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ERRATIC BEHAVIOR IN ECONOMIC MODELS

by

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An intellectually attractive but potentially frustrating feature of economic theory is that even seemingly simple models may involve mathematically complex behavior. Paradoxes abound, dynamical motion which intuitively should be "simple" may be highly erratic, and small changes in assumptions can lead to radically different conclusions. On first glance, this complexity has to be surprising. After all, much of economics is based on concepts of aggregation and optimization - concepts that might appear to introduce stability and predictability to conclusions of the resulting model. Yet, this need not be the case. Why?

The purpose of these notes is to shed light on a major source for the erratic consequences of economic models. I show that a basic and unifying explanation can be developed by modifying certain ideas from the modern theory of dynamical systems that explain "chaos." But, instead of emphasizing a technical development, my goal is to develop an intuitive approach so one can understand when and why such unexpected behavior may occur. Therefore, many of the concepts described here are introduced and illustrated with examples of voting and statistical paradoxes, problems with integer programming and allocation systems, and the erratic dynamics associated with optimal growth and price adjustment procedures. The technical material is motivated with informal arguments based on common examples from daily life such as the action of a bouncing ball. In this way I show why many of the unexpected outcomes from economics can be understood in terms of the properties of the inverse images of certain mappings. As an important corollary, I show that in any situation combining ***expansion and recurrence***, one must anticipate the accompanying economic behavior to be erratic.

Situations that combine expansion and recurrence are common to the social

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sciences. Of the two effects, recurrence is the more familiar as it is evident in business and other kinds of cycles. Recurrence effects are part of overlapping generation models, optimal growth models, and on and on. On the other hand, expansion effects may result from changes in discount rates, inflation and other parameters. Expansion and recurrence may arise as an unintended, but accompanying consequences of modelling. Whatever the reasons, one must expect expansion and recurrence to be combined in many, if not most, models of economics and the other social sciences. Consequently, we must anticipate the possibility that erratic, unpredictable outcomes are fundamental and common to the social sciences. "Order" may not exist in the manner researchers from these areas once believed.

Why should one believe that expansion and recurrence can cause highly erratic outcomes? While the development is started in Section 2.3, some intuition can be gained by considering the action of a bouncing ball. The motion of a ball dropped to a flat surface is quite predictable. Indeed, with experience, even a child understands where a dropped ball will go. Much of athletics is based on this predictable action -- a basketball player can dribble the ball rapidly down the court without concentrating on the basketball; a tennis player knows how to adjust the racket to direct the tennis ball to a desired position.

The predictability of the ball´s motion is lost should the surface be curved. Here the ball may rebound off the surface into any one of many different directions. This becomes a frustrating fact to anyone attempting a game of tennis with a warped racket. To see what can occur in the dynamics, think of dropping a ball on an inverted bowl. One way to get well behavior motion is to drop the ball with infinite precision directly over the top of the inverted bowl. The resulting motion is one where the ball bounces up and down along a fixed vertical line over the same point on the top of the bowl. However, if even the slightest error occurs when the ball is dropped, the ball hits the curved surface and, quite quickly, it bounces off of the bowl.

It is the curvature of the bowl that creates an expansion effect between the motion of the precise, theoretical orbit and the slightly altered, more realistic orbit of the ball. It is this expansion effect that forces the difference between the two motions to become large even if the initial dropping positions are close to one another. Because this expansion effect forces all nearby motion away from the theoretical equilibrium, the reference equilibrium orbit is called *unstable*. It is this dynamic causing the instability that allows one to argue that it is highly unlikely ever to observe the theoretical solution.

The instability argument only asserts that the expansion makes it unlikely for the motion of a bouncing ball to mimic the theoretically admissible motion of the ball bouncing forever over the top of the bowl. However, suppose that after the ball bounces off the bowl, it returns. Any condition permitting this to happen is a recurrence effect. For instance, the recurrence could occur because a second inverted bowl is placed next to the first one. With two bowls, there are two unstable regions. One can image how these two regions combine to provide an interactive motion with one another. The ball could bounce on the first bowl several times before it bounces off to hit the second bowl. Now the ball could bounce on this bowl several times before returning to the first bowl.

The two bowls admit many different scenarios. The ball may hit each bowl once before rebounding to hit the other bowl; it may hit each bowl several times before rebounding to the original bowl; it may bounce off one of the bowls to rebound off a third bowl; or it might bounce off of all the bowls to some other region. In other words, this combination of expansion and recurrence can create a complicated list of different kinds of highly erratic actions.

To review what causes the large number of possible dynamical scenarios, note that the recursion effect permits the motion to return near a starting point. The expansion effect introduces a large divergence in the behavior of two motions even though the initial points are near one another. With the continued expansion that is aided by the recurrence of the motion, we have that orbits starting near each other can end up being radically different. Thus the combination of recurrence and expansion can lead to an erratic dynamic characterized by i) a very large number of possible scenarios created by the two (or more) interacting regions of instability, and ii) where small changes in starting positions can change the motion from one scenario to a different one. These two properties are basic basic to dynamical chaos. Moreover, the bouncing ball description reasonably characterizes the mathematical explanation of this erratic behavior. So, when expansion and recurrence are combined, we must expect a large number of different kinds of outcomes.

As I show in Section 2, modifications of this story of the bouncing ball explain various puzzling phenomena ranging from price adjustment procedures to paradoxes of allocation methods. As it turns out, if one is interested in deriving properties of specified economic models, this type of analysis is a useful tool. In fact, by considering the global properties of the dynamics, additional information can be extracted from classical models that previously have been examined only with an equilibrium analysis. The global dynamics can provide new insight into the economics of the systems.

On the other hand, the conclusion that economic models may admit many different kinds of erratic outcomes runs against the basic objectives of a normative approach to economic theory. Here, the objective may be to understand how to control various kinds of behaviors. For instance, can the erratic motion of price adjustment procedures be tempered by using "speculation" within a model? Can convergence of a procedure be obtained by involving the "history" of past performance? In Section 3, I show how the same kind of mathematical techniques can be used to analyze this kind of issue.

For the most part, Sections 2 and 3 emphasize erratic dynamical behavior, and the illustrating examples only implicitly involve optimization and aggregation. To indicate how aggregation methods can lead to erratic behavior in economic models, in Section 4, I consider only the problems and paradoxes of aggregation procedures. This discussion is illustrated with voting and statistic processes. Here, a "paradox" is an outcome that is "unexpected" or "unanticipated."

To see how my discussion of aggregation procedures is related to the discussion of dynamics, recall that most often in the literature, paradoxes are analyzed in terms of specific examples. For instance, a counter intuitive election outcome may be described by specifying each voter´s rankings of the candidates; a surprising statistical outcome may be demonstrated in terms of a specific example. However, this piecemeal approach does not indicate what other kinds of outcomes can occur; it does not show how paradoxical outcomes from different subject areas are related, and it does not indicate the mathematical source of the unexpected outcomes. To address these concerns, in Section 4 a different approach is outlined to understand aggregation processes. This approach is based on discovering *everything that possibly can happen.* To realize this goal, I modify those ideas from dynamical systems which allows one to start with certain simple systems and then characterize, at least in a crude, qualitative sense, everything that can occur. This theme, this goal of computing a listing, or *dictionary*, is basic for my analysis of aggregation processes.

In these notes I borrow heavily from the ideas developed in dynamical systems. As I have already asserted, these notes do not constitute a short course on dynamical systems; there already exist several excellent texts on this topic both at the entry level (e.g., Devaney (1986)) as well as at a mathematically more sophisticated level. Instead my goal is to introduce with minimal technical arguments those central features that help us understand the surprising behavior of dynamics and of aggregation procedures. My goal in discussing dynamics is to develop the reader's intuition to understand and anticipate when the dynamics associated with economics can be highly erratic. In aggregation procedures the goal is to understand the kinds of paradoxes that should be expected.

To provide an unifying theme for these notes, I use a central topic in economics that is based both on optimization and aggregation and that exhibits both erratic dynamics and surprising consequences due to aggregation. This is tatonnement - the iterative procedure of how prices change according to the market pressures of supply and demand. However, while tatonnement serves as a motivating example throughout these notes, my main objective is to develop intuition why certain kinds of mathematical complexities arise in economics - not just in the tatonnement process. Therefore, once some of the basic consequences are determined, I illustrate these ideas with several other examples. Moreover, as true in Section 4, whenever there is a simpler example to describe certain properties, I use the simpler model.

2. Tatonnement

The description of tatonnement for a pure exchange economy is well known, particularly to economists. However to assist the reader unfamiliar with some of the concepts, I include a brief outline of the basic ideas in Appendix A. I now turn to the notation and basic assumptions.

2.1 Notation and the excess demand function

Assume there are $c \geq 2$ commodities and $a \geq 2$ agents in a simple exchange economy with a fixed amount of goods, where the *initial endowment* of the i^{th} agent is represented by the vector $\mathbf{w}^i = (w^i{}_1, \ldots, w^i{}_c) \in R^c{}_+$, ($R^c{}_+$ is the positive orthant of R^c). Here, $w^i{}_j$ represents the number of units of the j^{th} commodity that are held by the i^{th} agent. Assume that each agent's preferences are given by a smooth, concave utility function $U_i : R^c_+ \rightarrow R$ where all of the components of ∇U_i are positive. This assumption on ∇U_i captures the notion that "more is better" in the sense that if $\mathbf{x} > \mathbf{y}$ (each component of $\mathbf{y}$ is bounded below by the corresponding component of $\mathbf{x}$), then $U_i(\mathbf{y}) \geq U_i(\mathbf{x})$.

If $\mathbf{p} = (p_1,..,p_c)$ is the price vector, then the *budget constraint* is

2.1 $(\mathbf{p},\mathbf{x}) \leq (\mathbf{p},\mathbf{w}^i)$, or $(\mathbf{p},\mathbf{w}^i - \mathbf{x}) \geqq 0$ for $\mathbf{x} \in R^c_+$.

The budget plane is where equality is achieved. According to the assumptions, the i^{th} agent's demand is given by the point $\mathbf{x}^i$ where the level set of the utility function is tangent to the budget plane. Because of this tangency, it follows that $\nabla U_i(\mathbf{x}^i)$ is a scalar multiple of $\mathbf{p}$.

What agent i wants at prices $\mathbf{p}$ is $\mathbf{x}^i$, what he has is $\mathbf{w}^i$, so the *excess demand vector*, $\tau^i(\mathbf{p}) = \mathbf{x}^i - \mathbf{w}^i$, is what he want to trade. If a coordinate of $\tau^i(\mathbf{p})$ is negative, then the value indicates how much of this good the agent wants to sell; if it is positive, then the value indicates how much of this good the agent wants to buy. As this vector is in the budget plane, we have that

2.2 $(\mathbf{p}, \tau^i(\mathbf{p})) = 0.$

Whether the agent is able to do this depends upon whether the choices of the other agents -- is there a market for the goods? This leads to the definition of the *aggregate excess demand function*

2.3 $\tau(\mathbf{p}) = \Sigma_i \tau^i(\mathbf{p}).$

By virtue of Equation 2.2, we have the important Walras' law which asserts that

2.4 $(\mathbf{p}, \tau(\mathbf{p})) = 0.$

If the aggregate excess demand function, $\tau(\mathbf{p})$, is zero, then for each commodity the total of what people are will to sell equals what other people are willing to buy; the markets clear. But if $\tau(\mathbf{p}) \neq \mathbf{0}$, a new $\mathbf{p}$ needs to be adopted. The price adjustment problem is to figure out how to determine this new price. Namely, if the current price is $\mathbf{p}$, then how should the new price, $\mathbf{p}^*$, be determined? Most obviously, such a process must involve information about the market place as provided by the aggregate excess demand function $\tau(\mathbf{p})$. But, what kind of information is needed so that the process will lead to equilibrium? Second, it is unrealistic to expect an equilibrium price to be attained immediately. A more reasonable proposal is to create an iterative procedure; a process whereby eventually the price iterates begin to converge to an equilibrium price.

Question 1. Is there a price adjustment procedure that converges to a Walrasian equilibria for any choice of a pure exchange economy? If such a procedure exists, what kind of information about the economy does it require?

As a first, natural procedure, one should consider *tatonnement.* This adjustment procedure is based on the observation that if a component of the vector $\tau(\mathbf{p})$ is positive, then this commodity is in greater demand. To reflect this fact, the new price $\mathbf{p}^*$ should make that commodity more expensive by raising its price. Thus, it is natural to let the market pressures determine the price adjustment procedure by defining

2.5 $$\mathbf{p}^* = \mathbf{p} + h\tau(\mathbf{p})$$

where h is some modifying positive scalar. Unfortunately, as I show in the next section, this dynamic need not work. It turns out that for certain economies, the motion of the iteration process can become as erratic as one wishes. As I show, "chaos" can result even for economies with only two commodities!

Question 2. Why can erratic dynamical behavior arise from this price adjustment procedure? What does "erratic dynamical behavior" mean? What are some of the basic properties? Do the underlying reasons for this dynamic extend to other economic systems?

This question, along with Question 1, are the basic theme of the next two sections. As it turns out, a basic source of the erratic dynamic behavior is the over abundance of possible aggregate excess demand functions. Essentially, it turns out that if one needs a particular kind of aggregate excess demand function to illustrate a particular type of dynamical behavior, then this excess demand function exists. This remarkable fact is a corollary to the important Sonnenschein (1972), Mantel (1972), and Debreu (1974) Theorem. Loosely speaking, this theorem asserts that

if one chooses any continuous function $f(\mathbf{p})$ so that

i) $(\mathbf{p}, f(\mathbf{p})) = 0$ for all $\mathbf{p}$ with all components positive, and

ii) if p_i, the price for the i^{th} commodity, has a sufficiently small value, then $f_i(\mathbf{p})$ is positive (i.e., if the price is sufficiently small, then the aggregate demand for that good will be positive) *then for any $\epsilon > 0$, there exists an economy so that $\tau(\mathbf{p}) = f(\mathbf{p})$ for $\{\mathbf{p} \mid p_i \geq \epsilon$.*

In other words, if $f(\mathbf{p})$ satisfies Eq. 2.4 and indicates that a good is in strong demand when the price is sufficiently low, then f is the aggregate excess demand function for some economy.

Question 3. What is the reason for the Sonnenschein - Mantel - Debreu theorem? Is there an explanation that extends to other aggregation procedures?

Developing intuition for the answer to Question 3 forms the basis of Section 4.

To determine the dynamical properties of tatonnement, I use two standard reductions. The first uses that fact that if $\mathbf{p}$ is replaced with any positive scalar multiple of $\mathbf{p}$, the same theoretical development leading to Eq. 2.4 holds. This statement holds because the principal role of $\mathbf{p}$ is to determine the normal direction for the budget plane -- thus only the direction $\mathbf{p}/|\mathbf{p}|$ matters. We use this observation to assume that

2.6 $$\Sigma p_i = 1.$$

It now follows that we only need to worry about the first c-1 prices; the last price is determined from Eq. 2.6. In other words, each p_i, $i = 1,..,c-1$, can assume any value in $(0,1)$ so long as

2.7 $$p_c = 1-(p_1+..+p_{c-1}) > 0.$$

The next reduction involves eliminating the last term from the aggregate excess demand function. If the price $\mathbf{p}$ and the first c-1 components of $\tau(\mathbf{p})$ are known, then $\tau_c(\mathbf{p})$ is

2.8 $$\tau_c(\mathbf{p}) = (\Sigma\, p_i \tau_i(\mathbf{p}))/\{1-(p_1+..+p_{c-1})\}.$$

This means that we only need to consider the first c-1 components of $\mathbf{p}$ and of $\tau(\mathbf{p})$.

According to these two reductions, when $c = 2$, then all we need are the values of $p = p_1$ and $\tau(p) = \tau_1(p)$ where $\tau(p)$ is positive for values of p sufficiently close to zero and negative for values of p sufficiently close to unity. (This reflects a tacit assumption that the commodities are desired: as the price becomes arbitrarily small, the demand increases.) Using the Sonnenschein, Mantel, Debreu Theorem, it follows that any function satisfying these conditions serves as an aggregate excess demand function for some economy. Thus, tatonnement for $c = 2$ is given by

2.9 $$p_{n+1} = p_n + hf(p_n) = G_f(p_n)$$

where $f(x)$ is any smooth function defined on $[0,1]$ so that $f(0) > 0 > f(1)$. (Because of the reductions, the price is one-dimensional. Therefore, the subscript now corresponds to which iterate of this price is being considered.)

2.2 The dynamics of tatonnement - stability and instability.

The purpose of the iterative process is to find an equilibrium point, p, where $f(p) = 0$. At such a point the iterative process degenerates to $p_{n+1} = p_n$, so the process stops. Indeed, the processes stops and all future iterates agree with p_n if and only if $f(p_n) = 0$. Consequently the equilibria are identified with the points p where $G_f(p)$ $(= p + hf(p)) = p$, thus the set of equilibria are given by the intersection of the two lines $y = G_f(p)$ and the diagonal $y = p$. (See Figure 1.)

What happens to those points that are not equilibria? The iterative dynamic derives a new price, $p_{n+1} = G_f(p_n)$, based on the current price p_n. The obvious geometric description of locating this price on the graph of $y = G_f(p)$ is to start with the position of p_n on the horizontal axis, and then passing a vertical line $x = p_n$ through this point. The new price, p_{n+1}, is the intersection of this line with the graph $y = G_f(p)$.

The next price, p_{n+2}, is determined in a similar fashion. This means we need to find the location of p_{n+1} on the horizontal axis. To do this, use the horizontal line $y = p_{n+1}$ and its intersection with the diagonal line $y = p$. By definition, this intersection point has the coordinates (p_{n+1}, p_{n+1}), so a vertical projection of this point determines location of p_{n+1} on the horizontal axis. This is the intersection of the line $x = p_{n+1}$ with the x axis.

One more geometric description remains. Finding the price p_{n+2} involves using the vertical line $x = p_{n+1}$ twice. The first time is to find the location of the value p_{n+1} on the x-axis, and the second is to go from this point to the graph $y = G_f(p)$. Quite obviously, nothing is lost by ignoring the vertical projection as it is a redundant action. So, the process is achieved by first taking a vertical line passing through the price $y = p_n$ on the line $y = p$, and finding the intersection of this line with the graph $y = G(p)$. Next, pass a horizontal line through this intersection point, and find where the horizontal line intersects the diagonal $y = p$; this determines the value of p_{n+1}. The iterative process continues. See Figure 1.

While the above provides a geometric description for the iteration, it does not indicate how the system can evolve. To start, there are some situations where the dynamic is particularly well behaved. These are the situations whereby the various iterates quickly become indistinguishable from an equilibrium point. One can view such behavior as being similar to the motion of a rolling ball inside of a bowl. When the ball is at the bottom of the bowl, it remains there forever. This bottom position in the bowl corresponds to the equilibrium position. If the ball is slightly displaced, then the ball rolls back toward the bottom equilibrium position.

Equilibrium points that enjoy the above desirable characteristics are called (asymptotically) *stable.* A stable equilibrium is displayed in Figure 1.a. A characteristic of these equilibria is that once some iterate is sufficiently close to it, then the next iterate is even closer. In other words, such an equilibrium has the property of pulling near-by iterates into it. It is this absorption, or contraction process, that creates the situation where, eventually, one cannot distinguish between the equilibrium point and iterates of the dynamical process. By use of calculus, a sufficient condition for such equilibrium points can be found.

Proposition 1. If p^* is an equilibrium point and if

2.10 $$|G_f'(p^*)| < 1$$

then p^* is a stable point.

Condition 2.10 corresponds to a contraction, so Proposition 1 implies that contractions about an equilibrium should be identified with stability.

Outline of the proof. According to iterative process, $|p_{n+1} - p^*| = |G_f(p_n) - p^*| = |G_f(p_n) - G_f(p^*)|$ where the last term results from the equilibrium condition that $G_f(p^*) = p^*$. We have from the mean value theorem of calculus that $|G_f(p_n) - G_f(p^*)| = |G_f'(p')||p_n - p'|$ for some choice of p' between p_n and p^*. Thus

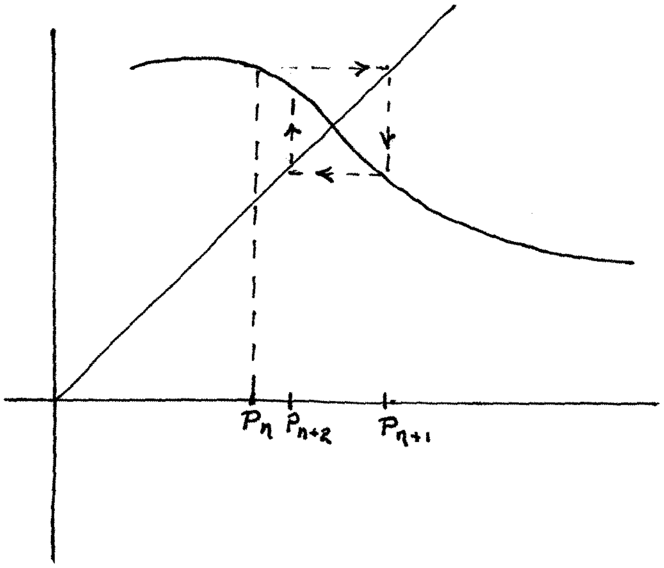


Fig. 1a.

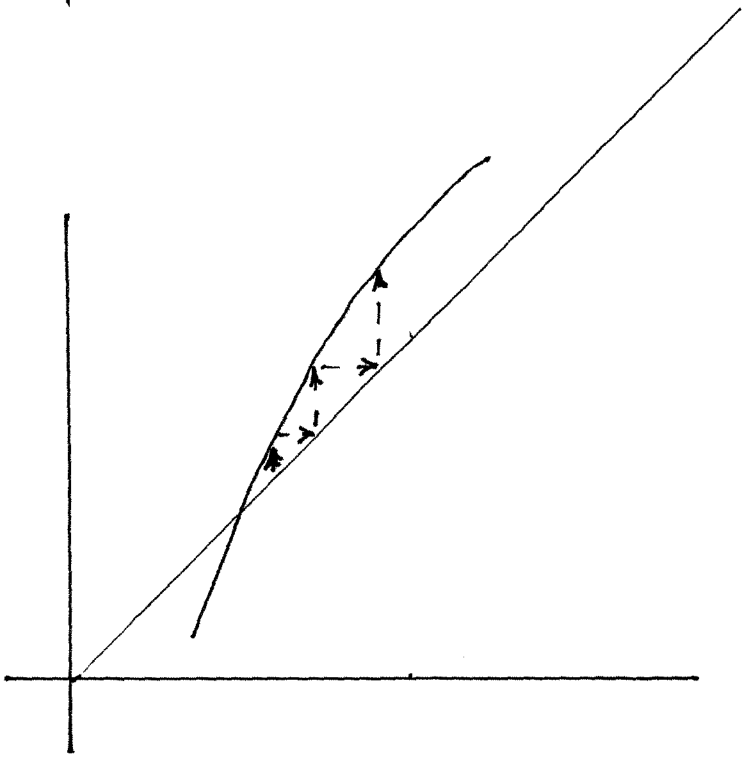


Fig. 1b.

2.11 $|p_{n+1} - p^*| = |G_f'(p')||p_n - p'|$.

If $|G_f'(p')| < 1$, then the assertion $|p_{n+1} - p^*| < |p_n - p^*|$ holds. However, because $|G_f'(p^*)| < 1$, this condition follows from the continuity of G_f with p_n sufficiently close to p^*.

(To make this proof complete, just note that there is a $k<1$ so that if p_{n+1} is sufficiently close to p^*, then $|G'(p')|< k$. In turn, this estimate can be used to show that $|p_{n+j} - p^*|< k^j|p_n-p|$. Because $k^j \rightarrow 0$ as $j \rightarrow \infty$, $p_{n+j} \rightarrow p^*$.)

At the other extreme from stable equilibria are the *unstable equilibrium points*. This is an equilibrium point where the dynamics forces iterates near the equilibrium *away* from the equilibrium. Such a dynamic is illustrated in Figure 1b. The importance of an unstable equilibrium is that it difficult to attain because if the equilibrium is not achieved precisely, then the dynamics moves away from it. This is much like the rolling motion of a ball on the surface of an inverted bowl. The equilibrium position for the ball is at the top. If the ball is precisely positioned, then it will remain there forever. However, with any small dislocation, the ball will roll away from this (unstable) equilibrium position. Thus, for reasons similar to why not much physical significance is accorded to this inverted bowl equilibrium position of a ball, one should view with skepticism the value of unstable equilibria in economic models.

A slight modification of the argument in the proof of Proposition 1 can be used to find a sufficient condition for an equilibrium to be unstable. Namely, p^* is unstable should

2.12 $|G_f'(p^*)| > 1$.

The only modification in the above argument is to change the inequality for the derivative. This change forces the inequality $|p_{n+1} - p^*| > |p_n - p^*|$ for p_n sufficiently close to p^*.

The derivative condition 2.12 indicates that there is an expansion. Therefore, an expansion about an equilibrium should be identified with the instability of the point. Consider an f that defines a G_f as indicated in Figure 2. For this choice of f, the equilibrium in interval b is stable. Indeed, because the slope of G_f at this point is nearly horizontal -- so $G_f'(p)$ is nearly zero -- Proposition 1 applies. Near this equilibrium, the contraction effects are strong. Indeed, by carrying out the geometric process described at the start of this section near this equilibrium, one sees how rapidly the iterates converge to the equilibrium. On the other hand, the equilibria in intervals a and c are unstable. To see how this follows from Eq. 2.12, notice that the graph $y = G_f(p)$ passes through the diagonal line $y = p$. For this to occur, the graph of $G_f(p)$ must be expanding faster than that of $y = p$. Thus, $G_f'(p) > (p)' = 1$. Again, a geometric representation of the dynamics near these two equilibrium points shows that the dynamics rapidly diverges from equilibrium.

2.3 The erratic dynamics of tatonnement.

The analysis of stable and unstable points provides only a local insight into the behavior of the dynamic processes. We want to know what happens on a more global basis. The ideal situation, of course, is if one always selects a starting point where the iterative process eventually converges to one of the equilibria. Such an outcome may result by an iterate landing precisely on an unstable equilibria (which is a rare occurrence), or landing sufficiently close to a stable equilibria so that the contraction dynamics of stability pull the successive iterates closer to the equilibrium. But, how common can such successful choices be made? Does the predictable, convergent motion of stability reasonably characterize the general situation? What else can happen?

One way to characterize the set of points that eventually converge to an equilibrium is to study those points that *never* converge. To do this, let the nonconvergent points be given by the set

2.13 $NC = \{p \mid$ if p is an initial price, then the dynamic defined by Eq. 2.9 never converges to a zero of f$\}$.

One would hope that this set of points is "small."

To see the kind of set *NC* that can occur in the price dynamics, consider a smooth function f that defines a G_f of the form given in Figure 2. The main features to note are that the four critical points marked by the dots define three regions on the price interval, and they are labelled a, b, c. The horizontal dotted lines define the boundaries of these regions. Namely, the upper horizontal line intersects the *y=x* diagonal precisely above the last critical point, while the lower horizontal line locates the left-most critical point. The feature which underlies and simplifies our argument is that in each labelled interval, the graph of G_f intersects both of the dotted lines. According to the Sonnenschein, Mantel,

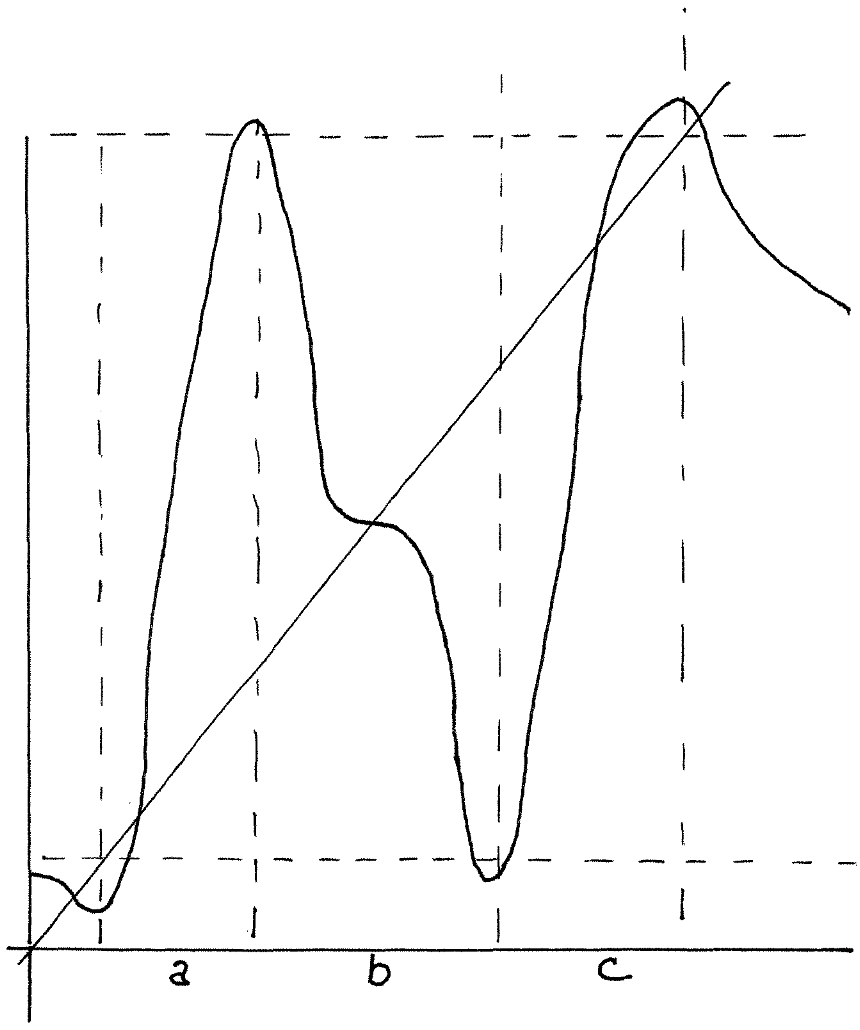


Fig. 2

Debreu Theorem, there exists economies that define such an f. The next statement shows that for an f of this kind, the set *NC* need not be small.

Theorem 1. For an f of the type given by Figure 2, the set *NC* is uncountable.

While this assertion is of interest, of greater importance for these notes is that I use the proof to show that the simple scheme given by Eq. 2.9 admits a dynamic that is as random as one wishes. To be more specific, first note that we are interested only in the those orbits that remain in the intervals a, b, and c. Our goal is to understand everything that can happen if such an orbit refuses to converge to one of the equilibria. Now, if p_1 is an initial iterate for such an orbit, this value and the dynamics defines a sequence of prices

2.14 $\{p_1, p_2, \ldots, p_n, \ldots\}$.

One way to describe this sequence is to determine the precise value of each iterate p_j. This is a difficult quantitative task! An alternative approach would be to sacrifice precision by replacing each iterate, p_j, with the much more crude information of which of the three intervals contains it. Namely, if $p_1 \in b$, $p_2 \in a$, $p_3 \in c$, ..., then the sequence given in 2.15 would be replaced by the sequences of "addresses" {b, a, c, ...}.

Let U({a,b,c}), the *universal set*, be the set of all sequences where the entries are one of the letters {a,b,c}. (This can be expressed as $\{a,b,c\}^N$, where N corresponds to the natural numbers {1,..,n,..}.) As I indicated above, a sequence of points defines a sequence in the universal set. My assertion is that the dynamics of the tatonnement for an f giving rise to Figure 2 can be so erratic that it may seem to be random. This behavior is specified in the next statement.

Theorem 2. Let f be a function that defines G_f as given in Figure 2, and let $S \in$ U({a,b,c}). There exists an initial iterate p_1 so that the k^{th} iterate of p_1, is in the interval denoted by the k^{th} symbol in S.

Theorem 2 means that *it is possible to specify* ***in advance*** *the total future of how the iterates of some point will bounce through the marked intervals, and there exists an initial point with this particular future.* An immediate implication of this theorem is that if the specified sequence of intervals is not eventually constant, then the corresponding iterative processes never remain in any fixed interval. In turn, this means that the dynamic cannot converge to a zero of f. As there are an uncountable number of sequences from the universal set that are not eventually constant, there must be an uncountable number of points where the dynamics cannot converge. It now follows that *NC* contains an uncountable number of points, so Theorem 1 is a consequence of Theorem 2.

The proof of Theorem 2 is fundamental for what else follows in these notes. This is because the basic objective of the proof is to characterize in a qualitative sense everything that can occur for this portion of the dynamics. To develop some intuition for the proof, consider a related problem of pool. On the pool table there is a white cue ball along with a large number of balls each with an identifying number. To be successful at this game, one must hit the cue ball in such a manner that a "specified future" will occur. Namely, once the cue ball is set in motion, it will hit some other ball where the struck ball now goes into motion and hits a third ball, and the recoiling action continues until motion stops. The idea, of course, is to choose how to hit the cue ball so that the subsequent motion does what one wants to have occur. This desired future corresponds to a listing of what ball hits what other ball. For instance, a listing (3,6,1,3,7,2,3,9) would mean that the cue ball hits the three-ball, the three-ball hits the six-ball, the six-ball hits the one-ball, the one-ball hits the three-ball, and so forth. (This listing should be identified with a sequence from the universal set.)

There are several ways to determine what type of dynamic behavior is possible. One approach would be to select a target point on the cue ball and carefully calculate what motion happens if the ball is hit there. This involves calculating where the cue ball will go and which balls, if any, would be hit. If the calculation of the orbit is a desired one, then that is the solution. If it is not a desired solution, then, in some fashion, a new target point needs to be selected and the solution associated with that point needs to be calculated. I call this process, which involves difficult computations and need not lead to success, the "precision computation" approach. Notice that this precision computation approach is not that dissimilar from standard techniques used in the social sciences, economics, and the physical sciences. A policy, an initial set of data is selected, and the consequences are determined after considerable computations.

The game of pool is not played with the precision computation technique. A more natural approach is to first find a *target region* where one should hit the cue ball so that it will hit the three-ball. By doing this, at least partial success will be attained; the motion realizes the first step of the specified future. However, once the three-ball is set in motion, it is not clear whether it will go anywhere near the six-ball. So, to get the three-ball to hit the six-ball, one has to find a *target region* on the three-ball. This is the region where if the cue ball hits the three-ball, then the three-ball will hit the six-ball. Next, we use this added information to refine where should hit the cue ball; the refined target region is where to hit the cue ball so that the cue ball will hit the target region on the three-ball. The process continues; it isn't sufficient for the three-ball to hit the six-ball; it must hit the six-ball in the appropriate position so that the six-ball will hit the one-ball. For this to occur, the appropriate target region is determined on the six-ball. This target region defines a refined target region on the three-ball, which in turn defines a refined target region on the cue ball. Thus, this natural procedure of creating *refined target regions* leads to an iterated inverse image approach of narrowing in on the precise point where one should hit the cue ball. This simple, refined targetting approach is central for all that is discussed in these notes.

Proof of Theorem 2. I use an iterative, refined targetting argument to characterize all of the points that share at least a finite part of the specified future. Toward this end, let a sequence S of specified intervals be given. Without loss of generality, assume that this sequence is S = {b, a, c, ...}. Let S_n be the listing that specifies the first n terms in S; e.g., S_2 = {b, a} and S_3 = {b,a,c}, etc. With this notation, the set

2.15 $C(S_n) = \{p \mid$ for $1 \le i \le n-1$, p_i is the i^{th} specified interval of S_n, and p_n is in the closure of the n^{th} specified interval of $S_n\}$,

consists of all of the initial points that satisfy the first n steps. For instance, $C(S_3) = C(\{b, a, c\}) = \{p_1 \mid \mathbf{p}_1 \in b,\ p_2 = G_f(\mathbf{p}_1) \in a$, and $p_3 = G_f(p_2) \in c^c$, where c^c is the closure of the interval c}.

To characterize the set $C(S_n)$, I use the fact that when G_f is restricted to any of the marked intervals, its image includes all three intervals. (The graph of G over each interval intersects both dotted lines. This special situation often is called the Markov property.) In particular, when G_f is restricted to b, which is denoted by $G_{f,b}$, then its image covers a^c. Obviously, $C(\{b, a\}) = C(S_2) = G_{f,b}^{-1}(a^c)$. As G is a continuous function and as a^c is a closed set, it follows that $C(S_2)$ is a non-empty closed subset of the interval b.

The set $C(S_2)$ is the initial target region for the interval b; it defines the initial points in b where the next iterate hits the interval a^c. However, the second iterate of a point in $C(S_2)$ need not hit region c. Therefore, to refine our argument, we need to define a target region in interval a; this is the set of points in interval a for which the next iterate is in region c. So, for the same reasons as used above, the target region in a, $C(\{a,c\}) = G_{f,a}^{-1}(c^c)$, is a non-empty, closed subset of the interval a, and it is the set of initial points in "a" where the next image is in c^c.

We want to refine the target region in b so that the next iterate not only hits interval a, but it hits the target region $C(\{a,c\})$. As $C(\{a,c\})$ is a subset of interval a, it follows that the refined target region in b, $C(S_3) = G_{f,b}^{-1}(G_{f,a}^{-1}(c^c))$, is a subset of $C(S_2)$ in the interval b. Indeed, it is that subset of initial iterates where p_1, p_2, and p_3 follow the specified future. Moreover, by the continuity of $G_{f,b}$ and $G_{f,a}$, it follows that $C(S_3)$ is a non-empty closed subset.

In general, the idea is the same. At each stage of the refined targetting approach, it follows that $C(S_n)$ is a closed subset of interval b, and that $C(S_{n+1})$ is a closed subset of $C(S_n)$. Therefore, this construction defines the sequence of nonempty, nested, closed sets

2.16 $$b \supset C(S_2) \supset C(S_3) \supset \dots \supset C(S_n) \supset \dots .$$

By definition, the set $C(S_n)$ consists of all of the initial iterates where the dynamics of tatonnement obey the first n steps of the specified sequence. Therefore, any point that lies in all of the sets of the sequence of Eq. 2.13 has the required property. Such points exist. This is because a countable intersection of compact, nested sets is nonempty.

Notice that the above construction holds for any specified sequence of intervals. If a sequence does not permit the trajectory to remain in any one interval, then the associated points are nonconverging. Because there are an uncountable number of sequences of this type, *NC* contains an uncountable number of points. This completes the proof of both theorems.

2.4 Dictionaries and properties of the motion

The refined targetting argument used in Section 2.3 is the key to "chaos." It is interesting to note that an important step in understanding this erratic dynamic is a willingness to sacrifice a precise description of the dynamics -- a statement about the precise location of each iterate -- by replacing it with a seemingly more crude, qualitative description where only the interval that contains each iterate is identified. The surprising payoff from this tradeoff is that now we are able to determine the qualitative behavior of an uncountable number of orbits. Indeed, in many situations, using this kind of approach permits one to characterize the qualitative behavior of everything that possibly can occur! Moreover, this approach provides new information about the dynamics of the system, so it is worth pausing to better understand these connections between the dynamics of tatonnement and the choice of sequence of intervals.

As described just prior to the statement of Theorem 2, each sequence of iterates -- each orbit as given by 2.14 -- can be identified with a sequence from U({a,b,c}). Let this identification process be given by the mapping h. As each initial iterate defines a unique sequence, h is a mapping from I, the union of the three intervals a, b, and c, to the universal set U({a,b,c}). Thus, h takes a point from I, an initial iterate, and assigns to the point a sequence of letters where the i^{th} letter identifies the interval containing the i^{th} iterated. It is reasonable to call each such sequence a *word*. The image set of h contains all possible words that identify the orbits of G_f, so this image set is called the *dictionary defined by G_f*, $D(G_f)$. With this notation, Theorem 2 asserts that for an f of the kind considered, $D(G_f) = U(\{a,b,c\})$, the dictionary equals the universal set.

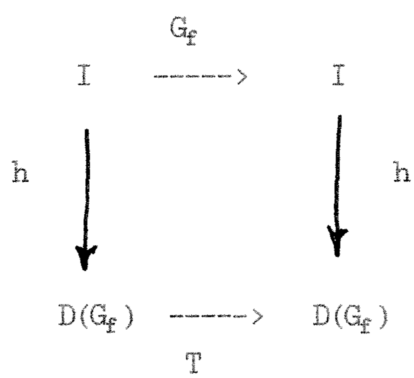
The concept of a dictionary is similar to the "tree structures" often used in economics to depict everything that can occur. A common example comes from game theory where the different outcomes are listed according to different combinations of the players' strategies. In other words, the tree specifies everything that can happen in the game. The main difference is that in the dynamics, there are an uncountable number of listings of "everything that can occur." Consequently the dictionary does not admit a simple, graphical representation. Indeed, a loose interpretation of chaos is that the dictionary --

the qualitative listing of everything that can occur -- admits an uncountable number of different possibilities.

There is a natural dynamic, the shift map, that is defined on a sequence from $D(G_f)$. This mapping, T, takes a sequence, deletes the first entry, and the image is the resulting sequence. For example, $T(\{b,a,c,d,...\}) = \{a,c,d,...\}$. The interest in this mapping is that it corresponds to what happens in the iterative dynamics. To see this, note that the sequence $\{p_1, p_2, p_3, p_4, ...\}$ describes the trajectory for the initial iterate p_1. However, if one started with point p_2 as the initial iterate, then the only difference in the trajectory is that the point p_1 is not included -- the new trajectory is $\{p_2, p_3, p_4, ...\}$. Moreover, if $h(p_1) = S$, then the sequence assigned to $h(p_2)$ must be the shifted sequence $T(S)$. This observation admits the representation

2.17 $$h(G_f(p)) = T(h(p))$$

which, in the literature, often is described by asserting that the mappings in the following diagram commute.



A dictionary need not agree with the universal set. Intuition supporting this statement comes from the motivating pool or billiard ball example. For instance, one can conceive of initial configurations of the pool balls where a specified listing of a future is impossible. To see this consider the sequence (3,1,5,2,7,...), and determine the target region on the one-ball so it can hit the five-ball. Suppose the balls have an initial positioning where the three-ball can never hit this target region on the one-ball; e.g., it may be that the target region on the one-ball is on the side opposite of where the three-ball can strike. If so, then this particular future never can occur. Using the notation of the proof of Theorem 2, this means that certain of the iterated inverse images of the

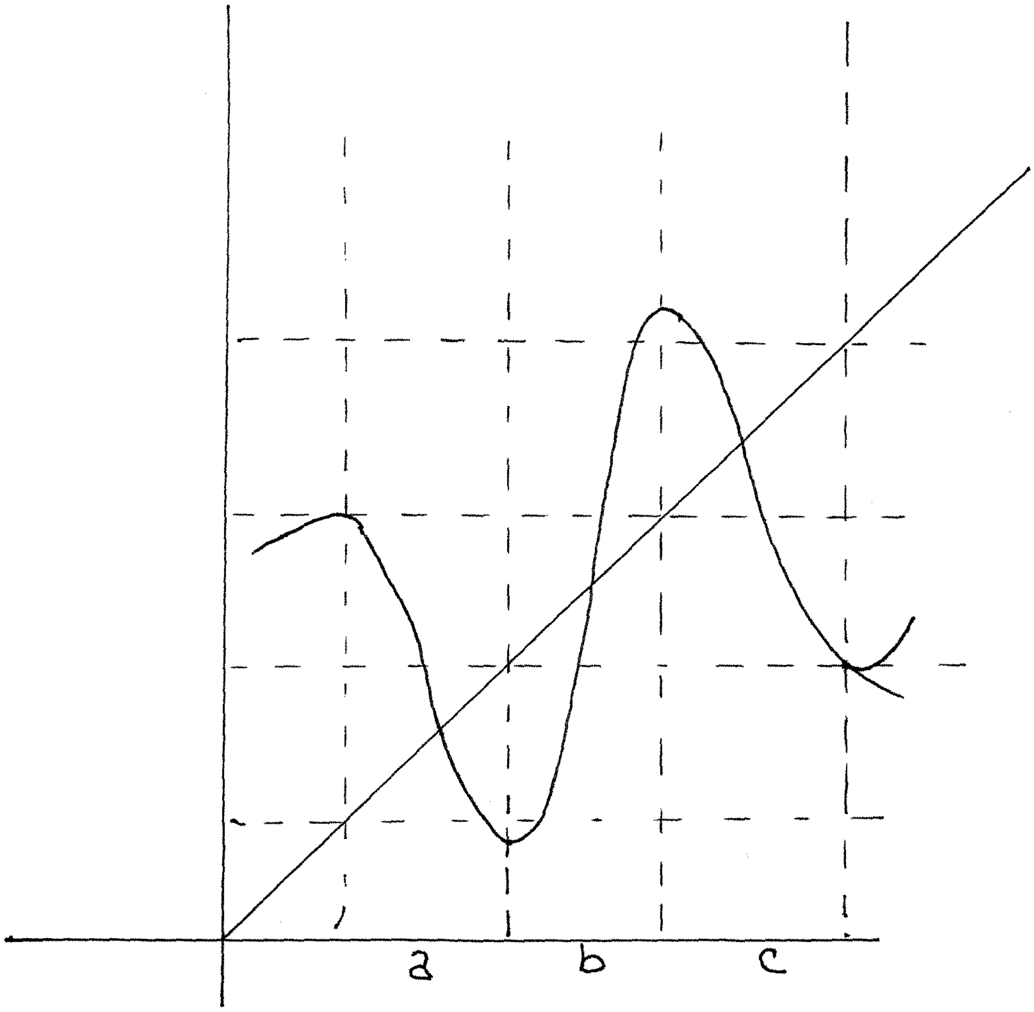


Fig 3.

target regions lead to empty sets.

From this argument, the reason the dictionary for the G_f map (Figure 2) is the universal set becomes clear. It is because the image set of the mapping G_f, when restricted to any of the three intervals, is the union of all three intervals, so all possible locations of target regions can be reached. By modifying the reasoning of the pool table example, it becomes clear that should G_f not have such a property, then it may be that some of the target regions cannot be reached. As with the pool balls, this assertion means that inverse image of such a target region under G_f is empty, so certain words cannot be in the dictionary of G_f. A more complicated situation is where only certain portions of various target regions are accessible. This raises the possibility that some further refinement due to the characterization of a target region on some other ball, is not attainable. For instance, it may be possible for the one-ball to strike the five ball, but not in a manner so that the five-ball strikes the two-ball. On the other hand, it may be that the sequence (3,7,5,2,7,..) is admissible.

To further illustrate how the ideas of motion on the pool table carry over to the motion of iterative dynamics, consider an f of the type given in Figure 3. Here the image of $G_{f,a}$ covers a and b, but not c, $G_{f,b}$ covers all three intervals, and $G_{f,c}$ covers only b and c, but it misses a. Because of these properties, $G_{f,a}^{-1}(c)$ is an empty set. This means that the kind of argument used to prove Theorem 2 fails should S be any sequence that allows the letter c to follow the letter a. A similar observation precludes the possibility of any sequence in the dictionary of G_f from having the letter a follow the letter c. It now is simple to characterize the dictionary $D(G_f)$; it contains all sequences where the letter a never follows c and c never follows a.

More complicated dictionaries can occur. For instance, suppose the f in Figure 3 is replaced with one that allows the image of $G_{f,a}$ to extend partly into region c. Now, depending on the magnitude of this extension, the dictionary may admit some words where c follows a. However, not all words of this kind would be admitted. For instance, it may be that $G_{f,c}^{-1}(a)$ happens to lie in that part of interval c which is outside of the image of $G_{f,a}$. If so, then sequences that admit orderings of the form ..,a,c,a,... would not be admitted in the dictionary.

The determination of the dictionary is the intuitive approach -- follow the iterates of the critical points of the mapping G_f to determine what kinds of iterated, inverse images are non-empty. Therefore, as f changes its geometric properties, new words may enter or former words may be dropped from the dictionary of the iterative process. Also, it is obvious that in the above examples, three intervals were selected only for illustrative purposes. Any number of intervals, determined by the critical point properties of G_f, can be used.

Sensitivity to initial conditions and Cantor sets

There are other consequence of the argument used in the proof of Theorem 2. An important characteristic of the erratic behavior demonstrated above is that *starting points can be selected arbitrarily close to one another where their subsequent trajectories are radically different; nearby starting points can have significantly different futures.* Such an assertion finds intuition from the billiard ball example. If the cue ball is hit precisely at a specified point, then the desired outcome may occur. But, if the cue ball is not hit precisely at this point, then the resulting behavior can differ radically from the intended one. This sensitivity of the future of the pool balls to where the cue ball is hit defines the radical difference between the game played by a competent player and mine.

To see mathematically why this assertion about sensitivity to initial conditions holds for G_f, notice that there are infinite number of different ways one can extend the partial sequence S_n into a full sequence. One extension, of course, remains true to the original sequence S, but many other sequences can be defined just by changing any number of the letters after the n^{th} entry. Indeed, there are an uncountable number of different sequences that extend S_n. Each extension of S_n defines a unique subset of $C(S_n)$; if two extensions, S' and S'', differ by even one symbol, then $C(S')$ and $C(S'')$ are disjoint subsets of $C(S_n)$. Therefore, contained in $C(S_n)$ are an uncountable number of disjoint sets, each of which leads to a different future. So, nearby points in $C(S_n)$ can be chosen so that they have as radically different futures as desired.

The above argument is predicated upon there being sequences S so that the points in $C(S_n)$ are close to one another for some value of n. But, the same argument shows that there are many choices. For example, the uncountable number of sequences starting with the symbol b all define disjoint sets that lie in the bounded interval b. Thus, most of these sets, $C(S)$, must have measure zero. Consequently, for any such sequence, S, there is an n so that the points in (each component of) $C(S_n)$ are close to one another. The above assertion of unpredictability now holds.

A third observation based on the construction of Section 2.3 is that *regions of stability and instability can be intertwined in complicated ways.* To see why this assertion is so, recall that the mapping G_f from Figure 2 admits a region of stability in interval b. Now, consider any sequence S_{n-1}. One way to extend S_{n-1} to S_n is to let the next symbol be b. By the construction of the refined targetting approach, each point in b has an iterated pre-image in the set $C(S_n)$. This means that there is an open set of points in $C(S_n)$ (and $C(S_{n-1})$) so that on the n^{th} iterate each is mapped to the stability region. But once an iterate is in this region of stability, all future iterates tend to the equilibrium point. On the other hand, there are many other ways to complete the sequence S_{n-1}, and for each of these possibilities there are points in $C(S_{n-1})$ with this specified future. Thus $C(S_{n-1})$, which may be a small set, contains both points that become eventually stable and points with a highly erratic future. To find just those points with a non-convergent dynamic, one would exclude the open set of points of eventual stability. Thus, for all choices of S_{n-1}, an open set, corresponding to the stable region in b, must be removed from $C(S_{n-1})$. The resulting set is called a Cantor set.

2.5 From abuse to the apportionment of Congressional seats

The erratic, chaotic dynamic of tatonnement illustrated in Section 2.3 is based on only the gross geometric properties of the graph of G_f. Indeed, all I used was an iterated inverse image argument to capture the idea of refined targetting. Consequently, any iterative process that admits a graph of the indicated kind must be accompanied by erratic dynamic behavior. To get a better idea of when such dynamics occur, note that the key element of the argument is an *expansion effect*; for the process given by Figure 2, when G_f is restricted to an interval, the image of G_f covered all three intervals. Indeed, without an expansion effect, one would not have any wild movement at all. Because of the *expansion* effect; G_f expands each interval by stretching it to cover all, or parts of the specified intervals, so a mixing of the locations of the iterates becomes possible.

Expansion, by itself, does not permit erratic dynamics to occur. For example, if G_f were monotonic, then the expansion effects would only move the iterates in on direction. Such monotone behavior does not contain any of the surprisingly erratic behavior indicated above. Thus, the other element leading to this behavior is recurrence. Should expansion and recurrence be combined, there is reason to believe that some version of the above iterative dynamics would hold. A word of caution, the combination of these two characteristics does not imply chaos occurs; it only asserts that the appropriate conditions that could cause such motion are there. Thus, one should interpret the coexistence of expansion and recurrence as a "erratic motion alert." (The actual existence of such motion requires a complete argument.)

For any process combining these effects of expansion and recurrence, one should investigate the possibility of chaotic motion. One way these two characteristics are combined is when a graph of the iterative process is multi-modal, as in Figures 2 and 3. Actually, as already suggested by the inverted bowl description, one only needs two regions to create recurrence. From this fact, it should be clear that such processes are common in economics. Indeed, many journal articles studying equilibrium effect of various economic processes are based on graphs of the which have all of the appropriate properties. For any such situation, the equilibrium analysis provides only a small portion of the total story! A more complete analysis requires considering the dynamics.

It now should be clear that "erratic behavior" should be considered the norm, not the unusual situation. Also, we can understand the effects of minor changes in assumptions. A minor change in economic assumptions can lead to a different "graph" of an iterative process. In turn, this can lead to significantly different dictionaries and dynamical behavior.

There is a potential "dark side" where this abundance of behaviors introduces the potential of "abuse." For instance, because so many different kinds of outcomes are possible, one can select among them to justify whatever one wants. However, we now understand that one orbit, one scenario, does not indicate the behavior of a system; a more complete, global analysis is required.

Another kind of potential abuse of these techniques is to "try to explain everything." For instance, once one has mastered the simple ideas associated with the dynamics of chaos and erratic behavior, it is fairly straightforward to adjust the properties of a graph so that the dynamics mimics past observed data. In no way does this mean that the created model explains the data; it just means that out of the many dynamical choices, by starting with data, a model has been created to "fit the data." The test of a model is not that it can be adjusted to fit past data, but that it can predict and explain future behavior. For instance, it probably is possible to modify one of the above figures so that, with an appropriate initial condition, the resulting dynamic mimics the performance of the stock market, some other "random" index, or another event over the past year. However, this fit does not mean that the graph explains the stock market; a test is that very few people would trust its ability to forecast enough to invest money. Nevertheless, it is reasonably certain that papers will appear in the literature that are based on such a data fitting approach and which claim to explain everything imaginable. Because of the "remarkable fit to data," the model may be advanced as an "explanation" of how the stock market, the economy, the firm, and various other organizations behave. This is bad economics and bad mathematics.

The refined targetting argument of Section 2.3 showing how recurrence and expansion can occur together is based on a simple iterated inverse image argument. This argument is not restricted to mappings from the line to the line; it holds equally well for mappings from the plane to the plane, from R^n to R^n, for set valued mappings, and on and on. Indeed, all one needs is the concept of the inverse image and a condition to allow the the intersection of nested, decreasing sets to be non-empty. This is of particular importance to economics and the social sciences because many of the models depend upon correspondences (i.e., set valued mappings) with various continuity properties. The same ideas used in Section 2.3 apply here.

Elsewhere (Saari, 1985) I have described how these ideas of expansion and recurrence combine in such models as optimal growth and messages systems associated with incentive problems. To conclude this section, I offer a new kind of example. The issue, raised in the US Constitution, requires the number of congressmen assigned to each state to depend on the fraction of the total US population residing in that state. As a simple illustration, suppose there are only three states where the population figures are given below.

State	Population	Fraction
A	4520	0.4520
B	4120	0.4120
C	1360	0.1360
Total	10,000	1.0000

The problem can be demonstrated when the house size is 25. Here state A is entitled to 11.3 representatives, B to 10.3 representative, and C to 3.4 representatives. This difficulty, where each state is entitled to a fractional portion of a representative, is the general situation. The question is to determine what kind of "rounding off" procedure should be applied.

One approach, which has been associated with Alexander Hamilton, is to first assign each state its integer value. This leads to A receiving 11 representatives, B receiving 10 representatives, and C receiving 3 representatives. The total number of representatives already assigned is 24; to reach the total of 25 representatives, one seat is to be assigned to some state. The Hamilton assignment procedure is the obvious one; just look at the fractional parts of the precise figure for each state, and then "round up" for that state with the largest fractional part. Here, state C has the largest fractional part of 0.4, so, according to this rule, C receives the extra representative. This is given in the following table.

	House size = 25			
State	Exact Rep	Integer	Frac Part	Assignment
A	11.3	11	0.3	11 + 0 = 11
B	10.3	10	0.3	10 + 0 = 10
C	3.4	3	0.4	3 + 1 = 4
Total	25	24	1.0	25

The Hamilton process, which was used in the USA for about a century, combines both expansion and recurrence. To see this, note that if p^j is the fraction of the total population in the j^{th} state, $j = 1,..,n$, then the exact apportionment for a house size of h is $h\mathbf{p} = h(p^1,..,p^n)$. Now, two different population figures give rise to two different choices of **p**, say **p** and **p**′. The difference in the exact assignments is h(**p**-**p**′), a vector that most clearly expands as the value of h grows.

To see where recurrence arises, notice that the assignment procedure only concerns the fractional part of each allocation. Thus, the fractional part grows from 0 to 1, and when it reaches unity, it starts over again at 0. This means that the important part of the assignment process -- the check on the fractional part -- combines an expansion and a recurrence effect. Consequently one should anticipate the procedure to exhibit erratic behavior.

To see what can happen, return to the above example for h = 26. Because the house now has one more representative, it is interesting to wonder which state receives this extra representative. Here the table is

House size = 26

State	Exact repr.	Integer Part	Frac Part	Final Assign
A	11.752	11	0.752	11 + 1 = 12
B	10.712	10	0.712	10 + 1 = 11
C	3.536	3	0.536	3 + 0 = 3
Total	26	24	2.000	26

In other words, the erratic effects of expansion and recurrence manifest themselves by *reducing* the representation for state C while increasing the representation for A and B. An extra representative gained or lost changes power within Congress, so this paradox is not just a point of curiosity. The example given above is not an isolated one; as I show in (Saari, 1978), if there are more than two states, then almost all population figures will, for some choice of h, have an effect of this sort. Moreover, such problems have occurred in practice. It was first observed when an increase in the house size of the USA Congress would have caused the state of Alabama to lose representation -- thus this kind of outcome is called an "Alabama paradox." After Alabama, the same problem affected other small states. Indeed, this erratic behavior is the reason the USA Congress has 435 representatives. Based on the population figures for 1910, Congress set the new size of Congress at a figure that would avoid this paradox. This number was 433. A seat each was reserved for each of the territories of Arizona and New

Mexico, which brings the total to 435. (For more details, see Balinski and Young, 1976.)

It is natural to wonder whether this allocation paradox can be avoided by using different assignment procedures. For example, start with h = 1 and allocate this first seat to the "most deserving" state. By induction, with h = n, assume that the n seats are assigned. When the house size increased to h = n+1 the extra seat is assigned to the "most deserving" state. Such an approach avoids the Alabama paradox, and this philosophy is widely used today where the differences reside in the adopted measure of "most deserving." Nevertheless, whatever method is adopted, the basic underlying system of "rounding off" remains subjected to the combined effects of expansion and recurrence. So, while there are assignment procedures that avoid the Alabama paradox, one might wonder whether a different kind of erratic behavior emerges. The answer is yes. If there are at least 4 states, then it may be that we round up or down by "too much." Namely, if the exact apportionment is 12.7, then a state is entitled either to 12 or 13 representatives. By using the above "house monotone" methods to avoid the Alabama paradox, there are situations where one rounds either down or up by two; i.e., the actual apportionment either is 11 or 14. This happens in practice, and the French elections are a good source of examples.

As explained in Saari, 1978, it is possible to avoid these paradoxes, but to do so, one must "look to the future." In other words, it turns out that with enough states any continuous procedure based on present and past allocations will have some kind of paradox. However, a procedure based on "future allocations" can avoid some of the difficulties.

3. Control of economic systems

The ideas of chaos and erratic behavior are important theoretical tools to understand what can happen in economic theory. However, an important normative issue is to understand how the various different solutions can be controlled to allow only "desired outcomes." If the goal for a process, such as tatonnement, is to converge to a certain outcome, then one must find ways of eliminating the contrary, erratic outcomes. But, what kind of models are required? What kind of information is needed? As I indicate next, the same kind of global approach can be used to address such issues. I demonstrate the ideas here with tatonnement.

It was shown in Section 2.3 that tatonnement can lead to chaos, even in simple two good trading societies. The issue now becomes to find some kind of price adjustment procedure that converges to an equilibrium. This procedure should be based on the market pressures as reflected by the aggregate excess demand function. Now, trying to develop a procedure can be a complicated issue in theory. This direct approach is complicated by the lack of guidance about what kind of information such a procedure needs so it can be successful. Therefore, I adopt a different approach.

The approach I use is much in the spirit of Arrow's Theorem from social choice. In social choice, one does not attempt to find a social choice function satisfying Arrow's axioms because we know that such a procedure does not exist. I use much the same philosophy. I specify certain basic properties for a large class of adjustment procedures, and the goal is to determine whether any adjustment procedure from this class can achieve convergence. If not, then any procedure based on such properties is doomed for failure. In this way, guidance is obtained to determine what kinds of procedures can, or cannot work.

To illustrate, suppose we are interested in finding a procedure of the form

3.1 $$p_{n+1} = p_n + M(f(p_n), f'(p_n)).$$

The mechanism is determined by the choice of the function M. For instance, if M does not depend upon the second variable, then the mechanism only depends upon the market pressures ($f(p)$), not upon the change in market pressures ($f'(p)$). If $M(u,v) = hu$, then the earlier process of tatonnement is recovered.

As stated, instead of specifying M, my goal is to determine minimal properties an acceptable M should possess. With an emphasis on "minimal properties," a larger set of processes is admitted. Of course, any selected M will have more specialized properties than the minimal list, but its behavior will be as described below.

In the minimal listing of properties, we want M to be i) defined over a set $(0,1)\times(R\setminus V)$ where V is a set of points and where the image is in $(0,1)$. Many choices of M may be defined over all values of the second variable (f'), but others, like $M = -u/v$ may not be defined at certain points, such as $v = 0$. (Notice that $M = -u/v$ is Newton's method for finding a zero of a function.) So, by admitting the possibility of M needing a set V, a larger class of mechanisms is allowed. Second, we require that where M is defined, ii) M is C^1 (i.e., the first partial derivatives of M are continuous). Finally, we want the process to stop at an equilibrium $f(p) = 0$, and only at an equilibrium. This assumes the form that iii) $M(u,v) = 0$ if and only if $u = 0$.

The properties i,ii,iii are sufficiently relaxed to admit models of speculation (based, say, on the values of $f'(p)$), on models of tatonnement where the value of the modifying parameter h is selected in an appropriate fashion based on, say, the values of $f(p)$ and $f'(p)$, and so forth. Thus, this class seems to be sufficiently large so that there should be at least one process where convergence always occurs. Unfortunately, this is not the case.

Theorem 3. For any choice of M satisfying the conditions i), ii), and iii), there exists an open set of economies, Ω, so that if $f \in \Omega$, then there is an open set of initial iterates for which the dynamic in Eq. 3.1 does not converge to a zero of f.

The theorem asserts that the conclusion holds for open sets of economies and starting positions. Thus this nonconvergence behavior is robust. This means that even if examples or starting positions demonstrating the conclusions of the theorem are perturbed, the modified example has the same kind of non-convergence property. Consequently, it follows from the theorem that any economic theory based on point information of the indicated type will not ensure convergence.

What kind of information is required to attain convergence? It is natural to inquire whether added information based on the past iterates and higher order derivatives will succeed. After all, one might develop a model of speculation based on the behavior of the last s iterates of the excess demand function. Such an expanded modelling requires only a slight change in our basic assumptions. So, let s be the number of iterates, k be the number of derivatives. Let iv) M be a C^1 mapping from $(0,1)^s x(R^k \setminus V)$ to $(0,1)$ where V is the finite union of smooth lower dimensional manifolds. Assume that v) $M(x^1,\ldots.x^s;-) = 0$ if and only if $x^1 = 0$. The dynamic defined by process M is given by

3.2 $$p_{n+1} = p_n + M(f(p_n),\ldots,f(p_{n-s});\ \ldots;f^{(k)}(p_n),\ldots,f^{(k)}(p_{n-s})).$$

Thus, if $s = k = 1$, the Eq. 3.2 becomes Eq. 3.1. One might expect that using all of this added information, a convergent procedure can be developed. This is not the situation.

Theorem 4. Let s, k be given positive integers. If M is a process satisfying iv) and v), then there exists an open set of economies, Ω, so that if $f \in \Omega$, then there is an open set of initial prices such that the dynamic given by Eq. 3.2 never converges to a zero of f.

It follows now, as before, that any finite amount of information of the above kind does not suffice for convergence. This means that robust examples can be found to frustrate any price adjustment theory based on this kind of point information. However, one might hope that even though the system refuses to converge, maybe the dynamic stays close enough to a zero that it suffices for practical purposes. This is not the case. As indicated in the proof, the motion can be bounded away from the equilibria.

The problem of non-convergence remains. Moreover, this same issue, for much the same mathematical reasons, occurs for several models involving message systems, incentives, etc. Therefore, the issue is fundamental, and one must find what can be done about it. Theorem 4 asserts that no theory based on finite point information will succeed. Consequently a successful theory must use more global kinds of information. This is the theme of Saari and Williams (1986) where we show one way to achieve convergence is to use several different choices of mechanisms. The basic idea can be described in terms of an analogy with social choice.

One way to avoid the consequences of Arrow's Theorem is to impose a restriction on the domain -- to define a social choice function satisfying specified properties for certain sets of profiles. So, different sets of profiles are associated with different choice functions. This suggests an approach to circumvent Arrow's theorem. Suppose it is possible to divide the space of all profiles into several different sets with the following property. Each profile is in at least one set, and for each set of profiles there is a social choice procedure that satisfies the specified properties. This means that several procedures are developed. To decide which procedure is the appropriate one, we need some crude information to determine how to assign a specific profile to an appropriate class of profiles. This assignment is equivalent to selecting the appropriate choice function. It is the choice function that gives the refined information about the selected alternative.

The same approach is used in Saari and Williams (1986). The space of economies is divided into (a finite number of) different sets. Over each set, there is a locally convergent mechanism. The "global information" concerns which set an economy belongs to. Once the global information is learned, then the adjustment mechanism is selected to find the refined information about the equilibrium. In other words, we now know that the goal of finding a universal mechanism cannot be realized. What can be realized is the slightly modified goal of finding a finite set of mechanisms that then cover most economies. Upon reflection, this approach seems natural. For example, consider price adjustment procedures. Why shouldn't we expect that different procedures are required for different circumstances? Why shouldn't one version work in Brazil, a second one in Western Europe, and a third in medieval times? This is suggested by the theoretical implications in the reference (Saari and Williams).

In Saari and Williams, it is shown that a finite number of sets of economies can be defined. But, in this reference it is not shown how to do so efficiently. In other words, what kind of information is required and how should it be used in this division process remains an open question. Our theorem only specifies the conditions under which such an approach works. It does not indicate how to take a specific model and divide the space of economies into the different sets. Chen (1989) has some interesting results in this direction for priced adjustment procedures. His characterization is in terms of the properties of the utility functions.

Of interest to these notes, the proof of Theorem 4 relies on concepts of stability and on modifications of the more global, geometric proof discussed earlier.

Outline of the proof of Theorem 3. The proof of this theorem is given in Saari (1985, 1987a) based on ideas from Saari and Urenko (1984). The basic ideas are described here. The idea of the proof is to use the notions of stability given in Section 2.2.

The function M is the adjustment process; it determines how much is added or subtracted from the price p_n. Find two sets of values (u^j, v^j) so that

3.3 $$0 < M(u^1, v^1) = -M(u^2, v^2) < 1/2.$$

That such points can be found involves an argument. The first part of the argument requires showing that M must have different signs. Once it is established that M has both signs, an intermediate value argument using the continuity of M near a zero can be used to show that the values of (u^j, v^j) can be found. To show that M has both signs, note that if M has only one sign, then the adjustment of prices is only in one direction. To show that such an M satisfies the negative conclusion of the theorem, choose any economy with a single equilibrium. For any initial iterate on the appropriate side of the equilibrium, the procedure must move the prices further away from the single price equilibrium.

If $f(p_1) = u^1$, $f'(p_1) = v^1$, then $p_2 - p_1 = M(u^1, v^1)$. Select any value for p_1, and define p_2 as above. Now, by choosing $f(p_2) = u^2$, $f'(p_2) = v^2$, we have that the next iterate, p_3 is given by $p_3 - p_2 = M(u^2, v^2) = -M(u^1, v^1) = -(p_2 - p_1)$. This means that $p_3 = p_1$ and that the dynamic forms a periodic orbit bouncing between the values of p_1 and p_2. It follows from the Sonnenschein, Mantel, Debreu theorem that such an f can be found.

This argument establishes that there are two initial points that refuse to converge to a zero of f because they form a period two orbit. Notice that there is considerable freedom in choosing this period two orbit; freedom that can be used to bound the orbit a fixed distance from the equilibrium. This freedom comes from the flexibility in the choice of the p_i's based on the choice of the values of u^j and the value of M.

What remains is to show that f can be selected so that the lack of convergence holds for an open set of initial iterates. This is done by using stability. Let $G(p) = p + M(f(p), f'(p))$. With the above choice of f, we have that $G(p_1) = p_2$ and $G(p_2) = p_1$. Thus, $G^2(p_1) = G(G(p_1)) = G(p_2) = p_1$. (Likewise $G^2(p_2) = p_2$.) This means that p_1 and p_2 are fixed points for the composed mapping G^2. If these equilibria of G^2 are stable, then it follows that all nearby points contract toward p_1. For the mapping G, this means that these orbits would approach the periodic orbit of p_1 and p_2. In other words, after sufficient number of iterates, one could not distinguish the orbits starting from an open set of nearby points with the periodic orbit.

To obtain the stability, it follows from Proposition 1 that it suffices to find conditions so that $|(G^2(p_1))'| < 1$. According to the chain rule, $(G^2(p_1))' = G'(G(p_1))G'(p_1) = G'(p_2)G'(p_1)$. Thus, the conclusion holds if it can be shown that $|G'(p_j)| < 1$ for $j = 1, 2$. However, $G'(p_j) = 1 + M_1(u^j, v^j)v^j + M_2(u^j, v^j)f''(p_j)$, where M_k denotes the partial derivative with respect to the k^{th} variable. The only free variable is the value of $f''(p_j)$, so choose these values so that the derivative condition on G′ is satisfied. Again, for any such choice, the Sonnenschein, Mantel, Debreu Theorem assures us that an economy exists with these properties. This completes the outline of the proof.

4. Paradoxes and aggregation procedures

By definition, a paradox is a surprising, unexpected outcome, so it can be viewed as being an "erratic outcome" of a particular system. One important, central tool of the social sciences where such outcomes occur with great frequency is in aggregation procedures. As such, one might wonder whether certain classes of aggregation "paradoxes" can be understood by using the iterated inverse image methods developed in Section 2. They can.

To illustrate, consider an election with the three candidates (a,b,c). Suppose a plurality election is held to determine who should be awarded the one tenure track position in a department, and suppose that the outcome is a>b>c. (In other words, a wins, b comes in second place, and c loses.) Next suppose that c withdraws from consideration, leaving only the candidates (a,b). It seems clear that since c was the bottom ranked candidate, nothing important has changed; a should still be declared the winner. Or, should she? Could it be that the sincere election outcome is b>a? Suppose it is b that withdraws from the election process. Could it be that the sincere election outcome now ranks c>a? In general, to understand the possible relationships among the election outcomes, one might wants to mimic what was done in dynamical systems by determining *everything that can occur.* I outline how this is done.

To start, consider how one would determine whether the outcome a>b>c and b>a could be a sincere consequence of voting. The standard approach used in the literature is to create an example illustrating this effect. Usually such examples are created by trial and error. One starts with a listing of a certain number of voters that have various kinds of ranking, and then determines whether the outcome is the desired one. If so, we are done; if not, then we need to modify the example and try again. This is the "precision computation" approach described earlier. Instead of using this technically difficult approach, I use a version of the iterated inverse image approach used for the dynamics.

The domain is the space of all listings of voter preferences. With the three candidates (a, b, c), there are six types of voters, where a voter type is determined by how the voter ranks the three candidates. In the following listing, x^j denotes the number of voters that are of the j^{th} type.

Type.	Ranking	Type.	Ranking
x^1	a>b>c	x^4	c>b>a
x^2	a>c>b	x^5	b>c>a
x^3	c>a>b	x^6	b>a>c

Now, an election can occur for any set of two or three candidates. The four possibilities are $S^1 = (a,b,c)$, $S^2 = (a,b)$, $S^3 = (a,c)$, and $S^4 = (b,c)$. If the tally of an election is expressed in vector format then the tally of a plurality election for S^1 is

4.1. S^1 $(x^1 + x^2,\ x^5 + x^6,\ x^3 + x^4)$

Here, the first coordinate, $x^1 + x^2$, represents the tally for a, the second for b, and the third for c. Indeed, in the tallies given for any set, let the component be determined in the order the candidates appear in the definition of the set. Thus, the majority vote tally for S^j is

4.2. S^2 $(x^1 + x^2 + x^3,\ x^4 + x^5 + x^6)$

4.3. S^3 $(x^1 + x^2 + x^6,\ x^3 + x^4 + x^5)$

4.4. S^4 $(x^1 + x^5 + x^6,\ x^2 + x^3 + x^4)$

The idea is the following. Start with the *complete indifference point* where $x^j = n$ for all choices of j. This causes a tie election for each of the four sets. Next, write down some listing of the election rankings for which one wants an example. To find a profile that realizes the specified election outcome, one just modifies -- add to or subtract from -- the coordinates of the complete indifference point. To illustrate, suppose the desired example is {a>b>c, b>a, c>a, c>b}. (This is the most difficult case.) To obtain the ranking for S^1, I need to

4.5a. add values e,f so $(x^1 + x^2) = 2n + e$, and $x^5 + x^6 = 2n + f$. To obtain the a>b>c ranking of S^1, we need

4.5.b $e > f$.

This choice is a partial characterization of the domain (the space of profiles), or the target region, which leads to a>b>c.

Next, turn to the ranking b > a, which is governed by (4.2). As asserted, the choices already made to obtain the ranking a>b>c should be thought of as part of the inverse image of this ranking. These choices change (4.2) to $(2n + e + x^3, x^4 + 2n + f)$. Because we are keeping the value $x^3 + x^4$ fixed at 2n (for S^1), let $x^3 = n - g$ and $x^4 = n + g$. Therefore, to obtain the b>a ranking for S^2, we need

4.6. $e - g < f + g$, or $e < f + 2g$.

These choices correspond to the second step of the iterated inverse image; I find the intersection of the inverse image of a>b>c and of b>a.

Now turn to the ranking c>a; this defines a target region for S^3. The expression for S^3, (3), becomes $(2n + e + x^6, 2n + x^5)$. To make the second coordinate larger, while respecting our earlier choice of $x^5 + x^6 = 2n + f$ (see 4.5), choose $x^6 = n - h$. This gives $(3n + e - h, 3n + f + h)$, so the restriction on h is

4.7. $e < f + 2h$

These restrictions correspond to a partial characterization of the inverse image of the first three rankings.

The last ranking defines the target region of c > b, which is given by the coordinates of (4.4). Our choices change this to $(x^1 + 2n + f, x^2 + 2n)$. To make the second component larger than the first, just give all of the increment of e to x^2. This makes the coordinates $(3n + f, 3n + e)$. As e>f, we get the ranking c>b.

In summary, any choice of $x^1 = n$, $x^2 = n + e$, $x^3 = n - g$, $x^4 = n + g$, $x^5 = n + f + h$, and $x^6 = n - h$ has the desired election outcome. The only restrictions on the four perturbation terms are that they are all positive, none larger than n, and that

4.8 $f < e < f + 2g$, and $e < f + 2h$.

These conditions can be viewed as describing a portion of the intersections of the inverse images of the four rankings. It is trivial, now, to find many different examples that yield this listing of election rankings. As an illustration, if g=h=n=e=2, f = 1, we get the example $(x^1, x^2, x^3, x^4, x^5, x^6) = (2,4,0,4,5,0)$.

Examples illustrating many other kinds of paradoxes can be constructed in a similar manner. (For a general description, see (Saari, 1988, 1987b).) As an example from statistics, consider the decision problem of choosing one of two urns, I´ or II´, that contain red and blue marbles. Once an urn is selected, then a marble is chosen at random. A red marble means the player wins a new car; a blue means the player gets the bill for the car, but no car. Now, with the added information $Pr(R|I') > Pr(R|II')$ (the probability of getting a red from urn I′ is greater than the probability of getting a red from II′), the decision is obvious -- choose urn I′. Next, suppose that a second set of urns, I" and II", is given with the similar information $Pr(R|I") > Pr(R|II")$. With either set of urns the answer is the same, choose the first urn. The aggregation problem arises when the contents of "good" urns I′ and I" are emptied into urn I, while those of the "bad" urns I" and II" are emptied into urn II. Which urn is better, I or II? The same approach described above can be used to show that there are situations where either outcome occurs. (An example where $Pr(R|I) < Pr(R|II)$ is if the contents of first four urns are, respectively, (9,24), (2,6), (3,6), and (11,24). Here, the first component of a tuple is the number of red marbles while the second is the total number of marbles in an urn.)

The above description creating the voting paradox uses nothing more than the iterated, inverse image argument of Section 2 where the target regions are defined by the specified rankings of the different sets. A natural question is whether this approach can provide a listing of everything that can occur. It can. To indicate the kind of results that are possible, assume there are $n \geq 3$ candidates given by the set $\{c_1, \ldots, c_n\}$. From this set of candidates, 2^n different subsets can be defined. However, one of these subsets is the empty set, and n of them consist of only one candidate. This means that there are $2^n-(n+1)$ subsets of candidates where it makes sense to hold an election. As above, list these subsets in some order as $S_1, \ldots, S_{2^n-(n+1)}$.

Each of these sets can be ranked in many different ways. So, let R_j consist of all possible rankings of the candidates of S_j. For instance, if $S_j = \{a,b\}$, then $R_j = \{a>b, b>a, a=b\}$. If S_j consists of 3 candidates, then R_j has 13 rankings -- 3! = 6 are rankings where there is no tie between candidates, 6 are rankings where there is a tie between two of the candidates, and the last one is the ranking where all three candidates are tied. Following the lead of Section 2, let $U^n = R_1 x \ldots x R_{2^n-(n+1)}$ be the *universal set*. In other words, an element of U^n is a listing of rankings; there is precisely one ranking for each subset of candidates. Moreover, all possible listings of rankings of this kind are in the universal set.

A *profile* is a listing of how many voters rank the candidates in a particular manner. For example, in the three candidate election discussed above, a profile, **p**, is any listing of $(x^1,\ldots,x^6)$ where x^j is a non-negative integer. For n candidates, there are n! ways the candidates can be listed (without ties). Thus a profile becomes a vector $\mathbf{p} = (x^1,\ldots,x^{n!})$ where x^j is a non-negative integer that denotes the number of voters that have the j^{th} ranking of the candidates.

For a given profile **p**, let $f_j(\mathbf{p})$ be the election ranking for the set of candidates S_j. Thus, if $f(\mathbf{p}) = (f_1(\mathbf{p}),\ldots,f_{2^n-(n+1)}(\mathbf{p}))$, then $f(\mathbf{p})$ is a listing of rankings where there is one ranking for each set of candidates. This means that $f(\mathbf{p}) \in U^n$; it is the listing of the plurality election rankings associated with the profile **p**.

To determine everything that can occur, we follow the lead of Section 2.3 to define a *dictionary*. The dictionary, as for dynamics, is the listing of everything that can occur for the specified system -- here the system is plurality voting. Therefore, call the set

4.9 $D^n = \{f(\mathbf{p}) \in U^n \mid \mathbf{p} \text{ is a profile}\}$

the *dictionary for plurality voting*. In this description of a dictionary, the profile **p** plays the role of the initial iterate in the price adjustment process. The precise election tallies for each set of candidates play the role of the subsequent iterates of the adjustment process. The listing of election rankings, $f(\mathbf{p})$, plays the role of the sequence of addresses for the iterates of the adjustment process.

As D^n is a subset of U^n, one wants to learn whether it is a large subset, or a small subset. The answer is given in the next statement.

Theorem 5. For any value of $n\geq 3$, $D^n = U^n$.

This theorem, which is proved and generalized to all positional voting methods in Saari (1989a,b), asserts that *anything can happen in sincere election rankings*. Because of this theorem, we now know that one can write down any paradox, any listing of election rankings, and there exists a profile so that the designed listing actually occurs! This means that the listing of voting paradoxes found in the literature (which were determined by what I call the "precision computation approach") does not even begin to suggest the complexity and the wealth of what can occur. Conversely, as this theorem specifies everything that can happen, it includes all possible paradoxes. As such, it eliminates a lucrative source of potential papers based on creating "still another interesting paradox." For instance, because D^6 has over 10^{60} entries, many lifetime careers could be based strictly on constructing 6 candidate plurality election examples.

The proof of this theorem is beyond the scope of these notes, but the basic idea is the one used throughout. It is the more global, iterated inverse image approach. As with the introductory example, one uses the ranking for each subset as a "target region". Starting with the inverse image of f_1, the set of profiles that achieve the first step of the future is determined. Next, a refinement is determined. This is the refinement of this subset of profiles that permits the target region for the second subset of candidates to be realized. This continues. The technical part of the proof is to demonstrate that the refined targetting approach leads to a non-empty intersection of profiles for any listing in U^n. Thus, this approach becomes most similar to that of Section 2.

. There are some differences in the proofs for aggregation procedures and the dynamics given earlier. As these differences admit intuition about when an aggregation process admits "paradoxes," I illustrate some of the basic ideas with the above statistical decision problem. For the four urns I´, II´, I", II", let x_1, x_2, x_3, x_4 denote, respectively, the fraction of red marbles in the urn. To determine $Pr(R|I)$ and $Pr(R|II)$, we need to know the fraction of all marbles in each urn, so let d_1, d_2, d_3, and d_4 denote these values. Thus, according to Bayes´ rule, $Pr(R|I) = (x_1d_1 + x_3d_3)/(d_1 + d_3)$ and $Pr(R|II) = (x_2d_2 + x_4d_4)/(d_2 + d_4)$.

Let $F(\mathbf{x},\mathbf{d}) = (x_1 - x_2, x_3 - x_4, Pr(R|I) - Pr(R|II))$. As the sign of each component determines which urn is preferred, the universal set, U, consists of all three-tuples where an entry is one of {+, 0, -}. There are 27 such tuples in U. Notice that F can be viewed as being a mapping from a seven dimensional space (the x_j´s define 4 degrees of freedom while the d_j´s define 3 degrees), to R^3. Each symbol from U defines which orthant (or portion of a coordinate plane) contains the precise value of F.

With the above representation, the dictionary for F can be computed. This is because the ranking $\mathbf{0} = (0,0,0)$ is a boundary point for each of the other regions. Now suppose an interior point $\mathbf{q}$ can be found in the domain where $F(\mathbf{q})$ =

0. If the Jacobian of F at **q** has maximal rank, then F takes an open set about **q** and maps it to an open set V about **0**. But V has to meet all of the regions of R^3. Consequently, the dictionary of F agrees with the universal set; all outcomes are possible. Indeed, because the domain still is larger dimensional than the range, one might expect that even more paradoxical behavior can occur. This is correct.

The main point of this argument is that the Jacobian of F replaces the geometric considerations of Section 2. With little thought, it becomes clear that the rank condition reflects a heterogeneity of the domain. As the domain most often models the space of profiles or preferences of the voters, we see that the heterogeneity can admit all possible kinds of paradoxes. This argument, based on the Jacobian of F, is described in greater detail in (Saari, 1987b, 1988) with extensions to discrete mappings (such as Arrow's Theorem) in (Saari, 1989c).

It appears that this approach of refined targetting, of the iterated inverse image, will provide much deeper insights into the behavior of aggregation procedures. Indeed, as a concluding comment, I argue that the Sonnenschein, Mantel, Debreu Theorem can be interpreted in this manner. In the setting of tatonnement, each person's "input" is given by his initial allocation and utility function. The output is the aggregated excess demand function. Thus, the natural question is to determine everything that can occur.

The program is first to find a universal space. Here U^n is the space of all functions f that satisfy Walras' law for n commodities (which is derived in the following appendix). The mapping, f, starts with the agents' initial endowments and utility functions as inputs and has the aggregate excess demand function as the output. Therefore, the dictionary D^n is the subset of U^n which contains all possible images of f. As true all through these notes, the issue is to determine D^n. The Sonnenschein, Mantel, Debreu Theorem asserts that with some technical conditions, $D^n = U^n$; a statement that the reader has learned can be a common conclusion for many different processes. As true with dynamics and voting, this assertion means that there is an infinite complexity in the listing of the outcomes.

With hindsight, one should expect this result characterizing the excess demand function. After all, this aggregation procedure has an infinite dimensional domain just for each agent. While there is only a finite degree of freedom in selecting the initial endowment, there is a infinite dimensional function space worth of choices for the concave utility functions. The aggregation process has to map this product of function spaces to the smaller space of demand functions, so one should expect the possibility of a conclusion of this sort. Moreover, proofs of the Sonnenschein, Mantel, Debreu Theorem, such as the one given in Debreu, are, at least conceptually, much the same as the basic ideas used here. In Section 2.3, we started with an arbitrary word from the universal set (i.e., an arbitrary listing determining the future iterates of the tatonnement process), and we showed that there exists an initial iterate leading to this future. Debreu starts with an arbitrary element of U^n -- a function satisfying Walras´ Law -- and he shows there exists an initial point (choices of agents´ initial allocations and utility functions) that gives rise to this excess demand function.

Appendix: Tatonnement

To start the description of tatonnement, assume there are $c \geq 2$ commodities with $a \geq 2$ agents in a simple exchange economy. The individual holdings, or *initial endowments* of the i^{th} agent are represented by a point $\mathbf{w}^i = (w^i_1, .., w^i_c)$. Here, w^i_j represents the number of units of the j^{th} commodity that are held by the i^{th} agent. The object is to determine under what circumstances would agent i be willing to exchange goods to obtain a more preferred holding, or *commodity bundle*, from R^c_+. Such a change requires agent i to prefer the new commodity bundle to the old one, so we need to impose an ordering - a preference relationship - on the points in R^c_+.

While there is a natural ordering of points on the line, there does not exist a natural ordering for points in a higher dimensional space. Instead, there are an infinite number of different approaches. Therefore, such a preference ordering must be adopted for each agent. A standard way to do this is to borrow the ordering from the real line by assuming an agent´s preferences are determined by a "utility function" $U_i: R^c_+ \text{-->} R$, where larger values of $U_i(\mathbf{x})$ imply a higher utility or desirability. The level sets of U_i determine an indifference relationship; agent i is indifferent among commodity bundles on a level set. Standard simplifying assumptions are that U_i is a smooth (i.e., it is differentiable), concave function where all of the components of ∇U_i are positive. This last assumption on ∇U_i captures the notion that "more is better" in the sense that if $\mathbf{x} > \mathbf{y}$ (i.e., if each component of $\mathbf{y}$ is bounded below by the corresponding component of $\mathbf{x}$), then $\mathbf{y}$ is preferred to $\mathbf{x}$ because $U_i(\mathbf{y}) \geq U_i(\mathbf{x})$.

Assume there is a fixed amount of goods in this economy where nothing is consumed or produced. Thus, the only way the i^{th} agent can obtain a more preferred commodity bundle is to trade with the other agents. This trading requires each agent to impose a relative weighting on his holdings to determine how much of certain goods he'll trade away in order to obtain more of other goods. Of course, this weighting depends on what is available from the other agents; i.e., it depends upon the relative weights adopted by the other agents. A way to coordinate this weighting among the agents is to impose a universal one in terms of *prices*. A price is given by a vector $\mathbf{p} = (p_1, .., p_c)$ where p_j is the price of one unit of the j^{th} commodity. With the prices, the worth or value to the i^{th} agent of the initial endowment $\mathbf{w}^i$ is given by the scalar product

$$(\mathbf{p}, \mathbf{w}^i) = \Sigma_j \; p_j w^i{}_j.$$

The optimization problem facing agent i is to get the best *affordable* deal. Thus, the *budget constraint* requires him to consider only those commodity bundles, $\mathbf{x} \in R^c{}_+$, for which

$(\mathbf{p}, \mathbf{x}) \leq (\mathbf{p}, \mathbf{w}^i)$, or $(\mathbf{p}, \mathbf{w}^i - \mathbf{x}) \leq 0$. The budget plane is where equality is achieved.

The budget plane has a particularly simple representation; it is the unique plane passing through the point $\mathbf{w}^i$ with $\mathbf{p}$ as a normal vector. Therefore, the affordable commodity bundles at price $\mathbf{p}$ are those on or below this plane. (See Figure 4.) The i^{th} agent's optimization problem of choosing his most preferred, affordable commodity bundle is to choose the bundle that maximizes his utility function subject to the budget constraint. Now, either by using elementary calculus and Lagrange multiplier techniques, or by using simple geometry as indicated in Figure 4, it follows that this maximum point, $\mathbf{x}^i$, is where the level set of the utility function is tangent to the budget plane. Because of this tangency, it follows that $\nabla U_i(\mathbf{x}^i)$ is a scalar multiple of $\mathbf{p}$.

What agent i wants at prices $\mathbf{p}$ is $\mathbf{x}^i$, what he has is $\mathbf{w}^i$, so the *excess demand vector*, $\tau^i(\mathbf{p}) = \mathbf{x}^i - \mathbf{w}^i$, is what he wants to trade. If a coordinate of $\tau^i(\mathbf{p})$ is negative, then the value indicates how much of this good the agent wants to sell; if it is positive, then the value indicates how much of this good the agent wants to buy. According to the optimization scheme, this vector is in the budget plane, so, by definition of the plane, $\tau^i(\mathbf{p})$ is orthogonal to the normal vector $\mathbf{p}$.

Namely,

$$(\mathbf{p},\ \tau^i(\mathbf{p})) = 0.$$

Whether the agent is able to do this depends upon whether the choices of the other agents -- is there a market for the goods? The obvious way to answer this question is to sum the individual excess demands, $\tau^i(\mathbf{p})$ to determine whether the total is zero (i.e., whether the markets will clear). This leads to the definition of the *aggregate excess demand function*

$$\tau(\mathbf{p}) = \Sigma_i \ \tau^i(\mathbf{p}).$$

The object is to find the price, or universal weighting, so that $\tau(\mathbf{p}) = \mathbf{0}$. By virtue of Equation 2.3, we have the important Walras´ law which asserts that for all choices of $\mathbf{p}$,

$$(\mathbf{p},\ \tau(\mathbf{p})) = 0.$$

As a final observation, note that all of the above holds if $\mathbf{p}$ is replaced with $\mathbf{p}/|\mathbf{p}|$. One reason is that $\mathbf{p}$ determines a weighting, but only the relative weighting between the commodities is needed. This is reflected by the role $\mathbf{p}$ plays in the definition of the budget plane; $\mathbf{p}$ is a normal vector. However, any scalar multiple of $\mathbf{p}$ also serves as a normal vector. This fact is used to reduce the system.

This scaling of $\mathbf{p}$ has occurred in practice. For instance, Brazil is one of the countries that during the 1980´s was subjected to an incredible high rate of inflation. As one of the less serious consequences, the Brazilian currency became severely devalued in just a short time. This change in the value of the currency created a practical problem; people would do everyday business, like buying groceries, with bills having a large number of zeros. In response, in 1984 the government just instructed the people to ignore a certain number of the zeros in a bill. Thus, the decree that a 10,000 cuzaro bill from now on will be considered a 10 cuzado bill corresponds to using a scalar multiple of 1/1,000.

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