IFN Working Paper No. 972, 2013

What Do Revealed Preference Axioms Reveal about Elasticities of Demand?

Per Hjertstrand and Barry E. Jones

Research Institute of Industrial Economics
P.O. Box 55665
SE-102 15 Stockholm, Sweden
info@ifn.se
www.ifn.se

What Do Revealed Preference Axioms Reveal about Elasticities of Demand?

Per Hjertstrand, *Research Institute of Industrial Economics*, Sweden[1]

Barry E. Jones, *Binghamton University* (SUNY), USA[2,3]

Abstract: It is well-known that observed data on prices and quantities of a set of goods is consistent with rational choice if the data satisfy revealed preference. In this paper, we derive estimators for demand and substitution elasticities at the observed data points for datasets satisfying the Strong version of the Strong Axiom of Revealed Preference (SSARP) from Chiappori and Rochet (1987). We find that these estimators are identified only up to a strictly positive parameter, which must be small enough that the utility function rationalizing the dataset satisfies certain properties. We show that the estimated elasticities of substitution approach zero in the limit as this parameter approaches zero. Thus, if the dataset satisfies SSARP, then it is consistent with negligible substitutability between any pair of goods at all observed data points. Our estimators are derived directly from results in Brown and Shannon (2000) and Brown and Kannan (2005).

JEL Codes: C61, D12.

Keywords: Strong Axiom of Revealed Preference, Afriat Inequalities, Elasticities of Demand

[1] Per Hjertstrand, Research Institute for Industrial Economics, Box 5565, SE-102, Stockholm, Sweden. Email: per.hjertstrand@ifn.se.

[2] Barry E. Jones, Department of Economics, Binghamton University (SUNY), PO Box 6000, Binghamton NY, 13902-6000. Email: bjones@binghamton.edu.

[3] The authors both wish to thank the Jan Wallander and Tom Hedelius Foundation for support of this research. We also thank Richard Anderson for many helpful comments on the paper.

1. Introduction

In this paper, we address a fundamental question in microeconomics: *What is the relationship between revealed preference axioms and estimators of demand and substitution elasticities?* We do so by deriving estimators of the elasticities of demand for observed data on prices and quantities of a set of goods satisfying the Strong version of the Strong Axiom of Revealed Preference (SSARP) from Chiappori and Rochet (1987). These estimators correspond to a smooth utility function, which rationalizes the observed data. A key result is that the elasticities are identified only up to a strictly positive parameter, which must be small enough that the utility function satisfies certain properties, but which is otherwise unrestricted. We show that the estimated elasticities of substitution approach zero in the limit as this parameter approaches zero. Thus, the estimated substitution elasticities are empirically indistinguishable from zero under SSARP. Our estimators are derived directly from results in Brown and Shannon (2000) and Brown and Kannan (2005). Our approach differs from approaches that have previously been investigated and which we will now quickly review.

It is well-known that a set of observed data on prices and quantities is consistent with rational choice if the data satisfy one of several revealed preference axioms. For example, Varian's (1982) formulation of Afriat's theorem says that the Generalized Axiom of Revealed Preference (GARP) is a necessary and sufficient condition for a data set to be rationalized by a non-satiated utility function, which is in turn a necessary and sufficient condition for the data to be rationalized by a continuous, concave and monotonic utility function. As he interpreted this result (p. 946), "… if some data can be rationalized by any nontrivial utility function at all it can in fact be rationalized by a very nice utility function. Or put another way, violations of continuity, concavity, or monotonicity cannot be detected with only a finite number of demand observations".[4] See Diewert (2012), Varian (2012) and Vermeulen (2012) for recent overviews. In a recent survey, Barnett and Serletis (2008) portray the non-parametric revealed preference approach as an alternative to parametric and semi-non-parametric demand analysis, where much of the focus is on the estimation of elasticities.

[4] Varian (1983) extends these results to the situation where the rationalizing utility function is weakly separable. See, Swofford and Whitney (1994), Fleissig and Whitney (2003, 2008) and Cherchye, Demuynck, De Rock and Hjertstrand (2012) for methods that can be used to test for weak separability.

Revealed preference axioms and separability conditions are sometimes verified in econometric studies prior to estimating a parametric demand system model, where the ultimate purpose is to calculate elasticities from the fitted functional form.[5] Chavas and Cox (1997) appear to be the first study, however, to actually calculate elasticities based directly on Afriat's theorem. As they point out (p. 76), "[w]hile the use of nonparametric techniques for testing utility maximization and/or separability is now fairly standard… nonparametric estimation of demand response (*e.g.* price and income elasticities) has apparently not appeared in the literature." Chavas and Cox's approach is based on two representations of preferences that bound the family of utility functions that rationalize a dataset under GARP (see Afriat, 1987). In an empirical application, they use these two representations to calculate demand responses from 20% changes in either total expenditure or in the prices of individual goods. Their approach requires sizeable changes in prices or income, since both representations of preferences are piecewise linear. Varian (2012, p. 333) credits Blundell, Browning and Crawford (2008) as carrying this research program forward. Specifically, they suggest a method to obtain non-parametric bounds on predicted consumer responses to price changes. The underlying idea of their approach is to take a dataset that satisfies a revealed preference axiom and consider potential responses to a new combination of prices and incomes that would not violate the axiom given the observed data.

In contrast, we establish a connection between revealed preference theory and the estimation of elasticities building on results from Brown and Shannon (2000) and Brown and Kannan (2005). These papers establish that a set of observed price and quantity data can be rationalized by a smooth, strictly quasiconcave and monotone utility function if and only if there exists a solution to a "dual strict" version of the well-known Afriat inequalities and that this is the case if and only if the data satisfy Chiappori and Rochet's SSARP axiom. We derive new estimators for some standard elasticities from the demand function that corresponds to Brown and Shannon's utility function. We show that both the Hicksian and Marshallian demands exhibit negative own-price elasticities at all observed data points and that the Morishima and Mundlak elasticities of substitution are positive at all observed data points.

[5] See, for examples, Barnhart and Whitney (1988) and Fisher and Fleissig (1997).

The elasticities are identified only up to a strictly positive parameter which must be small enough in order for Brown and Shannon's utility function to have the required properties, but which is otherwise unrestricted. The estimators for the elasticities of substitution tend to zero in the limit as this parameter approaches zero. An implication of this is that while the elasticities of substitution are always positive, they can be made arbitrarily close to zero. Thus, if the dataset satisfies SSARP, then it is consistent with negligible substitutability between any pair of goods at all observed data points. Or, stated more along Varian's lines, we cannot detect non-negligible substitution with only a finite number of demand observations.

A commonly cited weakness of the revealed preference literature is that it does not account for random features of the observed data including measurement error (See, for example, Barnett and Serletis, 2008, p. 218). If the observed data violate SSARP, our approach could be applied by calculating the minimal perturbation of the data that does not violate the axiom following Varian (1985) and Epstein and Yatchew (1985) and then estimating elasticities from the demand function that rationalizes the minimally perturbed data. In this case, the size of the minimal perturbation would indicate how closely this demand function approximates the observed data.

The remainder of the paper is organized as follows: Essential background and notation is introduced in Section 2. The dual strict version of the Afriat inequalities are introduced in Section 3 and Brown and Shannon's theorem and some essential expressions derived from it are discussed in Section 4. Our main results on elasticities are derived in Section 5. Section 6 concludes the paper.

2. Background and Notation

We begin by introducing some notation. We consider a dataset consisting of T observed price and quantity vectors for a group of N goods. Let $\boldsymbol{p}_t = (p_{1,t}, \dots, p_{N,t})$ denote an observed price vector and $\boldsymbol{x}_t = (x_{1,t}, \dots, x_{N,t})$ the corresponding observed quantity vector, where $t = 1, \dots, T$. We assume that all $p_{i,t}$ and $x_{i,t}$ are strictly positive. Let $I_t = \boldsymbol{p}_t \cdot \boldsymbol{x}_t$ denote expenditure and let $\overline{\boldsymbol{p}}_t = \boldsymbol{p}_t / I_t$ denote expenditure normalized prices.

Next, we state the standard revealed preference relations and some associated axioms (see Varian, 1982, p. 947): For any pair of observations t and s, $\boldsymbol{x}_t$ is *strictly directly revealed* *preferred* to $\boldsymbol{x}_s$ ($\boldsymbol{x}_t P^0 \boldsymbol{x}_s$) if $\boldsymbol{p}_t \cdot \boldsymbol{x}_t > \boldsymbol{p}_t \cdot \boldsymbol{x}_s$, $\boldsymbol{x}_t$ is *directly revealed preferred* to $\boldsymbol{x}_s$ ($\boldsymbol{x}_t R^0 \boldsymbol{x}_s$) if $\boldsymbol{p}_t \cdot \boldsymbol{x}_t \geq \boldsymbol{p}_t \cdot \boldsymbol{x}_s$ and $\boldsymbol{x}_t$ is *revealed preferred* to $\boldsymbol{x}_s$ ($\boldsymbol{x}_t R \boldsymbol{x}_s$) if there exist k additional observations $s_1, \ldots, s_k$ such that $\boldsymbol{x}_t R^0 \boldsymbol{x}_{s_1}, \boldsymbol{x}_{s_1} R^0 \boldsymbol{x}_{s_2}, \ldots, \boldsymbol{x}_{s_{k-1}} R^0 \boldsymbol{x}_{s_k}, \boldsymbol{x}_{s_k} R^0 \boldsymbol{x}_s$. The data ($\boldsymbol{p}_t, \boldsymbol{x}_t$), $t = 1, \ldots, T$, satisfy the *Generalized Axiom of Revealed Preference* (GARP) if $\boldsymbol{x}_t R \boldsymbol{x}_s$ implies not $\boldsymbol{x}_s P^0 \boldsymbol{x}_t$. The data satisfy the *Strong Axiom of Revealed Preference* (SARP) if $\boldsymbol{x}_t R \boldsymbol{x}_s$ and $\boldsymbol{x}_s \neq \boldsymbol{x}_t$ implies not $\boldsymbol{x}_s R^0 \boldsymbol{x}_t$.

A utility function, $U(\boldsymbol{x})$, *rationalizes* the data ($\boldsymbol{p}_t, \boldsymbol{x}_t$), $t = 1, \ldots, T$, if $U(\boldsymbol{x}_t) \geq U(\boldsymbol{x})$ for all $\boldsymbol{x}$ such that $\boldsymbol{p}_t \cdot \boldsymbol{x} \leq \boldsymbol{p}_t \cdot \boldsymbol{x}_t$ (Varian, 1982, p. 946). Afriat's theorem states that there exists a continuous, monotone and concave utility function rationalizing the data if and only if the data satisfy GARP (See Varian, 1982, building on Afriat, 1967). As part of Afriat's theorem, Varian (1982) showed that GARP is a necessary and sufficient condition for there to exist numbers U_t and $\tau_t > 0$, $t = 1, \ldots, T$, satisfying the *Afriat inequalities:*

$$U_t - U_s \leq \tau_s(\boldsymbol{p}_s \cdot \boldsymbol{x}_t - \boldsymbol{p}_s \cdot \boldsymbol{x}_s)$$

for all s and t and he developed an algorithm to construct numbers satisfying these inequalities.[6]

As defined by Chiappori and Rochet (1987, p. 688), the data ($\boldsymbol{p}_t, \boldsymbol{x}_t$), $t = 1, \ldots, T$, satisfy the *Strong version of the Strong Axiom of Revealed Preference* (SSARP) if it satisfies SARP and if $\boldsymbol{p}_s \neq \boldsymbol{p}_t$ implies $\boldsymbol{x}_s \neq \boldsymbol{x}_t$. Chiappori and Rochet (1987) prove a theorem, which states that if a set of data satisfy SSARP, then there exists a strictly increasing, infinitely differentiable and strongly concave utility function rationalizing the data. As part of their proof, they show that if the data satisfy SSARP, then $\psi = \max_{s,t} \frac{1}{2}\|\boldsymbol{x}_s - \boldsymbol{x}_t\|^2$ is strictly positive and for any $\varepsilon > 0$, there exist numbers U_t and $\tau_t > 0$, $t = 1, \ldots, T$, such that

$$U_t - U_s \leq \tau_s(\boldsymbol{p}_s \cdot \boldsymbol{x}_t - \boldsymbol{p}_s \cdot \boldsymbol{x}_s) - \varepsilon\psi < \tau_s(\boldsymbol{p}_s \cdot \boldsymbol{x}_t - \boldsymbol{p}_s \cdot \boldsymbol{x}_s) \qquad (1)$$

for all $s \neq t$. Chiappori and Rochet also modify Varian's (1982) algorithm to construct numbers satisfying (1), which Brown and Shannon (2000) refer to as the *strict Afriat inequalities*.

[6] Matzkin and Richter (1991) showed that there exists a continuous, strictly monotone and strictly concave utility function that rationalizes the data if and only if the data satisfy SARP.

3. The Dual Strict Afriat Inequalities

Our results in this paper are based on a dual strict form of the Afriat inequalities. As defined by Brown and Shannon (2000, p. 1532), a solution to the *dual strict Afriat inequalities* consists of numbers, V_t and $\lambda_t > 0$, $t = 1, \ldots, T$, such that

$$V_s - V_t < \lambda_s I_s(\overline{\boldsymbol{p}}_t \cdot \boldsymbol{x}_s - \overline{\boldsymbol{p}}_s \cdot \boldsymbol{x}_s) = \lambda_s I_s \left(\frac{\boldsymbol{p}_t \cdot \boldsymbol{x}_s}{\boldsymbol{p}_t \cdot \boldsymbol{x}_t} - 1\right) \tag{2}$$

for all $s \neq t$.[7] A theorem in Brown and Kannan (2005) establishes that the data $(\boldsymbol{p}_t, \boldsymbol{x}_t)$, $t = 1, \ldots, T$, satisfy SSARP if and only if there exists a solution to the dual strict inequalities (2).[8] We will now provide a simple demonstration of the equivalence of SSARP and the dual strict inequalities. This shows one way to obtain numbers satisfying the dual strict Afriat inequalities and provides some intuition. Alternatively, numbers satisfying those inequalities could be obtained by solving an appropriate linear programming problem along the lines of Diewert (1973), Diewert and Parkan (1985) or Fleissig and Whitney (2005). Readers may skip directly to Section 4 without much loss of continuity.

3.1 Equivalence of SSARP and the Dual Strict Afriat Inequalities

To demonstrate the equivalence of the dual strict Afriat inequalities and SSARP, we will first show that the dual strict inequalities imply SSARP and then we will show the converse.

Suppose that there exist numbers satisfying the dual strict Afriat inequalities (2). Let $t_1, \ldots, t_k$ be *distinct* observations, such that $\boldsymbol{p}_{t_k} \cdot \boldsymbol{x}_{t_k} \geq \boldsymbol{p}_{t_k} \cdot \boldsymbol{x}_{t_{k-1}}$, $\boldsymbol{p}_{t_{k-1}} \cdot \boldsymbol{x}_{t_{k-1}} \geq \boldsymbol{p}_{t_{k-1}} \cdot \boldsymbol{x}_{t_{k-2}}$, …, $\boldsymbol{p}_{t_2} \cdot \boldsymbol{x}_{t_2} \geq \boldsymbol{p}_{t_2} \cdot \boldsymbol{x}_{t_1}$. (2) would then imply that $V_{t_1} < V_{t_2} < \cdots < V_{t_{k-1}} < V_{t_k}$. Following this line of argument, we can see that $\boldsymbol{x}_t R \boldsymbol{x}_s$ and $\boldsymbol{x}_s \neq \boldsymbol{x}_t$ would imply $V_s < V_t$. But, this rules out the possibility that $\boldsymbol{x}_s R^0 \boldsymbol{x}_t$, since that would imply $V_t < V_s$. Thus, the data must satisfy SARP. The data in fact satisfy SSARP, since (2) also ensures that $\boldsymbol{p}_s \neq \boldsymbol{p}_t$ implies $\boldsymbol{x}_s \neq \boldsymbol{x}_t$.[9]

[7] For clarity, we note that Brown and Shannon (2000) make frequent use of the variable $\boldsymbol{q}_t$, which is defined so that $\boldsymbol{q}_t / I_t = -\lambda_t \boldsymbol{x}_t$ for all t.

[8] Brown and Kannan (2005) consider the case where we observe prices, aggregate demand and income distributions, but not the demands of individual consumers. They consider what they refer to as *strict Walrasian inequalities*. These include the strict Afriat inequalities for each consumer and the condition that the sum of the individual demands equals aggregate demand. In that context, the family of strict inequalities is non-linear, since individual demands are not observed.

[9] If this were not the case, then there would be some $t_1 \neq t_2$ such that $\boldsymbol{p}_{t_1} \neq \boldsymbol{p}_{t_2}$ and $\boldsymbol{x}_{t_1} = \boldsymbol{x}_{t_2}$. (2) would then imply $V_{t_1} - V_{t_2} < \lambda_{t_1} I_{t_1} \left(\frac{\boldsymbol{p}_{t_2} \cdot \boldsymbol{x}_{t_1}}{\boldsymbol{p}_{t_2} \cdot \boldsymbol{x}_{t_2}} - 1\right) = 0$ and $V_{t_2} - V_{t_1} < \lambda_{t_2} I_{t_2} \left(\frac{\boldsymbol{p}_{t_1} \cdot \boldsymbol{x}_{t_2}}{\boldsymbol{p}_{t_1} \cdot \boldsymbol{x}_{t_1}} - 1\right) = 0$, which is a contradiction.

Next, we show the converse. Suppose that the data $(\boldsymbol{p}_t, \boldsymbol{x}_t)$, $t = 1, \dots, T$, satisfy SSARP. We can then use Chiappori and Rochet's algorithm to obtain numbers U_t and $\tau_t > 0, t = 1, \dots, T$, satisfying the strict Afriat inequalities (1). Now, define a new dataset consisting of quantities, $\widehat{\boldsymbol{x}}_t = \overline{\boldsymbol{p}}_t$, and prices, $\widehat{\boldsymbol{p}}_t = \boldsymbol{x}_t$. For this dataset, expenditure is given by $\hat{I}_t = \widehat{\boldsymbol{p}}_t \cdot \widehat{\boldsymbol{x}}_t = \overline{\boldsymbol{p}}_t \cdot \boldsymbol{x}_t = 1$ for all t. So, for this dataset expenditure is always one and the prices, $\widehat{\boldsymbol{p}}_t$, are also the expenditure normalized prices. It is then easy to verify that $\hat{V}_t = -U_t$ and $\hat{\lambda}_t = \tau_t I_t$ are numbers satisfying the dual strict Afriat inequalities for the data $(\widehat{\boldsymbol{p}}_t, \widehat{\boldsymbol{x}}_t)$, $t = 1, \dots, T$ and, consequently, these data also satisfy SSARP (by the same argument that we used above). We can then apply Chiappori and Rochet's algorithm one more time to obtain numbers $\widehat{U}_t$ and $\hat{\tau}_t > 0$, $t = 1, \dots, T$, which satisfy the strict Afriat inequalities for the data $(\widehat{\boldsymbol{p}}_t, \widehat{\boldsymbol{x}}_t)$, so that

$$\widehat{U}_t - \widehat{U}_s < \hat{\tau}_s(\widehat{\boldsymbol{p}}_s \cdot \widehat{\boldsymbol{x}}_t - \widehat{\boldsymbol{p}}_s \cdot \widehat{\boldsymbol{x}}_s)$$

for all $s \neq t$. It follows that $V_t = -\widehat{U}_t$ and $\lambda_t = \hat{\tau}_t / I_t > 0$ are numbers satisfying (2), since

$$V_s - V_t = \widehat{U}_t - \widehat{U}_s < \hat{\tau}_s(\widehat{\boldsymbol{p}}_s \cdot \widehat{\boldsymbol{x}}_t - \widehat{\boldsymbol{p}}_s \cdot \widehat{\boldsymbol{x}}_s) = \lambda_s I_s(\overline{\boldsymbol{p}}_t \cdot \boldsymbol{x}_s - \overline{\boldsymbol{p}}_s \cdot \boldsymbol{x}_s)$$

for all $s \neq t$. Thus, there exists a solution to the dual strict Afriat inequalities.

4. Brown and Shannon's Theorem

In this section, we provide some key results from Brown and Shannon (2000), which we will need in the next section. We start with the following theorem:

> ***Theorem (Brown and Shannon, 2000, p. 1532)***:
>
> *There exists a strictly quasiconcave, monotone utility function that rationalizes the data $(\boldsymbol{p}_t, \boldsymbol{x}_t)$, $t = 1, \dots, T$, which is smooth on an open set containing $\boldsymbol{x}_t$ for all t such that the implied demand function is locally monotone at $(\boldsymbol{p}_t, I_t)$ for all t if and only if there exists a solution to the dual strict Afriat inequalities.*

In order to develop our results, we need to briefly recap some arguments from Brown and Shannon's proof (see p. 1535 of their paper). As they explain, if the data satisfy the dual strict Afriat inequalities, then there exists a "sufficiently small" $\varepsilon > 0$, such that

$$V_s - V_t - \lambda_s I_s(\overline{\boldsymbol{p}}_t \cdot \boldsymbol{x}_s - \overline{\boldsymbol{p}}_s \cdot \boldsymbol{x}_s) < -\varepsilon M \quad (3)$$

for all $s \neq t$, with $M = \max_{s,t} \|\bar{\boldsymbol{p}}_s - \bar{\boldsymbol{p}}_t\|^2$.[10] Brown and Shannon construct a homogeneous of degree zero function, $W(\boldsymbol{p}, I) = W(\bar{\boldsymbol{p}}, 1) = \phi(\bar{\boldsymbol{p}})$, which is C^∞, convex in $\bar{\boldsymbol{p}}$, strictly increasing in I and strictly decreasing in $\boldsymbol{p}$ having the following properties:

$$W(\boldsymbol{p}_t, I_t) = \phi(\bar{\boldsymbol{p}}_t) = V_t - \frac{\varepsilon}{2}\|\bar{\boldsymbol{p}}_t\|^2$$

$$\frac{\partial \phi(\bar{\boldsymbol{p}}_t)}{\partial \bar{p}_i} = -\lambda_t I_t x_{i,t} - \varepsilon \frac{p_{i,t}}{I_t}$$

$$\frac{\partial^2 \phi(\bar{\boldsymbol{p}}_t)}{\partial \bar{p}_i \partial \bar{p}_j} \frac{1}{I_t^2} = \frac{\partial^2 W(\boldsymbol{p}_t, I_t)}{\partial p_i \partial p_j} = 0$$

Finally, Brown and Shannon define the indirect utility function

$$V(\boldsymbol{p}, I) = \phi(\bar{\boldsymbol{p}}) + \frac{\varepsilon}{2}\|\bar{\boldsymbol{p}}\|^2 = \phi\left(\frac{\boldsymbol{p}}{I}\right) + \frac{\varepsilon \sum_{k=1}^N p_k^2}{2I^2} \tag{4}$$

which is C^∞, homogeneous of degree zero, strictly convex in $\bar{\boldsymbol{p}}$, strictly increasing in I and strictly decreasing in $\boldsymbol{p}$ over a compact, convex subset containing the observed expenditure normalized prices in its interior for sufficiently small ε. Let f_i denote the demand function for good i corresponding to V. Brown and Shannon show that V also has the following properties:

$$i)\ V(\boldsymbol{p}_t, I_t) = \phi(\bar{\boldsymbol{p}}_t) + \frac{\varepsilon}{2}\|\bar{\boldsymbol{p}}_t\|^2 = V_t$$

$$ii)\ \frac{\partial V(\boldsymbol{p}_t, I_t)}{\partial p_i} = \frac{\partial \phi(\bar{\boldsymbol{p}}_t)}{\partial \bar{p}_i}\frac{1}{I_t} + \frac{\varepsilon p_{i,t}}{I_t^2} = -\lambda_t x_{i,t}$$

$$iii)\ \frac{\partial V(\boldsymbol{p}_t, I_t)}{\partial I} = -\sum_{k=1}^N \frac{\partial \phi(\bar{\boldsymbol{p}}_t)}{\partial \bar{p}_k}\frac{p_{k,t}}{I_t^2} - \frac{\varepsilon \sum_{k=1}^N p_{k,t}^2}{I_t^3} = \lambda_t$$

$$iv)\ \frac{\partial^2 V(\boldsymbol{p}_t, I_t)}{\partial p_i \partial p_j} = \frac{\partial^2 \phi(\bar{\boldsymbol{p}}_t)}{\partial \bar{p}_i \partial \bar{p}_j}\frac{1}{I_t^2} + \delta_{i,j}\frac{\varepsilon}{I_t^2} = \delta_{i,j}\frac{\varepsilon}{I_t^2}$$

$$v)\ f_i(\boldsymbol{p}_t, I_t) = -\frac{\partial V(\boldsymbol{p}_t, I_t)}{\partial p_i} \Big/ \frac{\partial V(\boldsymbol{p}_t, I_t)}{\partial I} = x_{i,t}$$

for all t, where $\delta_{i,j}$ equals one if $i = j$ and zero otherwise.

[10] (2) implies that $M > 0$ provided that the dataset contains at least two distinct observations. To see this, suppose that $\bar{\boldsymbol{p}}_{t_1} = \bar{\boldsymbol{p}}_{t_2}$ such that $t_1 \neq t_2$, then (2) implies $V_{t_1} - V_{t_2} = V_{t_1} - V_{t_2} - \lambda_{t_1} I_{t_1}(\bar{\boldsymbol{p}}_{t_2} \cdot \boldsymbol{x}_{t_1} - \bar{\boldsymbol{p}}_{t_1} \cdot \boldsymbol{x}_{t_1}) < 0$ and $V_{t_2} - V_{t_1} = V_{t_2} - V_{t_1} - \lambda_{t_2} I_{t_2}(\bar{\boldsymbol{p}}_{t_1} \cdot \boldsymbol{x}_{t_2} - \bar{\boldsymbol{p}}_{t_2} \cdot \boldsymbol{x}_{t_2}) < 0$, which is a contradiction.

Properties i and iii indicate that V_t represents the value of the indirect utility function and that λ_t represents the marginal utility of income at each observation. Following Varian (1982, pp. 946), U_t and τ_t in the standard form of the Afriat inequalities can be interpreted similarly. This is consistent with the fact that $V_t > V_s$ whenever $\boldsymbol{x}_t R \boldsymbol{x}_s$ and $\boldsymbol{x}_s \neq \boldsymbol{x}_t$ (see Section 3.1). Properties ii and iii combine to give property v, which shows that the data are rationalized by the demand function corresponding to V as defined by (4).

We will also need the following additional properties, which are easily demonstrated:

$$vi)\ \frac{\partial^2 V(\boldsymbol{p}_t, I_t)}{\partial p_i \partial I} = -\sum_{k=1}^{N} \frac{\partial^2 \phi(\overline{\boldsymbol{p}}_t)}{\partial \bar{p}_k \partial \bar{p}_i} \frac{p_{k,t}}{I_t^3} - \frac{\partial \phi(\overline{\boldsymbol{p}}_t)}{\partial \bar{p}_i} \frac{1}{I_t^2} - \frac{2\varepsilon p_{i,t}}{I_t^3} = \frac{\lambda_t x_{i,t}}{I_t} - \frac{\varepsilon p_{i,t}}{I_t^3}$$

$$vii)\ \frac{\partial^2 V(\boldsymbol{p}_t, I_t)}{\partial I^2} = \sum_{j=1}^{N} \sum_{k=1}^{N} \frac{\partial^2 \phi(\overline{\boldsymbol{p}}_t)}{\partial \bar{p}_j \partial \bar{p}_k} \frac{p_{k,t} p_{j,t}}{I_t^4} + \sum_{k=1}^{N} \frac{\partial \phi(\overline{\boldsymbol{p}}_t)}{\partial \bar{p}_k} \frac{2p_{k,t}}{I_t^3} + \frac{3\varepsilon \sum_{k=1}^{N} p_{k,t}^2}{I_t^4} =$$

$$-\frac{2\lambda_t}{I_t} + \frac{\varepsilon \sum_{k=1}^{N} p_{k,t}^2}{I_t^4}$$

for all t. We can now proceed to our main results.

5. Calculating Elasticities at the Observed Data Points

If the data $(\boldsymbol{p}_t, \boldsymbol{x}_t)$, $t = 1, \dots, T$, satisfies SSARP, then there exists a solution to the dual strict Afriat inequalities. Consequently, we can calculate elasticities of demand at each observed data point from the demand function corresponding to the indirect utility function defined by (4). In this section, we will derive expressions for some standard elasticities. We find that both the Hicksian and Marshallian demands exhibit negative own-price elasticities at all observed data points and that the Morishima and Mundlak elasticities of substitution are positive at all observed data points. These expressions depend on the parameter ε described in the previous section. We show that ε can be made arbitrarily close to zero and explore the implications of this for the various elasticities we derive. We conclude the section by discussing how our results might be relevant even if the data violate SSARP.

5.1 Elasticity expressions for data that satisfy SSARP

Davis and Gauger (1996) summarize the elasticity formulas we will consider and the relationships between them. Let $E_i = \frac{\partial ln f_i}{\partial ln I}$ denote the *expenditure elasticity* of good i, let

$\eta^m_{i,j} = \frac{\partial ln f_i}{\partial ln p_j}$ be the *Marshallian cross-price elasticity* for good i with respect to the price of good j, and define the *Mundlak elasticity of substitution* as $U_{i,j} = -\frac{\partial ln(f_i/f_j)}{\partial ln(p_i/p_j)} = \eta^m_{j,i} - \eta^m_{i,i}$ (Davis and Gauger, 1996, p. 204).

Assuming that the data $(\boldsymbol{p}_t, \boldsymbol{x}_t)$, $t = 1, \ldots, T$, satisfy SSARP, there is a solution to the dual strict Afriat inequalities and, consequently, there exist numbers satisfying (3). We can, therefore, calculate elasticities from the demand function that rationalizes the data corresponding to the indirect utility function defined by (4). We use the notation $E_i(t)$ to denote the expenditure elasticity at a particular data point t and similarly for all other elasticities. To simplify some expressions, we let $w_{i,t} = p_{i,t}x_{i,t}/I_t$ denote the expenditure share for good i. The necessary derivatives for calculating the elasticities described above at the observed data points are provided by $ii - vii$, which gives us the following expressions for all t:

$$E_i(t) - 1 = \frac{\varepsilon}{\lambda_t I_t^2}\left(\frac{p_{i,t}}{x_{i,t}} - \frac{\sum_{k=1}^N p_{k,t}^2}{I_t}\right) \tag{5}$$

$$\eta^m_{i,i}(t) + w_{i,t} = \frac{\varepsilon p_{i,t}^2}{\lambda_t I_t^2}\left(\frac{1}{I_t} - \frac{1}{p_{i,t}x_{i,t}}\right) \tag{6}$$

and for $i \neq j$

$$U_{i,j}(t) = \frac{\varepsilon}{\lambda_t I_t^2}\frac{p_{i,t}}{x_{i,t}} \tag{7}$$

The expression for $U_{i,j}(t)$ is positive, since ε and λ_t are both strictly positive. The expression for the Marshallian own-price elasticities, given by (6), are negative, since $p_{i,t}x_{i,t} \leq I_t$.

Next, we consider Hicksian elasticities. Let $h_i(\boldsymbol{p}, u)$ denote the Hicksian demand function for good i. Let $\eta^h_{i,j} = \frac{\partial ln h_i}{\partial ln p_j} = \eta^m_{i,j} + w_j E_i$ be the *Hicksian cross-price elasticity* for good i with respect to the price of good j (Davis and Gauger, 1996, p. 204) and let $M_{i,j} = -\frac{\partial ln(h_i/h_j)}{\partial ln(p_i/p_j)} = \eta^h_{j,i} - \eta^h_{i,i}$ be the *Morishima elasticity of substitution* (Blackorby and Russell, 1989). It then follows from (5) and (6) that for all t

$$\eta^h_{i,i}(t) = \frac{\varepsilon}{\lambda_t I_t^2}\frac{p_{i,t}}{x_{i,t}}\left[2w_{i,t} - 1 - \left(\frac{x_{i,t}}{I_t}\right)^2 \sum_{k=1}^N p_{k,t}^2\right] \tag{8}$$

and for $i \neq j$

$$M_{i,j}(t) = \frac{\varepsilon}{\lambda_t I_t^2}\left[\frac{p_{i,t}}{x_{i,t}} - \left(\frac{p_{i,t}}{x_{i,t}} - \frac{p_{j,t}}{x_{j,t}}\right) w_{i,t}\right] \qquad (9)$$

The sign of $M_{i,j}(t)$ is determined by $\frac{p_{i,t}}{x_{i,t}} - \left(\frac{p_{i,t}}{x_{i,t}} - \frac{p_{j,t}}{x_{j,t}}\right) w_{i,t} = \frac{p_{i,t}}{x_{i,t}}\left(1 - w_{i,t}\right) + \frac{p_{j,t}}{x_{j,t}} w_{i,t}$, which is positive since $w_{i,t} \leq 1$. Consequently, the Morishma elasticities, like the Mundlak elasticities, are positive for all t. The Hicksian own price elasticities, given by (8), are negative, since the term in brackets equals $-\left(w_{i,t}^2 - 2w_{i,t} + 1\right) + \left(\frac{x_{i,t}}{I_t}\right)^2 \left(p_{i,t}^2 - \sum_{k=1}^{N} p_{k,t}^2\right) = -\left(w_{i,t} - 1\right)^2 - \sum_{k \neq i} p_{k,t}^2$, which is negative.

5.2 Limiting expressions

As discussed in the previous section, if a dataset satisfies SSARP, then for any $\varepsilon > 0$ there exists a solution to (3), which can be used to calculate elasticities at each observed data point from expressions (5) – (9). The fact that the right-hand sides of expressions (5) – (9) are all proportional to ε/λ_t has some interesting implications, which we explore here.

Suppose that the data satisfy SSARP. Let $\varepsilon^0 > 0$ and let V_t^0 and $\lambda_t^0 > 0$ be a corresponding set of numbers satisfying the inequalities (3) such that

$$V_s^0 - V_t^0 - \lambda_s^0 I_s(\overline{\boldsymbol{p}}_t \cdot \boldsymbol{x}_s - \overline{\boldsymbol{p}}_s \cdot \boldsymbol{x}_s) < -\varepsilon^0 M$$

for all $s \neq t$. The same numbers would then be a solution to (3) for any strictly positive ε such that $\varepsilon \leq \varepsilon^0$, since in that case

$$V_s^0 - V_t^0 - \lambda_s^0 I_s(\overline{\boldsymbol{p}}_t \cdot \boldsymbol{x}_s - \overline{\boldsymbol{p}}_s \cdot \boldsymbol{x}_s) < -\varepsilon^0 M \leq -\varepsilon M$$

for all $s \neq t$ (recall that by definition $M \geq 0$). Thus, if V_t^0 and $\lambda_t^0 > 0$ constitute a solution to (3) for $\varepsilon^0 > 0$, they also constitute a solution to (3) for any $\varepsilon > 0$ that is less than or equal to ε^0.

This property has the interesting implication that ε can be chosen so that ε/λ_t^0 is arbitrarily close to zero for all t without violating (3) from which (5) – (9) were derived. This, in turn, implies that we can choose ε so that $\eta_{i,i}^h(t)$, $M_{i,j}(t)$ and $U_{i,j}(t)$ are arbitrarily close to zero for all t and so that $\eta_{i,i}^m(t)$ is arbitrarily close to $-w_{i,t}$ and $E_i(t)$ is arbitrarily close to one for all t when evaluating (5) – (9) using ε/λ_t^0.

Thus, while both the Morishima and Mundlak elasticities of substitution are positive at all observations – indicating that pairs of goods are substitutes – they can in fact be made arbitrarily close to zero at all observed data points by choosing a small enough ε. Put differently, if the dataset satisfies SSARP, then it is consistent with negligible substitutability between any pair of goods at all observed data points. With respect to own-price elasticities, we find that the Hicksian own-price elasticities are negative at all data points, but they can also be made arbitrarily close to zero. Similarly, the Marshallian own-price elasticities can be made arbitrarily close to the corresponding good's expenditure share while the expenditure elasticities can all be made arbitrarily close to one at all data points.

5.3 What if the data violate SSARP?

Revealed preference axioms are not highly restrictive in some situations. For example, aggregate data often show strong trends in expenditure over time providing relatively few opportunities to violate GARP; See, for example, Varian (1982, p. 965). In such applications, SSARP is not likely to be much more restrictive than GARP. Nevertheless, it is reasonable to explore whether our results would have any practical implications when SSARP is violated. In our view, this is closely related to testing whether violations of revealed preference axioms can be attributed to measurement errors in the data.[11]

With respect to GARP, Varian (1985) and Epstein and Yatchew (1985) propose calculating the minimal perturbation of the observed quantities subject to the condition that the Afriat inequalities hold for the perturbed data. Such minimally perturbed data satisfy GARP by construction.[12] In the present context, if SSARP is violated, we could compute a minimal perturbation subject to the condition that the dual strict Afriat inequalities hold. We could then calculate elasticities for the demand function that rationalizes the minimally perturbed data. The minimally perturbed data is not an estimate of the true data, but the size of the minimal perturbation would indicate how closely the corresponding demand function approximates the observed data. Varian (1985) suggests a method to determine if violations of a revealed preference axiom are statistically significant, where the observed data are equal to the "true data" plus a random error term attributable to measurement errors or other factors. The minimal perturbation can be used to test the null hypothesis that the true data satisfy the revealed preference axiom; See Varian (1985) and Epstein and Yatchew (1985).

[11] See Jones and Edgerton (2009) for an overview.

[12] See Jones *et al.* (2005), Jones and de Peretti (2005) and Jones and Stracca (2008) for empirical applications.

Fleissig and Whitney (2005) propose an alternative approach to evaluating the significance of GARP violations. Their approach is based on solving a linear programming problem that minimizes the maximal violation of the Afriat inequalities. If the minimized objective is zero, then the observed data satisfy GARP. If not, then they propose an upper bound test to determine the significance of the violations; See also Fleissig and Whitney (2008). As mentioned previously, the same linear programming approach can be modified to find a solution to the dual strict Afriat inequalities. In the event that the inequalities are violated, the solution to the linear program would yield a solution that minimizes the maximal violation of those inequalities.

6. Conclusions

In this paper, we have shown that a direct connection can be established between elasticities of demand and Chiappori and Rochet's SSARP axiom. If a dataset consisting of observed prices and quantities for a set of goods satisfies SSARP, then we can derive expressions that estimate the elasticities of demand at all data points from a demand function corresponding to a smooth, strictly quasiconcave, monotone utility function that rationalizes the observed data. We find that the Morishima and Mundlak elasticities of substitution are always positive when calculated using our approach. The elasticity expressions depend, however, on a strictly positive parameter whose value is not determined by the theory. It must be small enough in order for the utility function to satisfy the required theoretical regularity properties, but it can be made arbitrarily close to zero.[13] An implication of this property is that the estimated elasticities of substitution approach zero in the limit as this parameter approaches zero, so that for a dataset that satisfies SSARP the substitution elasticities are empirically indistinguishable from zero at the observed data points.

If the observed data violate SSARP, then our approach could be implemented by computing minimally perturbed data that do not violate the axiom (along the lines of Varian, 1985 and Epstein and Yatchew, 1985) and then estimate elasticities from the demand function that rationalizes the minimally perturbed data. That demand function would more or less

[13] Speaking more generally, revealed preference methods do not yield a unique method for constructing a rationalizing utility function for a set of data satisfying a particular axiom of interest. Recall, for example, our discussion of Chavas and Cox's approach in the introduction.

closely approximate the observed data depending on the size of the minimal perturbation. In our view, this provides a fairly general answer to the question posed in the title of our paper. If the minimal perturbation is fairly small, then our estimators can be compared to the standard econometric approach to estimating elasticities from parametric functional forms fitted to the observed expenditure share system.[14]

[14] An important issue in that literature is regularity. Barnett and Serletis (2008) emphasize that certain locally flexible functional forms - which they refer to as "effectively globally regular" - such as the minflex Laurent, quadratic AIDS, and the general exponential form have much larger regularity regions than others. We refer the reader to their paper for an extensive discussion of the demand systems approach with an emphasis on the issue of regularity.

References

Afriat, S. N. (1967) "The Construction of Utility Functions from Expenditure Data", *International Economic Review* 8, 67–77.

Afriat, S. N. (1987) *Logic of Choice and Economic Theory* (Clarendon Press, Oxford).

Barnett, W. and A. Serletis (2008) "Consumer Preferences and Demand Systems", *Journal of Econometrics* 147, 210-224.

Barnhart, S. and G. Whitney (1988) "Nonparametic Analysis in Parametric Estimation: An Application to Translog Demand Systems", *Review of Economics and Statistics* 70 (1), 149-153.

Blackorby, C. and R. R. Russell (1989) "Will the Real Elasticity of Substitution Please Stand Up? (A Comparison of the Allen/Uzawa and Morishima Elasticities)" *American Economic Review* 79, 882-888.

Blundell, R., M. Browning and I. Crawford (2008). "Best Nonparametric Bounds on Demand Responses". *Econometrica* 76 (6), 1227-1262.

Brown, D. and R. Kannan (2005). "Decision Methods for Solving Systems of Walrasian Inequalities". Cowles Foundation Discussion Paper No. 1508.

Brown, D. and C. Shannon (2000) "Uniqueness, Stability, and Comparative Statics in Rationalizable Walrasian Markets", *Econometrica* 68 (6), 1529-1539.

Chavas, J.-P. and T. Cox (1997) "On Nonparametric Demand Analysis", *European Economic Review* 41, 75-95.

Cherchye, L., T. Demuynck, B. De Rock and P. Hjertstrand (2012). "Revealed Preference Tests for Weak Separability: An Integer Programming Approach". Working Paper.

Chiappori, P.-A. and J.-C. Rochet (1987) ''Revealed Preferences and Differentiable Demand", *Econometrica* 55 (3), 687-691.

Davis, G. and J. Gauger (1996) "Measuring Substitution in Monetary-Asset Demand Systems", *Journal of Business and Economic Statistics* 14 (2), 203-208.

Diewert, W. E. (1973) "Afriat and Revealed Preference Theory", *Review of Economic Studies* 40, 419-426.

Diewert, W. E. and C. Parkan (1985) "Tests for Consistency of Consumer Data", *Journal of Econometrics* 30 (1-2), 127-147.

Diewert, W. E. (2012) "Afriat's Theorem and Some Extensions to Choice Under Uncertainty", *Economic Journal* 122, 305-331.

Epstein, L. and A. Yatchew (1985) "Nonparametric Hypothesis Testing Procedures and Applications to Demand Analysis", *Journal of Econometrics* 30, 149-169.

Fisher, D. and A. Fleissig (1997) "Monetary Aggregation and the Demand for Assets", *Journal of Money, Credit and Banking* 29 (4), 458-475.

Fleissig, A. R. and G. A. Whitney (2003), "A New PC-Based Test for Varian's Weak Separability Conditions", *Journal of Business and Economic Statistics*, 21, 133-144.

Fleissig, A. R. and G. A. Whitney (2005) "Testing for the Significance of Violations of Afriat's Inequalities", *Journal of Business and Economic Statistics* 23, 355-362.

Fleissig, A. R. and G. A. Whitney (2008) "A Nonparametric Test of Weak Separability and Consumer Preferences", *Journal of Econometrics* 147, 275-281.

Jones, B., D. Dutkowsky and T. Elger (2005) "Sweep Programs and Optimal Monetary Aggregation", *Journal of Banking and Finance* 29, 483-508.

Jones, B. and P. de Peretti (2005) "A Comparison of Two Methods for Testing the Utility Maximization Hypothesis when Quantity Data are Measured with Error", *Macroeconomic Dynamics* 9 (5), 612-629.

Jones, B. and D. Edgerton (2009) "Testing Utility Maximization with Measurement Errors in the Data", *Advances in Econometrics* 24, 199-236.

Jones, B. and L. Stracca (2008) "Does Money Matter in the IS Curve: The Case of the UK," *The Manchester School* 76 (S1), 2008, 58-84.

Swofford, J. L. and G. A. Whitney (1994), "A Revealed Preference Test for Weakly Separable Utility Maximization with Incomplete Adjustment", *Journal of Econometrics* 60, 235-249.

Varian, H. (1982) "The Nonparametric Approach to Demand Analysis", *Econometrica* 50 (4), 945-973.

Varian, H. (1983) "Nonparametric Tests of Consumer Behavior", *Review of Economic Studies* 50, 99-110.

Varian, H. (1985) "Non-parametric Analysis of Optimizing Behavior with Measurement Error", *Journal of Econometrics* 30, 445-458.

Varian, H. (2012) "Revealed Preference and its Applications", *Economic Journal* 122, 332-338

Vermeulen, F. (2012) "Foundations of Revealed Preference: Introduction", *Economic Journal* 122, 287-294.