Advanced Microeconomic Theory EC104

Problem Set 1

1. Each of n farmers can costlessly produce as much wheat as she chooses. Suppose that the kth farmer produces W_k, so that the total amount of what produced is $W = W_1 + W_2 + \ldots + W_n$. The price p at which wheat sells is then determined by the demand equation $p = e^{-W}$.

1a. Show that the strategy of producing one unit of wheat strongly dominates all of a profit-maximizing farmer's other strategies. Check that the use of this strategy yields a profit of e^{-n} for a farmer.

The payoff for a single (i^{th}) farmer is:

$$\pi_i(W_i, W_{-i}) = [e^{-(W_1+\ldots+W_n)}]W_i$$

First-order condition gives:

$$\frac{\partial \pi_i}{\partial W_i} = 0 \Leftrightarrow e^{-W} + W_i(-1)(e^{-W}) = 0$$

and we easily obtain:

$$W_i^* = 1$$

Notice that $W_i^* = 1$ is the maximizing value irrespective of the actual value of W (or more specifically, W_{-i}). Hence, $W_i = 1$ must be the dominant strategy. If all n farmers set $W_i = 1$, then $W = (1 + \ldots + 1) = n \times 1 = n$. Hence, $\pi_i = e^{-n} \times 1 = e^{-n}$.

1b. Explain why the best of all agreements that treat each farmer equally requires each to produce only $\frac{1}{n}$ units of wheat. Check that a farmer 's profit would then be $\frac{1}{en}$. Why would such an agreement need to be binding (that is, signed as a legally binding contract) for it to be honored by a profit-maximizing farmers?

To find the "best of all agreements", we need to maximize the social surplus with respect to $W_1, \ldots, W_n$, i.e., we need to solve:

$$\max_{W_1,\ldots,W_n} (W_1 + \ldots W_j + \ldots + W_n)e^{-(W_1+\ldots W_j+\ldots+W_n)}$$

The First-order condition (FOC) with respect to the representative farmer's quantity choice w_j is:

$$(-1)(W_1 + \ldots + W_j + \ldots + W_n)e^{-(W_1+\ldots+W_j+\ldots+W_n)} + e^{-(W_1+\ldots W_j+\ldots+W_n)} = 0$$

By imposing the symmetry condition $W_j = W_i = W^S$ for any $i \neq j$, we have

$$(-1)NW^s + 1 = 0$$

Hence,

$$W^s = 1/n$$

It follows that:

$$\pi_i = \frac{1 \times e^{-1}}{n} = \frac{1}{ne}$$

Notice, however, that if each farmer other than i produces $\frac{1}{n}$ (< 1!!!), farmer $i's$ best response will be to produce...$W_i = 1$!!! That means each farmer will have an incentive to produce more. Hence we would need a binding contract to enforce this agreement

1c. Confirm that xe^{-x} is largest when $x = 1.$ Deduce that all the farmers would make a larger profit if they all honored the agreement rather than each producing one unit and so flooding the market.

We need to maximise

$$\max_x [xe^{-x}]$$

The F.O.C. is given by:

$$-xe^{-x} + e^{-x} = 0$$

which implies that

$$x = 1$$

To maximize the social surplus, we can also maximise We^{-W} with respect to the total quantity W. The socially optimum total output has to be 1 and since all farmers have equal share, each produces $\frac{1}{n}$. Again,

$$\pi_i = \frac{1 \times e^{-1}}{n} = \frac{1}{ne} > e^{-n}$$

1d. You would have realized what the exercise went through was version of the "tragedy of the commons". Why is such an n-player game a generalization of the Prisoners' Dilemma?

This wheat production game may considered a generalization of the Prisoner's Dilemma (PD) game because just as in a PD game the equilibrium strategy is a dominant strategy and yet the equilibrium outcome is clearly Pareto dominated (i.e., is inefficient).

2. Consider the following case of a *differentiated good Cournot model*. Firm i produces type i widgets at a constant unit cost of $c_i,\ i = 1,2$. If $q_1$and q_2 are the quantities of the two varieties produced, the respective prices for the two goods are determined by the demand equations $p_1 = M - 2q_1 - q_2$ and $p_2 = M - q_1 - 2q_2$. Adapt Cournot's duopoly model to this new situation, and find:

2a. The firms' reaction functions.

The profit function of each firm $i = 1,2$ is equal to:

$$\begin{aligned}\pi_i(q_i, q_j) &= [p_i(q_i + q_j) - c_i]q_i \\ &= [M - 2q_i + q_j - c_i]q_i \\ &= Mq_i - 2q_i^2 - q_iq_j - c_iq_i\end{aligned}$$

To find the reaction function of firm i simply set

$$
\begin{aligned}
\frac{\partial \pi_i}{\partial q_i} &= 0 \\
&\Leftrightarrow M - 4q_i - q_j - c_i = 0 \\
&\Leftrightarrow q_i(q_j) = \frac{M - q_j - c_i}{4}
\end{aligned}
$$

2b. The quantities produced in equilibrium and prices at which the goods are sold and the equilibrium profits.

Solving the reaction functions simultaneously, we easily obtain for $i = 1, 2$:

$$q_i^* = \frac{M}{5} + \frac{c_j - 4c_i}{15}$$

and thus

$$
\begin{aligned}
q_1^* &= \frac{M}{5} + \frac{c_2 - 4c_1}{15} \\
q_2^* &= \frac{M}{5} + \frac{c_1 - 4c_2}{15}
\end{aligned}
$$

The prices are then given by:

$$
\begin{aligned}
p_1^* &= M - 2q_1^* - q_2^* \\
&= \frac{2M}{5} + \frac{7c_1}{15} + \frac{2c_2}{15} \\
p_2^* &= M - q_1^* - 2q_2^* \\
&= \frac{2M}{5} + \frac{2c_1}{15} + \frac{7c_2}{15}
\end{aligned}
$$

Finally, the equilibrium profits are:

$$
\begin{aligned}
\pi_1^* &= (p_1^* - c_1)q_1^* \\
&= \frac{2}{225}(3M - 4c_1 + c_2)(3M - 4c_1 + c_2) \\
\pi_2^* &= (p_2^* - c_2)q_2^* \\
&= \frac{2}{225}(3M + c_1 - 4c_2)(3M + c_1 - 4c_2)
\end{aligned}
$$

3. Two firms set prices in a market whose demand curve is given by $Q = D(p)$, where $D(p)$ is a downward-sloping function and p is the lower of the two prices.The lower priced firm meets all of the demand; if the two firms post the same price, then they each get half the market. Assume costs of production are zero and that prices can only be quoted in dollar *discrete* units (0, 1, 2...).

3a. Show that if the rival firm charges a price above the monopoly price p_m, then the best response is to charge the monopoly price.

Call the price that the rival firm charges p (where $p \geq p_m$). Any price below p captures the market. By definition of a monopoly price, p_m dollars yields a higher profit than any of the alternatives. Therefore the best response to p is p_m.

3b. Show further that if the rival firm charges a price $p(> 1)$ at or below the monopoly price, then the best response is to charge a price below p.

The profits at p are $\frac{D(p)}{2}p$ whereas the profits at $p-1$ are $D(p-1)(p-1)$. Since demand is downward sloping, we have: $D(p-1) > D(p)$ while $p-1 \geq \frac{p}{2}$ since $p \geq 2$. Thus the best response is to charge a price below p.

3c. Conclude from the preceding arguments that the unique Nash Equilibrium price must be for each firm to price at 1 dollar.

So long as the rival charges a price $p \geq 2$, it will be a best response to undercut, and the rival's best response will be to further undercut. However, when $p = 1$, then the best response must be a price at \$1 (since at \$0, profits are zero). Thus, a price of \$1 for each firm is a Nash Equilibrium.

3d. What would be the Nash Equilibrium if there were 3 firms in this market? More generally, if there were n firms? Explain.

There is no change because the benefit to undercutting the rivals remain the same since the entire market is captured by the lowest price. But the profits from not undercutting are now $\frac{D(p)}{N}p$. Hence, each firm would undercut if

$$\frac{D(p)}{N}p < D(p-1)(p-1)$$

As before, $D(p-1) > D(p)$ and $p-1 \geq \frac{p}{N}$ as long as $p \geq \frac{N}{N-1}$. Thus, the Nash Equilibrium will still be for every firm to set a price equals to \$1.

4. (Osborne, Exercise 60.1) (Variant of Cournot's game, with market-share maximizing firms) Consider a standard Cournot game with two firms 1 and 2 competiting in quantities q_1 and q_2, where the marginal cost of production is constant and the same for both firms and denoted by c and the (inverse) market demand is given by:

$$P(Q) = \begin{cases} \alpha - Q & \text{if} \quad Q \leq \alpha \\ 0 & \text{if} \quad Q > \alpha \end{cases}$$

where $\alpha > 0$ and $c \geq 0$ are constant and $Q = q_1 + q_2$ is the market demand.

4a. Find the Nash equilibrium (equilibria?) of a variant of the standard Cournot's duopoly game (linear inverse demand, constant unit cost) that differs only in that one of the two firms chooses its output to maximize its market share subject to not making a loss, rather than to maximize its profit.

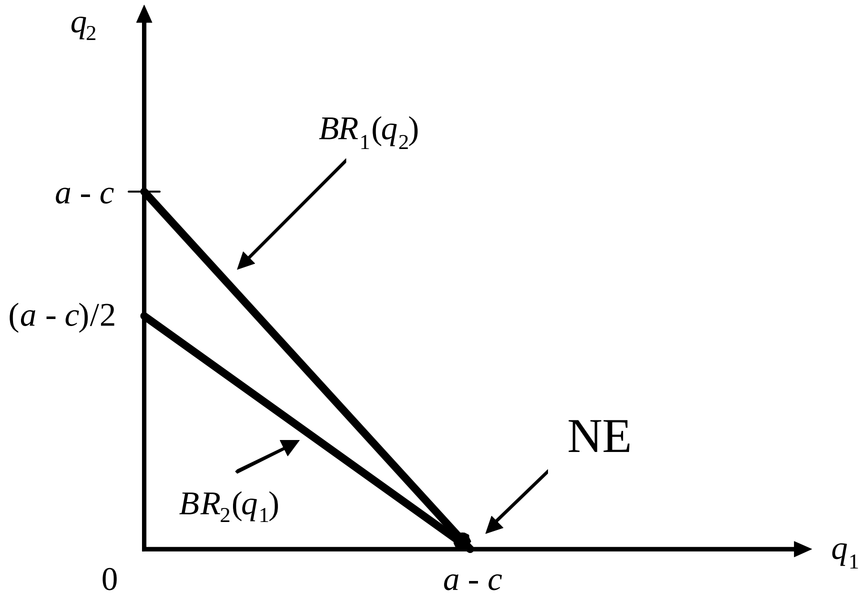
Let firm 1 be the market-share maximizing firm. If $q_2 > \alpha - c$, there is no output of firm 1 for which its profit is nonnegative. Thus its best response to such an output of firm 2 is $q_1 = 0$. If $q_2 \leq \alpha - c$, then firm 1 wants to choose its output q_1 large enough so that the price is c (and hence its profit is zero). Thus firm 1's best response to such a value of q_2 is $q_1 = \alpha - c - q_2$. We conclude that firm 1's best response function is:

$$b_1(q_2) = \begin{cases} \alpha - c - q_2 & \text{if} \quad q_2 \leq \alpha - c \\ 0 & \text{if} \quad q_2 > \alpha - c \end{cases}$$

Firm 2's best response function is the same as in the standard Cournot model (linear inverse demand, constant unit cost)

$$b_2(q_1) = \begin{cases} (\alpha - c - q_2)/2 & \text{if} \quad q_2 \leq \alpha - c \\ 0 & \text{if} \quad q_2 > \alpha - c \end{cases}$$

These best response functions are shown in the following figure.



The game has a unique Nash equilibrium, $(q_1^*, q_2^*) = (\alpha - c, 0)$, in which firm 2 does not operate. (The price is c, and firm 1's profit is zero.)

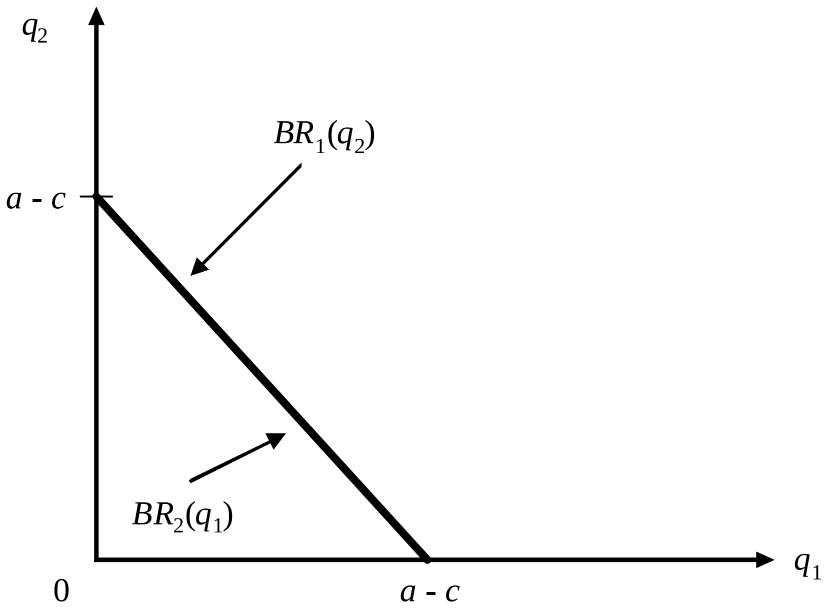
4b. What happens if each firm maximizes its market share?

If both firms maximize their market shares, their reaction functions are:

$$b_1(q_2) = \begin{cases} \alpha - c - q_2 & \text{if} \quad q_2 \leq \alpha - c \\ 0 & \text{if} \quad q_2 > \alpha - c \end{cases}$$

$$b_2(q_1) = \begin{cases} \alpha - c - q_1 & \text{if} \quad q_1 \leq \alpha - c \\ 0 & \text{if} \quad q_1 > \alpha - c \end{cases}$$

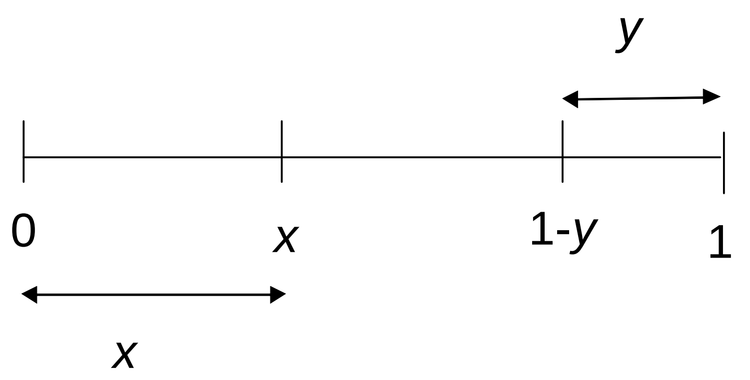
This means that the downward-sloping parts of their best response functions coincide in the analogue of the figure above.



Thus every pair (q_1, q_2) with $q_1 + q_2 = \alpha - c$ is a Nash equilibrium.

5. (Hotelling voting game with 3 candidates) An election has 3 candidates and takes place under the plurality rule. Voters are uniformly spread along an ideological spectrum from left to right whose extreme points are 0 (extreme left) and 1 (extreme right). Each voter votes for the candidate whose declared position is closest to the voter's own position. The candidates have no ideological attachment and take up any position along the line, each seeking only to maximize her share of votes.

Suppose you are one of the three candidates. The leftmost of the other two is at point x, and the rightmost is at point $(1 - y)$, where $x + y < 1$, so the rightmost candidate is a distance y from 1. The following figure illustrates this situation:



5a. Show that your best response is to take up the following positions under the given conditions:

(i) just slightly to the left of x if $x > y$ and $3x + y > 1$.

(ii) just slightly to the right of $(1 - y)$ if $y > x$ and $x + 3y > 1$.

(iii) exactly halfway between the other candidates if $3x + y < 1$ and $x + 3y < 1$.

Observe firs that:

(i) If you locate just to the left of the leftmost candidate, your vote total equals x.

(ii) If you locate just to the right of the rightmost candidate, your vote total equals y.

(iii) If you locate between the other two candidates, your vote total equals $(1 - y - x)/2$.

The conditions given in the question determine which of x, y, and (1 – y – x)/2 is biggest and thus which of these locations gives you the most votes.

(i) Note that $x > y$ is given. So to prefer locating just left of x, it has to be that it gives the most votes, that is:

$$x > \frac{1-y-x}{2}$$

This is equivalent to:

$$3x + y > 1$$

Thus, locating just to the left of the leftmost candidate gives the most votes.

(ii) Note that $y > x$ is given. We need to verify now that

$$y > \frac{1-y-x}{2}$$

This is equivalent to:

$$x + 3y > 1$$

Thus, locating just right of $(1 - y)$ gives the most votes.

(iii) We have to check that:

$$\frac{1-y-x}{2} > x$$

which is equivalent to:

$$3x + y < 1$$

We have also to check that

$$\frac{1-y-x}{2} > y$$

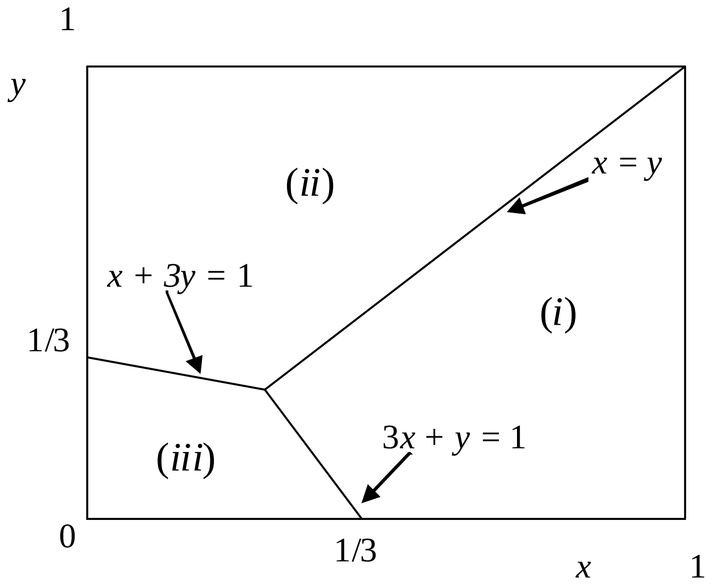
which is equivalent to:

$$x + 3y < 1$$

Thus, locating between the other two candidates gives the most votes.

5b. In a graph with *x* and *y* along the axes (*y* is on the vertical axis and *x* on the horizontal axis), show the areas (the combination of *x* and *y* values) where each of the reponse rules (*i*), (*ii*), and (*iii*) of question (5a), is best for you.

The graph is as follows:



5c. From your analysis, what can you conclude about the Nash equilibrium of the game where the three candidates each choose positions?

Given the goal of candidates (as stated in the question), a Nash equilibrium in locations can exist only if each candidate is in the location that maximizes his or her share of the vote, given the locations of the other candidates.

A candidate can always increase his vote share by moving. For example, any candidate who doesn't have another candidate located at a just slightly more moderate position can always gain votes by moving toward the center of the line. Thus, there is no Nash equilibrium in locations in the three-candidate game.

6. (Osborne Exercise 80.1) (Timing product release) Two firms are developing competing products for a market of fixed size. The longer a firm spends on development, the better its product. But the first firm to release its product has an advantage: the customers it obtains will not subsequently switch to its rival. Once a person starts using a product, the cost of switching to an alternative, even one significantly better, is too high to make a switch worthwhile.

A firm that releases its product first, at time t, captures the share $h(t)$ of the market, where h is a function that increases from time 0 to time T, with $h(0) = 0$ and $h(T) = 1$. The remaining market share is left for the other firm. If the firms release their products at the same time, each obtains half of the market. Each firm wishes to obtain the highest possible market share.

6a. Model this situation as a strategic game and plot the utility function u_i of firm i as a function of t_i for different values of t_j (i.e. consider three cases: $h(t_j) < 1/2$, $h(t_j) = 1/2$, and $h(t_j) > 1/2$).

A strategic game that models this situation is:

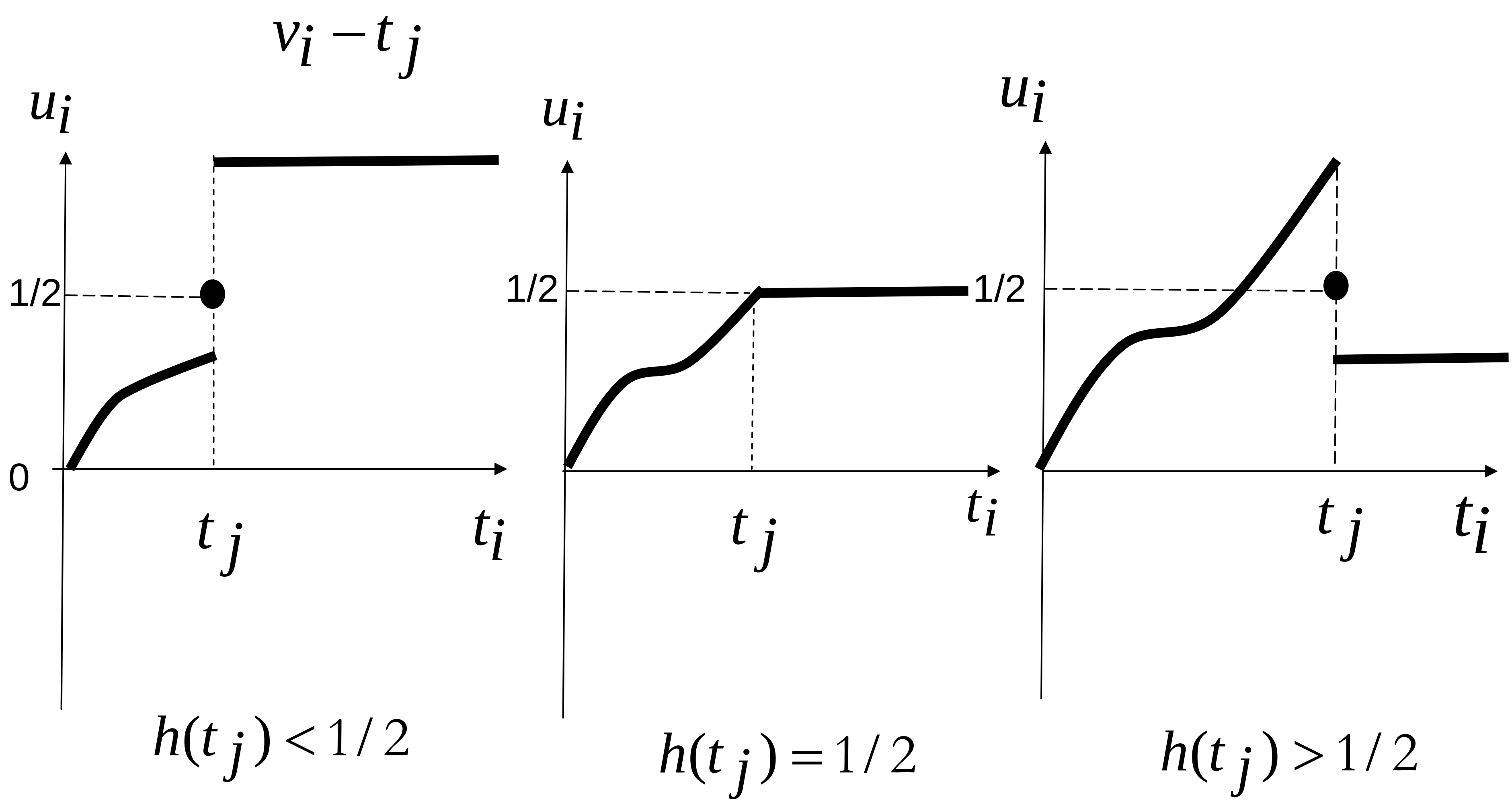
Players: The two firms.

Actions: The set of actions of each player is the set of possible release times, which we can take to be the set of numbers t for which

Preferences: Each firm's preferences are represented by its market share; the payoffs are:

$$u_i(t_1, t_2) = \begin{cases} h(t_i) & \text{if} \quad t_i < t_j \\ 1/2 & \text{if} \quad t_i = t_j \\ 1 - h(t_j) & \text{if} \quad t_i > t_j \end{cases}$$

Payoffs for different values of t_j



6b. Calculate the best-reponse functions of each player. Denote the time t for which $h(t_1) = h(t_2) = h(t^*) = 1/2$ by t^*. Observe that when finding firm i's best response to firm j's release time t_j, three cases must be considered:

(1) if t_j is such that $h(t_j) < 1/2$;
(2) if t_j is such that $h(t_j) = 1/2$;
(3) if t_j is such that $h(t_j) > 1/2$.

Based on the payoff functions, we can easily determine the BR functions. Indeed, from the payoff function, we see that

(1) if t_j is such that $h(t_j) < 1/2$, then the set of firm i's best responses is the set of release times after t_j.

(2) if t_j is such that $h(t_j) = 1/2$, then the set of firm i's best responses is the set of release times greater than or equal to t_j.

(3) if t_j is such that $h(t_j) > 1/2$, then firm i wants to release its product just before t_j. Since there is no latest time before t_j, firm i has no best response in this case. (It has good responses, but none is optimal.)

In summary, player i's best response function is given by:

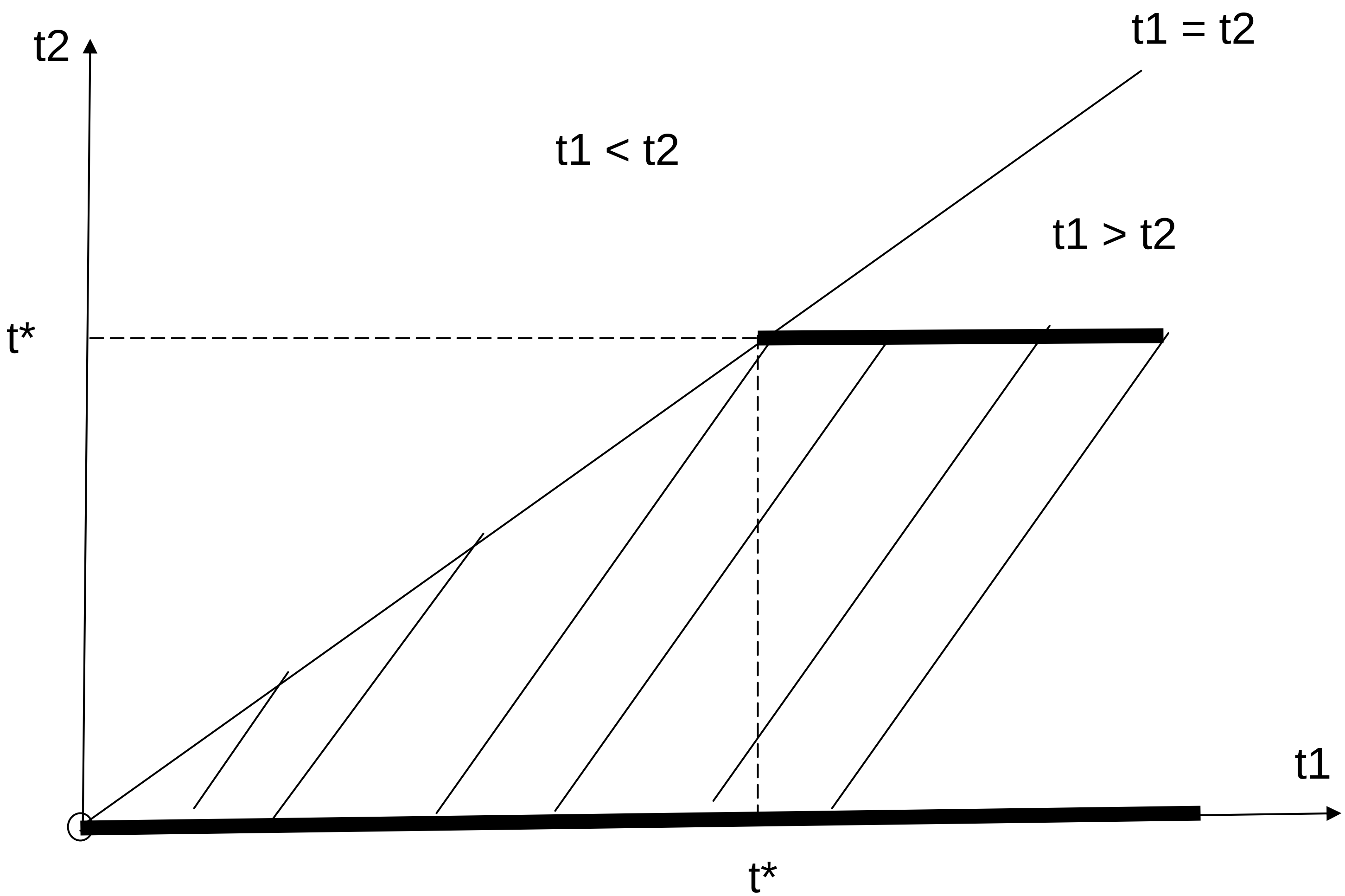
$$BR_i(t_j) = \begin{cases} \{t_i : t_i > t_j\} & \text{if} \quad h(t_j) < 1/2 \\ \{t_i : t_i \geq t_j\} & \text{if} \quad h(t_j) = 1/2 \\ \textit{No BR} & \text{if} \quad h(t_j) > 1/2 \end{cases}$$

Denote the time t for which $h(t_1) = h(t_2) = h(t^*) = 1/2$ by t^*, then we have:

$$BR_1(t_2) = \begin{cases} \{t_1 : t_1 > t_2\} & \text{if} \quad t_2 < t^* \\ \{t_1 : t_1 \geq t_2\} & \text{if} \quad t_2 = t^* \\ \textit{No BR} & \text{if} \quad t_2 > t^* \end{cases}$$

Figure: Player 1’s BR function

- Player 1’s BR

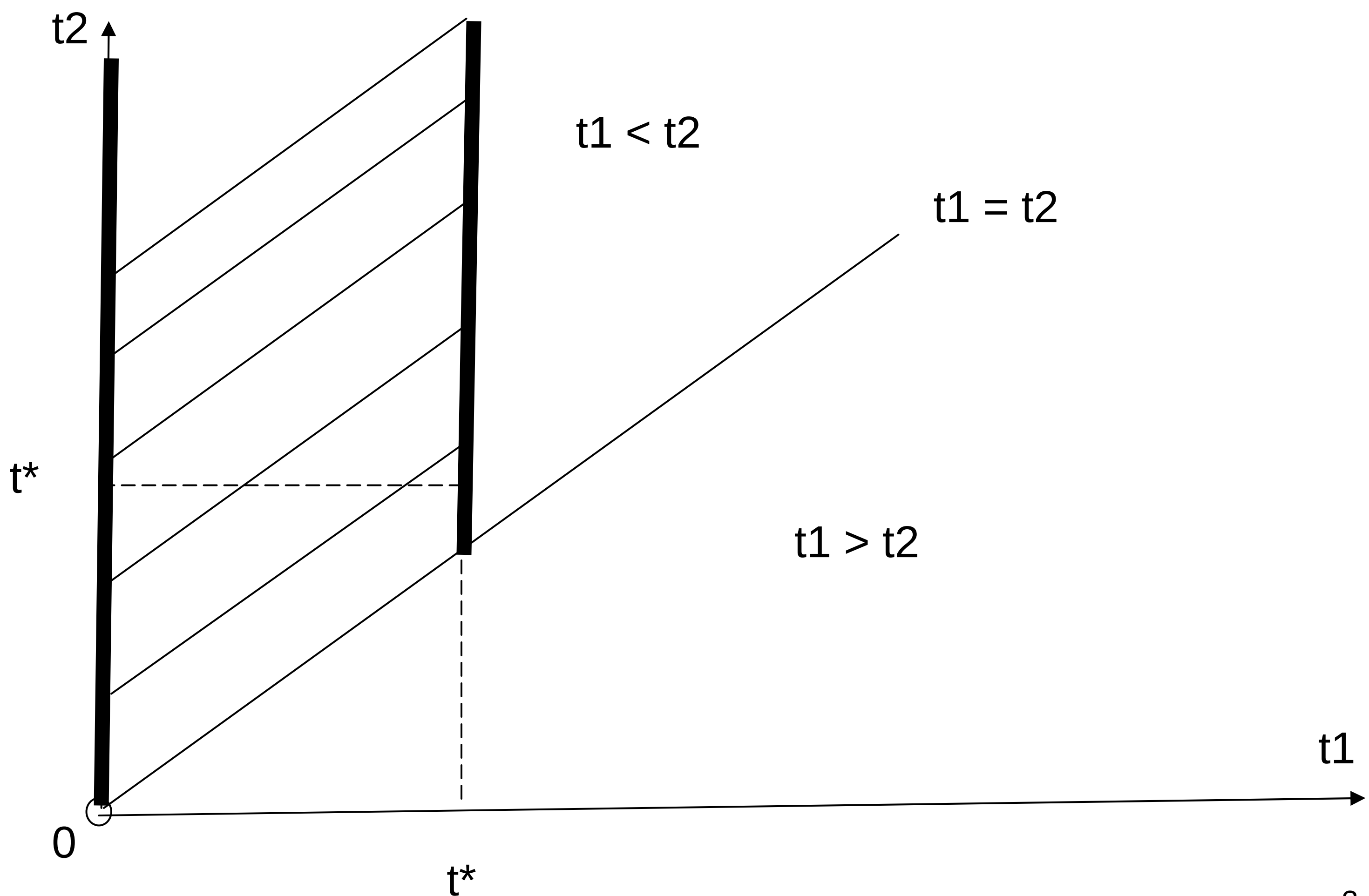


Similarly, for player 2, we have:

$$BR_{12}(t_1) = \begin{cases} \{t_2 : t_2 > t_1\} & \text{if} \quad t_1 < t^* \\ \{t_2 : t_2 \geq t_1\} & \text{if} \quad t_1 = t^* \\ \text{\textit{No BR}} & \text{if} \quad t_1 > t^* \end{cases}$$

Figure: Player 2's BR function

- Player 2's BR



6c. Show that there is unique a Nash equilibrium in which both firms release their product at time t^* where $h(t_1) = h(t_2) = h(t^*) = 1/2$.

Unique Nash equilibrium (t*,t*)

- NE