so that any choice of price vector (or quantity vector) can only clear the market for one shock vector. Thus the best response is ex-post optimal.

**Lemma 3** (Optimal response to residual demand)**.** *First Order Condition: A necessary condition for supply function $\mathbf{s}_n(\mathbf{p})$ of supplier $n$ to be a best response to its stochastic residual demand in (6) is that at every point $(\mathbf{p}, \mathbf{q}) = (\mathbf{p}, \mathbf{s}_n(\mathbf{p}))$ on the supply schedule, we have*

$$\mathbf{h}_n(\mathbf{p}, \mathbf{s}_n(\mathbf{p})) = \mathbf{0}, \tag{7}$$

*where*

$$\mathbf{h}_n(\mathbf{p}, \mathbf{q}) = \mathbf{q} + \frac{\partial \mathbf{d}_n(\mathbf{p})}{\partial \mathbf{p}} \left( \mathbf{p} - \frac{\partial \mathbf{c}_n(\mathbf{q})}{\partial \mathbf{q}} \right). \tag{8}$$

*Global Second Order Condition: A sufficient condition for $\mathbf{s}_n(\mathbf{p}^*)$ to be the best quantity to offer at price vector $\mathbf{p}^*$ is that it satisfies (7) and*

$$(\mathbf{p} - \mathbf{p}^*)^\top \mathbf{h}_n(\mathbf{p}, \mathbf{s}_n(\mathbf{p}^*)) \leq 0 \tag{9}$$

*for every $\mathbf{p} \in \mathcal{P}$.*

It follows from Lemma 3 that the first-order condition for multi-product markets is a generalization of the first-order condition for one divisible good by Klemperer and Meyer (1989).

## 4.2 Symmetric equilibrium

In this section we consider identical suppliers and look for symmetric SFE, i.e. where each firm $n \in \mathcal{N}$ submits identical supply schedules, $\mathbf{s}_n = \mathbf{s}$. It follows from Lemma 3 that the necessary first-order conditions are given by a single, vector-valued partial differential equation (PDE).

$$\mathbf{F}\left(\mathbf{p}, \mathbf{s}, \frac{\partial \mathbf{s}}{\partial \mathbf{p}}\right) = \mathbf{s} - \left((N-1)\frac{\partial \mathbf{s}}{\partial \mathbf{p}} - \frac{\partial \mathbf{d}}{\partial \mathbf{p}}\right)\left(\mathbf{p} - \frac{\partial \mathbf{c}}{\partial \mathbf{q}}\right) = \mathbf{0}. \tag{PDE}$$

Solutions $\mathbf{s}^{\text{PDE}}(\mathbf{p})$ to this PDE will be SFE if they satisfy (1) the bidding format and (2) the global second-order conditions. We can rewrite this PDE so that we can find a symmetric supply Jacobian for a given level of price $\mathbf{p}_0$ and quantity $\mathbf{q}_0$:

$$\frac{\partial \mathbf{s}(\mathbf{p}_0)}{\partial \mathbf{p}} = \boldsymbol{G}(\mathbf{p}_0, \mathbf{q}_0)$$

Here, the function $\boldsymbol{G}$ maps the price and quantity vector on a symmetric matrix representing the supply Jacobian. This formulation rewrites the differential equation in a standardized format.

Any supply Jacobian needs to be symmetric according to Assumption 2, even if the supply function is not part of an equilibrium. The symmetric supply Jacobian $\frac{\partial \mathbf{s}(\mathbf{p}_0)}{\partial \mathbf{p}}$ at price $\mathbf{p}_0$ has $I(I+1)/2$ degrees of freedom. The partial differential equation (PDE) imposes $I$ conditions on the supply Jacobian, one for each good. This might seem to suggest that the symmetric supply Jacobian is not uniquely defined and that there are $I(I-1)/2$ degrees of freedom. However, the proposition below shows that this is not the case, because related symmetry conditions need to be satisfied for derivatives of the supply Jacobian. This reduces the degrees of freedom and results in a unique supply Jacobian. The supply Jacobian is such that the separating bundles that (locally) diagonalize the demand Jacobian and cost Hessian, will also (locally) diagonalize the supply Jacobian. Hence, with the separating bundles the markets will separate, also strategically. If markets are separate, bidders have no incentive to condition their supply on the price of other bundles.

**Proposition 2.** *Let $(\mathbf{q}_0, \mathbf{p}_0)$ be a point in $\mathcal{Q} \times \mathcal{P}$. At this point there exists a unique symmetric, continuously differentiable supply Jacobian $\partial \mathbf{s}/\partial \mathbf{p}$ that solves (PDE). It satisfies the commuting relationship:*

$$\frac{\partial \mathbf{d}(\mathbf{p}_0)}{\partial \mathbf{p}} \frac{\partial c(\mathbf{q}_0)}{\partial \mathbf{q}^2} \frac{\partial \mathbf{s}(\mathbf{p}_0)}{\partial \mathbf{p}} = \frac{\partial \mathbf{s}(\mathbf{p}_0)}{\partial \mathbf{p}} \frac{\partial c(\mathbf{q}_0)}{\partial \mathbf{q}^2} \frac{\partial \mathbf{d}(\mathbf{p}_0)}{\partial \mathbf{p}}$$

*and is congruently diagonalized by the separating bundling matrix $\boldsymbol{A}$ that locally diagonalizes $\partial \mathbf{d}/\partial \mathbf{p}$ and $\left(\partial^2 c/\partial \mathbf{q}^2\right)^{-1}$at $(\mathbf{q}_0, \mathbf{p}_0)$.*

This proposition provides two practical methods to derive the supply Jacobian. It can be found by rewriting the SFE in separating bundles coordinates, calculate $I$ one-dimensional supply slopes for those bundles and transform the solution back into goods coordinates. An alternative is solving a set of equations: The FOC ($I$ conditions), symmetry of the Jacobian ($I^2/2 - I/2$ conditions) and symmetry of its derivatives ($I^2/2 - I/2$ conditions), which correspond to the commutative property in Proposition 2. This set of equations uniquely pins down the supply Jacobian.

$$\left.\begin{array}{r}\mathbf{F}\left(\mathbf{p}_0,\mathbf{q}_0,\frac{\partial\mathbf{s}(\mathbf{p}_0)}{\partial\mathbf{p}}\right)=0\\ \frac{\partial\mathbf{s}(\mathbf{p}_0)}{\partial\mathbf{p}}=\left(\frac{\partial\mathbf{s}(\mathbf{p}_0)}{\partial\mathbf{p}}\right)^{\mathsf{T}}\\ \frac{\partial\mathbf{d}(\mathbf{p}_0)}{\partial\mathbf{p}}\frac{\partial c(\mathbf{q}_0)}{\partial\mathbf{q}^2}\frac{\partial\mathbf{s}(\mathbf{p}_0)}{\partial\mathbf{p}}=\left(\frac{\partial\mathbf{d}(\mathbf{p}_0)}{\partial\mathbf{p}}\frac{\partial c(\mathbf{q}_0)}{\partial\mathbf{q}^2}\frac{\partial\mathbf{s}(\mathbf{p}_0)}{\partial\mathbf{p}}\right)^{\mathsf{T}}\end{array}\right\}\rightarrow\frac{\partial\mathbf{s}(\mathbf{p}_0)}{\partial\mathbf{p}}=G(\mathbf{p}_0,\mathbf{q}_0)$$

The results in Proposition 2 can be used to straightforwardly numerically compute non-linear, multi-product supply function equilibria, as illustrated in Appendix B.3. However, there is a singularity at points where the price equals the marginal cost. More advanced numerical tools are likely to be needed if one needs robust estimates of supply functions in the neighborhood of such singularities.

# 5 Quadratic cost and linear demand

Henceforth, we assume quadratic costs and linear demand, which implies that there is a single bundling transformation that separates markets globally. By applying the theory of single-good SFE, we can characterize all equilibrium supply schedules. One of the equilibrium supply schedules is linear. This linear schedule is the unique equilibrium when demand shocks are unbounded. Furthermore, in the linear setting, costs and net utility are homothetic in quantities. This makes it possible to describe the interactions between products based on demand preference and production cost through representative iso-utility and iso-cost curves. This facilitates the graphical illustration of separating bundles, as is shown in the Appendix B.2.

**Assumption 3** (Quadratic cost and linear demand)**.**

1. *The costs of suppliers are quadratic, common knowledge and denoted by* $c_n(\mathbf{q}) = \frac{1}{2}\mathbf{q}^\top \boldsymbol{C}\mathbf{q}$ *for all* $n \in \mathcal{N}$*, where* $\boldsymbol{C}$ *is a positive-definite matrix.*

2. *The demand is linear in price, with additive demand shocks expressed as* $\mathbf{d}(\mathbf{p}, \boldsymbol{\varepsilon}) = \boldsymbol{\varepsilon} - \boldsymbol{D}\mathbf{p}$*, where* $\boldsymbol{D}$ *is a positive-definite matrix.*

The cost Hessian and the demand Jacobian remain constant under Assumption 3. Proposition 2 implies that there exists one fixed bundling that separates the markets *for all* prices and quantities, which can be formalized in the next corollary.

**Corollary 1.** *Under Assumption 3 (quadratic costs and linear demand), the separating bundle* $\boldsymbol{A} = \left(\boldsymbol{B}^{-1}\right)^T$ *remains unchanged regardless of the values of* $(\mathbf{q}, \mathbf{p})$*. Therefore a single bundling transformation* $\boldsymbol{A}$ *globally separates markets. In this specific basis both matrices*

$$\tilde{\boldsymbol{C}} = \boldsymbol{A}^\top \boldsymbol{C}\boldsymbol{A} \quad \textit{and} \quad \tilde{\boldsymbol{D}} = \boldsymbol{B}^\top \boldsymbol{D}\boldsymbol{B}$$

*become diagonal while the supply Jacobian*

$$\frac{\partial \tilde{\mathbf{s}}}{\partial \tilde{\mathbf{p}}} = \boldsymbol{B}^{\top} \frac{\partial \mathbf{s}}{\partial \mathbf{p}} \boldsymbol{B}$$

*is diagonal for every price* $\mathbf{p} \in \mathcal{P}$.

With the markets separated, supply in each market can be optimized independently. Hence, it becomes a simple task to determine supply function equilibria, as we can apply the findings of KM for a single good market to each independent bundle-market.

Corollary 1 demonstrates that equilibrium outcomes are invariant to bundling. In other words, if supply functions are an SFE in one coordinate system, then transformed supply functions will also remain an SFE in the transformed coordinate system. Therefore, by characterizing equilibria for the separated markets, we are effectively characterizing equilibria for all possible bundles of those goods.

For two identical firms, KM characterize the set of SFEs, including non-linear SFE, for a single good market where the range of the demand shock is a bounded interval $[0, \bar{\varepsilon}]$. They establish that this set is a one-parameter family of symmetric SFEs, which forms a fan between the most competitive equilibrium (Bertrand outcome when the shock is at the upper bound) and the least competitive equilibrium (Cournot outcome when the shock is at the upper bound). They also find that any symmetric upward sloping supply function that satisfies the one-dimensional version of the first-order condition in (PDE) is an SFE. The results of KM can easily be extended to $N$ firms, and also involve negative demand shocks. For negative shock values, prices and quantities become negative, indicating that firms are purchasing from consumers. By a reflection of the values through the origin we can use the KM methodology to derive SFEs. If the function $s(q)$ is an SFE for the negative shock interval $[\underline{\varepsilon}, 0]$, then the function $-s(-q)$ is an SFE on the positive shock interval $[0, -\underline{\varepsilon}]$. The fact that all SFEs have the same slope through the origin allows us to independently define the positive and negative sections of the SFE curve. Consequently, we can define the set of all SFEs in a linear market for a single good as a two-parameter family.

**Definition 1.** Let $\mathfrak{S}(\underline{\varepsilon}, \bar{\varepsilon}, \delta, \gamma, N)$ denote the set of SFE, $s(p)$, in an $N$-player game with a single good characterized by a linear stochastic demand $d(p, \varepsilon) = \varepsilon - \delta p$, with demand shock support $[\underline{\varepsilon}, \bar{\varepsilon}]$, and quadratic cost $c(q) = \frac{1}{2}\gamma q^2$.

From this definition, it becomes straightforward to characterize all SFE, including non-linear SFE, in multi-product markets with quadratic costs and linear demand.

**Corollary 2.** *Let* $\boldsymbol{A} = \left(\boldsymbol{B}^{-1}\right)^T$ *be a bundling transformation that globally separates markets, with* $\tilde{\boldsymbol{D}} = \boldsymbol{B}^\top \boldsymbol{D}\boldsymbol{B}$ *and* $\tilde{\boldsymbol{C}} = \boldsymbol{A}^\top \boldsymbol{C}\boldsymbol{A}$ *the corresponding diagonal matrices for bundles. Let* $\tilde{d}_{ii}$ *and* $\tilde{c}_{ii}$ *be the i-th diagonal entries of* $\tilde{\boldsymbol{D}}$ *and* $\tilde{\boldsymbol{C}}$*, then all symmetric SFE Nash equilibria take the form (and any symmetric supply function with this form is a symmetric SFE)*

$$s_j(\mathbf{p}) = \sum_{i \in \mathcal{I}} b_{ji} \cdot \tilde{s}_i \left( \sum_{k \in \mathcal{I}} b_{ki} p_k \right)$$

*where each* $\tilde{s}_i(\cdot)$ *belongs to the set of one-dimensional SFE for bundle i. One-dimensional SFE of different bundle markets can be combined in any way such that*

$$\{\tilde{s}_1(\cdot), ..., \tilde{s}_i(\cdot), ..., \tilde{s}_I(\cdot)\} \in \prod_i \mathfrak{S}_i$$

*with*

$$\mathfrak{S}_i = \mathfrak{S}\left(\tilde{\varepsilon}_i^{\min}, \tilde{\varepsilon}_i^{\max}, \tilde{d}_{ii}, \tilde{c}_{ii}, N\right).$$

*For each separating bundle i, we get maximum and minimum demand shocks from the feasible separating shocks* $\tilde{\mathcal{E}}$*:*

$$\begin{aligned} \tilde{\varepsilon}_i^{\min} &= \min\left\{\tilde{\varepsilon}_i \,|\, \tilde{\varepsilon} \in \tilde{\mathcal{E}}\right\} \\ \tilde{\varepsilon}_i^{\max} &= \max\left\{\tilde{\varepsilon}_i \,|\, \tilde{\varepsilon} \in \tilde{\mathcal{E}}\right\}. \end{aligned} \tag{10}$$

Hence, the set of SFEs is a $2I$ parameter family of supply schedules, which correspond to the purchasing and supply schedules for each of the separating bundles, in $I$ markets.

We now confine ourselves to a special class of SFE with *linear schedules*, denoted as $\mathbf{s}(\mathbf{p}) = \boldsymbol{S}\mathbf{p}$. For such a schedule, the supply Jacobians $\partial \mathbf{s}/\partial \mathbf{p} = \boldsymbol{S}$ are constant and equal to $\boldsymbol{S}$. First, we show the existence and uniqueness of such an SFE. We then show that when demand shocks are unbounded, this linear schedule is the unique SFE.

**Lemma 4.** *A linear schedule* $\mathbf{s}(\mathbf{p}) = \boldsymbol{S}\mathbf{p}$ *is an SFE if it solves the algebraic Riccati equation.*

$$\boldsymbol{S} - (\boldsymbol{D} + (N-1)\boldsymbol{S})(\boldsymbol{I} - \boldsymbol{C}\boldsymbol{S}) = \mathbf{0}. \tag{11}$$

*There is a unique positive-definite* $\boldsymbol{S}$ *that solves this equation. If the demand shock has full support,* $\mathcal{E} = \mathbb{R}^I$*, then this linear schedule is the unique symmetric SFE.*

The first part of the proof follows from rewriting the first-order conditions ((PDE)) for linear supply schedules. The Algebraic Riccati equation is multi-dimensional version of a quadratic equation which has $2^I$ roots, only one which is positive definite. [13] If the support of the shock for goods $\mathcal{E}$ is unbounded, $\mathcal{E} = \mathbb{R}^I$, then the support of the shock for separating bundles $\tilde{\mathcal{E}}$ is unbounded as well $\tilde{\mathcal{E}} = \mathbb{R}^I$. Hence, for each separating bundle the demand shock has full support, and KM's result for single goods applies: the unique SFE is the linear supply schedule.

Armstrong and Vickers (2018) demonstrate that for Cournot models, the supply gradient vector is equal to the cost gradient vector plus a mark-up vector: $\mathbf{p} = \boldsymbol{S}\mathbf{q} = \boldsymbol{C}\mathbf{q} + \boldsymbol{D}^{-1}\mathbf{q}/N$. The mark-up decreases with the number of firms $N$ in the market. Such a straightforward linear relationship does not exist for Supply Function Equilibria since the intensity of competition depends not only on the demand slope $\boldsymbol{D}$ but also on the supply slopes of competitors $\boldsymbol{S}$, which are endogenous. An SFE requires a quadratic equation to be solved. The mark-ups are examined in greater depth in the following section. Rather than absolute mark-ups, we shall consider relative mark-ups, which allow for more convenient expressions for linear SFE.

# 6 Welfare analysis

## 6.1 Multi-dimensional mark-ups and welfare.

To characterize the mark-ups in the linear SFE we introduce the *Lerner matrix*

$$\boldsymbol{L} = \boldsymbol{I} - \boldsymbol{C}\boldsymbol{S}. \tag{12}$$

This matrix maps an equilibrium price vector $\mathbf{p}$ onto the price-cost mark-up ($\Delta\mathbf{p}^{\text{Mark-up}}$). That is:

$$\boldsymbol{L}\mathbf{p} = \Delta\mathbf{p}^{\text{Mark-up}} \equiv \mathbf{p} - \frac{\partial c}{\partial \mathbf{q}}. \tag{13}$$

It gives information about the relative price distortions due to market power in the multi-good market for any given price vector. The Lerner matrix is dimensionless.

An additional measure of market performance is the *(oligopoly) pass-through matrix*, which is defined as

$$\boldsymbol{\rho} = (\boldsymbol{I} + \tfrac{1}{N}\boldsymbol{S}^{-1}\boldsymbol{D})^{-1}. \tag{14}$$

Analogous to Weyl and Fabinger (2013), the pass-through matrix describes how an

[13] This is straightforward for separated markets. We know from Klemperer and Meyer (1989) that there are two roots for each separated market, one with positive slope and one with negative slope. Hence, there are $2^I$ ways in which they can be combined, but there is only one combination that has positive slopes in each separated market. We know from Section 3 that bundling does not change definiteness of demand and supply Jacobians.

increase in per unit taxes $\mathbf{t}$ on goods (or an increase in marginal costs) would affect prices on the margin, $d\mathbf{p} = \boldsymbol{\rho} d\mathbf{t}$ in an imperfectly competitive market with industry supply curve $N\boldsymbol{S}\mathbf{p}$. It also maps a demand shock $\boldsymbol{\varepsilon}$ onto the equilibrium outcome $\mathbf{q}$. That is, $\mathbf{q} = \boldsymbol{\rho}^\top \boldsymbol{\varepsilon}$.[14]

We define the *competitive pass-through matrix* $\boldsymbol{\rho_0}$ as the pass-through matrix with competitive bids $\boldsymbol{S} = \boldsymbol{C}^{-1}$:

$$\boldsymbol{\rho_0} = (\boldsymbol{I} + \tfrac{1}{N}\boldsymbol{C}\boldsymbol{D})^{-1}. \tag{15}$$

This matrix is dimensionless and aggregates some primitives of the model: the industry cost Hessian $\boldsymbol{C}/N$ and the demand Jacobian $\boldsymbol{D}$. It does not depend on firms' strategic decisions. In essence, the competitive pass-through matrix is a compact measure of industry characteristics.

The *pass-through distortion matrix* $\boldsymbol{R}$ is the ratio of the oligopoly and competitive pass-through matrices:

$$\boldsymbol{R} = \boldsymbol{\rho}\boldsymbol{\rho}_0^{-1}$$

It measures to which extent market power changes the pass-through matrix.

In a perfectly competitive setting with linear demand and quadratic costs, expected welfare can be expressed as a function of the fundamentals of the model (demand Jacobian $\boldsymbol{D}$, the competitive pass-through matrix $\boldsymbol{\rho}_0$, and the second moment of the shock distribution $\boldsymbol{\Sigma} + \boldsymbol{\mu}\boldsymbol{\mu}^\top$). Welfare losses in an oligopoly setting depend additionally on the pass-through distortion $\boldsymbol{R}$.

**Proposition 3** (Welfare Loss Decomposition)**.** *Let $\boldsymbol{W}_0 = \boldsymbol{\rho}_0\boldsymbol{D}^{-1}/2$ denote the competitive welfare matrix, and define the welfare-weighted shock importance matrix $\boldsymbol{M}$ as*

$$\boldsymbol{M} := \boldsymbol{W}_0(\boldsymbol{\Sigma} + \boldsymbol{\mu}\boldsymbol{\mu}^\top),$$

*with $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ the mean vector and co-variance matrix of the demand shock $\boldsymbol{\varepsilon}$. Then the following statements hold:*[15]

1. ***Competitive welfare** $w_0$ **for given demand shock** $\boldsymbol{\varepsilon}$ is a quadratic function.*

$$w_0 = \boldsymbol{\varepsilon}^\top \boldsymbol{W}_0 \boldsymbol{\varepsilon}$$

2. ***Oligopoly welfare losses** $w_L$ **for given demand shock** $\boldsymbol{\varepsilon}$ is a quadratic function*

$$w_L = \boldsymbol{\varepsilon}^\top (\boldsymbol{I} - \boldsymbol{R})^2 \boldsymbol{W}_0 \boldsymbol{\varepsilon}.$$

[14]The clearing price for a shock $\boldsymbol{\varepsilon}$ can be determined from $\boldsymbol{\varepsilon} - \boldsymbol{D}\mathbf{p} = N\boldsymbol{S}\mathbf{p}$. Total output is given by $N\boldsymbol{S}\mathbf{p}$.

[15]Note that the trace function, which sums the diagonal elements of a matrix, is denoted by $tr(\cdot)$.

3. ***Expected competitive welfare***

$$\mathbb{E}[w_0] = \text{tr}(\boldsymbol{M}).$$

4. ***Expected welfare loss in oligopoly***

$$\mathbb{E}[w_L] = \text{tr}\left[(\boldsymbol{I} - \boldsymbol{R})^2 \boldsymbol{M}\right]$$

5. ***Decomposition of relative welfare loss into level and dispersion effects****:Define the weighted average relative pass-through as*

$$\bar{r} := \frac{\text{tr}(\boldsymbol{MR})}{\text{tr}(\boldsymbol{M})}.$$

*Then:*

$$\text{RelLoss} = \frac{\mathbb{E}[w_L]}{\mathbb{E}[w_0]} = (1 - \bar{r})^2 + \frac{\text{tr}\left[(\boldsymbol{R} - \bar{r}\boldsymbol{I})\boldsymbol{M}(\boldsymbol{R} - \bar{r}\boldsymbol{I})\right]}{\text{tr}(\boldsymbol{M})}.$$

Given the quadratic nature of the objective function, the expressions only depend on the second-order moments of the shock distribution and not on the precise shape of the probability distribution. The matrix $\boldsymbol{M}$ contains all the information needed to compute expected welfare under perfect competition: the magnitude of the demand shocks (with larger shocks corresponding to higher welfare) and the trading surplus available for each given shock. The pass-through distortion matrix $\boldsymbol{R}$ explains how market power reduces sales volume for the different products.

The next lemma establishes some general properties of the matrices and their eigenvalues.

**Lemma 5.** *Let $\boldsymbol{L}$, $\boldsymbol{\rho}$, $\boldsymbol{\rho}_0$, $\boldsymbol{R}$, $\boldsymbol{M}$ and $\boldsymbol{W}_0$ be the Lerner, pass-through, competitive pass-through, pass-through distortion, welfare-weighted shock importance matrices and the competitive welfare matrix. Then,*

- *Changing from goods to bundle space corresponds to a similarity transformation of the matrices $\boldsymbol{L}$, $\boldsymbol{\rho}$, $\boldsymbol{\rho}_0$, $\boldsymbol{R}$ and $\boldsymbol{M}$ (and their inverses):*

$$\tilde{\boldsymbol{L}} = \boldsymbol{B}^{-1}\boldsymbol{L}\boldsymbol{B} \quad \tilde{\boldsymbol{\rho}} = \boldsymbol{B}^{-1}\boldsymbol{\rho}\boldsymbol{B} \quad \tilde{\boldsymbol{\rho}}_{\boldsymbol{0}} = \boldsymbol{B}^{-1}\boldsymbol{\rho}_{\boldsymbol{0}}\boldsymbol{B} \quad \tilde{\boldsymbol{R}} = \boldsymbol{B}^{-1}\boldsymbol{R}\boldsymbol{B}\,, \quad \tilde{\boldsymbol{M}} = \boldsymbol{B}^{-1}\boldsymbol{M}\boldsymbol{B}\,,$$

- *$\boldsymbol{L}$, $\boldsymbol{\rho}$, $\boldsymbol{\rho}_0$ and $\boldsymbol{R}$ (and their inverses) have the same set of eigenvectors which correspond to the separating bundles. Hence, they are jointly diagonalizable by the separating bundles. They also commute with respect to multiplication.*

- *The eigenvalues and trace of the matrices* $\boldsymbol{L}$, $\boldsymbol{\rho}$, $\boldsymbol{\rho}_0$, $\boldsymbol{R}$ *and* $\boldsymbol{M}$ *(and their inverses and products) are invariant to bundling.*

- *Changing from goods to bundle space corresponds to a congruence transformation of* $\boldsymbol{W}_0$*:*

$$\tilde{\boldsymbol{W}}_0 = \boldsymbol{A}^T \boldsymbol{W}_0 \boldsymbol{A}$$

- *The matrix* $\boldsymbol{W}_0$ *is symmetric, positive definite and diagonalizable by the separating bundles.*

It can be noted that the matrices $\boldsymbol{L}$, $\boldsymbol{\rho}$, $\boldsymbol{\rho}_0$, $\boldsymbol{R}$, $\boldsymbol{M}$ and $\boldsymbol{W}_0$ are representations of invariant underlying objects, which can be referred to as tensors.[16] In particular, Lerner, pass-through, pass-through distortion and welfare-weighted shock importance tensors are mixed tensors or linear mappings, for which representative matrices have some invariant properties. Lemma 5 shows that the trace and eigenvalues of $\boldsymbol{L}$, $\boldsymbol{\rho}$, $\boldsymbol{\rho}_0$, $\boldsymbol{R}$ and $\boldsymbol{M}$ (and their products) are independent of bundling. Hence, they are fundamental and coordinate-invariant properties of multi-product markets. For example, the results in Proposition 3 are invariant to bundling. The commutative property of the matrices $\boldsymbol{L}$, $\boldsymbol{\rho}$, $\boldsymbol{\rho}_0$ and $\boldsymbol{R}$ is a strong property, which implies that we can more easily manipulate expressions.[17] Moreover, these four matrices are all diagonalized by the separating bundles. For a diagonalized matrix $\boldsymbol{L}$, $\boldsymbol{\rho}$, $\boldsymbol{\rho}_0$ or $\boldsymbol{R}$, the diagonal elements are equal to the matrix' (invariant) eigenvalues. We index these diagonal elements (eigenvalues) by $\lambda$. The matrix $\boldsymbol{W}_0$ represents a bilinear form, which has different tensor properties. The eigenvalues of the matrix $\boldsymbol{W}_0$ are not invariant to bundling, but the matrix is diagonalized by the separating bundles. We can index these diagonal elements by $\lambda$ as well. The matrix $\boldsymbol{M}$ depends on stochastic properties of the demand shock, and may not be diagonalized by the separating bundles. Still, we can index the diagonal elements of this matrix, and the matrix $\boldsymbol{\Sigma} + \boldsymbol{\mu}\boldsymbol{\mu}^\top$, by $\lambda$ when the matrices have been transformed by the separating bundle. $\boldsymbol{R}$ and $\boldsymbol{W}_0$ are diagonalized for separating bundles. Hence, for the separating bundles, it will only be the diagonal elements of $\boldsymbol{M}$ and $\boldsymbol{\Sigma} + \boldsymbol{\mu}\boldsymbol{\mu}^\top$ that matters inside the trace function.

[16] A tensor is a mathematical object that generalizes scalars and vectors to higher dimensions. Vectors in physics are used to depict essential physical attributes, such as velocity or force. A vector is invariant in itself, but it can be expressed through a set of coordinates, which do depend on the specific coordinate system in use. Tensors too, are utilized to depict key physical properties. For instance, permittivity and permeability tensors characterize the electric and magnetic traits of materials, while the Cauchy stress tensor represents forces at work within objects. As our analysis shows, tensors can also be used to characterize key properties of multi-product markets.

[17] The commutative property follows from the fact that the tensors have the same eigenvectors. Matrices with different eigenvectors do not commute. For example, it would typically be the case that $\boldsymbol{CD} \neq \boldsymbol{DC}$.

**Corollary 3.** *[Bundle-by-Bundle Decomposition]* $\boldsymbol{R}$ *and* $\mathbf{W}_0$ *are jointly diagonalizable by the separating bundles. Let:*

- $r_\lambda := \rho_\lambda / \rho_{0,\lambda}$ *be the pass-through distortion in separating bundle* $\lambda$*,*
- $m_\lambda := w_{0,\lambda}(\tilde{\boldsymbol{\Sigma}} + \tilde{\boldsymbol{\mu}}\tilde{\boldsymbol{\mu}}^\top)_{\lambda\lambda}$ *the weight of bundle* $\lambda$ *and,*
- $\bar{r} := \sum_\lambda m_\lambda r_\lambda / \sum_\lambda m_\lambda$ *the average relative pass-through.*

*Then:*

1. ***Absolute welfare loss:***
$$\mathbb{E}[w_L] = \sum_\lambda m_\lambda (1 - r_\lambda)^2.$$

2. ***Decomposition of Relative welfare loss:***
$$\text{RelLoss} = \sum_\lambda m_\lambda (1 - r_\lambda)^2 / \sum_\lambda m_\lambda = (1 - \bar{r})^2 + \sum_\lambda m_\lambda (r_\lambda - \bar{r})^2 / \sum_\lambda m_\lambda.$$

For separated markets, bundles can be studied independently. For such bundles, the diagonal element $\lambda$ of a Lerner or pass-through matrix becomes the well-known Lerner index and pass-through rate for the bundle $\lambda$. Figure 2 illustrates the Lerner index and pass-through rate for a single good market with linear supply and demand. The Lerner index $L$ multiplied by the price $p$ is the price-cost mark-up $(p - c)$. The pass-through rate $\rho$ measures to which extent the mark-up leads to an increase of the consumer price, as compared to the competitive price: $p - p^{comp} = \rho L p$. The competitive pass-through multiplied by the demand shock, $\rho_0 \varepsilon$, is competitive industry output when the demand shock is $\varepsilon$; and $\rho\varepsilon$ is industry output in the oligopoly situation. Hence, $1 - \rho/\rho_0$ is the fraction of output that is withheld.

From Figure 2 it is clear that for perfectly-competitive, separated markets welfare is given by a triangle with base $D^{-1}\varepsilon$ and height $\rho_0\varepsilon$. For a given demand shock $\varepsilon$, welfare becomes $\frac{1}{2}\varepsilon\rho_0 D^{-1}\varepsilon$ , which is consistent with Proposition 3. The triangle of deadweight losses have the same shape but dimensions are scaled with the withholding ratio, $1 - \rho\rho_0^{-1}$, which is also consistent with Proposition 3.

For separated markets, the pass-through rates can be expressed as a function of price elasticity of demand $\epsilon_{D_\lambda}$ and industry supply $\epsilon_{S_\lambda}$. Equivalently, the competitive pass-through rate depends on the price elasticity of competitive industry supply $\epsilon_{C_\lambda^{-1}}$.

$$\rho = \frac{\epsilon_{S_\lambda}}{\epsilon_{S_\lambda} + \epsilon_{D_\lambda}} \qquad \rho_{0,\lambda} = \frac{\epsilon_{C_\lambda^{-1}}}{\epsilon_{C_\lambda^{-1}} + \epsilon_{D_\lambda}}.$$

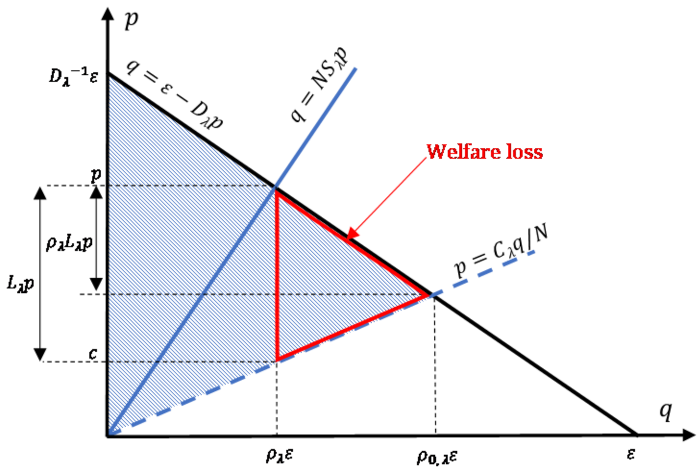
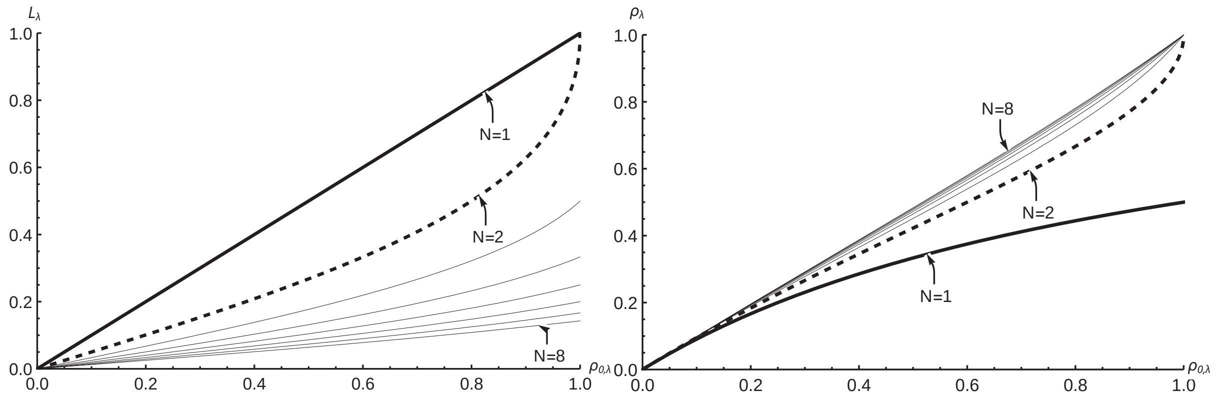



**Figure 2:** Illustration of the Lerner index and (competitive) pass-through in separated markets. The graph represents industry oligopoly supply curve, industry competitive supply curve and demand. The blue shaded area represents total surplus with perfect competition, the area inside the red line is the deadweight loss with oligopoly.

The competitive pass-through is large ($\rho_o \to 1$), if competitive supply is elastic ($\epsilon_{C_\lambda^{-1}} \to \infty$) and demand is inelastic ($\epsilon_{D_\lambda} \to 0$). It is small ($\rho_o \to 0$), when the competitive supply is inelastic ($\epsilon_{C_\lambda^{-1}} \to 0$) and demand is elastic ($\epsilon_{D_\lambda} \to \infty$).

## 6.2 Comparative statics

The next proposition links the market structure parameters – namely the competitive pass-through matrix and the number of firms – with the resulting market performance measures given by the Lerner matrix and the oligopoly pass-through matrix. Changing the number of firms $N \to N^*$ while keeping competitive pass-through $\boldsymbol{\rho}_0$ constant, requires an adjustment of the cost Hessian $\boldsymbol{C}$, the demand Jacobian $\boldsymbol{D}$, or both, such that $\boldsymbol{CD}/N = \boldsymbol{C}^*\boldsymbol{D}^*/N^*$. This guarantees that properties of the competitive market do not change. With a constant $\boldsymbol{D} = \boldsymbol{D}^*$, this requires a rescaling of the cost Hessian $\boldsymbol{C}$ such that the competitive aggregate industry supply curve remains unchanged $\boldsymbol{C}/N = \boldsymbol{C}^*/N^*$. This corresponds to a policy that reallocates production assets between firms, i.e. forced divestiture. With a constant $\boldsymbol{C} = \boldsymbol{C}^*$, i.e. new firms identical to the old ones are added, this requires a rescaling of the demand Jacobian $\boldsymbol{D}$ such that $\boldsymbol{D}/N = \boldsymbol{D}^*/N^*$ is constant. In policy terms this can correspond to improving competition by removing trade barriers between two autarkic markets. Figure 3 and Proposition 4 show that such policy measures will make the market more competitive. Eigenvalues of the Lerner matrix will decrease and eigenvalues of the oligopoly pass-through matrix will increase.




**(a)** Lerner Index

**(b)** Pass-through

**Figure 3:** Lerner index and pass through as a function of the competitive pass-through, $\rho_{0,\lambda}$, and number of firms $N = 1, .., 8$.

**Proposition 4.** *Let* $\boldsymbol{L}$*,* $\boldsymbol{\rho}$ *and* $\boldsymbol{\rho}_0$ *be the equilibrium Lerner, the pass-through and the competitive pass-through matrices and* $L_\lambda$*,* $\rho_\lambda$ *and* $\rho_{0,\lambda}$ *be corresponding eigenvalues that belong to the same eigenvector. Then,*

1. $\boldsymbol{L}$ *and* $\boldsymbol{\rho}$ *depend only on* $\boldsymbol{\rho}_0$ *and the number of firms* $N$*.* $\boldsymbol{L}$ *is the stable solution (=eigenvalues smaller than 1) of the Algebraic Riccati equation:*

$$(N-1)\,\boldsymbol{L}^2 - N\boldsymbol{\rho}_0^{-1}\boldsymbol{L} + \boldsymbol{I} = \boldsymbol{0}, \tag{16}$$

*and* $\boldsymbol{\rho}$ *can be calculated from:*

$$\boldsymbol{\rho} = (\boldsymbol{L}\boldsymbol{\rho}_0 - \boldsymbol{I})^{-1}(\boldsymbol{L}\boldsymbol{\rho}_0 - \boldsymbol{\rho}_0). \tag{17}$$

2. *For a monopoly (*$N = 1$*),* $\boldsymbol{L}$ *and* $\boldsymbol{\rho}$ *are determined by:*

$$\boldsymbol{L} = \boldsymbol{\rho}_0 \qquad \boldsymbol{\rho} = \boldsymbol{\rho}_0(\boldsymbol{\rho}_0 + \boldsymbol{I})^{-1}$$

3. *For many firms* $(N \to \infty)$*,* $\boldsymbol{L}$ *and* $\boldsymbol{\rho}$ *can be approximated as,*

$$\boldsymbol{L} \approx \frac{\boldsymbol{\rho}_0}{N} \qquad \boldsymbol{\rho} \approx \boldsymbol{\rho_0} - \boldsymbol{\rho}_0^2 \frac{\boldsymbol{I} - \boldsymbol{\rho}_0}{N}$$

4. *For small eigenvalues of the competitive pass-through matrix (*$\rho_{0,\lambda} \to 0$*),* $\boldsymbol{L}$ *and* $\boldsymbol{\rho}$ *can be approximated as,*

$$\boldsymbol{L} \approx \frac{\boldsymbol{\rho}_0}{N} \qquad \boldsymbol{\rho} \approx \boldsymbol{\rho}_0$$

5. *The eigenvalues of the pass-through matrices are between zero and one. Moreover, the eigenvalues of* $\boldsymbol{\rho}$ *are smaller than the corresponding eigenvalues of* $\boldsymbol{\rho}_0$*, i.e.* $\rho_\lambda \leq \rho_{0,\lambda}$.

6. *For* $N \geq 2$*, the eigenvalues of the Lerner matrix are non-negative and bounded from above by*
$$L_\lambda \leq \frac{1}{N-1}.$$
*This limit is reached when the eigenvalues of the competitive pass-through matrix converge to* $1$*,* $\rho_{0,\lambda} \to 1$.

7. *Increasing the number of firms N while keeping competitive pass-through* $\boldsymbol{\rho}_0$ *constant makes markets more competitive (Lerner eigenvalues decrease), and increases the pass-through eigenvalues.*
$$\frac{\partial L_\lambda}{\partial N} < 0 \qquad \frac{\partial \rho_\lambda}{\partial N} > 0$$

8. *Increasing an eigenvalue of the competitive pass-through matrix* $\boldsymbol{\rho}_0$ *makes markets less competitive (Lerner eigenvalue increases) and increases the pass-through matrix* $\rho$
$$\frac{\partial L_\lambda}{\partial \rho_{0,\lambda}} > 0 \qquad \frac{\partial \rho_\lambda}{\partial \rho_{0,\lambda}} > 0$$

9. *For* $N \geq 2$*, the ratio of the oligopolistic and the competitive pass-through* $\rho_\lambda / \rho_{0,\lambda}$ *is U-shaped in* $\rho_{0,\lambda}$*. The ratio is 1 for* $\rho_{0,\lambda} = 0$ *and* $\rho_{0,\lambda} = 1$*, and reaches its minimum* $2(N-1)\left(\sqrt{\frac{N}{N-1}} - 1\right)$ *at* $\rho_{0,\lambda} = \frac{1}{2}\sqrt{\frac{N}{N-1}}$.

When the market is relatively competitive, that is, with many firms or a competitive pass-through matrix with small eigenvalues, results (3) and (4) show that the Lerner matrix is approximately proportional to the competitive pass-through matrix:[18]

$$\boldsymbol{L} \approx \frac{\boldsymbol{\rho}_0}{N}.$$

This implies that mark-ups for individual products are inversely proportional to the number of firms.

When all eigenvalues of the competitive pass-through matrix are close 1 and hence, the competitive pass-through matrix converges to the identity matrix $\boldsymbol{\rho}_0 \to \boldsymbol{I}$, result (6) shows that the Lerner matrix becomes a diagonal matrix with the same relative

[18] Note that the formula also holds for the monopoly case, $N = 1$, but the formula is an inaccurate approximation for an intermediate number of firms, unless the competitive pass-through matrix has small eigenvalues.

mark-ups for all products,

$$\boldsymbol{L} = \frac{\boldsymbol{I}}{N-1} \text{ and } \boldsymbol{\rho} = \boldsymbol{\rho}_0.$$

The mark-ups are the same for any product and are approximately inversely proportional to the number of firms. This can be compared with a related equiproportionality result in the multi-product Cournot model by Armstrong and Vickers (2018) for the case with constant returns to scale.

In relative terms, dead weight losses are small when there is little withholding, $\rho_\lambda/\rho_{0,\lambda} \to 1$. Figure 3b and Proposition 4 show that is the case when markets are competitive and mark-ups are low ($N \to \infty$, or $\rho_{0,\lambda} \to 0$) or when demand is inelastic in an oligopoly setting ($\rho_{0,\lambda} \to 1$ for $N > 2$). The intuition for the latter is the following: Although price mark-ups are high, $L_\lambda \to \frac{1}{N-1}$, output withholding and deadweight loss are small because consumers do not respond to prices. Relative output reductions are largest for intermediate values of $\rho^*_{0,\lambda} = \frac{1}{2}\sqrt{\frac{N}{N-1}}$.

Welfare does not depend on bundling. Hence, it follows from Corollary 3 and Proposition 4 that if we keep $N$ fixed, then relative welfare losses are maximized (irrespective of the weights $m_\lambda$) when $\rho_{0,\lambda} = \rho^*_{0,\lambda}$ for all separating bundles. Based on results in Corollary 3 and Proposition 4, it is straightforward to show the following:

**Corollary 4.** *For given $N \geq 2$, relative welfare losses have an upper bound:*

$$\frac{\mathbb{E}[w_L]}{\mathbb{E}[w_0]} \leq \frac{1}{4\left(N\left(1+\sqrt{1-1/N}\right)-1/2\right)^2}.$$

.

## 6.3 Comparisons with Cournot

The results in Section 6.1 are valid for any market structure with linear demand, linear aggregate supply, and quadratic aggregate costs. For example, the results can be applied to a Cournot model with symmetric firms and where shocks are observed before outputs are chosen. It is straightforward to show that the properties 2, 4, 5, 7 and 8 in Proposition 4 also hold for such a Cournot model. However, there are also significant differences.

**Proposition 5.** *Let $\boldsymbol{L}^C$ and $\boldsymbol{\rho}^C$ be the equilibrium Lerner matrix, the pass-through matrix for a Cournot model and let $L^C_\lambda$ and $\rho^C_\lambda$ be corresponding eigenvalues that belong to the same eigenvector (separating bundle). Then,*

1. *$\boldsymbol{L}^C$ and $\boldsymbol{\rho}^C$ depend only on $\boldsymbol{\rho}_0$ and the number of firms N and is given by*

$$\boldsymbol{L}^C = \frac{\boldsymbol{\rho}_0}{N}\left(\boldsymbol{I} - \frac{N-1}{N}\boldsymbol{\rho}_0\right)^{-1}$$
$$\boldsymbol{\rho}^C = \boldsymbol{\rho}_0\left(\tfrac{1}{N}\boldsymbol{\rho}_0 + \boldsymbol{I}\right)^{-1}.$$

2. *For many firms ($N \to \infty$) and $\rho_{0,\lambda} < 1$ for $\lambda = 1, \ldots, I$, $\boldsymbol{L}^C$ and $\boldsymbol{\rho}^C$ can be approximated as,*

$$\boldsymbol{L}^C \approx \frac{\boldsymbol{\rho}_0}{N}\left(\boldsymbol{I} - \boldsymbol{\rho}_0\right)^{-1} \qquad \boldsymbol{\rho}^C \approx \boldsymbol{\rho}_0 - \frac{\boldsymbol{\rho}_0^2}{N}.$$

3. *The eigenvalues of the Lerner matrix are non-negative and bounded from above by*

$$L^C_\lambda \leq 1.$$

   *Irrespective of N, this limit is reached when the eigenvalues of the competitive pass-through matrix converge to 1, $\rho_{0,\lambda} \to 1$.*

4. *For $N \geq 2$, the ratio of the oligopolistic and the competitive pass-through $\rho^C_\lambda/\rho_{0,\lambda}$ is decreasing in $\rho_{0,\lambda}$. The ratio is 1 for $\rho_{0,\lambda} = 0$ and $\frac{N}{N+1}$for $\rho_{0,\lambda} = 1$.*

It is straightforward to show that eigenvalues of the Cournot model's Lerner matrix and oligopoly pass-through matrix are higher and lower, respectively, in comparison to the linear SFE model. In particular, relative mark-ups can be huge in the Cournot case when $\rho_{0,\lambda} \to 1$, also for many firms. Armstrong and Vickers (2018) demonstrate that for Cournot model the absolute mark-up is linear in the number of firms, but the Lerner matrix, which is a relative mark-up is not.

It follows from Corollary 3 and Proposition 5 that if we keep $N$fixed, then relative welfare losses are maximized in the Cournot model when $\rho_{0,\lambda} = 1$ for all separating bundles. Based on results in Corollary 3 and Proposition 5, it is straightforward to show the following:

**Corollary 5.** *For given $N \geq 2$, relative welfare losses in the Cournot model have an upper bound:*

$$\frac{\mathbb{E}[w_L]}{\mathbb{E}[w_0]} \leq \frac{1}{(N+1)^2}.$$

In the Cournot case, at least 9 symmetric firms are needed to ensure that relative welfare losses are at or below 1%.[19] For the linear SFE model, it is enough to have 4 symmetric firms, which follows from Corollary 4.

[19] Corchón (2008) finds a similar result for single-good Cournot markets with constant returns to scale.

# 7 Conclusions

We study a single-round procurement auction for multiple heterogeneous divisible goods. Symmetric firms have costs that are common knowledge and offer a vector of supply functions where supply of a good depends on the prices of all goods. Demand has additive demand shocks. Clearing in each market determines equilibrium prices and production volumes. Optimal bidding behavior is determined by a set of partial differential equations, one for each firm. A firm's supply offer for one good may depend on the price of the other because (1) there are (dis)economies of scope in production, (2) goods are substitutes or complements, (3) demand shocks are correlated across markets, and (4) firms link markets for strategic considerations.

We simplify the multi-dimensional problem by choosing a suitable coordinate system, which might be non-orthogonal. In our model, a coordinate transformation corresponds to bundling of goods. We consider divisible and complete bundles, so that bundling does not restrict what combinations of the underlying goods can be traded. Moreover, we consider linear prices, so that there is no discrimination in the market. We show that payoffs, consumer surplus and the allocation of underlying goods are invariant to bundling. Hence, bundling does not fundamentally change market outcomes. Moreover, we show that our model assumptions – for instance, definiteness of the cost Hessian, demand Jacobian and supply Jacobian – do not change with bundling.

For quadratic costs, linear demand, and symmetric firms, bundles can be chosen such that both marginal costs and the demand for bundles are independent. That is, the marginal cost for bundle 1 does not depend on the production level for bundle 2, and the demand for bundle 1 does not depend on the price for bundle 2 (if we hold the demand shock in market 1 fixed). If demand and costs are independent, we show that a firm's bidding strategies would also be independent: Each firm will submit a bid function for which supply of bundle 1 does not depend on the price of bundle 2. This is true even if demand shocks are correlated across markets. Hence, 'smart' bundling allows us to consider the procurement for each bundle separately. We can then use the KM model (Klemperer and Meyer, 1989) for a single divisible-good market to characterize equilibrium offers. There is a unique symmetric equilibrium, which is linear, when demand shocks are unbounded. For the linear equilibrium, relative prices are non-distorted along a separating bundle, i.e. the same as in a competitive market. This is a property that could potentially be used to identify separating bundles empirically.

To characterize properties of the linear equilibrium, we introduce Lerner and pass-through matrices, which generalize the Lerner index and pass-through rate used for single-good markets. The pass-through matrix can be used to identify the effect of a vector of unit taxes. But we use the matrix to calculate pass through from demand shocks to cleared prices, outputs, mark-ups and welfare losses. We show that eigenvalues of the Lerner and pass-through matrices are fundamental properties of multi-product markets. They are invariant to bundling. We show that the competitive pass-through matrix and the number of firms provide sufficient information to estimate mark-ups in an imperfectly competitive market. We also apply this approach to a multi-product Cournot market with symmetric firms.

When computing welfare losses, we find it useful to work with two pass-through matrices: one for the case where producers make competitive offers and another for the case where they make strategic offers. We show that the relative difference between these matrices determines the welfare loss. In particular, for separated markets the relative welfare loss for each product equals the square of the relative difference between the two pass-through matrices. For each number of firms, we can estimate an upper bound for the relative welfare loss. At least four symmetric firms ensure that the relative welfare loss is always at or below 1 %. At least nine firms are needed in a corresponding Cournot model.

For non-linear problems, separating bundles can be used to separate markets locally. In this paper, we illustrate how local bundling can be used to numerically compute multi-dimensional SFE for non-linear cases. Local independence transformations can potentially also be useful when designing an auction in practice. It is often more straightforward to trade independent bundles in separate auctions instead of interdependent goods in a multi-product auction. In practice it is mainly of interest to minimize the interaction around the expected clearing price, so local market separation should be sufficient. If it is not possible to find locally separating bundles in practice, it would still be possible to find bundles that locally minimize the interdependence between markets.

# 8 References

Adachi, T. and M. Fabinger (2022). Pass-through, welfare and incidence under imperfect competition. *Journal of Public Economics* 211, 104589.

Ahlqvist, V., P. Holmberg and T. Tangerås (2022). A survey comparing centralized and decentralized electricity markets. *Energy Strategy Reviews* 40, 100812.

Anderson, E. J. and A. B. Philpott (2002). Optimal offer construction in electricity markets. *Mathematics of Operations Research* 27, 82–100.

Armstrong, M. (2016). Nonlinear pricing. *Annual Review of Economics* 8, 583-614.

Armstrong, M. and J. Vickers (2018). Multiproduct pricing made simple. *Journal of Political Economy* 126(4), 1444-1471.

Ausubel, L. and P. Cramton (2004). Auctioning Many Divisible Goods, *Journal of the European Economic Association* 2(2-3), 480-493.

Ausubel, L., P. Cramton and P. Milgrom (2006). The clock-proxy auction: A practical combinatorial auction design. In Peter Cramton, Yoav Shoham, and Richard Steinberg (eds.), *Combinatorial Auctions*, MIT Press, Chapter 5, 115-138.

Back, K. and J. F. Zender (1993). Auctions of divisible goods: On the rationale for the treasury experiment. *The Review of Financial Studies* 6(4), 733-764.

Baldwin, E. and P. Klemperer (2019). Understanding Preferences: 'Demand Types', and The Existence of Equilibrium with Indivisibilities. *Econometrica* 87(3), 867-932.

Bizzarri, M. (2022). Supply and demand function competition in general equilibrium: endogenous market power in input-output networks. Working paper no. 648, Center for Studies in Economics and Finance (CESF), University of Naples Federico II.

Carvajal, A. M., M. Rostek and M. Weretka (2010). Bundling without price discrimination, Working paper No. 936, Department of Economics, University of Warwick.

Corchón, L. C. (2008). Welfare losses under Cournot competition. *International Journal of Industrial Organization* 26(5), 1120-1131.

Finster, S. (2020). Strategic bidding in product-mix, sequential, and simultaneous auctions. Nuffield College, University of Oxford.

Giese, J. and C. Grace (2023). An evaluation of the Bank of England's ILTR operations: comparing the product-mix auction to alternatives. Working Paper No. 1044. Bank of England.

Graf, C., F. Quaglia and F. A. Wolak (2020). Simplified electricity market models with significant intermittent renewable capacity: Evidence from Italy. Working Paper No. 27262. National Bureau of Economic Research.

Green, R. J. and D. M. Newbery (1992). Competition in the British Electricity Spot Market. *Journal of Political Economy* 100(5), 929-953.

Gribik, P. R., W. W. Hogan and S. L. Pope (2007). Market-clearing electricity prices and energy uplift. Working paper.

Herrero, I., P. Rodilla and C. Batlle (2020). Evolving bidding formats and pricing schemes in USA and Europe day-ahead electricity markets, *Energies* 13(19), 5020.

Holmberg, P. and D. Newbery (2010). The supply function equilibrium and its policy implications for wholesale electricity auctions, *Utilities Policy* 18(4), 209-226.

Holmberg, P. and Philpott, A. B. (2018). On supply-function equilibria in radial transmission networks. *European Journal of Operational Research* 271(3), 985-1000.

Horn, R. A. and C. R. Johnson (2013). Matrix Analysis. Second edition. Cambridge: Cambridge University Press.

Hortaçsu, A. and D. McAdams (2010). Mechanism Choice and Strategic Bidding in Divisible Good Auctions: An Empirical Analysis of the Turkish Treasury Auction Market, *Journal of Political Economy* 118(5), 833-865.

Hortaçsu, A. and S. Puller (2008). Understanding strategic bidding in multi-unit auctions: a case study of the Texas electricity spot market. *The RAND Journal of Economics* 39(1), 86–114.

Johnson, J. P. and D. P. Myatt (2003). Multiproduct quality competition: Fighting brands and product line pruning, *American Economic Review* 93(3), 748-774.

Johnson, J. P. and D. P. Myatt (2006a). On the simple economics of advertising, marketing, and product design, *American Economic Review* 96(3), 756-784.

Johnson, J. P. and D. P. Myatt (2006b). Multiproduct cournot oligopoly. *The RAND Journal of Economics* 37(3), 583-601.

Kastl, J. (2011). Discrete bids and empirical inference in divisible good auctions. *The Review of Economic Studies*, 78(3), 974-1014.

Klemperer, P. D. (2008). A New Auction for Substitutes: Central Bank Liquidity Auctions, the U.S. TARP, and Variable Product-Mix Auctions. Mimeo: Oxford University.

Klemperer, P. D. (2010). The product-mix auction: A new auction design for differentiated goods. *Journal of the European Economic Association* 8(2-3), 526-536.

Klemperer, P. D. and M. A. Meyer (1989). Supply function equilibria in oligopoly under uncertainty, *Econometrica* 57, 1243–1277.

Lancaster, K. J. (1966). A new approach to consumer theory. *Journal of Political Economy* 74(2), 132-157.

Magnus, J. R. and H. Neudecker (2019). Matrix differential calculus with applications in statistics and econometrics. Third edition. New York: John Wiley & Sons.

McCartin, B. J. (2013). A Matrix Analytic Approach to the Conjugate Diameters of an Ellipse. *Applied Mathematical Sciences* 7(36), 1797-1810.

Milgrom, P. (2009). Assignment messages and exchanges. *American Economic Journal: Microeconomics* 1(2), 95-113.

O'Neill, R. P., P. M. Sotkiewicz, B. F. Hobbs, M. H. Rothkopf and W. R. Stewart Jr (2005). Efficient market-clearing prices in markets with nonconvexities. *European Journal of Operational Research* 164(1), 269-285.

Pekeč, A. and M. H. Rothkopf (2003). Combinatorial auction design. *Management Science* 49(11), 1485-1503.

Reguant, M. (2014). Complementary bidding mechanisms and startup costs in electricity markets. *The Review of Economic Studies* 81(4), 1708-1742.

Ritz, R. A. (2024). Does competition increase pass-through? *The RAND Journal of Economics* 55(1), 140-165.

Rostek, M. and M. Weretka (2012). Price inference in small markets, *Econometrica* 80(2), 687-711.

Rostek, M. and J. H. Yoon (2021). Exchange design and efficiency, *Econometrica* 89(6), 2887-2928.

Rostek, M. and J. H. Yoon (2025). Financial product design in decentralized markets. *Journal of Political Economy* 133(3), 888-934.

Rostek, M. and J. H. Yoon (2024a). Innovation in decentralized markets: technology versus synthetic products. *American Economic Journal: Microeconomics* 16(1), 63-109.

Rostek, M. J. and J. H. Yoon (2024b). Imperfect competition in financial markets: Recent developments. Forthcoming in *Journal of Economic Literature.*

Rothkopf, M. H., A. Pekeč and R. M. Harstad (1998). Computationally manageable combinational auctions. *Management Science* 44(8), 1131-1147.

Ruddell, K. (2018). *Supply Function Equilibrium over a Constrained Transmission Line I: Calculating Equilibria,* IFN Working Paper 1208, Research Institute of Industrial Economics, Stockholm.

Shilov, G. E. (1961). *An introduction to the theory of linear spaces.* Englewood Cliffs, NJ: Prentice-Hall, Inc.

Sioshansi, R. and S. S. Oren (2007). How good are supply function equilibrium models: an empirical analysis of the ERCOT balancing market. *Journal of Regulatory Economics* 31(1), 1–35.

Theil, H. (1983). *Linear Algebra and Matrix Methods in Econometrics.* In Handbook of Econometrics, Volume I (pp. 3-65).

Varian, H. R. (1975). A Third Remark on the Number of Equilibria of an Economy, Econometrica 43(5/6), 985-86.

Vives, X. (2011), Strategic supply function competition with private information, *Econometrica* 79(6), 1919–1966.

Weyl, E. G. and M. Fabinger (2013). Pass-Through as an Economic Tool: Principles of Incidence under Imperfect Competition, *Journal of Political Economy* 121(3), 528-583.

Wilson, R. (1979). Auctions of shares. *Quarterly Journal of Economics* 93(4), 675–689.

Wilson, R. (2008). Supply function equilibrium in a constrained transmission system. *Operations Research* 56(2), 369–382.

Wittwer, M. (2021). Connecting disconnected financial markets? *American Economic Journal: Microeconomics* 13(1), 252-282.

Wolak, F. A. (2007). Quantifying the Supply-Side Benefits from Forward Contracting in Wholesale Electricity Markets. *Journal of Applied Econometrics* 22(7), 1179–1209.

Wood, A. J. and B. F. Wollenberg (1996). Power Generation, Operation and Control. New York: John Wiley & Sons, inc.

# A Proofs of Lemmas and Propositions

**Proof of Lemma 1: Congruence transformations of cost Hessian and demand/supply Jacobians.**

*Proof.* Taking the second partial derivative of the cost function for bundles $\tilde{c}_n(\tilde{\mathbf{q}}) = c_n(\boldsymbol{A}\tilde{\mathbf{q}})$ with respect to $\tilde{q}_i$ and $\tilde{q}_j$, and using the definition of the bundling matrix $\boldsymbol{A}$, gives the following expressions for the elements of the cost Hessian:

$$\frac{\partial^2 \tilde{c}_n}{\partial \tilde{q}_i \partial \tilde{q}_j} = \sum_{kl} \frac{\partial c_n}{\partial q_k \partial q_l} \frac{\partial q_k}{\partial \tilde{q}_i} \frac{\partial q_l}{\partial \tilde{q}_j} = \sum_{kl} \frac{\partial c_n}{\partial q_k \partial q_l} A_{ki} A_{lj}.$$

$\boldsymbol{A}$ is a non-singular matrix, so we recognize this as the congruence transformation of a bilinear form (Horn and Johnson, 2013), which proves the first part of the Lemma. We have from (5) that $\tilde{\mathbf{d}}\left(\tilde{\mathbf{p}}\right) = \boldsymbol{B}^\top \mathbf{d}(\boldsymbol{B}\tilde{\mathbf{p}})$. Hence,

$$\tilde{d_i}\left(\tilde{\mathbf{p}}\right) = \sum_k d_k(\boldsymbol{B}\tilde{\mathbf{p}}) B_{ki}$$

Taking derivatives with respect to $\tilde{p}_j$ gives the following elements of the demand Jacobian:

$$\frac{\partial \tilde{d_i}}{\partial \tilde{p}_j} = \sum_{kl} \frac{\partial d_k}{\partial p_l} B_{ki} B_{lj}.$$

This congruence transformation proves the second part of the Lemma. The derivation for the supply Jacobian is analogous. □

**Proof of Proposition 1: Equilibrium conditions are preserved under bundle transformations.**

*Proof.* The proof follows from the strong equivalence of the game in bundles and good space. In Section 3, we defined the following bijections:

$$\mathbf{q} = \boldsymbol{A}\widetilde{\mathbf{q}} \quad \widetilde{\mathbf{q}} = \boldsymbol{B}^T\mathbf{q}$$

$$\boldsymbol{\varepsilon} = \boldsymbol{A}\widetilde{\boldsymbol{\varepsilon}} \quad \widetilde{\boldsymbol{\varepsilon}} = \boldsymbol{B}^T\boldsymbol{\varepsilon}$$

$$c_n\left(\mathbf{q}\right) = \widetilde{c}_n\left(\boldsymbol{B}^T\mathbf{q}\right) \quad \widetilde{c}_n\left(\widetilde{\mathbf{q}}\right) = c_n\left(\boldsymbol{A}\widetilde{\mathbf{q}}\right)$$

$$\mathbf{p} = \boldsymbol{B}\widetilde{\mathbf{p}} \quad \widetilde{\mathbf{p}} = \mathbf{A}^\mathbf{T}\mathbf{p}$$

$$\mathbf{s}_n(\mathbf{p}) = \boldsymbol{A}\tilde{\mathbf{s}}_n(\boldsymbol{A}^\top\mathbf{p}) \quad \tilde{\mathbf{s}}_n(\tilde{\mathbf{p}}) = \boldsymbol{B}^T\mathbf{s}_n(\boldsymbol{B}\tilde{\mathbf{p}})$$
$$\mathbf{d}(\mathbf{p}) = \boldsymbol{A}\tilde{\mathbf{d}}(\boldsymbol{A}^\top\mathbf{p}) \quad \tilde{\mathbf{d}}(\tilde{\mathbf{p}}) = \boldsymbol{B}^T\mathbf{d}(\boldsymbol{B}\tilde{\mathbf{p}}),$$

where $\boldsymbol{B}^T = \boldsymbol{A}^{-1}$. Note that we consider non-singular bundling matrices. From the bijections above, we realize (by left-multiplying the market clearing condition for goods by $\boldsymbol{A}^{-1}$) that the clearing conditions for the two markets are equivalent

$$\mathbf{d}(\mathbf{p}) + \boldsymbol{\varepsilon} = \sum_{n \in N} \mathbf{s}_n(\mathbf{p}) \quad \Longleftrightarrow \quad \widetilde{d}(\tilde{\mathbf{p}}) + \widetilde{\boldsymbol{\varepsilon}} = \sum_{n \in N} \widetilde{\mathbf{s}}_n(\tilde{\mathbf{p}}).$$

One implication of this is that payoffs will be the same in the market for goods and bundles for corresponding shocks, if producers choose equivalent supply function strategies in the two markets.

$$\mathbf{p}^T\mathbf{s}_n(\mathbf{p}) - c_n(\mathbf{s}_n(\mathbf{p})) = (\boldsymbol{B}\widetilde{\mathbf{p}})^T \boldsymbol{A}\tilde{\mathbf{s}}_n(\tilde{\mathbf{p}}) - c_n(\boldsymbol{A}\tilde{\mathbf{s}}_n(\tilde{\mathbf{p}})) = \widetilde{\mathbf{p}}^T\tilde{\mathbf{s}}_n(\tilde{\mathbf{p}}) - \widetilde{c}_n(\widetilde{\mathbf{q}}),$$

because $\boldsymbol{B}^T\boldsymbol{A} = \boldsymbol{I}$. Hence, if a set of supply functions for goods is a Nash equilibrium, then the corresponding set of supply functions will be a Nash equilibrium for the bundles, and vice versa. □

**Proof of Lemma 2: Existence of a basis that jointly diagonalizes cost Hessian and demand Jacobian.**

*Proof.* The matrix $\boldsymbol{C}$ is positive-definite and $\boldsymbol{D}^{-1}$ is symmetric, so by a fundamental result in the theory of quadratic forms, found in e.g. Theil (1983; Chapter 1), there exists a matrix $\boldsymbol{A}$ that congruently diagonalizes both $\boldsymbol{D}^{-1}$ and $C$. That is,

$$\boldsymbol{A}^\top\boldsymbol{D}^{-1}\boldsymbol{A} \quad \text{and} \quad \boldsymbol{A}^\top C\boldsymbol{A}$$

are both diagonal. Next, we want to show that this diagonalization is unique up to scalar multiplication and permutations. Let $\boldsymbol{A}$ be some matrix that congruently diagonalizes both $\boldsymbol{D}^{-1}$ and $\boldsymbol{C}$ as above. Congruence transformations preserve definiteness. Hence, we can invert the first term above and multiply by the latter, which gives another diagonal matrix

$$\boldsymbol{A}^{-1}\boldsymbol{D}\left(\boldsymbol{A}^\top\right)^{-1}\boldsymbol{A}^\top C\boldsymbol{A} = \boldsymbol{A}^{-1}\boldsymbol{D}C\boldsymbol{A} = \boldsymbol{\Lambda}.$$

Multiplying this with $\boldsymbol{A}$ on the left gives $\boldsymbol{D}C\boldsymbol{A} = \boldsymbol{A}\boldsymbol{\Lambda}$, so the columns of $\boldsymbol{A}$ are eigenvectors of $\boldsymbol{D}C$. The eigenvalues of $\boldsymbol{D}C$ are all distinct (Assumption 1.c), its eigenvectors are unique up to scalar multiplication and permutations. Hence, we realize that the bundling that jointly diagonalizes $C$ and $\boldsymbol{D}$ must also be unique up to scalar multiplication and permutations. □

**Proof of Lemma 3: Characterization of first- and second-order optimality conditions**

*Proof.* For a given supply function $s_n(\mathbf{p})$ and for the well-behaved residual demand in (6), there is a bijective mapping between the demand shocks $\varepsilon$ and the clearing prices $\mathbf{p}$ realized in the market. This mapping is determined implicitly from the market-clearing condition

$$s_n(\mathbf{p}) = \mathbf{d}_n(\mathbf{p}) + \varepsilon.$$

Supplier $n$ maximizes its expected profit in equation 3. Because of the one-to-one correspondence between shocks and prices, this expectation can be maximized ex-post by finding the price that maximizes profit for each outcome of the shock. Differentiate $\pi_n(\mathbf{p}, \mathbf{d}_n(\mathbf{p}) + \varepsilon)$, as defined in (1), with respect to $\mathbf{p}$ to obtain the vector-valued first-order condition (7).

Consider the function

$$g(t) = \pi_n\left(\mathbf{p}^* + t\left(\mathbf{p} - \mathbf{p}^*\right), \mathbf{d}_n\left(\mathbf{p}^* + t\left(\mathbf{p} - \mathbf{p}^*\right), \varepsilon^*\right)\right),$$

where $\varepsilon^*$is the shock vector that gives the price vector $\mathbf{p}^*$ for the supply function $s_n(\mathbf{p})$. A sufficient condition for $g(t)$ to obtain a maximum at $t = 0$, and to ensure that the payoff is higher for $\mathbf{p}^*$than for $\mathbf{p}$, is that $g'(t) \cdot t \leq 0$. Applying the chain rule yields the condition (9). □

**Proof of Proposition 2: Uniqueness and strategic separation of Symmetric SFE Jacobian**

*Proof.* The bid functions $\mathbf{s}(\mathbf{p})$ are determined by the system of first-order conditions. To simplify notation we will use subscript $,i$ to denote partial derivatives with respect to $p_i$ or $q_i$ for the rest of the proof. The supply function for good $i$ has to satisfy the first-order condition:

$$s_i = \sum_k (p_k - c_{,k})\left((N-1)\, s_{k,i} - d_{k,i}\right).$$

This condition is valid for all values of $\mathbf{p}$ along the supply function $\mathbf{s}(\mathbf{p})$, and it follows from Assumptions 1 and 2 that the condition is continuously differentiable. Consequently, the derivatives are also identities. We differentiate the identity with respect to

$p_j$:

$$s_{i,j} = (N-1)\, s_{j,i} - d_{j,i} - (N-1) \sum_{kl} c_{,kl} s_{l,j} s_{k,i} + \sum_{kl} c_{,kl} s_{l,j} d_{k,i} + \sum_{k} (p_k - c_{,k}) \left( (N-1)\, s_{k,ij} - d_{k,ij} \right).$$

The derivative $s_{j,i}$ can be found by interchanging the indices $i$ and $j$.

$$s_{j,i} = (N-1)\, s_{i,j} - d_{i,j} - (N-1) \sum_{kl} c_{,kl} s_{l,i} s_{k,j} + \sum_{kl} c_{,kl} s_{l,i} d_{k,j} + \sum_{k} (p_k - c_{,k}) \left( (N-1)\, s_{k,ji} - d_{k,ji} \right).$$

It follows from Assumptions 1 and 2 that the cost Hessian as well as the demand and supply Jacobians are symmetric. Hence, the two expressions above should be equal and that the equality can be simplified to:

$$\sum_{kl} c_{,kl} s_{l,i} d_{k,j} = \sum_{kl} c_{,kl} s_{l,j} d_{k,i}$$

This expression needs to be satisfied for all combinations of $(i, j)$. In matrix format, this expression requires that the matrix $\boldsymbol{DCS}$ is symmetric:

$$\boldsymbol{DCS} = \boldsymbol{SCD}. \tag{18}$$

The matrix $\boldsymbol{DCS}$ is symmetric if and only if the matrices $\boldsymbol{DC}$ and $\boldsymbol{SC}$ commute. That is:

$$(\boldsymbol{DC})(\boldsymbol{SC}) = (\boldsymbol{SC})(\boldsymbol{DC})$$

This follows directly from right-multiplying equation 18 by the positive definite matrix $\boldsymbol{C}$. The commutative property of matrix $\boldsymbol{SC}$ with the matrix $\boldsymbol{DC}$, where the latter has distinct eigenvalues, implies that they share the same eigenvector space:

$$\boldsymbol{A} = \text{eigenvector}(\boldsymbol{DC}) = \text{eigenvector}(\boldsymbol{SC})$$

We have not shown that $\boldsymbol{SC}$ has distinct eigenvalues. Still, we can use part of the proof in Lemma 2 to show that $\boldsymbol{S}$ and $\boldsymbol{C}$ are jointly diagonalized by the same vectors as $\boldsymbol{D}$ and $\boldsymbol{C}$, which correspond to the bundling matrix $\boldsymbol{A}$.

Hence, for separating bundles and given price and quantity vectors $\tilde{\mathbf{p}}^{\mathbf{0}}$ and $\tilde{\mathbf{q}}^{\mathbf{0}}$, the supply Jacobian can be uniquely determined from the following equations.

$$\tilde{q}_i^0 = \sum_k \left(\tilde{p}_k^0 - \tilde{c}_{,k}(\tilde{\mathbf{q}}^0)\right)\left((N-1)\,\tilde{s}_{k,i}(\mathbf{p}^0) - \tilde{d}_{k,i}(\tilde{\mathbf{p}}^0)\right) \quad \forall i$$

$$\tilde{s}_{i,j}(\mathbf{p}^0) = \tilde{s}_{j,i}(\mathbf{p}^0) = 0 \quad \forall i \neq j$$

□

**Proof of Lemma 4: Linear supply function is an SFE if and only if Riccati equation holds.**

*Proof.* **Existence:** For linear supply schedules, (PDE) simplifies to the Riccati equation in (11). Hence, it is a necessary condition for linear SFE. Given Proposition 1 and Corollary 1 we may, without loss of generality, take $\boldsymbol{C}$ and $\boldsymbol{D}$ to be diagonal, i.e. we consider separating bundles. For this bundling, it follows from the Corollary that also the supply Jacobian $\boldsymbol{S}$ will be diagonalized. Setting the off-diagonal elements of $\boldsymbol{S}$ to zero, we obtain independent quadratic equations from the Riccati equation

$$(N-1)\,c_{ii}s_{ii}^2 - (N-2-c_{ii}d_{ii})\,s_{ii} - d_{ii} = 0 \qquad \text{for } i = 1, \ldots, I. \tag{19}$$

By the quadratic formula this has two roots

$$s_{ii} = \frac{N-2-c_{ii}d_{ii} \pm \sqrt{(N-2-c_{ii}d_{ii})^2 + 4\,(N-1)\,c_{ii}d_{ii}}}{2\,(N-1)\,c_{ii}},$$

only one of which is positive. Taking the positive root for each good gives an $\boldsymbol{S}$ with positive diagonals. This is the only positive-definite solution of the Riccati equation. The solution satisfies the first-order condition. Hence, it follows from KM and Corollary 2 that it constitutes an SFE, a linear SFE. Also, it follows from KM and Corollary 2 that the linear SFE is the unique SFE for unbounded demand shocks.

□

**Proof of Proposition 3: Expected welfare losses and decomposition**

*Proof.* When the auction clears with total output $\mathbf{q}$, the net consumer surplus is

$$\frac{1}{2}\mathbf{q}^\top \boldsymbol{D}^{-1}\mathbf{q} \tag{20}$$

and the net total profit is

$$\mathbf{p}^{\top}\mathbf{q} - \frac{1}{2N}\mathbf{q}^{\top}\boldsymbol{C}\mathbf{q} = \frac{1}{2}\mathbf{q}^{\top}\left(2\boldsymbol{S}^{-1}/N - \boldsymbol{C}/N\right)\mathbf{q}, \tag{21}$$

since $\mathbf{q} = N\boldsymbol{S}\mathbf{p}$. Adding (20) gives total welfare

$$\frac{1}{2}\mathbf{q}^{\top}\left(\boldsymbol{D}^{-1} + 2\boldsymbol{S}^{-1}/N - \boldsymbol{C}/N\right)\mathbf{q}$$

$$\frac{1}{2}\mathbf{q}^{\top}\left(\boldsymbol{I} + 2\boldsymbol{S}^{-1}\boldsymbol{D}/N - \boldsymbol{C}\boldsymbol{D}/N\right)\boldsymbol{D}^{-1}\mathbf{q} = \frac{1}{2}\mathbf{q}^{\top}\left(2\boldsymbol{\rho}^{-1} - \boldsymbol{\rho}_0^{-1}\right)\boldsymbol{D}^{-1}\mathbf{q}.$$

Output is a linear function of the demand shock

$$\mathbf{q} = N\boldsymbol{S}\left(\boldsymbol{D} + N\boldsymbol{S}\right)^{-1}\varepsilon = \rho^{T}\varepsilon.$$

It is straightforward to show that $\boldsymbol{D}^{-1}\rho^{T} = \rho\boldsymbol{D}^{-1}$, so total welfare can be written

$$\frac{1}{2}\varepsilon^{\top}\boldsymbol{\rho}\left(2\boldsymbol{\rho}^{-1} - \boldsymbol{\rho}_0^{-1}\right)\boldsymbol{\rho}\boldsymbol{D}^{-1}\varepsilon$$

$$\frac{1}{2}\varepsilon^{\top}\left(2\boldsymbol{\rho}\boldsymbol{\rho}_0^{-1} - \boldsymbol{\rho}\boldsymbol{\rho}_0^{-1}\boldsymbol{\rho}\boldsymbol{\rho}_0^{-1}\right)\boldsymbol{\rho}_0\boldsymbol{D}^{-1}\varepsilon$$

$$\varepsilon^{\top}\left(2\boldsymbol{R} - \boldsymbol{R}^2\right)\boldsymbol{W}_0\varepsilon$$

$$\varepsilon^{\top}\boldsymbol{W}_0\varepsilon - \varepsilon^{\top}\left(\boldsymbol{I} - \boldsymbol{R}\right)^2\boldsymbol{W}_0\varepsilon.$$

The first term corresponds to competitive welfare, for which $\boldsymbol{R} = \boldsymbol{I}$, when the demand shock is $\varepsilon$. The second term is the welfare loss. The expressions for expected welfare and welfare losses follows from the expectation formula for quadratic forms (Magnus and Neudecker, 2019).

The decomposition of the welfare loss can be explained as follows. The trace of a product of matrices remains unchanged under cyclic permutations of the matrices. Hence,

$$\mathbb{E}[w_L] = \text{tr}\left[(\boldsymbol{I} - \boldsymbol{R})^2\boldsymbol{M}\right] = \text{tr}\left[(\boldsymbol{I} - \boldsymbol{R})\boldsymbol{M}(\boldsymbol{I} - \boldsymbol{R})\right]$$

Let

$$\boldsymbol{I} - \boldsymbol{R} = \boldsymbol{U} - \boldsymbol{V},$$

where

$$\boldsymbol{U} = (1 - \bar{r})\,\boldsymbol{I}$$

and

$$\boldsymbol{V} = \boldsymbol{R} - \bar{r}\boldsymbol{I}.$$

Hence,

$$(\boldsymbol{I} - \boldsymbol{R})\boldsymbol{M}(\boldsymbol{I} - \boldsymbol{R}) = \boldsymbol{UMU} - \boldsymbol{UMV} - \boldsymbol{VMU} + \boldsymbol{VMV}.$$

Trace remains unchanged under transpose. Hence,

$$\text{tr}\left[(\boldsymbol{I} - \boldsymbol{R})\boldsymbol{M}(\boldsymbol{I} - \boldsymbol{R})\right] = \text{tr}\left[\boldsymbol{UMU} - 2\boldsymbol{UMV} + \boldsymbol{VMV}\right].$$

The decomposition result now follows as $\text{tr}\,(\boldsymbol{UMU}) = (1 - \bar{r})^2\,\text{tr}\,(\boldsymbol{M})$and $\text{tr}\,(\boldsymbol{VMV}) = \text{tr}\left[(\boldsymbol{R} - \bar{r}\boldsymbol{I})\boldsymbol{M}(\boldsymbol{R} - \bar{r}\boldsymbol{I})\right]$. Moreover, it can be shown that the trace of the cross term is zero.

$$\text{tr}\,(\boldsymbol{UMV}) = (1 - \bar{r})\,\text{tr}\,(\boldsymbol{MV}) = (1 - \bar{r})\,[\text{tr}\,(\boldsymbol{MR}) - \bar{r}\,\text{tr}\,(\boldsymbol{M})] = 0,$$

which gives the decomposed welfare loss. □

**Lemma 5: Transformation properties of matrices.**

*Proof.* Using the definition of the Lerner matrix in bundle coordinates, the transformation properties of the matrices $\boldsymbol{C}$ and $\boldsymbol{S}$, and the relationship $\boldsymbol{AB}^T = \boldsymbol{I}$, we can easily see that:

$$\begin{aligned}
\tilde{\boldsymbol{L}} &= \boldsymbol{I} - \tilde{\boldsymbol{C}}\tilde{\boldsymbol{S}} \\
&= \boldsymbol{I} - (\boldsymbol{A}^\top \boldsymbol{CA})(\boldsymbol{B}^\top \boldsymbol{SB}) \\
&= \boldsymbol{I} - \boldsymbol{B}^{-1}\boldsymbol{CSB} \\
&= \boldsymbol{B}^{-1}(\boldsymbol{I} - \boldsymbol{CS})\boldsymbol{B} \\
&= \boldsymbol{B}^{-1}\boldsymbol{LB}
\end{aligned}$$

With the separating bundles, the matrices $\tilde{\boldsymbol{S}}$ and $\tilde{\boldsymbol{C}}$ are diagonal matrices. As the product of two diagonal matrices is diagonal, the matrix $\tilde{\boldsymbol{L}}$ is diagonal:

$$\begin{aligned}
\tilde{\boldsymbol{L}} &= \boldsymbol{I} - \tilde{\boldsymbol{C}}\tilde{\boldsymbol{S}} \\
&= \boldsymbol{I} - \boldsymbol{\Lambda}_C\boldsymbol{\Lambda}_S \\
&= \boldsymbol{\Lambda}_L
\end{aligned}$$

Hence, Lerner matrix in bundle coordinates is given by the similarity transformation

$$\boldsymbol{\Lambda}_L = \boldsymbol{B}^{-1}\boldsymbol{LB}$$

Hence, the columns of the matrix $\boldsymbol{B}$, which correspond to the separating bundles, are eigenvectors of the Lerner matrix $\boldsymbol{L}$. The diagonal elements of $\boldsymbol{\Lambda}_L$ correspond to the eigenvalues of the Lerner matrix:

$$\boldsymbol{L}\boldsymbol{B} = \boldsymbol{B}\boldsymbol{\Lambda}_L$$

Eigenvalues are invariant to a similarity transformation (Horn and Johnson, 2013). Hence, the eigenvalues do not change with bundling. Neither does the trace of a matrix that transforms with a similarity transformation change with bundling (Horn and Johnson, 2013). The argument is analogous for the matrices $\boldsymbol{\rho}$, $\boldsymbol{\rho}_0$, $\boldsymbol{R}$, and the inverses of these matrices. The columns of the matrix $\boldsymbol{B}$ are eigenvectors of these matrices (and their inverses) as well. Next, we show that $\boldsymbol{L}$ and $\boldsymbol{\rho}^{-1}$ commute, by using the fact that they have the same set of eigenvectors (i.e. are jointly diagonalizable). The commutative property of matrix multiplication follows from having identical eigenvector spaces.

$$\begin{aligned}
\boldsymbol{L}\boldsymbol{\rho}^{-1} &= \boldsymbol{B}\boldsymbol{\Lambda}_L\boldsymbol{B}^{-1}\boldsymbol{B}\boldsymbol{\Lambda}_\rho^{-1}\boldsymbol{B}^{-1} \\
&= \boldsymbol{B}\boldsymbol{\Lambda}_L\boldsymbol{\Lambda}_\rho^{-1}\boldsymbol{B}^{-1} \\
&= \boldsymbol{B}\boldsymbol{\Lambda}_\rho^{-1}\boldsymbol{\Lambda}_L\boldsymbol{B}^{-1} \\
&= \boldsymbol{B}\boldsymbol{\Lambda}_\rho^{-1}\boldsymbol{B}^{-1}\boldsymbol{B}\boldsymbol{\Lambda}_L\boldsymbol{B}^{-1} \\
&= \boldsymbol{\rho}^{-1}\boldsymbol{L}.
\end{aligned}$$

The same argument can be made for any two matrices that have the same set of eigenvectors.

It follows from (15) that $\boldsymbol{W_0} = (\boldsymbol{D} + \frac{1}{N}\boldsymbol{DCD})^{-1}/2$. By assumption, $\boldsymbol{C}$ is a symmetric and positive definite matrix, so $\boldsymbol{DCD}$ is a congruence transformation, which preserves the symmetry and positive definiteness properties of $\boldsymbol{C}$. We also have that $\boldsymbol{D}$ is symmetric and positive definite. Addition and the inverse also preserve these properties, so $\boldsymbol{W_0}$ inherits them. The matrices $\boldsymbol{C}$ and $\boldsymbol{D}$ are diagonalized by the separating bundles. Hence, $\boldsymbol{W}_0$ will also be diagonalized by these bundles. We can use Lemma 1 to show that the matrix $\boldsymbol{W_0}$transforms with a congruence transformation.

$$\widetilde{\boldsymbol{W_0}} = (\boldsymbol{B^T}\boldsymbol{DB} + \tfrac{1}{N}\boldsymbol{B}^T\boldsymbol{DBA}^T\boldsymbol{CAB}^T\boldsymbol{DB})^{-1}/2$$

$$\widetilde{\boldsymbol{W_0}} = (\boldsymbol{B^T}\boldsymbol{DB} + \tfrac{1}{N}\boldsymbol{B}^T\boldsymbol{DCDB})^{-1}/2$$

$$\widetilde{\boldsymbol{W_0}} = \left(\boldsymbol{B^T}(\boldsymbol{D} + \tfrac{1}{N}\boldsymbol{DCD})\boldsymbol{B}\right)^{-1}/2$$

$$\widetilde{W}_0 = A^T \left(D + \tfrac{1}{N} DCD\right)^{-1} A/2 = A^T W_0 A.$$

The matrix $M$ transforms by a similarity transformation.

$$\widetilde{M} = A^T W_0 A \left(A^{-1} \Sigma \left(A^{-1}\right)^T + A^{-1-T} \left(A^{-1}\right)^T\right)$$

$$\widetilde{M} = B^{-1} W_0 \left(\Sigma + ^{-T}\right) B = B^{-1} M B.$$

Hence, eigenvalues and the trace of $M$ are invariant to bundling. Still, $M$ has different properties from $L$, $\rho$, $\rho_0$ and $R$. Normally, $M$ and $\Sigma + ^{-T}$ are not diagonalized by separating bundles. Hence, the eigenvectors of $M$ will generally differ from the separating bundles, so that $M$ does not commute with $L$, $\rho$, $\rho_0$ and $R$. □

**Proof of Proposition 4: Comparative statics.**

- The algebraic Riccati equation which determines $S$ is given by:

  $$S - (D + (N-1)S)(I - CS) = 0$$

  Left multiplying this equation by $C$ gives:

  $$CS - (CD + (N-1)CS)(I - CS) = 0$$

  We now use the definition of $L = I - CS$ and $\rho_0^{-1} = I + \frac{1}{N} CD$ to find the quadratic equation in (16). We show below that there is a unique solution with eigenvalues smaller than 1.

- The relation in (17) can be written as:

  $$\rho_0^{-1} - L = (I - L)\rho^{-1}.$$

  Using the definitions $L$,$\rho$ and $\rho_0$, we find

  $$CS + \frac{1}{N} CD = CS(I + \frac{1}{N} S^{-1} D),$$

  which can easily be multiplied out to find an identity.

- Claims 2-4 are proved further down. Next we prove claims 5-9. In these proofs, we will use the result in Lemma 5 that eigenvalues of the Lerner and pass-through matrix do not depend on bundling. Hence, we can, without loss of generality, calculate eigenvalues for separating bundles. Our first result is that the eigenvalues of $\boldsymbol{\rho}_0$ are positive and less than one. This follows from the definitions of the matrices that link them to the eigenvalues of $\boldsymbol{C}$ and $\boldsymbol{D}$ for separated markets:

$$\rho_{0,\lambda} = \frac{1}{1 + \frac{1}{N} C_\lambda D_\lambda}$$

The eigenvalues of $\boldsymbol{L}$ are given by:

$$L_\lambda = 1 - C_\lambda S_\lambda$$

which is obviously smaller than 1 as both $C_\lambda$ and $S_\lambda$ are positive. This follows from the assumption that the cost Hessian and supply Jacobian are positive definite.

Considering separated markets, we realize that the first-order condition in (16) can be written as follows:

$$(N-1) L_\lambda^2 - N \frac{L_\lambda}{\rho_{0,\lambda}} + 1 = 0.$$

Any root of this expression must be positive $L_\lambda \geq 0$, as the first and third term of the sum are positive. This proves part of claim 6, and that $C_\lambda S_\lambda \leq 1$. The quadratic equation has two positive roots, one root, $\lambda_L^+$, which is smaller than 1 and one root, $L_\lambda^-$, which is larger than 1, and is irrelevant.

$$\begin{aligned} L_\lambda^+ &= \frac{\rho_{0,\lambda}}{N} \frac{2}{\left(1 + \sqrt{1 - 4(N-1)\left(\frac{\rho_{0,\lambda}}{N}\right)^2}\right)} < 1 \\ L_\lambda^- &= \frac{\rho_{0,\lambda}}{N} \frac{2}{\left(1 - \sqrt{1 - 4(N-1)\left(\frac{\rho_{0,\lambda}}{N}\right)^2}\right)} > 1 \end{aligned} \tag{22}$$

We realize that $L_\lambda^- > 1$. Otherwise, we must have that

$$\sqrt{N^2 - 4(N-1)\rho_{0,\lambda}{}^2} < N - 2\rho_{0,\lambda}.$$

Squaring both sides of this expression and simplifying gives us the condition $\rho_{0,\lambda}(1 - \rho_{0,\lambda}) < 0$, which is never satisfied as $0 < \rho_{0,\lambda} < 1$.

For $L_\lambda^+ < 1$ we must have that:

$$\sqrt{N^2 - 4\,(N-1)\,\rho_{0,\lambda}{}^2} > 2\rho_{0,\lambda} - N.$$

Squaring both sides and simplifying gives us the condition $\rho_{0,\lambda}(1-\rho_{0,\lambda}) > 0$, which is always satisfied. This is the stable solution in claim 1.

- The eigenvalues of $\rho$ are positive and less than the corresponding eigenvalues of $\rho_0$. This follows from that $C_\lambda S_\lambda \leq 1$.

  $$\rho_\lambda = \frac{1}{1+\frac{1}{N}\frac{D_\lambda}{S_\lambda}} \leq \frac{1}{1+\frac{1}{N}C_\lambda D_\lambda} = \rho_{0,\lambda} \leq 1.$$

  This partly proves claim 5.

- It follows from (22) that the derivative of the Lerner eigenvalue (for constant $\rho_{0,\lambda}$) is given by:

  $$\frac{\partial L_\lambda}{\partial N} = \frac{N - 2(N-1)\rho_{0,\lambda}^2 - \sqrt{N^2 - 4(N-1)\rho_{0,\lambda}^2}}{\rho_{0,\lambda}(N-1)^2\sqrt{N^2-4(N-1)\rho_{0,\lambda}^2}}$$

  The nominator is positive, so the slope depends on the sign of the denominator. So $\frac{\partial L_\lambda}{\partial N} < 0$ if

  $$N - 2(N-1)\rho_{0,\lambda}^2 < \sqrt{N^2-4(N-1)\rho_{0,\lambda}^2}$$

  Squaring both sides gives

  $$N^2 - 4N(N-1)\rho_{0,\lambda}^2 + 4(N-1)^2\rho_{0,\lambda}^4 < N^2 - 4(N-1)\rho_{0,\lambda}^2$$

  Collecting terms gives:

  $$\rho_{0,\lambda}^2(\rho_{0,\lambda}^2 - 1) < 0$$

  which is always satisfied. This proves the first part of claim 7.

- It follows from (17) that

  $$\rho_\lambda = (1-L_\lambda)/(1/\rho_{0,\lambda} - L_\lambda) = 1 - \frac{1/\rho_{0,\lambda} - 1}{1/\rho_{0,\lambda} - L_\lambda}.$$

  Above, we showed that $\frac{\partial L_\lambda}{\partial N} < 0$. From this follows that $\frac{\partial \rho_\lambda}{\partial N} > 0$ and that $\frac{\partial r_\lambda}{\partial N} = \frac{\partial(\rho_\lambda/\rho_{0,\lambda})}{\partial N} > 0$, which proves the remainder of claim 7.

- It follows from (22) that the derivative of the Lerner eigenvalue with respect to

$\rho_{0,\lambda}$for constant $N$ is:

$$\frac{\partial L_\lambda}{\partial \rho_{0,\lambda}} = \frac{-N + \frac{N^2}{\sqrt{N^2-4(N-1)\rho_{0,\lambda}^2}}}{2(N-1)\rho_{0,\lambda}^2}$$

The nominator is positive, so the derivative is $\partial L_\lambda / \partial \rho_{0,\lambda} > 0$ if the denominator is positive. This condition can be rewritten as:

$$N > \sqrt{N^2 - 4(N-1)\rho_{0,\lambda}^2}$$

Squaring both sides and simplifying shows this is the case. This proves parts of claim 8. We have $\partial L_\lambda / \partial \rho_{0,\lambda} > 0$, so to find an upper bound for the Lerner eigenvalue $L_\lambda$, for $N \geq 2$, we can take the limit of $\rho_{0,\lambda} \to 1$ :

$$L_\lambda = \frac{2}{N + \sqrt{N^2 - 4\,(N-1)}} = \frac{1}{N-1},$$

which corresponds to claim 6.

- To complete claim 8, we need to show that $\rho_\lambda$ is an increasing function of $\rho_{0,\lambda}$. The derivative of $\rho_\lambda$with respect to $\rho_{0,\lambda}$for constant $N$ is:

$$\frac{\partial \rho_\lambda}{\partial \rho_{0,\lambda}} = \frac{\left(2 + \frac{4\rho_{0,\lambda}(N-1)}{\sqrt{N^2-4\rho_{0,\lambda}^2(N-1)}}\right)(2+2\rho_{0,\lambda}) - 2(N + 2\rho_{0,\lambda} - \sqrt{N^2 - 4\rho_{0,\lambda}^2(N-1)})}{(2+2\rho_{0,\lambda})^2}, \tag{23}$$

which can be simplified to

$$\frac{\partial \rho_\lambda}{\partial \rho_{0,\lambda}} = 2\frac{4\rho_{0,\lambda}(N-1) - (N-2)\sqrt{N^2 - 4\rho_{0,\lambda}^2(N-1)} + N^2}{\sqrt{N^2 - 4\rho_{0,\lambda}^2(N-1)}(2+2\rho_{0,\lambda})^2}.$$

The expression is positive as $\left(N^2 + 4\rho_{0,\lambda}(N-1)\right)^2 = N^4 + 8\rho_{0,\lambda}(N-1)N^2 + 16\rho_{0,\lambda}^2(N-1)^2$ is strictly larger than $(N-2)^2\left(N^2 - 4\rho_{0,\lambda}^2(N-1)\right)$.

- For constant $N$,we get

$$r_\lambda = \frac{\rho_\lambda}{\rho_{0,\lambda}} = \frac{1 - L_\lambda}{1 - \rho_{0,\lambda} L_\lambda}. \tag{24}$$

Hence,

$$\frac{\partial r_\lambda}{\partial \rho_{0,\lambda}} = \frac{\frac{\partial L_\lambda}{\partial \rho_{0,\lambda}}\left(\rho_{0,\lambda} - 1\right) + \left(1 - L_\lambda\right) L_\lambda}{\left(1 - \rho_{0,\lambda} L_\lambda\right)^2} \tag{25}$$

From the Riccati equation $(N-1)L_\lambda^2 - N\frac{L_\lambda}{\rho_{0,\lambda}} + 1 = 0$,it follows that

$$(N-1)\,2L_\lambda \frac{\partial L_\lambda}{\partial \rho_{0,\lambda}} - N\frac{\frac{\partial L_\lambda}{\partial \rho_{0,\lambda}}\rho_{0,\lambda} - L_\lambda}{\rho_{0,\lambda}^2} = 0,$$

which gives

$$\frac{\partial L_\lambda}{\partial \rho_{0,\lambda}} = \frac{NL_\lambda}{\rho_{0,\lambda}\left(N - 2(N-1)L_\lambda \rho_{0,\lambda}\right)}.$$

Hence, (25) can be rewritten as follows:

$$\frac{\partial r_\lambda}{\partial \rho_{0,\lambda}} = L_\lambda \frac{N(\rho_{0,\lambda}-1) + (1-L_\lambda)\rho_{0,\lambda}\left(N-2(N-1)L_\lambda\rho_{0,\lambda}\right)}{\rho_{0,\lambda}\left(N-2(N-1)L_\lambda\rho_{0,\lambda}\right)(1-\rho_{0,\lambda}L_\lambda)^2}. \tag{26}$$

From the Riccati equation in (16), we get.

$$\rho_{0,\lambda} = \frac{NL_\lambda}{(N-1)L_\lambda^2 + 1}. \tag{27}$$

Partially substituting this equality into (26) as well as simplifications and factorization gives

$$\frac{\partial r_\lambda}{\partial \rho_{0,\lambda}} = \rho_{0,\lambda}\frac{(N-1)(L_\lambda-1)^2\left(L_\lambda+1-\sqrt{\frac{N}{N-1}}\right)\left(L_\lambda+1+\sqrt{\frac{N}{N-1}}\right)\left(L_\lambda^2+1/(N-1)\right)}{N^2L_\lambda\left(1/(N-1)-L_\lambda^2\right)(1-\rho_{0,\lambda}L_\lambda)^2}.$$

We have from claim 6 that $L_\lambda \leq 1/(N-1)$ and from claim 8 that $L_\lambda$ is increasing with respect to $\rho_{0,\lambda}$. Hence, $\frac{\partial r_\lambda}{\partial \rho_{0,\lambda}}$ is negative for $L_\lambda \in (0, L_\lambda^*)$ and positive for $L_\lambda \in \left(L_\lambda^*, 1/(N-1)\right)$, where $L_\lambda^* = \sqrt{\frac{N}{N-1}} - 1$. Thus, $r_\lambda$ has a minimum at $L_\lambda^*$. Using (27), we realize that this corresponds to:

$$\rho_{0,\lambda}^* = \frac{N\left(\sqrt{\frac{N}{N-1}}-1\right)}{(N-1)\left(\sqrt{\frac{N}{N-1}}-1\right)^2+1} = \frac{\sqrt{\frac{N}{N-1}}}{2}$$

and

$$r_\lambda = \frac{2-\sqrt{\frac{N}{N-1}}}{1-\frac{\sqrt{\frac{N}{N-1}}}{2}\left(\sqrt{\frac{N}{N-1}}-1\right)} = 2(N-1)\left(\sqrt{\frac{N}{N-1}}-1\right).$$

It follows from the Riccati equation and (24) that $L_\lambda = 0$ and $r_\lambda = 1$ when $\rho_{0,\lambda} \to 0$. Similarly, it follows that $L_\lambda = 1/(N-1)$ and $r_\lambda = 1$ when $\rho_{0,\lambda} \to 1$. This proves claim 9.

- Now, we prove claims 2-4. For the monopoly case in claim 2, we set $N = 1$ in (16) which gives $\boldsymbol{L} = \boldsymbol{\rho}_0$. From (17) we get:

$$\boldsymbol{\rho} = (\boldsymbol{\rho}_0 - \boldsymbol{\rho}_0^{-1})^{-1}(\boldsymbol{\rho}_0 - \boldsymbol{I}).$$

Note that $(\boldsymbol{\rho}_0 - \boldsymbol{\rho}_0^{-1})^{-1} = \boldsymbol{\rho}_0(\boldsymbol{\rho}_0 - \boldsymbol{I})^{-1}(\boldsymbol{\rho}_0 + \boldsymbol{I})^{-1}$, which results in:

$$\boldsymbol{\rho} = \boldsymbol{\rho}_0(\boldsymbol{\rho}_0 + \boldsymbol{I})^{-1}$$

- With $\mu = 1/N$, the Riccati equation can be written as follow:

$$(1-\mu)\,\boldsymbol{L}^2 - \boldsymbol{\rho}_0^{-1}\boldsymbol{L} + \mu\boldsymbol{I} = \boldsymbol{0}.$$

For $\mu = 0$ (many firms in claim 3) we have: $\boldsymbol{L}(\boldsymbol{L} - \boldsymbol{\rho}_0^{-1}) = 0$. This expression has two solutions: $\boldsymbol{L} = 0$ (with zero eigenvalues) and $\boldsymbol{L} = \boldsymbol{\rho}_0^{-1}$ (with eigenvalues larger than 1). The latter solution is irrelevant. Taking the differential of the first-order condition (for a fixed $\boldsymbol{\rho}_0$) and rewriting the expression we find:

$$\frac{\partial \boldsymbol{L}}{\partial \mu} = \left(2(1-\mu)\boldsymbol{L} - \boldsymbol{\rho}_0^{-1}\right)^{-1}(\boldsymbol{L}^2 - \boldsymbol{I})$$

For $\mu \to 0$, we have $\boldsymbol{L} = 0$, and this simplifies to

$$\frac{\partial \boldsymbol{L}}{\partial \mu} = \boldsymbol{\rho}_0$$

Using a Taylor expansion of $\boldsymbol{L}$ around $\mu = 0$, we find:

$$\boldsymbol{L}(\mu) = \boldsymbol{L}(0) + \frac{\partial \boldsymbol{L}(0)}{\partial \mu}\mu + \mathcal{O}\left(\mu^2\right),$$

which establishes the first part of claim 3.

- The pass-through matrix is given by:

$$(\boldsymbol{L} - \boldsymbol{\rho}_0^{-1})\boldsymbol{\rho} = (\boldsymbol{L} - \boldsymbol{I}).$$

For $\mu \to 0$ and $\boldsymbol{L} = \boldsymbol{0}$ we get

$$\boldsymbol{\rho} = \boldsymbol{\rho}_{\boldsymbol{0}}.$$

Taking the full differential of the relationship defining $\boldsymbol{\rho}$ gives:

$$\boldsymbol{\rho} d\boldsymbol{L} + (\boldsymbol{L} - \boldsymbol{\rho}_0^{-1}) d\boldsymbol{\rho} = d\boldsymbol{L}$$

Evaluating this at $\boldsymbol{\mu} \to 0$, $\boldsymbol{L} \to 0$ and $\boldsymbol{\rho} \to \boldsymbol{\rho}_0$ gives:

$$\frac{\partial \boldsymbol{\rho}}{\partial \mu} = \boldsymbol{\rho}_0(\boldsymbol{\rho}_0 - I)\frac{\partial \boldsymbol{L}}{\partial \mu}$$

Given that $\partial \boldsymbol{L}/\partial \mu = \boldsymbol{\rho}_0$, we find the following Taylor expansions of the pass-through matrix:

$$\boldsymbol{\rho}(\mu) = \boldsymbol{\rho}_0 + \boldsymbol{\rho}_0^2(\boldsymbol{\rho}_0 - \boldsymbol{I})\mu + \mathcal{O}\left(\mu^2\right),$$

which proves the second part of claim 3.

- For small $\boldsymbol{\rho}_0$, the first part of claim 4 follows immediately from (16). We get the second part of the claim by implementing the approximation for $\boldsymbol{L}$into (17) and then neglecting second-order terms of $\boldsymbol{\rho}_0$.

**Proof of Proposition 5: Lerner and Pass-Through under Cournot Competition**

*Proof.* In the Cournot setting, producers observe shocks before deciding on their output. We are solving for Cournot equilibria, where the equilibrium output is proportional to the price, $\mathbf{q} = \boldsymbol{S}^C\mathbf{p}$. Still, after shocks have been observed, each producer's output is fixed, it does not depend on the price. Hence, the Riccati equation in (11) simplifies as follows:

$$\boldsymbol{S}^C - \boldsymbol{D}\left(\boldsymbol{I} - \boldsymbol{C}\boldsymbol{S}^C\right) = \mathbf{0}.$$

Left multiplying this equation by $\boldsymbol{C}$ and using definitions of the Lerner and pass-through matrices gives

$$\boldsymbol{I} - \boldsymbol{L}^C - N\left(\boldsymbol{\rho}_0^{-1} - \boldsymbol{I}\right)\boldsymbol{L}^C = \mathbf{0},$$

which establishes the first part of claim 1. Next, we can show from the simplified Riccati equation and the definition of $\boldsymbol{\rho}_0$ in (15) that:

$$\left(\boldsymbol{S}^C\right)^{-1}\boldsymbol{D}/N = (\boldsymbol{I} + \boldsymbol{C}\boldsymbol{D})/N = \boldsymbol{\rho}_0^{-1} - (N-1)\,\boldsymbol{I}/N.$$

Plugging this into the definition of $\boldsymbol{\rho}^C$ in (14) establishes the second part of claim 1.

The first part of Claim 2 follows straightforwardly from claim 1. In the second part, we use that the Taylor series expansion for $(1+\boldsymbol{X})^{-1} = 1 - \boldsymbol{X} + O\left(\|\boldsymbol{X}\|^2\right)$ (Horn and

Johnson (2013).

The expression for $\boldsymbol{L}^C$ in claim 1 also holds for separating bundles, which gives an expression for $L^C_\lambda$. It follows from this expression that $L^C_\lambda$ is increasing with respect to $\rho_{0,\lambda}$. The maximum of $L^C_\lambda$ is reached at $\rho_{0,\lambda} = 1$,which gives $L^C_\lambda = 1$. This proves claim 3.

It follows from the expression for $\boldsymbol{\rho}^C$ in claim 1 that

$$r^C_\lambda = \frac{\rho^C_\lambda}{\rho_{0,\lambda}} = \frac{N}{\rho_{0,\lambda} + N},$$

which establishes claim 4. □

# B Numerical Computations and Visual Illustrations

## B.1 Two-good numerical example with separating bundles

**Example 1. (Simultaneous diagonalization**) Suppose we have a price $\mathbf{p}_0$ at which demand has slope

$$\frac{\partial \mathbf{d}}{\partial \mathbf{p}}(\mathbf{p}_0, \varepsilon) = -\boldsymbol{D} = -\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

and a quantity $\mathbf{q}_0$ at which the cost Hessian is

$$\frac{\partial^2 c}{\partial \mathbf{q}^2}(\mathbf{q}_0) = \boldsymbol{C} = \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}.$$

Then

$$\boldsymbol{DC} = \begin{bmatrix} 1 & 2 \\ 0 & 5 \end{bmatrix}$$

has eigenvalues $\lambda_1 = 1,\ \lambda_2 = 5$ with eigenvectors $[1, 0]^\top$ and $[1, 2]^\top$.[20] Thus

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \qquad \boldsymbol{B} = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}$$

diagonalize $\boldsymbol{C}$ and $\boldsymbol{D}$ to

$$\tilde{\boldsymbol{C}} = \boldsymbol{A}^\top \boldsymbol{C} \boldsymbol{A} = \begin{bmatrix} 2 & 0 \\ 0 & 10 \end{bmatrix} \qquad \text{and} \qquad \tilde{\boldsymbol{D}} = \boldsymbol{B}^\top \boldsymbol{D} \boldsymbol{B} = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

**Example 2. (Unique SFE slope)** Let there be $N = 2$ symmetric suppliers in the market. Suppose that quantity $\mathbf{q}_0 = [4, 6]^\top$ and price $\mathbf{p}_0 = [3, 14]^\top$ in a market with additive demand shocks, where demand has slope

$$\boldsymbol{D} = -\frac{\partial \mathbf{d}}{\partial \mathbf{p}}(\mathbf{p}_0, \varepsilon) = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix},$$

[20] It can be noted that the two eigenvectors are non-orthogonal, so the coordinate transformation from the unbundled market to the market for bundles will be non-orthogonal. The reason for this is that $\boldsymbol{DC}$ is non-symmetric, which is the case unless $\boldsymbol{C}$ and $\boldsymbol{D}$ happen to be congruent matrices, so that $\boldsymbol{DC} = \boldsymbol{CD}$.

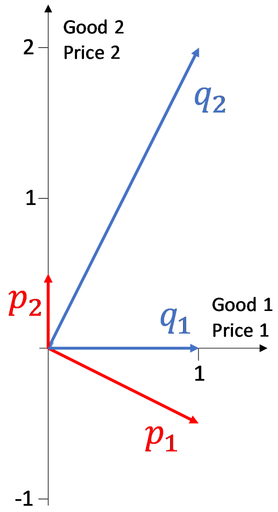



**Figure 4:** Result of Example 1. The columns of matrix $\boldsymbol{A} = [\mathbf{q}_1, \mathbf{q}_2]$ corresponds to the separating bundles and the columns of matrix $\boldsymbol{B} = [\mathbf{p}_1, \mathbf{p}_2]$ corresponds to the corresponding orthogonal price bundles.

and where cost has gradient $\partial c/\partial \mathbf{q} = [4, 15]^\top$ and Hessian

$$\boldsymbol{C} = \frac{\partial^2 c}{\partial \mathbf{q}^2}(\mathbf{q}_0) = \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}.$$

Hence, the bundling matrix $\boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ from Example (1) will separate the market locally. We know from Proposition 2 that there is a unique Jacobian $\partial \mathbf{s}/\partial \mathbf{p}$ that solves the PDE, and that this Jacobian will be diagonalized by the separating bundling matrix $A$. Using $\tilde{\boldsymbol{C}}$ and $\tilde{\boldsymbol{D}}$ from Example (1), and solving (PDE) for separating bundles gives:

$$\frac{\partial \widetilde{\mathbf{s}}}{\partial \widetilde{\mathbf{p}}} = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}.$$

Next, we transform this Jacobian back to the underlying goods. Hence, the unique Jacobian of a symmetric SFE passing through $(\mathbf{q}_0, \mathbf{p}_0)$ is

$$\frac{\partial \mathbf{s}}{\partial \mathbf{p}} = \boldsymbol{A}\frac{\partial \widetilde{\mathbf{s}}}{\partial \widetilde{\mathbf{p}}}\boldsymbol{A}^T = \begin{bmatrix} 3 & 5 \\ 5 & 10 \end{bmatrix}.$$

**(Non-linear SFE in a linear market).** Suppose that the market is linear with two suppliers ($N = 2$). The matrices that define the cost and demand functions, represented by $\boldsymbol{C}$ and $\boldsymbol{D}$ respectively, are in accordance with Example 1. Let us assume that

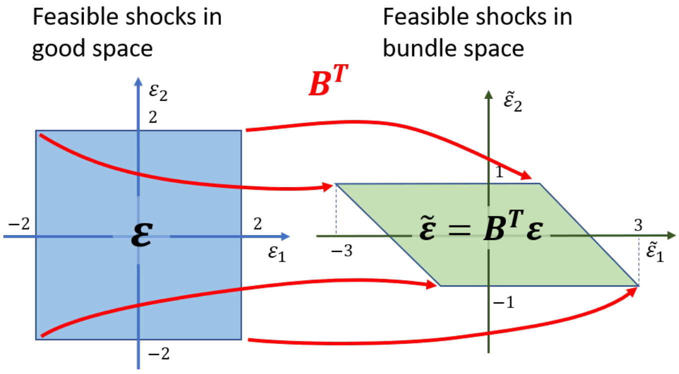



**Figure 5:** Illustration of Example 3: Transformation of demand shock support from good-space into bundle-space.

the demand shock $\varepsilon$ has a range of values within a rectangle, as visualized in Figure 5 denoted as:

$$\mathcal{E} = [-2, 2] \times [-2, 2] .$$

The bundling

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \qquad \boldsymbol{B} = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}$$

separates the markets globally, and provides ranges for the shocks in bundle coordinates

$$\tilde{\varepsilon}_1 \in [-3, 3] , \quad \tilde{\varepsilon}_2 \in [-1, 1] .$$

The transformation of the shock space due to change of basis resulting from bundling is depicted in Figure 5.

**Example 3. (Linear multi-good SFE).** Consider the same market as in Example 1. This gives us

$$\boldsymbol{\rho}_0 = (\boldsymbol{I} + \tfrac{1}{N}\boldsymbol{CD})^{-1} = \begin{bmatrix} 2/3 & 0 \\ -4/21 & 2/7 \end{bmatrix} .$$

The eigenvalues of this matrix are $2/3$ and $2/7$, and they are invariant (see Lemma 5). Hence, for separating bundles we get

$$\tilde{\boldsymbol{\rho}}_0 = \begin{bmatrix} 2/3 & 0 \\ 0 & 2/7 \end{bmatrix} = \begin{bmatrix} \tilde{\rho}_{0,X} & 0 \\ 0 & \tilde{\rho}_{0,Y} \end{bmatrix} .$$

The Riccati equations separate and become two quadratic equations:

$$\tilde{L}_j^2 - \frac{2}{\tilde{\rho}_{0,j}} \cdot \tilde{L}_j + 1 = 0 \quad j = X, Y$$

The relative price cost mark-ups for bundles $X$ and $Y$ are:

$$\tilde{L}_X = \frac{2}{3+\sqrt{5}} \approx 38.2\% \text{ and } \tilde{L}_Y = \frac{2}{7+3\sqrt{5}} \approx 14.6\%.$$

The slopes of the linear SFE for separating bundles are given by

$$\tilde{s}_X = \frac{1-\tilde{L}_X}{\tilde{c}_X} = \frac{-1+\sqrt{5}}{4} \text{ and } \tilde{s}_Y = \frac{1-\tilde{L}_Y}{\tilde{c}_Y} = \frac{-1+\frac{3}{5}\sqrt{5}}{4}.$$

Transforming back to goods gives the Lerner matrix in goods:

$$\boldsymbol{L} = \boldsymbol{B}\begin{bmatrix}\tilde{L}_X & 0\\ 0 & \tilde{L}_Y\end{bmatrix}\boldsymbol{B}^{-1} = \begin{bmatrix}\tilde{L}_X & 0\\ \frac{\tilde{L}_Y-\tilde{L}_X}{2} & \tilde{L}_Y\end{bmatrix} \approx \begin{bmatrix}38.2\% & 0\%\\ 11.8\% & 14.6\%\end{bmatrix}$$

and the supply Jacobian in good coordinates is

$$\boldsymbol{S} = \boldsymbol{A}\begin{bmatrix}\tilde{s}_1 & 0\\ 0 & \tilde{s}_1\end{bmatrix}\boldsymbol{A}^\top = \begin{bmatrix}\frac{-1+\frac{4}{5}\sqrt{5}}{2} & \frac{-1+\frac{3}{5}\sqrt{5}}{2}\\ \frac{-1+\frac{3}{5}\sqrt{5}}{2} & -1+\frac{3}{5}\sqrt{5}\end{bmatrix} \approx \begin{bmatrix}0.39 & 0.17\\ 0.17 & 0.34\end{bmatrix}.$$

The pass-through matrix is:

$$\boldsymbol{\rho} = (\boldsymbol{I}+\tfrac{1}{N}\boldsymbol{S}^{-1}\boldsymbol{D})^{-1} = \begin{bmatrix}1-\frac{\sqrt{5}}{5} & 0\\ \frac{-\sqrt{5}}{15} & 1-\frac{\sqrt{5}}{3}\end{bmatrix} \approx \begin{bmatrix}0.55 & 0\\ -0.15 & 0.25\end{bmatrix}.$$

The pass-through distortion matrix is:

$$\boldsymbol{R} = \boldsymbol{\rho}\boldsymbol{\rho_0^{-1}} = \begin{bmatrix}3/2-3\sqrt{5}/10 & 0\\ 1-13\sqrt{5}/30 & 7/2-7\sqrt{5}/6\end{bmatrix} \approx \begin{bmatrix}0.83 & 0\\ 0.031 & 0.89\end{bmatrix}.$$

Assume that the demand shock has the following mean and covariance:

$$\boldsymbol{\mu} = \mathbb{E}(\boldsymbol{\varepsilon}) = \begin{bmatrix}7 & 10\end{bmatrix}^\top \text{ and } \boldsymbol{\Sigma} = Var(\boldsymbol{\varepsilon}) = \begin{bmatrix}4 & 4\\ 4 & 9\end{bmatrix}$$

From the pass-through matrices we get:

$$\boldsymbol{W}_0 = \boldsymbol{\rho_0}\boldsymbol{D}^{-1}/2 = \begin{bmatrix}2/3 & -1/3\\ -1/3 & 5/21\end{bmatrix}$$

and

$$\boldsymbol{M} = W_0 \left( \boldsymbol{\Sigma} + \boldsymbol{\mu}\boldsymbol{\mu}^T \right) = \begin{bmatrix} 32/3 & 13 \\ -1/21 & 9/7 \end{bmatrix}.$$

It now follows from Proposition 3 that expected welfare for competitive bids would be:

$$\operatorname{tr}(\boldsymbol{M}) = \frac{251}{21} \approx 11.95.$$

The expected welfare loss in the oligopoly market is:

$$\operatorname{tr}\left( (\boldsymbol{I} - \boldsymbol{R})^2 \boldsymbol{M} \right) = \frac{13175}{126} - \frac{140\sqrt{5}}{3} \approx 0.21.$$

## B.2 Illustrating separating bundles as conjugate vectors

Following Armstrong and Vickers (2018), we express net consumer surplus $U(\mathbf{q})$ in terms of quantities, which benefits our analysis of multi-product markets and the intuitive understanding of our results. In the linear setting, net utility can be described as a quadratic form:

$$U(\mathbf{q}) = \frac{1}{2}\mathbf{q}^\top \boldsymbol{D}^{-1}\mathbf{q},$$

where $\mathbf{q}$ represents a vector of cleared quantities. The production costs are also quadratic: $c(\mathbf{q}) = \frac{1}{2}\mathbf{q}^\top \boldsymbol{C}\mathbf{q}$. The iso-cost and iso-net-utility curves form ellipses, which are homothetic with respect to the origin.[21]

The shape and orientation of the iso-cost ellipse vary depending on the nature of the cost interaction between the products, as illustrated in Figure 6. If there are no cost interactions then the ellipse's axes align with the two main axes. In the presence of cost synergies (economies of scope), the major axis of the ellipse lies in the first and third quadrants, while production rivalry (diseconomies of scope) places it in the second and the fourth quadrants. The shape and orientation of the iso-net-utility ellipse, on the other hand, depend on whether goods are complements or substitutes. If product demand is independent, the axes of the iso-net-utility ellipse align with the coordinate system. In the case of complementary goods, the major axis of the net-utility ellipse resides in the first and third quadrants, but for substitute goods the major axis is found in the second and the fourth quadrants.

For two bundles, $\mathbf{q}_i$ and $\mathbf{q}_j$, it follows from the congruence transformation in Lemma 1 that element $i, j$ of matrix $\tilde{\boldsymbol{C}}$ is given by $\tilde{C}_{ij} = \mathbf{q}_i^\top \boldsymbol{C}\mathbf{q}_j$. Hence, the production of bun-

[21]Net utility is homothetic, but not gross utility, $V(\mathbf{q}) = \frac{1}{2}\mathbf{q}^\top \boldsymbol{D}^{-1}\mathbf{q} + \mathbf{q}^\top \boldsymbol{D}^{-1}\left('' - \mathbf{q}\right)$, unless $'' = 0$. In this sense, the demand side of our linear model has similarities to assumptions made in Armstrong and Vickers (2018). They find that homothetic net utility simplifies the solution of Cournot equilibria, even if gross utility is not homothetic.

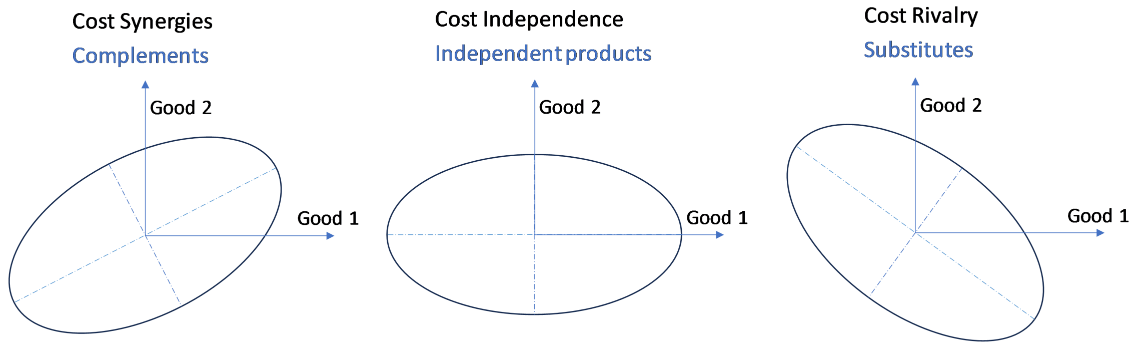



**Figure 6:** The shape and orientation of the iso-cost curve and iso-net consumer surplus depend on the nature of the cost and demand interaction between products.

dles $\mathbf{q}_i$ and $\mathbf{q}_j$ are rivalry when $\mathbf{q}_i^\top \boldsymbol{C}\mathbf{q}_j > 0$, there are cost synergies when the bilateral form is negative $\mathbf{q}_i^\top \boldsymbol{C}\mathbf{q}_j < 0$, and production costs are independent when $\mathbf{q}_i^\top \boldsymbol{C}\mathbf{q}_j = 0$. Often it is possible to choose bundles such that products with cost synergies is changed to new products with cost rivalry, or vice versa. Hence, cost synergy is not invariant to bundling. Analogously, the sign of $\mathbf{q}_i^\top \boldsymbol{D}^{-1}\mathbf{q}_j$ determines whether bundles are complements or substitutes, and these properties can depend on how bundles are chosen.

To diagonalize the cost Hessian $\boldsymbol{C}$ one must identify conjugate vectors $\mathbf{q}_1$ and $\mathbf{q}_2$ with respect to the matrix $\boldsymbol{C}$, such that $\mathbf{q}_2^T \boldsymbol{C}\mathbf{q}_1 = 0$. In the case of a linear model, this condition has a straightforward graphical interpretation, as the cost Hessian $\boldsymbol{C}$ maps a quantity $\mathbf{q}$ on the cost gradient $\boldsymbol{C}\mathbf{q}$. If vector $\mathbf{q}_2$ is conjugate to vector $\mathbf{q}_1$, it will be orthogonal to the marginal cost gradient at $\mathbf{q}_1$, i.e. $\boldsymbol{C}\mathbf{q}_1$. This is illustrated in Figure 6. Alternatively, vector $\mathbf{q}_1$ will be orthogonal to the marginal cost gradient at $\mathbf{q}_2$, i.e. $\boldsymbol{C}\mathbf{q}_2$. Consequently, a pair of conjugate vectors $(\mathbf{q}_1, \mathbf{q}_2)$ corresponds to the concept of conjugate diameters of the iso-cost ellipse $\frac{1}{2}\mathbf{q}^\top \boldsymbol{C}\mathbf{q}$ (Shilov, 1961; McCartin, 2013). Note that for any vector $\mathbf{q}_1$ there exists a corresponding conjugate vector $\mathbf{q}_2$, resulting in infinitely many pairs of conjugate diameters. In general, the conjugate diameters are not orthogonal, $\mathbf{q}_1^\top \mathbf{q}_2 \neq 0$ except for the pair that corresponds to the main axes of the ellipse, which are the eigenvectors of $\boldsymbol{C}$.[22] This situation is depicted on the right side of Figure 7. Similarly, to diagonalize $\boldsymbol{D}^{-1}$, one must find bundles that are conjugate vectors with respect to the matrix $\boldsymbol{D}^{-1}$.

For a pair of vectors $(\mathbf{q}_1, \mathbf{q}_2)$ to diagonalize both matrices$\boldsymbol{C}$ and $\boldsymbol{D}^{-1}$, the two vectors must be conjugate with respect to both $\boldsymbol{C}$ and $\boldsymbol{D}^{-1}$. In other words, vector $\mathbf{q}_2$ must be orthogonal to both gradients at $\mathbf{q}_1$, $\boldsymbol{C}\mathbf{q}_1$ and $\boldsymbol{D}^{-1}\mathbf{q}_1$. This implies that those gradients are aligned and equal up to a scalar

[22]The exception to this is when the ellipse is degenerate and becomes a circle, occurring when $\boldsymbol{C}$ has repeated eigenvalues. Then all conjugate diameters are orthogonal.

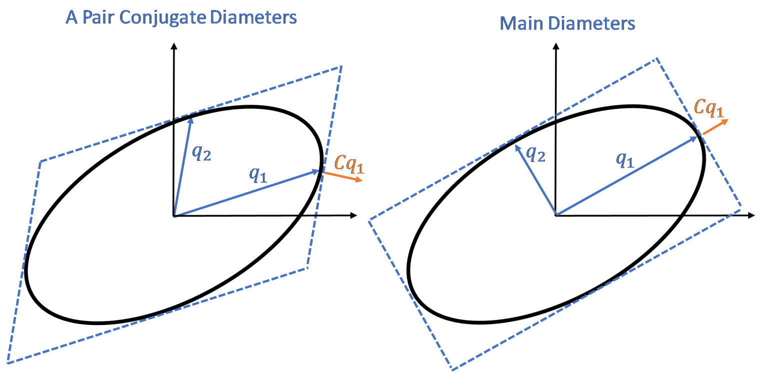



**Figure 7:** The graph depicts conjugate vector pairs for the iso-cost ellipse. The figure on the left demonstrates how, for a given vector $\mathbf{q}_1$, the conjugate vector may be obtained as the one that is orthogonal to the cost gradient and therefore parallel to the iso-cost line in $\mathbf{q}_1$. The primary diameters are a specific case of conjugate vector pairs, and they are orthogonal to each other, as seen in the figure on the right.

$$\boldsymbol{C}\mathbf{q}_1 = \lambda \boldsymbol{D}^{-1}\mathbf{q}_1.$$

Hence, the iso-cost and iso net-utility curves have tangency points at $\mathbf{q}_1$, as illustrated in Figure 8. It follows that $\mathbf{q_1}$ is a generalized eigenvector (and $\lambda$ a generalized eigenvalue) with respect to $\boldsymbol{C}$ and $\boldsymbol{D}^{-1}$. The relationship above can be rewritten as follows:

$$\boldsymbol{DC}\mathbf{q}_1 = \lambda \mathbf{q}_1.$$

This confirms that $\mathbf{q_1}$ is an eigenvector of $\boldsymbol{DC}$, as claimed in Lemma 2. [23]

If we limit ourselves to linear SFE, the reported cost function $\frac{1}{2}\mathbf{q}^\top \boldsymbol{S}^{\mathbf{-1}}\mathbf{q}$ can also be represented as an ellipse. Figure 9 illustrates iso-net utility (blue), iso-cost (red) and the iso-reported-cost (black) ellipses for a production level $\mathbf{q}$. The gradients at $\mathbf{q}$ represent marginal net utility, marginal cost, and the equilibrium price. In general, they are not parallel.

According to Corollary 1 separating bundles will diagonalize the supply Jacobian. As a result, the pair of mutual conjugate vectors $(\mathbf{q}_1, \mathbf{q}_2)$ of $\boldsymbol{C}$ and $\boldsymbol{D}^{-1}$ is likewise a pair of conjugate vectors of $\boldsymbol{S}^{-1}$. Figure 9 shows that the three ellipses have a pair of conjugate diameters in common, along the vectors $\mathbf{q}_1$ and $\mathbf{q}_2$. Along the separating bundle $\mathbf{q}_2$ (green) , the gradients of all plotted ellipses are orthogonal to the separating bundle $\mathbf{q}_1$, and vice versa.

[23]When the matrix $\boldsymbol{DC}$ has identical eigenvalues (= repeated eigenvalues), then the levels of constant net utility and cost are homothetic to each other. The matrices $\boldsymbol{D}^{-1}$ and $\boldsymbol{C}$ have the same set of conjugate vectors, all of which separate the markets. In the paper we rule out this complication by assuming that $\boldsymbol{DC}$ does not have repeated eigenvalues.

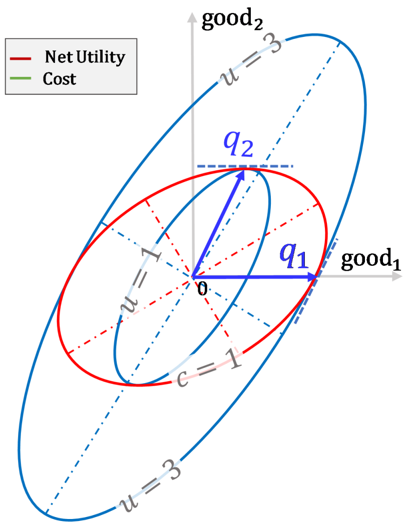



**Figure 8:** Matrices $\boldsymbol{C}$ and $\boldsymbol{D}$ are simultaneously diagonalized by a pair of bundles $(\mathbf{q}_1, \mathbf{q}_2)$ that are conjugate diameters in the iso-cost ellipse $\frac{1}{2}\mathbf{q}^\top \boldsymbol{C}\mathbf{q}$ and the iso-net-utility ellipse $\frac{1}{2}\mathbf{q}^\top \boldsymbol{D}^{-1}\mathbf{q}$. Data examples in Appendix B.1 are used in this figure.

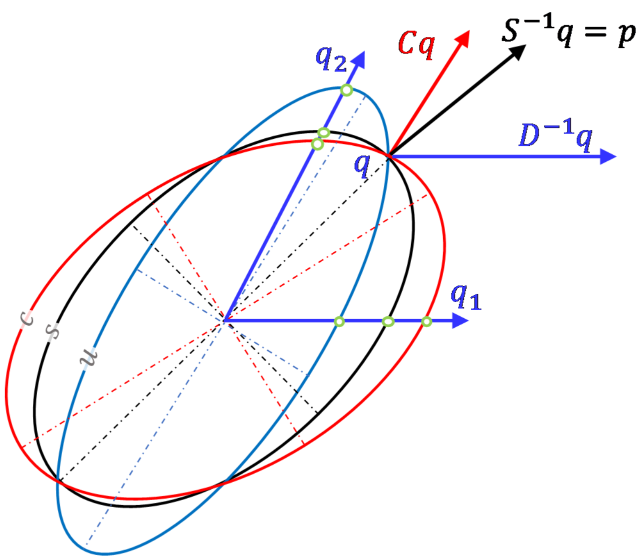



**Figure 9:** The iso-net utility (blue), iso-cost (red) and the iso-reported-cost (black) ellipses for a production level $\mathbf{q}$. The three ellipses have a pair of conjugate diameters in common which are proportional to $\mathbf{q}_1$ and $\mathbf{q}_2$.

**Algorithm 1** Outline of algorithm that numerically computes supply $\mathbf{q} = \mathbf{s}(\mathbf{p})$ with starting point $(\mathbf{q}_0, \mathbf{p}_0)$. The function JACOBIAN uses the equations in Subsection 4.2 to determine the supply Jacobian at each price in the grid. $S_i$and $I_i$ denote the $i$th column of the Supply Jacobian and identity matrix, respectively. Note that columns and rows are indexed from 0.

**procedure** SFE($c(\cdot), \mathbf{d}(\cdot), \mathbf{q}^0, \mathbf{p}^0$)
    **global** $N, N_1, N_2, \mathbf{I}, \Delta p_1, \Delta p_2$ ▷ grid size and step size
    $\mathbf{q}^{(0,0)} := \mathbf{q}^0$ ▷ Initialize $\mathbf{q}^{(0,0)}$
    $\mathbf{p}^{(0,0)} := \mathbf{p}^0$ ▷ Initialize $\mathbf{p}^{(0,0)}$
    **for** $i = 0, N_1$ **do**
        $\mathbf{c}^{(i,0)} := \partial_{\mathbf{q}} c(\mathbf{q}^{(i,0)})$
        $C^{(i,0)} := \partial^2_{\mathbf{q}^2} c(\mathbf{q}^{(i,0)})$
        $D^{(i,0)} := -\partial_{\mathbf{p}} \mathbf{d}(\mathbf{p}^{(i,0)})$
        $S^{(i,0)} :=$JACOBIAN$(N, C^{(i,0)}, D^{(i,0)}, \mathbf{q}^{(i,0)}, \mathbf{p}^{(i,0)}, \mathbf{c}^{(i,0)})$
        $\mathbf{q}^{(i+1,0)} := \mathbf{q}^{(i,0)} + \Delta p_1 S_0^{(i,0)}$
        $\mathbf{p}^{(i+1,0)} := \mathbf{p}^{(i,0)} + \Delta p_1 \mathbf{I}_0$
▷ Calculate $\mathbf{q}^{(1,0)}, \mathbf{q}^{(2,0)}, \mathbf{q}^{(3,0)}, ...$
        **for** $j = 1, N_2$ **do**
            $\mathbf{c}^{(i,j)} := \partial_{\mathbf{q}} c(q^{(i,j)})$
            $C^{(i,j)} := \partial^2_{\mathbf{q}^2} c(q^{(i,j)})$
            $D^{(i,j)} := -\partial_{\mathbf{p}} \mathbf{d}(\mathbf{p}^{(i,0)})$
            $S^{(i,j)} :=$JACOBIAN(N,$C^{(i,j)}, D^{(i,j)}, \mathbf{q}^{(i,j)}, \mathbf{p}^{(i,j)}, \mathbf{c}^{(i,j)}$)
            $\mathbf{q}^{(i,j+1)} := \mathbf{q}^{(i,j)} + \Delta p_2 S_j^{(i,j)}$
            $\mathbf{p}^{(i,j+1)} := \mathbf{p}^{(i,j)} + \Delta p_2 \mathbf{I}_j$
▷ Calculate $\mathbf{q}^{(i,1)}, \mathbf{q}^{(i,2)}, \mathbf{q}^{(i,3)}, ...$
        **end for**
    **end for**
**end procedure**

## B.3 Pseudocode for computing SFE using local separation.

A consequence of having a unique supply Jacobian, as shown in Proposition 2, is that we are able to use numerical integration to solve for a $\mathbf{q}^{\text{PDE}}(\mathbf{p})$, for a given starting point $(\mathbf{q}^0, \mathbf{p}^0)$.[24] This idea is illustrated in Algorithm 1. It calculates the vector of supply functions by forward integration (Euler method) on a rectangular price grid $\mathbf{p}^{(i,j)}$ where $i \in \{0, 1, ..., N_1\}$ and $j = \{0, 1, ..., N_2\}$ are the node indices, and $\Delta p_1$ and $\Delta p_2$ denote the grid spacings for goods 1 and 2. This grid is illustrated in Figure 10.

[24]Often one needs a curve as an initial condition to solve a PDE. In our case, a single starting point is enough. The reason is that the supply Jacobian is restricted to be symmetric in our problem.

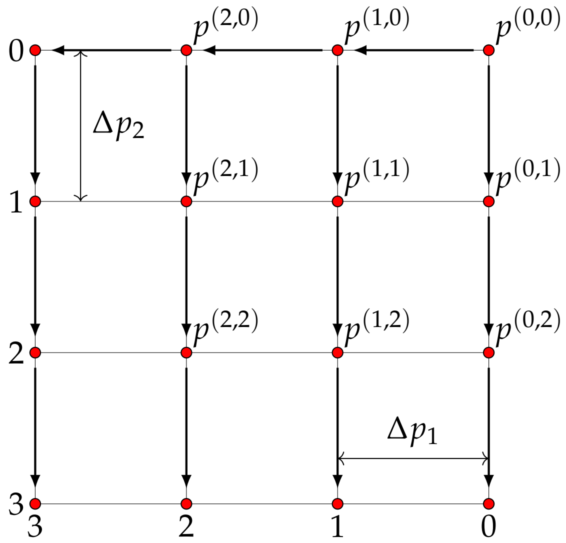



**Figure 10:** Price grid for numerical integration of $\mathbf{s}^{\mathrm{PDE}}(\mathbf{p})$.