Industriens Utredningsinstitut
THE INDUSTRIAL INSTITUTE FOR ECONOMIC AND SOCIAL RESEARCH

A list of Working Papers on the last pages

Forthcoming in *Journal of Economic Theory.*

No. 427, 1994

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December 1994
Revised, August 1995

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THE MASS-ACTION INTERPRETATION OF NASH EQUILIBRIUM

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August 25, 1995

ABSTRACT. Nash's [16] "mass action" interpretation of his equilibrium concept for non-cooperative games, boundedly rational players are repeatedly and randomly drawn from large populations to play the game, one population for each player position. The players are assumed to base their strategy choice on the strategies' observed "relative advantage." This note formally examines this interpretation in terms of a few classes of population dynamics based on imitative and innovative adaptation, and innovative adaptation with memory, respectively. Extending some results in evolutionary game theory, connections between long-run aggregate behavior and Nash equilibrium are established.

1. THE 'MASS ACTION' INTERPRETATION

In his unpublished Ph.D. dissertation, John Nash in fact provided two interpretations of his equilibrium concept for non-cooperative games. In the first interpretation, which became the conventional, one imagines that the game in question is played only once, that the participants are "rational," and that they know the full structure of the game, in fact that this is common or mutual knowledge (see e.g. Aumann and Brandenburger [1]). However, Nash comments: "It is quite strongly a rationalistic and idealizing interpretation" ([16], p. 23).

The second interpretation, which Nash calls the *mass-action* interpretation, was until recently largely unknown (Leonard [11], Weibull [23], Björnerstedt and Weibull [3]). Here Nash imagines that the game in question is played over and over again by participants who are not necessarily "rational" and who need not know the structure of the game:

*This is a revised version of a paper presented in the seminar *On the Importance of Nash's Contribution to Game Theory*, held at the Swedish Academy of Sciences on December 8, 1994. I thank Immanuel Bomze, Harald Lang and Assar Lindbeck for helpful comments to the earlier version. The current version was presented at the conference *Dynamic Evolutionary Game Theory in Biology and Economics*, Wilfrid Laurier University, August 15-19, 1995. The research was sponsored by the Industrial Institute for Economic and Social Research (IUI), Stockholm.

> "It is unnecessary to assume that the participants have full knowledge of the total structure of the game, or the ability and inclination to go through any complex reasoning processes. But the participants are supposed to accumulate empirical information on the relative advantages of the various pure strategies at their disposal.
>
> To be more detailed, we assume that there is a population (in the sense of statistics) of participants for each position of the game. Let us also assume that the 'average playing' of the game involves n participants selected at random from the n populations, and that there is a stable average frequency with which each pure strategy is employed by the 'average member' of the appropriate population.
>
> Since there is to be no collaboration between individuals playing in different positions of the game, the probability that a particular n-tuple of pure strategies will be employed in a playing of the game should be the product of the probabilities indicating the chance of each of the n pure strategies to be employed in a random playing" ([16], pp. 21-22.)

Nash notes that if s_i is a population distribution over the pure strategies $\alpha \in A_i$ available to the i'th player position, then $s = (s_i)$ is formally identical with a mixed strategy profile, and the expected payoff to any pure strategy α in a random matching between an n-tuple of individuals, one from each player population, is identical with the expected payoff $\pi_{i\alpha}(s)$ to this strategy when played against the mixed strategy profile s:

> "Now let us consider what effects the experience of the participants will produce. To assume, as we did, that they accumulated empirical evidence on the pure strategies at their disposal is to assume that those playing in position i learn the numbers $\pi_{i\alpha}(s)$. But if they know these they will employ only optimal pure strategies, i.e., those pure strategies [...] such that $\pi_{i\alpha}(s) = \max_\beta \pi_{i\beta}(s)$. Consequently, since s_i expresses their behavior, s_i attaches positive coefficients only to optimal pure strategies, [...]. But this is simply a condition for s to be an equilibrium point.
>
> Thus the assumption we made in this 'mass-action' interpretation lead to the conclusion that the mixed strategies representing the average behavior in each of the populations form an equilibrium point." (op. cit., p. 22)[1]

Nash makes the reservation, though, that

[1] Nash denotes payoffs with a Roman p instead of, as here, a Greek π.

"Actually, of course, we can only expect some sort of approximate equilibrium, since the information, its utilization, and the stability of the average frequencies will be imperfect." (op. cit., p. 23).

These remarks suggest that Nash equilibria could be identified as stationary, or perhaps dynamically stable, population states in dynamic models of boundedly rational strategy adaptation in large strategically interacting populations. In spirit, this interpretation is not far from Friedman's [5] subsequent "as if" defence of microeconomic axioms. For just as Nash argued that boundedly rational players will adapt toward strategic optimality, Friedman argued that only profit maximizing firms will survive in the long run under (non-strategic) market competition. Moreover, the view that games are played over and over again by individuals who are randomly drawn from large populations was later independently taken up by evolutionary biologists (Maynard Smith and Price [13], Taylor and Jonker [21]). Certain recent contributions to evolutionary game theory can be said to theoretically examine the link from evolutionary selection to economic rationality in well-defined strategic environments. This paper tries to draw this picture and to go somewhat beyond the current research frontier.

2. Notation and Preliminaries

Consider a finite n-player game G in normal (or strategic) form. Let $I = \{1,..,n\}$ be the set of *player positions* in the game, A_i the pure-strategy set of player position i, S_i its mixed-strategy simplex, and $S = \times_{i \in I} S_i$ the polyhedron of mixed-strategy profiles. For any player position i, pure strategy $\alpha \in A_i$ and mixed strategy $s_i \in S_i$, let $s_{i\alpha}$ denote the probability assigned to α. A strategy profile s is called *interior* (or *completely mixed*) if *all* pure strategies are used with positive probability. The expected payoff to player position i when a profile $s = (s_1,..,s_n) \in S$ is played will be denoted $\pi_i(s)$, while $\pi_{i\alpha}(s)$ denotes the payoff to player position i when the player in this position uses pure strategy $\alpha \in A_i$ against the profile $s \in S$. Hence, the game G can be summarized as the triplet (I, S, π), and a strategy profile $s \in S$ is a *Nash equilibrium* of G if and only if $s_{i\alpha} > 0$ implies $\pi_{i\alpha}(s) = \max_{\beta \in A_i} \pi_{i\beta}(s)$.

In the spirit of the "mass action" interpretation, imagine that the game is played over and over again by individuals who are randomly drawn from (infinitely) large populations, one population for each player position i in the game. A *population state* is then formally identical with a mixed-strategy profile $s \in S$, but now each component $s_i \in S_i$ represents the distribution of pure strategies in player population i, i.e., $s_{i\alpha}$ is the probability that a randomly selected individual in population i will use pure strategy $\alpha \in A_i$ when drawn to play the game G. A pure strategy $\alpha \in A$ will be said to be *extinct* in population state if its population share $s_{i\alpha}$ is zero. A state in which no pure strategy is extinct is called *interior*.

In this interpretation $\pi_{i\alpha}(s)$ is the (expected) payoff to an individual in player population i who uses pure strategy α, an "α-strategist," and $\pi_i(s) = \sum_\beta s_{i\beta}\pi_{i\beta}(s)$ is the average (expected) payoff in player population i, both quantities being evaluated in population state s.

Suppose that every now and then (say, according to a statistically independent Poisson process) each individual reviews her strategy choice. By the law of large numbers the aggregate process of strategy adaptation may then be approximated by deterministic flows, and these may be described in terms of ordinary differential equations. We will consider a few novel classes of such population dynamics.

3. Imitative Adaptation

First assume that individuals switch only to strategies that are already in use, and that they do so only on the basis of these strategies' current performance. Technically, this means that we have a population dynamics in the form

$$\dot{s}_{i\alpha} = g_{i\alpha}(s)\, s_{i\alpha} \tag{1}$$

for some functions $g_{i\alpha} : S \to \mathbb{R}$. The quantity $g_{i\alpha}(s)$ thus represents the *growth rate* of the population share of α-strategists in player population i when the overall population state is s. All growth rate functions $g_{i\alpha}$ are assumed to be Lipschitz continuous, implying the existence and uniqueness of a solution to the system of ordinary differential equations (1) through any initial population state $s \in S$. Moreover, we assume that $\sum_{\alpha\in A_i} g_{i\alpha}(s)s_{i\alpha} = 0$ for all player positions i and population states s. This implies that all solutions starting in S stay forever in S at all times. Moreover, one can show that the solution through any interior population state remains interior at all times.

Note that if all individuals in each player population initially happen to use only one pure strategy then they will all continue doing so forever, in any dynamics in the form (1) and irrespective of whether some extinct strategy yields a higher payoff or not. Thus contrary to the suggestion in Nash's mass action interpretation, stationarity in dynamics in the form (1) is not a sufficient condition for a population state to constitute a Nash equilibrium.

A prime example of a dynamics in the form (1) is the *replicator dynamics* used in evolutionary biology (Taylor and Jonker [21], Taylor [22]). In this literature, pure strategies represent genetically programmed behaviors, reproduction is asexual, each offspring inherits its parent's strategy, and payoffs represent reproductive fitness. Thus $\pi_{i\alpha}(s)$ is the number of (surviving) offspring to an α-strategist in population i, and $\pi_i(s)$ is the average number of (surviving) offspring per individual in the same population. In the standard version of this dynamics (Taylor [22], Hofbauer and Sigmund [9]), each pure strategy's growth rate is proportional to the absolute difference between its current payoff and the current average payoff in its player population:

$$g_{i\alpha}(s) = \pi_{i\alpha}(s) - \pi_i(s). \tag{2}$$

An alternative version (Maynard Smith [14], Hofbauer and Sigmund [9]) presumes $u_i(s) > 0$ and has growth rates proportional to the relative difference between its current payoff and the current average payoff in its player population:

$$g_{i\alpha}(s) = \pi_{i\alpha}(s)/\pi_i(s) - 1. \tag{3}$$

We will here consider a class of dynamics (1) that contains both versions of the replicator dynamics, and a wide range of dynamics that arise in social evolution based on imitation processes (see Weibull [23], Björnerstedt and Weibull [3], Weibull [24]). The defining property for the class in question is that if there exists a (extinct or non-extinct) pure strategy that gives a payoff above average in its player population, then *some* such pure strategy has a positive *growth rate*. In particular, if *all* such strategies are non-extinct, then some such population share will actually grow in population share. This class of population dynamics will here be called *imitative* (*weakly payoff positive* in Weibull [24]):

Formally, for any population state s and player position i, let $B_i(s)$ denote the (possibly empty) subset of *better-than-average* pure strategies,[2]

$$B_i(s) = \{\alpha \in A_i : \pi_{i\alpha}(s) > \pi_i(s)\} \ , \tag{4}$$

and call a dynamics (1) *imitative* if its growth rate functions $g_{i\alpha}$ satisfying the following axiom:

[IM]: If $B_i(s) \neq \emptyset$, then $g_{i\alpha}(s) > 0$ for some $\alpha \in B_i(s)$.

Note that this condition requires no knowledge on behalf of the individuals in one player position about the payoffs to other player positions, nor is any detailed knowledge of the payoffs to one's own strategy set necessary. It is sufficient that individuals *on average* tend to switch toward *some* of better-than-average performing strategies, granted these are non-extinct. Clearly both versions of the replicator dynamics are imitative in this sense: *all* pure strategies earning above average in their player population have positive growth rates in these two dynamics.

In order to state the next result we need to define stationarity, stability, and reachability in any dynamics in the form (1). A population state s is *stationary* if the

[2] The set $B_i(s)$ is a (possibly empty) subset of the (nonempty) set $\gamma_i(s) = \{\alpha \in A_i : \pi_{i\alpha}(s) \geq \pi_i(s)\}$ of weakly better replies, studied in Ritzberger and Weibull [18].

solution through s stays put at s forever. Technically, this means that $g_{i\alpha}(s)s_{i\alpha} = 0$ for all player positions $i \in I$ and pure strategies $\alpha \in A_i$. Likewise, a populations state s is *(Lyapunov) stable* if small perturbations of the state do not lead the population state away, i.e., if every neighborhood V of s contains a neighborhood U of s such that all solutions starting in U remain forever in V. Following Ritzberger and Weibull we call a population state s *reachable from the interior* if it is the limit to some interior solution, i.e., if there exists some initial population state s^o with no pure strategy extinct such that the solution through s^o converges to s as $t \to +\infty$. (Note that we do not require s itself to be interior.) It is not difficult to show that stationarity is necessary for both stability and reachability.

We are now in a position to establish the following properties of all imitative dynamics: (a) If no pure strategy is extinct in a stationary population state, then the state constitutes a Nash equilibrium, (b) If a population state is the limit to some interior solution, then it constitutes a Nash equilibrium, (c) If a population state is dynamically stable, then it constitutes a Nash equilibrium. Claim (b) generalizes a result due to Nachbar [15] and (c) is a generalization of a result due to Bomze [4] for the single-population version of the replicator dynamics as applied to symmetric two-player games.

Proposition 1. *Consider any imitative population dynamics (1).*

(a) All interior stationary states are Nash equilibria.

(b) All states reachable from the interior are Nash equilibria.

(c) All stable states are Nash equilibria.

(For a proof, see Weibull [24]).

Note that each of the claims (a) and (b) involves a non-extinction hypothesis: the first claim presumes that the population state itself is interior and the second claim presumes that it can be reached from an interior state. Indeed, these two claims are otherwise not generally valid. In contrast, claim (c) does not explicitly involve any non-extinction hypothesis. However, such a hypothesis is implicit in the definition of dynamic stability. For this criterion asks what happens when a population state is slightly perturbed - in particular, when currently extinct strategies enter the population in small population shares. Thus, all three results directly or indirectly involve some non-extinction hypothesis. This is quite natural: dynamics in the form (1) do not allow extinct strategies to become used, no matter how high payoffs these would yield, while, in contrast, Nash equilibrium requires that only optimal strategies be used. The next section studies a class of population dynamics which allows extinct strategies to become non-extinct.

4. Innovative Adaptation

Now consider population dynamics in the more general form

$$\dot{s}_{i\alpha} = f_{i\alpha}(s) \tag{5}$$

for some functions $f_{i\alpha} : S \to \mathbb{R}$. The quantity $f_{i\alpha}(s)$ thus represents the net increase per time unit of the population share of α-strategists in player population i when the overall population state is s. In order to guarantee the existence and uniqueness of a solution to (5) through any state in S we assume that each function $f_{i\alpha}$ is Lipschitz continuous. Moreover, we assume that $\sum_{\alpha \in A_i} f_{i\alpha}(s) = 0$ for all player states i and population states s, and that $s_{i\alpha} = 0 \Rightarrow f_{i\alpha}(s) \geq 0$. Together these conditions imply that all solutions starting in S remain forever in S at all future times. Such a function f will be called a *vector field* for (5).

The class of population dynamics in the form (5) clearly contains as a sub-class all population dynamics in the form (1). However, in contrast to the latter, the former allow for the possibility that currently extinct pure strategies enter the population; it is not excluded that $\dot{s}_{i\alpha} > 0$ although $s_{i\alpha} = 0$. Dynamics in the more general form (5) may thus contain an "innovative" element; some individuals may begin using earlier unused strategies, either intentionally - by way of experimentation or calculation - or unintentionally - by way of mistakes or mutations.[3]

Indeed, a certain degree of inventiveness is easily seen to be sufficient for all stationary population states to constitute Nash equilibria - as suggested in the above quotes from Nash [16]. The requirement on the dynamics (5) is simple and very close in spirit to the above imitation axiom [**IM**]: if there is some (extinct or non-extinct) pure strategy which yields a payoff above the current average payoff in the player population in question, then *some* such pure strategy will grow in population share. The difference as compared with [**IM**] is that now we require growth also of extinct strategies. Such dynamics will be called *innovative.*

Formally, for any population state $s \in S$ and player position $i \in I$, let $B_i(s)$ denote the set of better-than-average pure strategies defined in equation (4), and call a dynamics (5) *innovative* if its vector field f satisfies the following axiom:

[**IN**]: If $B_i(s) \neq \emptyset$, then $f_{i\alpha}(s) > 0$ for some $\alpha \in B_i(s)$.

Just as [**IM**] this condition requires no knowledge on behalf of the individuals in one player position about the payoffs to other player positions, nor is any detailed knowledge of the payoffs to one's own strategy set required. It is sufficient that

[3] In dynamics in the form (1) such "innovative" elements are only considered indirectly by way of dynamic stability criteria.

individuals *on average* tend to switch toward *some* of the better-than-average performing strategies - even if these happen to be extinct. This condition is, for instance, met if reviewing individuals move toward the *best* replies to the current population state. However, no imitative dynamics is innovative: extinct strategies, no matter how good, do not grow in those dynamics. Hence, innovative and imitative dynamics, respectively, constitute two disjoint classes of population dynamics.

Innovative dynamics being more "rationalistic" than imitative dynamics, weaker dynamic properties of a population state suffices for it to constitute a Nash equilibrium:[4]

Proposition 2. *All stationary states in innovative dynamics (5) constitute Nash equilibria.*

Proof: If $s \in S$ is stationary, then all $f_{i\alpha}(s)$ are zero, and thus, by **[IN]**, all subsets $B_i(s)$ are empty. Hence $\pi_{i\alpha}(s) \leq \pi_i(s)$ for all player positions $i \in I$ and pure strategies $\alpha \in A_i$, implying that s is a Nash equilibrium. **End of proof.**

The three claims in Proposition 1 thus hold *a fortiori* for all innovative dynamics: all interior stationary states are Nash equilibria, and so are all states that are reachable from the interior, as well as all dynamically stable states. In fact, for innovative dynamics "reachability from the interior" can be weakened to mere "reachability" from any initial state, irrespective if this is interior or not. The reason is simply that if a solution converges to some state, then this limit state must be stationary (this is also true for imitative dynamics), and by Proposition 2 stationarity implies Nash equilibrium (which is not true for imitative dynamics).

The two versions of the replicator dynamics are examples of imitative dynamics and so are not innovative. So-called best reply dynamics (see Gilboa and Matsui [6], Matsui [12] and Hofbauer [10]) do not belong to the present class of innovative dynamics either, for the technical reason that these are generated from (set-valued) differential inclusions rather than from differential equations.

As an example of an innovative dynamics (5) consider $\dot{s}_{i\alpha} = f^+_{i\alpha}(s)$ where

$$f^+_{i\alpha}(s) = \frac{\pi^+_{i\alpha}(s)}{1+\pi^+_i(s)} - \frac{\pi^+_i(s)}{1+\pi^+_i(s)} s_{i\alpha}, \tag{6}$$

and

$$\pi^+_{i\alpha}(s) = \max\{\pi_{i\alpha}(s) - \pi_i(s), 0\} \quad \text{and} \quad \pi^+_i(s) = \sum_{\beta \in A_i} \pi^+_{i\beta}(s) \tag{7}$$

Here $\pi^+_{i\alpha}(s)$ is the *excess payoff* to pure strategy α over the average payoff in its player population, defined to be zero for strategies earning below average, and $\pi^+_i(s)$ is

[4] A population state s is *stationary* in a dynamics (5) if $f_{i\alpha}(s) = 0$ for all $i \in I$ and $\alpha \in A_i$.

the aggregate excess payoff. It is as if all individuals in each player population revised their strategy choice at the same rate $\pi_i^+(s)/(1+\pi_i^+(s))$, and all revising individuals switch to pure strategies with probabilities $\pi_{i\alpha}^+(s)/\pi_i^+(s)$, i.e., proportional to each strategy's excess payoff. Note that the common revision rate in a player population is an increasing function of the aggregate excess payoff to the population; this rate essentially measures, in terms of payoffs, how far the population is from a best reply to the current population state. In a state that constitutes a Nash equilibrium, and only then, all excess payoffs are zero, and so all revision rates are zero and the state is stationary.

Given these interpretations it may not come as a surprise that the innovative population dynamics $\dot{s}_{i\alpha} = f_{i\alpha}^+(s)$ is closely related to the iteration mapping introduced in Nash's [17] famous existence proof for Nash equilibrium in finite games, a mapping that was soon afterwards adopted in general equilibrium theory in existence proofs for Walrasian equilibria (Arrow and Debreu [2]). Nash used the mapping $F : S \to S$ defined by

$$F_{i\alpha}(s) = \frac{s_{i\alpha} + \pi_{i\alpha}^+(s)}{1 + \sum_{\beta \in A_i} \pi_{i\beta}^+(s)} \quad . \tag{8}$$

In discrete time, the iteration goes $s(t+1) = F(s(t))$ for $t = 0, 1, 2, \ldots$, and Nash [17] notes that the fixed points under the continuous mapping F are precisely the Nash equilibria of the game. The existence of such points then follows immediately from Brouwer's Fixed Point Theorem.

To see the connection with (6), define the time derivative $\dot{s}_{i\alpha}$ by way of the linear interpolation $\dot{s}_{i\alpha} = F_{i\alpha}(s) - s_{i\alpha} = f_{i\alpha}^+(s)$.

It is not difficult to verify that f^+ meets the requirements for a vector field of a dynamics (5). To see that axiom **[IN]** is met, suppose $B_i(s) \neq \emptyset$ and consider the sum $\sum_{\alpha \in B_i(s)} f_{i\alpha}^+(s)$. By (6), this sum is positive. Hence $f_{i\alpha}^+(s)$ is positive for at least one strategy $\alpha \in B_i(s)$.

From the above-mentioned connection with Nash's existence proof it follows that the set of stationary states under this particular innovative dynamics in fact is *identical* with the set of Nash equilibria:

$$f^+(s) = 0 \quad \Leftrightarrow \quad F(s) = s \quad \Leftrightarrow \quad s \text{ is a Nash equilibrium} \tag{9}$$

It is clear from the above analysis that we can generate a whole family of innovative dynamics in this fashion, spanning from the "Nash dynamics" $\dot{s}_{i\alpha} = f_{i\alpha}^+(s)$ to virtually best-reply dynamics as follows. For any $\sigma \geq 1$, let

$$\dot{s}_{i\alpha} = \frac{\left[\pi_{i\alpha}^+(s)\right]^\sigma - s_{i\alpha} \sum_{\beta \in A_i} \left[\pi_{i\beta}^+(s)\right]^\sigma}{1 + \sum_{\beta \in A_i} \left[\pi_{i\beta}^+(s)\right]^\sigma} \tag{10}$$

For $\sigma = 1$ the vector field of this dynamics is simply f^+. As $\sigma \to +\infty$, the inflow to non-best replies to s becomes smaller and smaller. Moreover, for any $\sigma \geq 1$ the right-hand side in (10) defines a vector field that meets [**IN**], and the set of stationary states in any such dynamics coincides with the set of Nash equilibria of the game.

5. Innovative Adaptation with Memory

The two classes of population dynamics studied so far may be said to be too restrictive in a number of respects to properly represent the kind of process indicated in Nash's mass action interpretation. Important neglected aspects are memory and expectation formation - "the participants are supposed to accumulate empirical information on the relative advantages of the various pure strategies at their disposal."

A standard technique to incorporate memory in a dynamic model is to expand the state space so that the results for memory-less processes can be applied to the appropriately enriched but again memory-less model. A suggestion in this direction is given in Swinkels [20] for a class of deterministic continuous-time models of dynamic strategy adaptation. Likewise, Young [25] elaborates a stochastic discrete-time model of strategy adaptation in which individuals have memory of finite length. Expanding the state space accordingly, Young obtains a Markov chain to which classical techniques can be applied. Hurkens [10] studies a related stochastic model of strategy adaptation with memory. The first model of dynamic strategy adaptation with memory, however, is the Brown-Robinson model of *fictitious play* (Robinson [19]) in which players always play best replies to the empirical time average over strategies used up to date.

We will here introduce two classes of innovative adaptation dynamics with memory/expectations formation, one related to Nash's text and another related to fictitious play.

5.1. Perceived Payoffs. Here we add an element representing the above-mentioned "relative advantage" of alternative pure strategies. All individuals in each player population are assumed to have the same perception in this respect. More exactly, for each player position $i \in I$, pure strategy $\alpha \in A_i$ and point in time $t \in \mathbb{R}$, let $p_{i\alpha}(t)$ be the currently *perceived payoff* associated with this pure strategy. We assume that the previous population dynamics (5) is accordingly augmented to the following pair or equations:

$$\dot{s}_{i\alpha} = f_{i\alpha}(s, p) \tag{11}$$

$$\dot{p}_{i\alpha} = h_{i\alpha}(s, p), \tag{12}$$

where s, as before, is the population distribution of pure strategies, and p is the vector of perceived payoffs of the pure strategies in the game. The current flow of strategy adjustment may thus depend on the currently perceived payoffs. In this sense the perceived payoffs represent the individuals' expectations of future payoffs to the pure strategies available in their player position.

In analogy with the case of population dynamics in the form (5) f is assumed to be Lipschitz continuous, to meet $\sum_{\alpha \in A_i} f_{i\alpha}(s,p) = 0$ for all player positions i and states (s,p), and to satisfy the implication $s_{i\alpha} = 0 \Rightarrow f_{i\alpha}(s,p) \geq 0$. As for the function h, we assume that each component function $h_{i\alpha}$ is Lipschitz continuous and satisfies the two implications

$$\pi_{i\alpha}(s) > p_{i\alpha} \quad \Rightarrow \quad h_{i\alpha}(s,p) > 0,$$

$$\pi_{i\alpha}(s) < p_{i\alpha} \quad \Rightarrow \quad h_{i\alpha}(s,p) < 0. \tag{13}$$

In other words, the perceived payoff to a strategy increases (decreases) if its current payoff exceeds (falls short of) its currently perceived payoff. This does not mean that all individuals necessarily know the payoffs to all pure strategies, it just means that the representative individual in each player population gradually adjusts her perception of the pure strategies' payoffs in the direction of their current payoffs.[5]

The above assumptions together imply that the system (11,12) has a unique solution through any state (s,p), and that all solutions with s initially in S will have s remaining in S at all future times. Such a pair (f,h) of functions will be called a *vector field* for (11,12), and the induced dynamics will be called a *population-payoff-perception (PPP) dynamics.*

In order to connect such dynamics with the notion of Nash equilibrium it suffices to adapt the earlier requirement **[IN]** of inventiveness: if *some* pure strategy has a perceived payoff above the average of the currently perceived payoff in its player population, then *some* such pure strategy will increase its population share. Intuitively, it is as if strategy revising individuals "asked around" in their player population. Formally, for any state (s,p) and player position i, let

$$B_i^*(s,p) = \{\alpha \in A_i : p_{i\alpha} > p_i\} \ . \tag{14}$$

where $p_i = \sum_{\beta \in A_i} s_{i\beta} p_{i\beta}$. Inventiveness with respect to perceived payoffs can then be formalized as follows:

[5]In contrast to Nash's text, where "the participants are supposed to accumulate *empirical* information on the relative advantages ...," the present formalization presumes that the individuals playing the game perceive the payoffs even to those pure strategies that are not in use.

[**IN′**]: If $B_i^*(s,p) \neq \emptyset$, then $f_{i\alpha}(s,p) > 0$ for some $\alpha \in B_i^*(s,p)$.

A dynamics (11,12) where f is innovative in this sense, and f and h satisfy the above conditions, will be called an *innovative* PPP-dynamics. A state (s,p) will be called a *Nash equilibrium state* if s is a Nash equilibrium of the underlying game and $p = \pi(s)$. The following extension of Proposition 2 is straight-forward:

Proposition 3. *All stationary states under innovative PPP dynamics (11,12) are Nash equilibrium states.*

Proof: Stationarity of (s,p) implies $f_{i\alpha}(s,p) = h_{i\alpha}(s,p) = 0$. By [**IN′**] all sets $B_i^*(s,p)$ are then empty, and by (13) $p_{ia} = \pi_{i\alpha}(s)$ for all $i \in I$ and $\alpha \in A$, so $\pi_{i\alpha}(s) = p_{ia} \leq \sum_{\beta \in A_i} s_{i\beta} p_{i\beta} = \pi_i(s)$ for all $i \in I$ and $\alpha \in A_i$. Hence s is a Nash equilibrium and $p = \pi(s)$. **End of proof.**

As an example of a function h with the above properties suppose that the perceived payoff to a pure strategy is its discounted average actual payoff over the entire infinite past:

$$p_{i\alpha}(t) = \delta \int_{-\infty}^{t} \pi_{i\alpha}\left[s(\tau)\right] e^{\delta(\tau - t)} d\tau, \tag{15}$$

for some $\delta > 0$. Differentiation with respect to t gives

$$\dot{p}_{i\alpha}(t) = \delta\left(\pi_{i\alpha}\left[s(t)\right] - p_{i\alpha}(t)\right), \tag{16}$$

i.e., $h_{i\alpha}(s,p) = \delta\left[\pi_{i\alpha}(s) - p_{i\alpha}\right]$.

Combining this memory/expectations formation rule with the "generalized Nash dynamics" (10) one obtains innovative PPP dynamics of the form

$$\dot{s}_{i\alpha} = \frac{\left[p_{i\alpha}^+(s)\right]^\sigma - s_{i\alpha} \sum_{\beta \in A_i} \left[p_{i\beta}^+(s)\right]^\sigma}{1 + \sum_{\beta \in A_i} \left[p_{i\beta}^+(s)\right]^\sigma} \tag{17}$$

$$\dot{p}_{i\alpha} = \delta\left[\pi_{i\alpha}(s) - p_{i\alpha}\right], \tag{18}$$

where $\delta > 0$, $\sigma \geq 1$, and $p_{i\alpha}^+(s) = \max\left\{p_{i\alpha}(s) - p_i(s), 0\right\}$.

5.2. Perceived Population Shares. An alternative approach to memory/expectation formation is to suppose that the individuals playing the game know the payoff function of the game and form beliefs about the population shares associated with all pure strategies available to all other player positions in the game. Clearly this is a more demanding requirement on the individuals' information and computational capacity. However, it is close in spirit to learning dynamics in economics, and formally this alternative approach is very similar to the one developed above.

For each player position i, pure strategy α, and time t, let $q_{i\alpha}(t)$ be the currently *perceived population share* using the strategy. Again, this may be viewed as the strategy-revising individuals' expectations about the future. We now assume that the population dynamics (5) is augmented to the following pair or equations:

$$\dot{s}_{i\alpha} = f_{i\alpha}(s, q) \tag{19}$$

$$\dot{q}_{i\alpha} = h_{i\alpha}(s, q), \tag{20}$$

where s, as before, is the population distribution of pure strategies, and q is the vector of perceived strategy shares.

In analogy with the case of PPP-dynamics f is assumed to be Lipschitz continuous, to satisfy $\sum_{\alpha \in A_i} f_{i\alpha}(s, q) \equiv 0$ and $s_{i\alpha} = 0 \Rightarrow f_{i\alpha}(s, q) \geq 0$. As for the function h, we assume that each component function $h_{i\alpha}$ is Lipschitz continuous and satisfies the two implications

$$s_{i\alpha} > q_{i\alpha} \quad \Rightarrow \quad h_{i\alpha}(s, q) > 0,$$

$$s_{i\alpha} < q_{i\alpha} \quad \Rightarrow \quad h_{i\alpha}(s, q) < 0. \tag{21}$$

In other words, the perceived population share of a strategy increases (decreases) if its current population share exceeds (falls short of) its currently perceived share. Moreover, we require perceived population shares to be consistent in the sense that $q(t) \in S$ at all times t. This follows if the function h has the same qualitative properties as f: $\sum_{\alpha \in A_i} h_{i\alpha}(s, q) \equiv 0$ and $q_{i\alpha} = 0 \Rightarrow h_{i\alpha}(s, q) \geq 0$

The above assumptions together imply that the system (19,20) has a unique solution through any state $(s, q) \in S^2$, and that all solutions initially in S^2 will remain in S^2 at all future times. Such a dynamics will be called a *population-strategy-perception (PSP) dynamics.*

In order to connect such dynamics with the notion of Nash equilibrium it suffices to apply the earlier requirement [**IN**] of inventiveness to the perceived rather than actual population state: if *some* pure strategy has a payoff above average, computed on the basis of the currently perceived population shares, then *some* such pure strategy will increase its (true) population share. Formally, inventiveness with respect to perceived population shares means that f satisfies

[**IN″**]: If $B_i(q) \neq \emptyset$, then $f_{i\alpha}(s,q) > 0$ for some $\alpha \in B_i(q)$.

A dynamics (11,12) where f is innovative in this sense, and f and h satisfy the above conditions, will be called an *innovative* PSP dynamics. A state (s,q) will be called a *Nash equilibrium state* if s is a Nash equilibrium of the underlying game and $q = s$. The following extension of Proposition 2 is straight-forward:

Proposition 4. *All stationary states under innovative PSP dynamics (19,20) are Nash equilibrium states.*

Proof: Stationarity of (s,q) implies $f_{i\alpha}(s,q) = h_{i\alpha}(s,q) = 0$. By (21) $q_{ia} = s_{i\alpha}$ for all $i \in I$ and $\alpha \in A_i$. By [**IN″**] all sets $B_i(s)$ are then empty, and by (21) $q_{ia} = s_{i\alpha}$ for all $i \in I$ and $\alpha \in A_i$, so $\pi_{i\alpha}(s) \leq \pi_i(s)$ for all $i \in I$ and $\alpha \in A_i$. Hence s is a Nash equilibrium and $q = s$. **End of proof.**

Again all three implications in Proposition 1 follow: interior stationarity implies Nash equilibrium, and so do reachability and stability, respectively.

As an example of a function h with the above properties suppose that the perceived population share to a pure strategy is its discounted average population share over the entire infinite past:

$$q_{i\alpha}(t) = \delta \int_{-\infty}^{t} s_{i\alpha}(\tau) e^{\delta(\tau - t)} d\tau, \tag{22}$$

for some $\delta > 0$. Differentiation with respect to t gives

$$\dot{q}_{i\alpha} = \delta \left(s_{i\alpha} - q_{i\alpha} \right). \tag{23}$$

This memory process is related to the memory process used in fictitious play. To see this, note that fictitious play sets a pure strategy's perceived population share equal to its (undiscounted) time average from some initial time $t = 0$ onward:

$$q_{i\alpha}(t) = \frac{1}{t} \int_{0}^{t} s_{i\alpha}(\tau) d\tau \tag{24}$$

Differentiation with respect to time t results in[6]

$$\dot{q}_{i\alpha} = \frac{1}{t}\left(s_{i\alpha} - q_{i\alpha}\right). \tag{25}$$

Thus, the memory process in fictitious play is a gradual slow-down of a memory process of type (16), the constant factor δ is replaced by the decreasing factor $1/t$.

6. Conclusion

The hitherto largely unknown mass-action interpretation of Nash equilibrium suggests a perspective on strategic interaction in stark contrast to the conventional rationalistic perspective, but which is close to the perspective of evolutionary game theory. The formalizations of this interpretation given here lend support to Nash's informal claim that "stable" aggregate behavior forms a Nash equilibrium.

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[6] In order to render this vector field locally Lipschitz continuous on an open time interval containing 0, one may add a (small) positive constant ε to t in the denominator in (25). Technically this is equivalent with a mere rescaling of time. If one requires the initially recalled payoff vector $p(0)$ to equal the initial payoff $\pi[s(0)]$, then it is as if the initial population believes that the initial payoffs have been constantly earned during the (arbitrarily short) "prehistoric" time interval $(-\varepsilon, 0)$.

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