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SUPPLY FUNCTION EQUILIBRIA IN OLIGOPOLY UNDER UNCERTAINTY

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We model an oligopoly facing uncertain demand in which each firm chooses as its strategy a "supply function" relating its quantity to its price. Such a strategy allows a firm to adapt better to the uncertain environment than either setting a fixed price or setting a fixed quantity; commitment to a supply function may be accomplished in practice by the choice of the firm's size and structure, its culture and values, and the incentive systems and decision rules for its employees. In the absence of uncertainty, there exists an enormous multiplicity of equilibria in supply functions, but uncertainty, by forcing each firm's supply function to be optimal against a range of possible residual demand curves, dramatically reduces the set of equilibria. Under uncertainty, we prove the existence of a Nash equilibrium in supply functions for a symmetric oligopoly producing a homogeneous good and give sufficient conditions for uniqueness. We perform comparative statics with respect to firms' costs, the industry demand, the nature of the demand uncertainty, and the number of firms, and sketch the extension to differentiated products. Firms' equilibrium supply functions are steeper with marginal cost curves that are steeper relative to demand, fewer firms, more highly differentiated products, and demand uncertainty that is relatively greater at higher prices. The steeper are the supply functions firms choose in equilibrium, the more closely competition resembles the Cournot model (which exogenously imposes vertical supply functions—fixed quantities); with flatter equilibrium supply functions, competition is closer to the Bertrand model (which exogenously imposes horizontal supply functions—fixed prices).

KEYWORDS: Supply functions, oligopoly, uncertainty, Cournot, Bertrand.

1. INTRODUCTION

ECONOMISTS HAVE LONG DEBATED whether it makes more sense to think of firms as choosing prices or quantities as strategic variables.[2] In a world with uncertainty, however, a firm may not want to commit to either of these simple types of strategy, nor can all decisions be deferred until the resolution of the uncertainty. Adjustment costs and the problems of organizational communication mean that decisions about the size and structure of the organization, the organization's values, and the decision rules to be followed by lower-level managers must be made in advance. These decisions implicitly determine a *supply function* that relates the quantity the firm will sell to the price the market will bear. Such a supply function allows the firm to adapt better to changing conditions than does a simple commitment to a fixed price or a fixed quantity, come what may.

For example, if a consulting firm sticks to a fixed rate per hour, it is fixing a price (perhaps subject to a capacity constraint). In fact, however, even when firms quote fixed rates, the real price often varies. When business is slack, more hours

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[2]The literature begins with Bertrand's (1883) criticism of Cournot (1838). The more recent literature includes Singh and Vives (1984), Cheng (1985), and Klemperer and Meyer (1986).

are worked on projects than are reported, but when the office is busy, marginally related training, travel time, and the time spent originally negotiating the project may all be charged to the client. Top management in effect commits to a supply function by choosing the number of employees and the rules and organizational values that determine how both the real price and the number of hours supplied adjust to demand—some firms hold the real price very close to the quoted one by choosing very rigid rules about accurately reporting the hours worked to the client, while others allow individual managers far more discretion. It is the individual managers, bidding for staff resources within the firm, who move the firm along its supply schedule and bring the firm into equilibrium with the market demand. As another example, airlines use computer systems to adjust the number of discount seats they offer according to current reservation levels. Here management chooses a particular supply function by its choice of computer program. Running the program finds the price-quantity pair on this supply function that lies on the actual market demand curve. Grossman (1981) (in a model without uncertainty) suggests that firms can commit to supply functions by signing contracts with customers, specifying the quantity to be supplied as a function of the market price. Alternatively, firms could sign contracts with suppliers of inputs (e.g. workers), tying production and factor payments to the output price. When competing for government procurement contracts, firms often submit as bids functions specifying the amounts they are prepared to supply at any given price.[3]

Grossman (1981) and Hart (1985) studied supply function equilibria in the absence of uncertainty. We consider this approach problematic for two reasons. First, as we show in Section 2, an enormous multiplicity of outcomes can be supported by Nash equilibria in supply functions in the absence of uncertainty. Second, in the absence of uncertainty, the motivation for modeling firms as competing via supply functions is not compelling. Without uncertainty, a firm knows its equilibrium residual demand with certainty, and it therefore has a single profit-maximizing point, which it could achieve by choosing either a fixed price or a fixed quantity. It gains nothing from the ability to choose a more general supply function, so the use of these more general strategies would not be robust even to arbitrarily tiny costs of maintaining the greater flexibility they embody.

With exogenous uncertainty about, say, market demand, however, a firm has an uncertain residual demand even in equilibrium and so has a set of profit-maximizing points, one corresponding to each possible realization of its residual demand. In this case, a supply function provides valuable flexibility, because it

[3] We are focusing on the choice of supply functions by suppliers in the output market. However, the share contracts considered by Weitzman (1984) are formally similar: they specify the wage-employment combinations that firms, as buyers of labor, are willing to accept. Also related is Wilson's (1979) "demand function equilibrium" in a game in which bidders for shares submit demand schedules to an auctioneer who determines the market-clearing price and the allocation of a fixed supply of shares.

can be chosen to coincide with this set of optimal price-quantity pairs. It thus commits the firm in advance of the realization of the uncertainty to achieving its ex post profit-maximizing outcome. The only flexibility the firm gives up is the flexibility to produce at price-quantity pairs at which, given the other firms' equilibrium behavior, it will *never* be optimal to produce.

Explicit consideration of the uncertainty firms face not only explains why firms wish to adopt supply functions as strategic variables; we will show that it also dramatically shrinks the set of Nash equilibria relative to a world without uncertainty, perhaps to a unique equilibrium. Furthermore, given the nature and the support of one-dimensional uncertainty (for example, a common additive shock to all firms' demands), the equilibria are independent of the distribution of the uncertainty.[4]

Below we develop a formal model of supply function competition in oligopoly under demand uncertainty. Before the demand shock is realized, each firm commits to a function specifying the quantity it will produce as a function of its price. We assume, for simplicity, that firms' cost curves are independent of their choices of supply functions. After the shock is realized, all markets clear: each firm produces at the point on its supply function which intersects its realized residual demand curve. We do *not* model the process by which markets are brought into equilibrium. Formally, we may think of each firm as observing the demand shock ex post and then choosing the point on its supply function that will clear the market (given its rivals' chosen supply functions). We note that since in the equilibria we study a firm which in fact achieves its *unconstrained* ex post optimum, it would have no incentive to deviate from this behavior even if it were not committed to choosing a point on its supply function (provided that it believes that its rivals are committed to their supply functions). In practice, we believe that firms have mechanisms that allow them to (approximately) implement supply functions *without* directly observing the realization of the random shock. For example, some management consulting firms have "internal market mechanisms" in which managers compete with each other for staff and other resources at the same time as appropriately adjusting the prices they are quoting in negotiating fees with clients. This process brings internal supply into (approximate) balance with external demand, without any single decision maker directly observing the demand realization. Similarly, airlines' computerized reser-

[4] Of course, introducing uncertainty is not the only way to enrich the model and so choose among the Nash equilibria. Different supply functions may have different costs of implementation. For example, choosing a vertical supply function (fixed quantity) may offer a cost advantage relative to choosing a flatter supply function and having to retain a flexible production technology until the realization of demand is known; on the other hand, choosing a horizontal supply function is equivalent to fixing a price and may be organizationally easy to implement relative to a more complex functional form. Grossman (1981) restricts supply functions to lie everywhere above the average cost curve and shows that when there is free entry to a technology with fixed costs and strictly convex variable costs, then the Nash equilibrium in (upward-sloping) supply functions will be the competitive equilibrium. Vives' (1986) model of capacity choice followed by competitive pricing is formally very similar to a model of competition in supply functions in which the slope of supply functions is fixed by the value of an exogenous technological coefficient.

vation systems approximately implement a supply function without complete observations of demand.[5]

Any equilibrium concept requires some story about how firms adjust to uncertainty. For example, in a stochastic Cournot game, firms must adjust their prices to the realization of demand, and in a stochastic Bertrand game, firms must adjust their quantities. Only in a supply function equilibrium, however, do firms adjust to the uncertainty in an *optimal* manner given their competitors' behavior—with stochastic Cournot or Bertrand, firms would wish to alter their behavior after learning something about demand.

To summarize, our goal in this paper is to provide as realistic as possible a model of oligopolistic competition in a *static* (*one-shot*) *setting*. Our belief is that the model we provide better represents oligopoly behavior than standard single-period Bertrand or Cournot models.[6]

Section 2 shows why there is a multiplicity of equilibria in supply functions in the absence of uncertainty.[7] Section 3 is the core of the paper. It presents the formal analysis of supply function equilibria under uncertainty. We begin with symmetric duopolists producing a homogeneous good in a market where an exogenous one-dimensional shock causes a horizontal translation of the demand

[5]The *Wall Street Journal* (August 24, 1984) quotes Delta's manager of system marketing as saying, "We don't have to know if a balloon race in Albuquerque or a rodeo in Lubbock is causing an increase in demand for a flight." A computer program chooses quantities of different categories of discount seats, gets an estimate of demand by comparing reservation levels with historical patterns, adjusts the quantities of discount seats according to the estimate, re-estimates demand, etc. Here sales are made in the process of adjusting to the demand shock. However, we know (Myerson (1981), Bulow and Roberts (1987)) that any (single- or multi-stage) mechanism that a monopolist uses to sell a fixed number of tickets to the highest-reservation-price buyers (and that gives any non-buyer zero expected surplus) yields the same expected revenue. Therefore, provided the oligopolists' ticketing systems are symmetrically programmed (in equilibrium), any process by which they sell a total of Q tickets to the Q highest-reservation-price customers yields each firm the same expected revenue as if they had sold all Q tickets in a $Q+1$th price auction and shared the revenues. Thus, from firms' point of view, the approximation of the ticketing process to the implementation of a supply function equilibrium in which sales are made only at the market-clearing prices may be a good one.

[6]A more complete model of oligopoly should probably involve a multistage game. Even with a single time period in which goods are delivered (e.g., single airline flight or period in which consulting services are provided), adjustment to uncertainty may involve sequential sales and production decisions in response to (possibly imperfect) observations of competitors' behavior.

[7]In the model of Section 2, competition in supply functions is equivalent to competition in reaction functions: any supply function for i determines the output for i that would clear the market given an output for j, and any such reaction function, coupled with the demand curve, determines a relationship between i's quantity and price. (See Hart (1985).) Also representable as models of supply function competition are models in which owners design incentive contracts relating their managers' compensation to competitors' profits and/or sales as well as to the firm's own profits and perhaps sales. (A manager will maximize his salary given his incentive contract and is assumed to know the contracts of the rival manager at the time he makes production or pricing decisions. In this setting, owner i's choice of incentive contract for his manager simply determines the output manager i will choose as a function of the other firm's output, that is, the contract determines firm i's reaction function.) Although uniqueness of equilibrium can be obtained under certainty by restricting the form of contracts (see Vickers (1985) and Fershtman and Judd (1987)), it follows from our analysis in Section 2 that under certainty, an enormous multiplicity of outcomes are equilibria if there are no restrictions on the form of contracts. (See also Fershtman, Judd, and Kalai (1987) on this point.) Under uncertainty, however, supply functions, reaction functions, and incentive contracts are not equivalent strategic variables.

curve. We demonstrate existence of a supply function equilibrium for unbounded support of the uncertainty and show for this case that in any equilibrium, supply functions are symmetric across firms and upward-sloping. We give conditions under which equilibrium is unique; an example of uniqueness is provided by the case of linear demand and linear marginal cost.[8] We present general comparative statics propositions (one of which extends our results to a symmetric oligopoly of any size) and illustrate them using the linear model. Changes in the environment affect not only the levels but also the *slopes* of firms' supply functions, and hence affect whether competition is more like Cournot competition (in which firms are constrained to compete with fixed quantities, that is, vertical supply functions) or Bertrand competition (in which firms compete with fixed prices, that is, horizontal supply functions). The extension to differentiated products is sketched in Section 4. Section 5 presents some interpretations and concluding observations.

2. SUPPLY FUNCTION EQUILIBRIA WITHOUT UNCERTAINTY

We begin by showing why there is a multiplicity of Nash equilibria in continuous, twice differentiable supply functions in an undifferentiated products duopoly in the absence of uncertainty.

The industry demand curve is $Q = D(p)$. Define $\hat{p}$ as the price such that $D(\hat{p}) = 0$, and assume that $D(\cdot)$ is twice continuously differentiable, strictly increasing, and concave on $(0, \hat{p})$. The firms have identical cost functions $C(\cdot)$, with $C'(q) \geq 0$ and $C''(q) \geq 0$ for all $q \geq 0$. A strategy for firm k $(k = i, j)$ is a twice continuously differentiable function mapping price into a level of output for k: $S^k : [0, \hat{p}) \rightarrow (-\infty, \infty)$. Allowing firms (in principle) to choose supply functions specifying negative quantities makes no difference to our results but will slightly simplify the analysis in Section 3 by making residual demand curves everywhere differentiable.

Firms i and j choose supply functions simultaneously. Provided that there is a unique price p^* such that $D(p^*) = S^i(p^*) + S^j(p^*)$, that is, industry demand matches the total amount firms are willing to supply, firms sell $S^i(p^*)$ and $S^j(p^*)$ at p^*, earning profits $p^*S^i(p^*) - C(S^i(p^*))$ and $p^*S^j(p^*) - C(S^j(p^*))$. We assume that if a market-clearing price does not exist, or is not unique, then no production takes place and firms' profits are zero. This assumption ensures that such an outcome will not arise in equilibrium, but the assumption does not constrain firms' behavior in any important way—the equilibria we examine remain equilibria under any reasonable alternative assumption about firms'

[8] The spirit of this analysis is similar to that of our (1986) paper: we show there that the presence of exogenous uncertainty can give firms strict preferences between setting price and setting quantity ex ante and, as a consequence, can lead to a unique equilibrium in a game in which firms can choose between these two types of strategic variable. Other examples of introducing uncertainty to refine a continuum of Nash equilibria include Nash's (1953) and Binmore's (1987) selection of a unique equilibrium in Nash's demand game and Saloner's (1982) and Matthews and Mirman's (1983) selection of an equilibrium in a limit-pricing game.

payoffs when the market-clearing price is not unique.[9] We confine attention to pure strategy Nash equilibria in supply functions: such an equilibrium consists of a pair of functions $S^i(p)$ and $S^j(p)$ such that $S^k(\cdot)$ maximizes k's profits given that m chooses $S^m(\cdot)$, $k, m = i, j$, $m \neq k$.

Consider a strictly positive output pair $(\bar{q}_i, \bar{q}_j)$ at which the market price $\bar{p} = D^{-1}(\bar{q}_i + \bar{q}_j)$ exceeds each firm's marginal cost $C'(\bar{q}_k)$, $k = i, j$. To support this point as an equilibrium outcome, we seek a pair of supply functions $S^i(\cdot)$ and $S^j(\cdot)$ passing through $(\bar{p}, \bar{q}_i)$ and $(\bar{p}, \bar{q}_j)$, respectively, and such that $\bar{q}_i$ is a profit-maximizing point along i's residual demand curve and similarly for j. Given $S^j(p)$, i finds its profit-maximizing price along its residual demand curve by solving $\max_p p[D(p) - S^j(p)] - C(D(p) - S^j(p))$, yielding the first-order condition

$$D(p) - S^j(p) + [p - C'(D(p) - S^j(p))][D'(p) - S^{j\prime}(p)] = 0.$$

In order for $\bar{p}$ to solve this equation, we must have

$$S^{j\prime}(\bar{p}) = \frac{\bar{q}_i}{\bar{p} - C'(\bar{q}_i)} + D'(\bar{p}),$$

where we have used the fact that $S^j(\bar{p}) = \bar{q}_j$, so $\bar{q}_i = D(\bar{p}) - S^j(\bar{p})$. The second derivative of i's profit with respect to p is

$$\begin{aligned}\pi^i_{pp}(p; S^j(\cdot)) = 2[&D'(p) - S^{j\prime}(p)] \\ &- C''(D(p) - S^j(p))[D'(p) - S^{j\prime}(p)]^2 \\ &+ [p - C'(D(p) - S^j(p))][D''(p) - S^{j\prime\prime}(p)].\end{aligned}$$

If $S^{j\prime}(\bar{p})$ satisfies the equation above and if $S^{j\prime\prime}(\bar{p}) \geq 0$, then the local second-order conditions are satisfied at $\bar{p}$. A similar argument shows that if

$$S^{i\prime}(\bar{p}) = \frac{\bar{q}_j}{\bar{p} - C'(\bar{q}_j)} + D'(\bar{p}) \qquad \text{and} \qquad S^{i\prime\prime}(\bar{p}) \geq 0,$$

then $\bar{p}$ is a local profit-maximum for j along j's residual demand. To complete our construction, it remains only to extend $S^i(\cdot)$ and $S^j(\cdot)$ over the whole domain of prices $[0, \hat{p})$ in such a way that (i) $\bar{p}$ is a *global* profit-maximum for each firm and (ii) $\bar{p}$ is the *only* market-clearing price.

Consider the case in which

$$\frac{\bar{q}_k}{\bar{p} - C'(\bar{q}_k)} + D'(\bar{p}) \geq 0, \qquad k = i, j,$$

so both supply functions are required to have nonnegative slopes at $\bar{p}$. Extending $S^i(\cdot)$ and $S^j(\cdot)$ linearly over $[0, \hat{p})$ makes $\bar{p}$ a global profit-maximum for each firm. Global profit maximization at $\bar{p}$ is also ensured by any increasing, twice

[9] In the absence of uncertainty, a firm can always achieve its profit-maximizing point along its residual demand curve by choosing a supply function that crosses the residual demand curve only at that point. See Section 3 for further discussion.

continuously differentiable supply functions that at $\bar{p}$ are tangent to the linear supply functions and that for all other prices specify outputs that are positive and larger than the linear ones. Furthermore, since when both firms use increasing supply functions, the market-clearing price is unique, condition (ii) is satisfied as well. Thus, the output pair $(\bar{q}_i, \bar{q}_j)$ can be supported by an infinite number of supply function pairs with each element of a pair a best response to the other.

This result can be generalized to other output pairs by modifying the way in which the supply functions that at $\bar{p}$ are locally best responses to each other are extended over the domain $[0, \hat{p})$. While we do not have a general algorithm for satisfying conditions (i) and (ii), we can satisfy them and support with families of supply functions any pair $(\bar{q}_i, \bar{q}_j)$ that is symmetric or not too asymmetric.[10,11] Furthermore, our construction generalizes straightforwardly to more than two firms or to differentiated products.

The multiplicity of equilibria in supply functions stems from the fact that the slope of $S^j(\cdot)$ through $(\bar{p}, \bar{q}_j)$ ensures that $(\bar{p}, \bar{q}_i)$ is the point along i's residual demand where i's marginal revenue equals its marginal cost. Since in the absence of exogenous uncertainty, i's residual demand is certain, then as long as the global second-order conditions are satisfied, any supply function for i that intersects its residual demand just once, at $(\bar{p}, \bar{q}_i)$, is an optimal response to $S^j(\cdot)$. Thus, i is willing to choose a supply function with a slope through $(\bar{p}, \bar{q}_i)$ that ensures that $(\bar{p}, \bar{q}_j)$ is profit-maximizing along j's residual demand. Given this, j is willing to choose $S^j(\cdot)$. In the next section we introduce exogenous uncertainty about demand, which makes firms' residual demands uncertain and so gives them strict preferences over the outputs specified by their supply functions for a range of prices; consequently, the set of outcomes supportable by supply function equilibria is dramatically reduced.

3. SUPPLY FUNCTION EQUILIBRIA UNDER UNCERTAINTY

In a world with exogenous uncertainty about, say, demand, a firm has a set of profit-maximizing points—one for each realization of the uncertainty—even when it knows its competitor's (pure strategy) equilibrium behavior. In this setting, a firm can generally achieve higher expected profits by committing to a supply function than by committing to a fixed price or a fixed quantity, because a

[10]An algorithm for symmetric pairs $(\bar{q}, \bar{q})$ is the following: Consider the linear supply function, S^l, through $(\bar{p}, \bar{q})$ for which $\bar{p}$ satisfies each firm's first-order condition. If $S^l(\cdot)$ intersects $\frac{1}{2}D(\cdot)$ at a price $\tilde{p} < \bar{p}$, choose $\varepsilon > 0$, let $\delta(\varepsilon) = \frac{1}{2}D(\tilde{p} + \varepsilon) - S^l(\tilde{p} + \varepsilon)$, and define $S^e(p) = \frac{1}{2}D(p) - \delta(\varepsilon)$ for $p \leqslant \tilde{p} + \varepsilon$, $S^e(p) = S^l(p)$ otherwise. To check that $S^e(\cdot)$ forms a symmetric equilibrium for sufficiently small ε, observe first that $\bar{p}$ is i's global profit-maximizing price if j chooses $S^l(\cdot)$; therefore, i prefers $(\bar{p}, \frac{1}{2}D(\bar{p}))$ to $(\tilde{p}, \frac{1}{2}D(\tilde{p}))$ and so also (using the concavity of $D(\cdot)$) to all points below $\tilde{p}$ on $\frac{1}{2}D(\cdot)$. Hence, $\bar{p}$ is i's global profit-maximizing price if j chooses $S^e(\cdot)$, and $\bar{p}$ is the unique market-clearing price if both firms choose $S^e(\cdot)$. If $S^l(\cdot)$ intersects $\frac{1}{2}D(\cdot)$ at a price $\tilde{\tilde{p}} > \bar{p}$, proceed analogously, bringing $S^e(\cdot)$ just inside $\frac{1}{2}D(\cdot)$ at $\tilde{\tilde{p}}$ and above. If $S^l(\cdot)$ intersects $\frac{1}{2}D(\cdot)$ only at $\bar{p}$, then $S^l(\cdot)$ is itself a symmetric supply function equilibrium supporting $(\bar{q}, \bar{q})$.

[11]Note, additionally, that *every* point $(\bar{q}_i, \bar{q}_j)$ at which both firms earn nonnegative profits can be supported as a supply function equilibrium outcome using *discontinuous* supply functions such as $S^k(D^{-1}(\bar{q}_i + \bar{q}_j)) = \bar{q}_k$, $S^k(p) = M$ for all $p \neq D^{-1}(\bar{q}_i + \bar{q}_j)$ for some $M > D(0)$, $k = i, j$.

supply function allows better adaptation to the uncertainty. Furthermore, by reducing the set of optimal supply functions for each firm to a single element, uncertainty can dramatically pare the set of supply function equilibria, even to a unique equilibrium.

We now define and characterize equilibria in supply functions in the presence of exogenous demand uncertainty for the symmetric undifferentiated products duopoly model of Section 2. Our analysis is generalized to the case of $n \geq 2$ firms in Propositions 8a and 8b. By restricting attention to symmetric firms, we can show that asymmetric equilibria do not exist and can characterize symmetric equilibria by solving a single differential equation; with asymmetric firms, characterization of equilibria would require solving a system of coupled differential equations. Differentiated products are discussed in Section 4.

Let industry demand be subject to an exogenous shock ε, where ε is a scalar random variable with strictly positive density everywhere on the support $[\underline{\varepsilon}, \bar{\varepsilon}]$: $Q = D(p, \varepsilon)$, where for all (p, ε), $-\infty < D_p < 0$, $D_{pp} \leq 0$, and $D_\varepsilon > 0$. (Later, we will specialize to the case $D_{p\varepsilon} = 0$ for all (p, ε).) Since $D_\varepsilon > 0$, we can invert the demand curve and write $e(Q, p)$ for the value of the shock ε for which industry demand is Q at price p, that is, $e(Q, p)$ satisfies $Q = D(p, e(Q, p))$. Since for $\varepsilon < e(0,0)$, there is no point on $D(p, \varepsilon)$ with $p \geq 0$ and $Q \geq 0$, we assume that the support of ε is a subset of $[e(0,0), \infty)$. The firms have identical cost functions $C(\cdot)$, with $C'(q) > 0$, $\forall q > 0$, and $0 < C''(q) < \infty$, $\forall q \geq 0$. Without loss of generality, let $C'(0) = 0$. If $C'(0) = \alpha > 0$, then the analysis below can be applied to solve for supply functions expressed in terms of $\tilde{p} = p - \alpha$.

A strategy for firm k is a function mapping price into a level of output for k: S^k: $[0, \infty) \to (-\infty, \infty)$.[12] Firms choose supply functions simultaneously, without knowledge of the realization of ε. After the realization of ε, supply functions are implemented by each firm producing at a point $(p^*(\varepsilon), S^k(p^*(\varepsilon)))$ such that $D(p^*(\varepsilon)) = S^i(p^*(\varepsilon)) + S^j(p^*(\varepsilon))$, that is, demand matches total supply, provided a unique such price $p^*(\varepsilon)$ exists. As in Section 2, we assume that if a market-clearing price does not exist, or is not unique, then firms earn zero profits. Again this assumption ensures that such an outcome will not arise in equilibrium, but the assumption is not an important constraint on firms' behavior. Our assumptions about demand and costs ensure that the set of ex post profit-maximizing points for each firm as its residual demand curve varies, when the other firm is using a potential equilibrium strategy, can be described by a supply function which intersects each residual demand curve once and only once. Hence, as long as an alternative assumption about the consequences of multiple intersections between supply functions and residual demand curves does not give firms higher profits than in the most profitable of the points of intersection, our equilibria remain equilibria. Furthermore, there are no supply functions tracing through ex post profit-maximizing points as ε varies that are equilibria under an

[12] Restricting firms to choosing supply functions specifying nonnegative quantities at all prices, but not insisting on differentiability at the price axis, would lead to the same results but would slightly complicate the analysis by permitting the possibility that residual demand curves are not differentiable everywhere.

alternative assumption but not under our assumption. Therefore, modifying our assumption about multiple intersections would not affect the set of equilibria we examine in our propositions.

We focus on pure strategy Nash equilibria in supply functions: such an equilibrium consists of a pair of functions $S^i(p)$ and $S^j(p)$ such that $S^k(\cdot)$ maximizes k's expected profits, where the expectation is taken with respect to the distribution of ε, given that m chooses $S^m(\cdot)$, $k, m = i, j$, $m \neq k$. We confine attention to twice continuously differentiable supply functions.

Firm i's residual demand at any price is the difference between industry demand and the quantity that j is willing to supply at that price. Thus if j is committed to the supply function $S^j(p)$, i's residual demand curve is $D(p, \varepsilon) - S^j(p)$. Since ε is a scalar, the set of profit-maximizing points along i's residual demand curves as ε varies is a one-dimensional curve in price-quantity space. If this curve can be described by a supply function $q_i = S^i(p)$ that intersects each realization of i's residual demand curve once and only once, then by committing to $S^i(\cdot)$, i can achieve ex post optimal adjustment to the shock. In this case, $S^i(\cdot)$ is clearly i's unique optimal supply function in response to $S^j(\cdot)$. We assume now that the set of ex post profit-maximizing points for i *can* be described by a supply function and show later that under our hypotheses there exist equilibria in which this is indeed the case.

Given this assumption, in solving for i's optimal supply function, we can replace the maximization of expected profits by the maximization with respect to p of profits for each realization of ε. Firm i solves

$$\max_p p\left[D(p, \varepsilon) - S^j(p)\right] - C\left(D(p, \varepsilon) - S^j(p)\right). \tag{1}$$

The first-order condition is

$$\begin{aligned} D(p, \varepsilon) - S^j(p) & \\ + \left[p - C'\left(D(p, \varepsilon) - S^j(p)\right)\right]&\left[D_p(p, \varepsilon) - S^{j\prime}(p)\right] = 0. \end{aligned} \tag{2}$$

If (1) is globally strictly concave in p (second-order conditions will be checked later), then (2) implicitly determines i's unique profit-maximizing price $p_i^0(\varepsilon)$ for each value of ε; the corresponding profit-maximizing quantity is $D(p_i^0(\varepsilon), \varepsilon) - S^j(p_i^0(\varepsilon)) \equiv q_i^0(\varepsilon)$. The functions $p_i^0(\varepsilon)$ and $q_i^0(\varepsilon)$ represent in parameterized form i's set of ex post optimal points as its residual demand curve shifts; if $p_i^0(\varepsilon)$ is invertible (invertibility will also be checked later), this locus can be written as a function from price to quantity: $q_i = S^i(p) \equiv q_i^0((p_i^0)^{-1}(p))$. Since $D_\varepsilon > 0$, no two realizations of i's residual demand curve can intersect; this condition, together with uniqueness of $p_i^0(\varepsilon)$ for each ε, implies that $S^i(p)$ intersects i's residual demand curve once and only once for each ε, at $p_i^0(\varepsilon)$. Hence $S^i(p)$ is i's optimal supply function in response to $S^j(p)$.

Let us rewrite (2) so that it implicitly defines the function $S^i(p)$. Replace $q_i^0(\varepsilon) \equiv D(p_i^0(\varepsilon), \varepsilon) - S^j(p_i^0(\varepsilon))$ by $S^i(p)$ and use $e(Q, p)$ as defined above to replace $D_p(p_i^0(\varepsilon), \varepsilon)$ by $D_p(p, e(S^i(p) + S^j(p), p))$, so (2) becomes

$$(3) \qquad S^i(p) + [p - C'(S^i(p))] \left[D_p\left(p, e\left(S^i(p) + S^j(p), p\right)\right) - S^{j\prime}(p)\right] = 0.$$

We begin by considering symmetric equilibria and will then show that no asymmetric equilibria exist. In a symmetric equilibrium $S^i(p) = S^j(p) \equiv S(p)$, so (3) becomes, after some rearrangement,

$$(4) \qquad S'(p) = \frac{S(p)}{p - C'(S(p))} + D_p(p, e(2S(p), p)) \equiv f(p, S(p)).$$

Henceforth, except where indicated otherwise, we assume that $D_{p\varepsilon} = 0$ for all (p, ε), i.e., demand is translated horizontally by the shock ε. Then if we write $D_p(p, e(2S(p), p))$ simply as $D_p(p)$, (4) becomes the differential equation

$$(5) \qquad S'(p) = \frac{S}{p - C'(S)} + D_p(p) \equiv f(p, S).$$

Given a function $S(\cdot)$ that intersects $\frac{1}{2}D(p, \varepsilon)$ at a unique point for each $\varepsilon \in [\underline{\varepsilon}, \bar{\varepsilon}]$, define $\underline{p}$ as the solution to $S(p) = \frac{1}{2}D(p, \underline{\varepsilon})$ and $\bar{p}$ as the solution to $S(p) = \frac{1}{2}D(p, \bar{\varepsilon})$. In any equilibrium involving symmetric, differentiable supply functions $S(\cdot)$ that trace through each firm's ex post profit-maximizing points, $S(\cdot)$ must satisfy (5) for all $p \in [\underline{p}, \bar{p}]$.[13] Conversely, if for all $\varepsilon \in [\underline{\varepsilon}, \bar{\varepsilon}]$, there is a unique price at which $2S(p) = D(p, \varepsilon)$ and if $S(\cdot)$ satisfies (5) for all $p \in [\underline{p}, \bar{p}]$ as well as appropriate second-order conditions, then $S(\cdot)$ is a supply function equilibrium (SFE).

The differential equation (5) is just the symmetric version of the pair of first-order conditions developed in Section 2 to construct a supply function equilibrium in the absence of uncertainty. The role of uncertainty is simply to necessitate that the first-order conditions hold at *every* price which for some realization of ε clears the market.

The nonautonomous first-order differential equation (5) can be rewritten as

$$(6) \qquad p'(S) = r(S, p) \equiv \frac{1}{f(p, S)}$$

or as the two-dimensional autonomous system

$$(7) \qquad S'(t) = S + D_p(p)(p - C'(S)),$$
$$p'(t) = p - C'(S).$$

While the derivation of (5) above assumed that firms use as strategies functions from price to quantity, the system of equations (7) would in fact emerge from a more general analysis which allowed firms to choose any one-dimensional manifold relating price to quantity. Each trajectory solving (5) (or (6) or (7)) is a locus

[13] For all solutions to (5) above the marginal cost curve, $\underline{p} < \bar{p}$. We are suppressing the dependence of $\underline{p}$ and $\bar{p}$ upon S for notational simplicity.

of points satisfying each firm's first-order condition for profit maximization as ε varies, when its rival is committed to producing somewhere along the same locus. Some of these trajectories can be expressed as functions $S(p)$, while others cannot. Any trajectory that solves the differential equation in the region corresponding to possible realizations of the demand curve, and that satisfies appropriate second-order conditions as well, is a Nash equilibrium in one-dimensional price-quantity manifolds.

Our description of our model as representing competition in supply functions, rather than competition in supply manifolds, is justified by the following result, demonstrated in Proposition 1 below. Any trajectory solving (5) that cannot be expressed as a function $S(p)$ will, for large enough uncertainty about demand, violate the second-order conditions. Therefore, for unbounded support of the uncertainty, not only are our equilibria in supply functions equilibria in supply manifolds, but they are the only equilibria in supply manifolds that trace through each firm's ex post optimal points.

We now characterize the solutions to (5) through a series of claims, whose proofs are in the Appendix.

CLAIM 1: The locus of points satisfying $f(p, S) = 0$ is a continuous, differentiable function, $S = S^0(p)$, satisfying

(i) $$S^0(0) = 0,$$

(ii) $$S^0(p) < (C')^{-1}(p), \quad \forall p > 0,$$

(iii) $$S^{0\prime}(p) > 0, \quad \forall p \geqslant 0,$$

(iv) $$S^{0\prime}(0) < \frac{1}{C''(0)}.$$

CLAIM 2: The locus of points satisfying $f(p, S) = \infty$ is a continuous, differentiable function, $S = S^\infty(p) \equiv (C')^{-1}(p)$. Hence $S^\infty(0) = 0$ and $0 < S^{\infty\prime}(p) < \infty$ for all $p \geqslant 0$.

CLAIM 3: For all points (p, S) between the $f = 0$ and the $f = \infty$ loci, $0 < f(p, S) < \infty$. For all points in the first quadrant above the $f = 0$ locus or below the $f = \infty$ locus, $0 > f(p, S) > -\infty$.

The results of Claims 1, 2, and 3 are illustrated in Figure 1. Trajectories solving (5) cross the $f = 0$ locus vertically and the $f = \infty$ locus horizontally.

CLAIM 4: For any $(p_0, S_0) \neq (0,0)$, there exists a unique solution to (5) passing through (p_0, S_0), and this solution is a continuous function in an open neighborhood of (p_0, S_0).

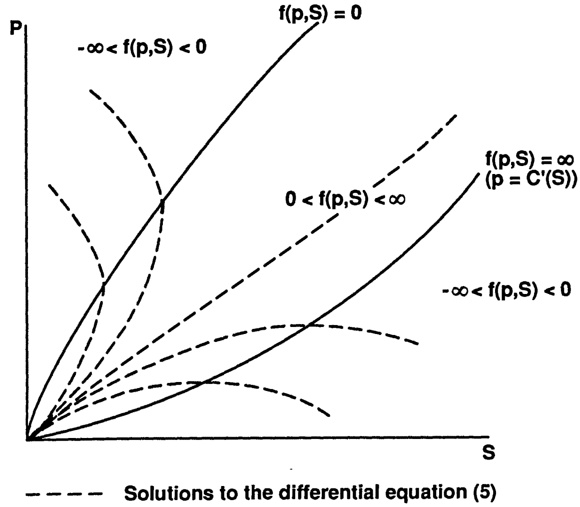

FIGURE 1.—Trajectories satisfying differential equation.

CLAIM 5: For any $(p_0, S_0) \neq (0,0)$ in the positive quadrant, the (unique) solution to (5) passing through (p_0, S_0) also passes through $(0,0)$.

CLAIM 6: For any $(p_0, S_0) \neq (0,0)$ in the positive quadrant, the (unique) solution to (5) passing through (p_0, S_0) has slope at the origin

$$\frac{1}{2}\left[D_p(0) + \sqrt{\left(D_p(0)\right)^2 - \frac{4D_p(0)}{C''(0)}}\right] > 0.$$

We now examine the second-order conditions for profit maximization.

CLAIM 7: If $S^j(p)$ solves (5), the second derivative of i's profit with respect to p for a given ε is

$$\begin{aligned}(8)\qquad \pi^i_{pp}(p, \varepsilon; S^j(\cdot)) = {} & \left[D_p(p) - S^{j\prime}(p)\right] \\ & \times\left[1 + C''\left(D(p) + \varepsilon - S^j(p)\right)S^{j\prime}(p)\right] \\ & - C''\left(D(p) + \varepsilon - S^j(p)\right) \\ & \times\left[D_p(p) - S^{j\prime}(p)\right]^2 - S^{j\prime}(p).\end{aligned}$$

PROOF: See Appendix.

For bounded support of the uncertainty $[\underline{\varepsilon}, \bar{\varepsilon}]$, any solution $S(\cdot)$ to (5) that is upward-sloping everywhere in the region of possible realizations of the (residual) demand curve is part of a SFE. To confirm this, observe that if $S'(p) > 0$ on $[\underline{p}, \bar{p}]$ (where $[\underline{p}, \bar{p}]$ is determined from $[\underline{\varepsilon}, \bar{\varepsilon}]$ and $S(\cdot)$ as described above), then (8) implies $\pi^i_{pp}(p, \varepsilon; S(\cdot)) < 0$ for all $p \in [\underline{p}, \bar{p}]$ and for all $\varepsilon \in [\underline{\varepsilon}, \bar{\varepsilon}]$. Thus $S(\cdot)$ satisfies the first- and the local second-order conditions for profit maximization for $p \in [\underline{p}, \bar{p}]$. The global second-order conditions, as well as the requirement that the market-clearing price be unique for each ε, can be satisfied by extending $S(\cdot)$ according to (5) for $p < \bar{p}$ and by extending $S(\cdot)$ linearly with slope $S'(\bar{p})$ for $p > \bar{p}$ (see (8) and (A1) in the Appendix).

With bounded support, for any $\hat{\varepsilon} \in (\underline{\varepsilon}, \bar{\varepsilon})$, then, there is a connected set of points along $Q = D(p, \hat{\varepsilon})$ that are market outcomes in a SFE. However, in contrast to the case with no uncertainty examined in Section 2, here the supply function $S(\cdot)$ supporting a point $(\hat{p}, D(\hat{p}, \hat{\varepsilon}))$, say, is unique over an interval of prices containing $\hat{p}$—the need for firms to adapt to realizations of ε that are smaller or larger than $\hat{\varepsilon}$ determines the slope of the equilibrium supply function everywhere on $[\underline{p}, \bar{p}]$ and not just at $\hat{p}$.

As $\bar{\varepsilon}$ is increased, more and more of the solutions $S(\cdot)$ to (5) cross either the $f = 0$ or the $f = \infty$ locus, and hence become negatively sloped, in the region of possible realizations of the (residual) demand curve, i.e. at a point (p_0, S_0) such that $e(2S_0, p_0) \in [\underline{\varepsilon}, \bar{\varepsilon}]$. We now show that when ε has unbounded support, the only solutions to (5) that satisfy the second-order conditions, and hence the only symmetric SFE's, are those that have strictly positive slope everywhere.

PROPOSITION 1 (Characterization): *If ε has full support ($\underline{\varepsilon} = e(0,0)$, $\bar{\varepsilon} = \infty$), $S(\cdot)$ is a symmetric SFE tracing through ex post optimal points if and only if for all $p \geqslant 0$, $S(\cdot)$ satisfies* (5) *and* $0 < S'(p) < \infty$.

PROOF: *Sufficiency*: Since $0 < S'(p) < \infty$ for all $p \geqslant 0$, total supply intersects total demand at a unique point for each ε. Since $S(\cdot)$ satisfies (5) for all $p \geqslant 0$, the first-order condition for ex post profit maximization is satisfied everywhere along $S(\cdot)$ when the other firm commits to the same supply function. The conditions on $S(\cdot)$ and $S'(\cdot)$ together imply, using (10), that $\pi^i_{pp}(p, \varepsilon; S(\cdot)) < 0$ for all $p \geqslant 0$ and for all $\varepsilon \geqslant e(0,0)$, so the global second-order conditions for ex post profit-maximization are satisfied everywhere along $S(\cdot)$. Therefore, $S(\cdot)$ is a symmetric SFE tracing through ex post optimal points.

Necessity: Satisfaction of (5) for all $p \geqslant 0$ is a necessary condition for a supply function defined for all $p \geqslant 0$ to trace through ex post optimal points when the other firm commits to the same supply function. To show that $0 < S'(p) < \infty$ is also a necessary condition, we show that if, for some p, $S(\cdot)$ ever crosses either $f = 0$ from below or $f = \infty$ from the left, $S(\cdot)$ must eventually violate the second-order conditions.

Once a trajectory $S(\cdot)$ crosses $f = 0$ from below, S' becomes and stays negative (by Claims 1 and 3) and, from (A2) in the Appendix, S'' also becomes and stays negative (for $S \geqslant 0$). Therefore the trajectory will eventually intersect the $S = 0$ axis at a point $(p_0, 0)$ with $p_0 > C'(0)$, where by (5), $S'(p_0) = f(p_0, 0) = D_p(p_0)$. Substitution into (10) shows that $\pi^i_{pp}(p_0, \varepsilon; S(\cdot)) = -S'(p_0) > 0$. Thus, for $\varepsilon = e(0, p_0)$ (the value of ε for which $(p_0, 0)$ satisfies i's first-order condition), $(p_0, 0)$ is a local minimum-profit point for i. Therefore, $S(\cdot)$ cannot be a SFE.

Once a trajectory $S(\cdot)$ crosses $f = \infty$ from the left, S' becomes and stays negative, so the trajectory must eventually go below firm i's (upward-sloping) average cost curve and so include points at which i's profits are negative. Such points cannot be optimal for any values of ε, so the trajectory cannot represent an equilibrium. (Note that such a trajectory cannot in any case be represented as a supply function, but this argument establishes that it could not even be part of an equilibrium in supply manifolds.) *Q.E.D.*

With this characterization of a symmetric SFE, we can now prove an existence result.

PROPOSITION 2 (Existence): *Assume ε has full support. There exists a SFE tracing through ex post optimal points. The set of symmetric equilibria consists either of a single trajectory or of a connected set of trajectories, i.e. a set such that, for any two trajectories in the set, all trajectories (solving (5)) that lie everywhere between these trajectories are also in the set.*

PROOF: Given any $\underline{S} > 0$, consider a vertical line segment at $\underline{S}$ connecting the $f = 0$ and the $f = \infty$ loci. Suppose, for contradiction, that there exists a lowest point, X, on this line segment that is on a trajectory solving (5) that crosses $f = 0$, and let the corresponding trajectory cross $f = 0$ at the point $Y = (\bar{p}, \bar{S})$. From Claims 4 and 5, any point $Z = (p, S)$ on $f = 0$ for which $S > \bar{S}$ must lie on a unique trajectory solving (5) that emerges from $(0, 0)$, never touches the trajectory through Y except at $(0, 0)$, and remains between $f = 0$ and $f = \infty$ until passing through Z. Therefore the trajectory through Z must cross the vertical line segment at a point below X, contradicting the supposition. A similar argument establishes that there does not exist a highest point on the line segment that is on a trajectory solving (5) that crosses $f = \infty$.

Since by Claims 4 and 5 trajectories solving (5) do not cross except at $(0, 0)$, if a point on the line segment is on a trajectory crossing $f = 0$ ($f = \infty$), then all points of the segment above (below) it are on trajectories crossing $f = 0$ ($f = \infty$). These results imply the existence on the line segment of a set of points, which must be either a single point or a closed interval, such that the trajectories through these points cross neither $f = 0$ nor $f = \infty$. By Proposition 1, the trajectories through these points constitute (the only) symmetric supply function equilibria, and since trajectories never cross except at $(0, 0)$, the equilibrium set has the form claimed. *Q.E.D.*

Our focus on *symmetric* supply function equilibria is justified by the following result:

PROPOSITION 3 (Symmetry of Equilibria): *If $\underline{\varepsilon} = e(0, 0)$, no asymmetric supply function equilibrium exists involving supply functions that trace through ex post optimal points.*

PROOF: The first-order conditions for the optimal supply functions in an asymmetric equilibrium are the system of differential equations:

$$S^{i\prime}(p) = \frac{S^j(p)}{p - C'(S^j(p))} + D_p(p), \tag{9a}$$

$$S^{j\prime}(p) = \frac{S^i(p)}{p - C'(S^i(p))} + D_p(p). \tag{9b}$$

If in a solution to this system, $S^i(0) = 0$ and $S^j(0) = 0$, then $S^{i\prime}(0)$ and $S^{j\prime}(0)$ solve the following pair of equations, derived from (9a) and (9b) using L'Hôpital's Rule, as in the proof of Claim 6:

$$S^{i\prime}(0) = \frac{S^{j\prime}(0)}{1 - C''(0)S^{j\prime}(0)} + D_p(0),$$

$$S^{j\prime}(0) = \frac{S^{i\prime}(0)}{1 - C''(0)S^{i\prime}(0)} + D_p(0).$$

These equations yield $S^{i\prime}(0) = S^{j\prime}(0) = \bar{S}^+$ and $S^{i\prime}(0) = S^{j\prime}(0) = \bar{S}^-$, where $\bar{S}^+$ and $\bar{S}^-$ (defined in the proof of Claim 6) are the positive and negative slopes at the origin of solutions to the differential equation (5). As before, no equilibrium supply function can have the negative slope $\bar{S}^-$ at the origin. Now suppose that there is an asymmetric equilibrium in which both $S^i(p)$ and $S^j(p)$ pass through the origin with positive slope $\bar{S}^+$. For any $p > 0$ such that $0 < S^i(p) < S^j(p) < (C')^{-1}(p)$, (9a) and (9b) imply that $S^{j\prime}(p) < S^{i\prime}(p)$. But this is inconsistent with the supposition. The argument is identical when the roles of i and j are reversed.

Thus if there is an asymmetric SFE tracing through ex post optimal points, then either $S^i(0) \neq 0$ or $S^j(0) \neq 0$. Now consider the unique solution to (9a) and (9b) passing through any (p_0, S_0^i, S_0^j), where $p_0 > 0$, $0 \leqslant S_0^i \leqslant (C')^{-1}(p_0)$, $0 \leqslant S_0^j \leqslant (C')^{-1}(p_0)$, and $S_0^i \neq S_0^j$. As p decreases, at some $\check{p} \in (0, p_0]$, some firm's supply function must either cross below the average cost curve or cross into the region where $S < 0$. When this happens, profits will become negative for that firm (and prices below and sufficiently close to $\check{p}$ will clear the market for some realizations of ε, since $\underline{\varepsilon} = e(0,0)$). *Q.E.D.*

Section 2 showed that a range of asymmetric supply function equilibria, supporting asymmetric outputs, exists in the absence of uncertainty. Without uncertainty, an outcome $(\bar{p}, \bar{q}_i, \bar{q}_j)$ is supported by supply functions with slopes at $\bar{p}$, $S^{i\prime}(\bar{p})$ and $S^{j\prime}(\bar{p})$, determined by evaluating the right-hand sides of (9a) and (9b) at $\bar{p}$ (noting that $S^i(\bar{p}) = \bar{q}_i$ and $S^j(\bar{p}) = \bar{q}_j$); at all other prices, the supply functions need only ensure that $\bar{p}$ is profit-maximizing for both firms and is the unique market-clearing price. In contrast, with demand uncertainty whose support has lower bound $e(0,0)$, equilibrium supply functions must satisfy (9a) and (9b) over a range of prices extending down to zero. Since no asymmetric solutions to (9a) and (9b) are profit-maximizing over this whole range of prices, no asymmetric SFE exists.

Contrast with Cournot and Bertrand Equilibria

The SFE's for unbounded support of the uncertainty have finite positive slope and so are distinct from the vertical supply functions that are exogenously imposed on firms in the Cournot model and from the horizontal supply relations exogenously imposed in the Bertrand model. A fixed price or a fixed quantity can never be an equilibrium strategy when marginal costs are upward sloping. To understand this, observe that if firm j were to commit to a fixed quantity, i's residual demand for any ε would simply be the industry demand curve translated horizontally. As the additive shock varies, the set of i's residual demands coincides with the set of market demands for a monopolist. For a monopolist, the locus of profit-maximizing points as ε varies is precisely the $f = 0$ locus (as can be checked from the monopolist's first-order condition); by Claim 1 this locus has positive slope. Similarly, if firm j were to commit to a fixed price $\bar{p}_j > 0$, the set of residual demand curves for i would all be flat at $\bar{p}_j$, zero above $\bar{p}_j$, and identical to the set of industry demand curves below $\bar{p}_j$. Firm i's best response supply function would then be identical to that of a monopolist for low realizations of demand, and so have finite positive slope for $p < \bar{p}_j$. (If $\bar{p}_j = 0$, firm i cannot earn positive profits, so given that $C''(0) > 0$, i prefers to commit to a quantity of zero.) We know from Proposition 3 that no asymmetric equilibria exist.

In the limit as the marginal cost curve becomes flat, Bertrand behavior becomes an equilibrium: if firm j commits to a fixed price equal to the constant marginal cost, firm i cannot earn positive profits so can do no better than to adopt the same strategy. In the limit as the marginal cost curve becomes steeper and approaches the vertical axis, supply function equilibrium behavior approaches a fixed quantity of zero. Under the assumptions of our model, which rule out these extremes, neither a fixed price nor a fixed quantity can ever represent an equilibrium adaptation to uncertainty.

In any SFE for unbounded support of ε, the outcome corresponding to any given value of ε is intermediate between the outcome that would arise if firms learned ε and then competed in the Cournot fashion and the outcome that would arise if they observed ε and then competed using Bertrand strategies. This claim is demonstrated using Figure 2, which plots the demand curve for a given realization of ε, $\hat{\varepsilon}$. The point C, at the intersection of the $f = 0$ locus with $\frac{1}{2}D(p, \hat{\varepsilon})$, represents the price and quantity for each firm in the unique Cournot equilibrium of the game when firms know that $\varepsilon = \hat{\varepsilon}$. This follows since C is the profit-maximizing point for each firm when $\varepsilon = \hat{\varepsilon}$, given that the other firm uses as its supply function the solution to (5), $S^C(p)$, which is vertical through C and therefore locally identical to a fixed quantity of q^C. On the other hand, the point B, at the intersection of the $f = \infty$ locus with $\frac{1}{2}D(p, \hat{\varepsilon})$, is the Bertrand equilib-

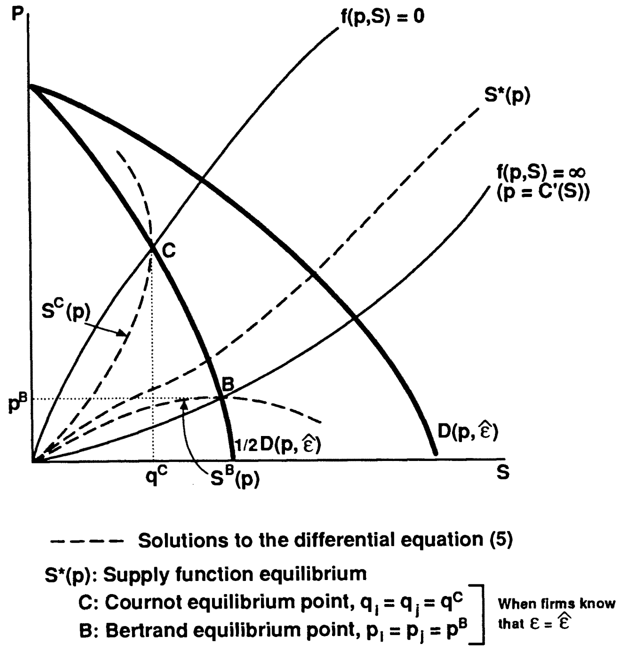

FIGURE 2.—Comparison of supply function equilibrium with Cournot and Bertrand equilibria.

rium of the game when it is known that $\varepsilon = \hat{\varepsilon}$.[14] Each firm's profits are maximized at B when $\varepsilon = \hat{\varepsilon}$, if the other firm's strategy is the solution to (5), $S^B(p)$, which is horizontal through B and therefore locally identical to a fixed price of p^B.[15]

Since any SFE for unbounded support of ε intersects $\frac{1}{2}D(p, \hat{\varepsilon})$ between points C and B (by Propositions 1 and 3), price and quantity in any SFE are intermediate between the Cournot and Bertrand equilibrium levels, for any realized value of ε. Since industry profits along $\frac{1}{2}D(p, \hat{\varepsilon})$ are strictly concave and are maximized at a price above the Cournot price, profits in a SFE are also intermediate between Cournot and Bertrand profits.

[14] Strictly, an interval of outcomes are Bertrand equilibria with perfect substitutes, because each firm's residual demand is discontinuous against a fixed price. (There is a range of prices p at which each firm prefers to serve half the market demand, rather than no demand at $p + \varepsilon$ or the whole demand at $p - \varepsilon$, for any $\varepsilon > 0$.) However the point B is singled out by two different limiting arguments. First, it is the limit of competition when firms are restricted to choosing linear supply functions of slope $S'(p) = k$ as $k \to \infty$. Second, in the differentiated products model of Section 4, B is the limit of the Bertrand equilibria (when firms know that $\varepsilon = \hat{\varepsilon}$) as the degree of product differentiation goes to zero. In addition, B is the unique Bertrand equilibrium at which price equals marginal cost.

[15] More generally, the unique Nash equilibrium in linear supply functions of exogenously determined slope $S'(p) = k$ is the point along $\frac{1}{2}D(p, \hat{\varepsilon})$ where $f = k$. The quantity-setting equilibrium corresponds to $k = 0$, while the price-setting equilibrium corresponds to $k = \infty$. Vives (1986) constructs a model that is formally very similar to a model of competition in supply functions of exogenously fixed slope and shows that Nash equilibrium outputs are relatively closer to Cournot (Bertrand) outputs the steeper (flatter) are firms' supply functions.

Whether or not the SFE is unique depends on the behavior of the marginal cost and demand curves for *large* S and p. Changing $C'(S)$ over a bounded domain $[\underline{S}, \bar{S}]$ or changing $D_p(p)$ over a bounded domain $[\underline{p}, \bar{p}]$ does not change the evolution of the solutions to (5) for $S > \bar{S}$ or $p > \bar{p}$, respectively. Therefore, neither of these changes affects the existence of a unique trajectory that never crosses either $f = 0$ or $f = \infty$.

Linear Example

For a market in which the demand and marginal cost curves are both globally linear, we now explicitly compute the solutions to (5) and show the existence of a unique SFE when ε has full support.

Industry demand is $D(p, \varepsilon) = -mp + \varepsilon$, where $m > 0$ and ε has (full) support $[0, \infty)$. Firms have identical quadratic cost curves $C(q) = cq^2/2$, where $c > 0$. The differential equation (5) becomes

$$S'(p) = \frac{S}{p - cS} - m, \tag{10}$$

which in parametric form is

$$\begin{pmatrix} \dfrac{dS}{dt} \\ \dfrac{dp}{dt} \end{pmatrix} = M\begin{pmatrix} S \\ p \end{pmatrix}, \quad \text{where} \quad M = \begin{pmatrix} 1 + mc & -m \\ -c & 1 \end{pmatrix}.$$

Let λ_i designate the eigenvalues and $\binom{v_i}{w_i}$ the corresponding eigenvectors of M $(i = 1, 2)$. Direct computation yields

$$\lambda_i = \frac{(2 + mc) \pm \sqrt{m^2c^2 + 4mc}}{2} \quad \text{and}$$

$$\frac{v_i}{w_i} = \frac{1 - \lambda_i}{c} = \frac{-m \mp \sqrt{m^2 + \dfrac{4m}{c}}}{2}.$$

Since the eigenvalues are real and unequal, the solution to the differential equation is

$$\begin{pmatrix} S \\ p \end{pmatrix} = A_1 e^{\lambda_1 t}\begin{pmatrix} v_1 \\ w_1 \end{pmatrix} + A_2 e^{\lambda_2 t}\begin{pmatrix} v_2 \\ w_2 \end{pmatrix},$$

where A_1, A_2 are arbitrary constants. Letting λ_1 denote the larger eigenvalue, we have $\lambda_1 > 1$, $0 < \lambda_2 < 1$ and $(v_1/w_1) < 0$, $(v_2/w_2) > 0$. As $t \to -\infty$, $S \to 0$ and $p \to 0$ for all A_1, A_2, so all solutions to (10) pass through the origin. Moreover as $t \to -\infty$, as long as $A_2 \neq 0$, $(S/p) \to (v_2/w_2) > 0$, so all solutions but one have a common positive slope at the origin (as we knew from Claim 6). As $t \to \infty$, if $A_1 \neq 0$, $(S/p) \to (v_1/w_1) < 0$, so all trajectories with $A_1 \neq 0$ eventually leave the region between the $f=0$ and the $f=\infty$ loci. The single solution with $A_1=0$ is the unique linear solution to (10) with positive slope at the origin:

$$S(p)=\left(\frac{v_2}{w_2}\right)p=\frac{1}{2}\left(-m+\sqrt{m^2+\frac{4m}{c}}\right)p. \tag{11}$$

Since $S'(p)>0$ for all p, this is the unique SFE.

The slope of this supply function and (for every ε) the market prices and outputs are intermediate between the Cournot and Bertrand cases (in which firms choose quantities and prices, respectively, after seeing the realization of the uncertainty).[16] It follows that firms' profits are also intermediate between these cases. However, it is not hard to check that firms' expected profits may be higher in the SFE than in either the stochastic Cournot or the stochastic Bertrand case (in which firms choose quantities and prices before seeing ε), because only in the SFE do firms adjust optimally to the uncertainty given their opponent's behavior.

Turnbull (1983), generalizing a result of Robson (1981), has shown for the market of this example that the function $S(p)=(v_2/w_2)p$ is the unique SFE when firms are restricted to choosing *linear* supply functions; with the restriction to linear supply functions, the uniqueness result is independent of the support of ε (as long as it is nondegenerate). Our analysis above shows that for bounded support of ε, there exists a continuum of nonlinear SFE's, but as the upper endpoint of the support increases, eventually all nonlinear solutions to (10) cease to be equilibria. For unbounded support, there exists a unique SFE, and it is linear.

While assuming global linearity of the demand and marginal cost curves allowed us to show uniqueness of the SFE by computing all of the solutions to (5), uniqueness can in fact be proved under the weaker assumption that demand and marginal cost are linear for p and S sufficiently large.

PROPOSITION 4 (Uniqueness): *Let ε have full support $[e(0,0),\infty)$. Assume that there exists S_1 such that for $S\geqslant S_1$, $C'(S)=a+bS$, with $b>0$, and that there exists p_1 such that for $p\geqslant p_1$ and for all ε, $D(p,\varepsilon)=-\mu p+\varepsilon$, with $\mu>0$. Then there is a unique SFE tracing through ex post optimal points.*

PROOF: See Appendix.

We note that given the nature of the uncertainty and its support, the SFE are independent of the distribution of the uncertainty, remaining unchanged even as the distribution becomes more and more sharply peaked at a specific point and approaches the no-uncertainty case. When a SFE under a natural kind of uncertainty is unique, therefore, it may be a natural candidate for the correct SFE in the limiting case of no uncertainty.

[16] The Cournot outcome has each firm selling $\hat{\varepsilon}/(3+mc)$ and the Bertrand outcome has each firm charging a price $\hat{\varepsilon}c/(2+mc)$, where $\hat{\varepsilon}$ is the known value of ε. In fact, there is a range of Nash equilibria in prices, but $\hat{\varepsilon}c/(2+mc)$ is the Bertrand price selected by the arguments of footnote 14.

Comparative Statics

We present a series of propositions detailing the comparative statics of the SFE and use the linear model to illustrate the results.

For Propositions 5 through 8b, assume that ε has full support and that the SFE is unique. With quantity (S) on the horizontal axis and price (p) on the vertical axis, we will say that a function is *steeper* at S if $|p'(S)|$ is larger or, equivalently, $|S'(p)|$ is *smaller*.

PROPOSITION 5: *Suppose that for $S \in (S_0, S_1)$, where $S_1 < \infty$, the marginal cost curve $C'(S)$ is shifted upwards (downwards), so that $C''(S)$ remains strictly positive for all S, and suppose that $C'(S)$ remains unchanged everywhere else. Then the SFE remains unique. If $S_0 \neq 0$, it emerges from the origin with unchanged slope, is steeper (flatter) than and lies strictly above (below) the original SFE for $S \in (0, S_0]$, lies strictly above (below) the original SFE for $S \in (S_0, S_1)$, and coincides with the original SFE for $S \geqslant S_1$. If $S_0 = 0$ and $C''(0)$ is unchanged (increased, decreased), then the slope of the SFE at the origin, $S'(0)$, is unchanged (decreased, increased), and the results for $S > S_0$ are as above. (See Figure 3a.)*

PROOF: Since S_1 is finite, for $S \geqslant S_1$ the evolution of the trajectories solving (5) is unaffected by the change in marginal costs below S_1, so there is a unique trajectory which for $S \geqslant S_1$ does not cross either the $f = 0$ or the $f = \infty$ locus: this trajectory coincides with the original SFE for $S \geqslant S_1$. Since the change in $C'(S)$ leaves the $f = 0$ and $f = \infty$ loci upward-sloping, this trajectory emerges from the origin and is *everywhere* between these two loci (by Claims 1, 2, and 3). Therefore, this trajectory is the unique SFE in the perturbed model (by Proposition 1).

Suppose that $C'(S)$ is increased on (S_0, S_1). For $S \in (S_0, S_1)$, the new SFE must lie everywhere strictly above the original SFE: if at some $\hat{S} \in (S_0, S_1)$ it were on or below the original equilibrium trajectory, then, since $C'(\hat{S})$ is now higher, we know from (5) that the new trajectory would be flatter at $\hat{S}$ than the old one. It follows from (5) that the new trajectory would be strictly below the old one at S_1, and since the original SFE is unique, the new trajectory would eventually cross the $f = \infty$ locus, thus invalidating it as an equilibrium.

Suppose $S_0 \neq 0$. For $S \leqslant S_0$, the trajectories solving (5) are unaffected by the change in $C'(S)$ above S_0. Since the new equilibrium lies strictly above the old one for $S \in (S_0, S_1)$, it must also be strictly higher at S_0, and therefore for $S \in (0, S_0]$, the new equilibrium trajectory must be higher and steeper (smaller S') than the original. Since $S_0 > 0$, $C''(0)$ is unchanged, so the slope of the SFE at the origin is unchanged (by Claim 6).

The arguments for a downwards shift in the marginal cost curve are parallel.

If $S_0 = 0$, then by Claim 6 the slope of the SFE at the origin, $S'(0)$, is unchanged, decreased, or increased according as $C''(0)$ is unchanged, increased, or decreased. *Q.E.D.*

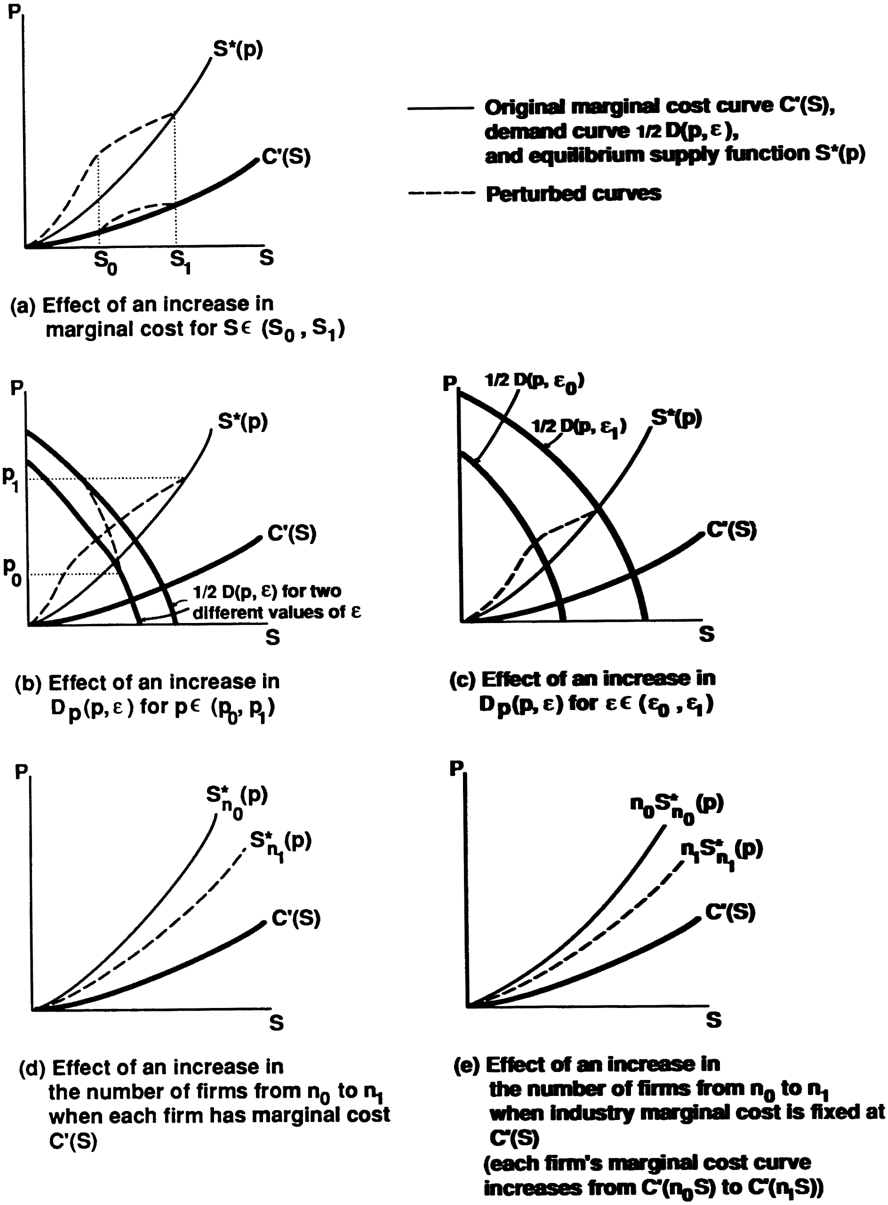

FIGURE 3.—Comparative statics.

PROPOSITION 6: *Suppose that for $p \in (p_0, p_1)$, where $p_0 \geqslant 0$ and $p_1 < \infty$, $D_p(p)$ is increased (decreased), so that $D_{pp}(p)$ remains nonpositive for all p, and suppose that $D_p(p)$ remains unchanged everywhere else. Then the SFE remains unique. It emerges from the origin with unchanged slope, is steeper (flatter) than and lies strictly above (below) the original SFE for $p \in (0, p_0]$, lies strictly above (below) the original SFE for $p \in (p_0, p_1)$, and coincides with the original SFE for $p \geqslant p_1$. (See Figure 3b.)*

PROOF: Similar to proof of Proposition 5—see Appendix.

For Proposition 7, we relax the assumption that $D_{p\varepsilon} \equiv 0$ and consider what happens to the SFE when, for an interval of ε values, the effect of the shock is to rotate the demand curve as well as to translate it horizontally. For this more general environment, the fundamental differential equation is now (4).

PROPOSITION 7: *Suppose that for $\varepsilon \in (\varepsilon_0, \varepsilon_1)$, where $\varepsilon_0 > e(0,0)$ and $\varepsilon_1 < \infty$, $D_p(p, \varepsilon)$ is increased (decreased) for all p, so that $D_{pp}(p, \varepsilon)$ remains nonpositive and $D_\varepsilon(p, \varepsilon)$ strictly positive for all (p, ε), and suppose that $D_p(p, \varepsilon)$ remains unchanged and independent of ε for $\varepsilon \notin (\varepsilon_0, \varepsilon_1)$. Then if for $\varepsilon \in (\varepsilon_0, \varepsilon_1)$ and for all p, $|D_{p\varepsilon}|/D_\varepsilon$ is sufficiently small, there remains a unique SFE. It emerges from the origin with unchanged slope, lies strictly above (below) the original SFE at points (S, p) such that $e(2S, p) \in (e(0,0), \varepsilon_1)$, and coincides with the original SFE at points (S, p) such that $e(2S, p) \geqslant \varepsilon_1$. (See Figure 3c.)*

PROOF: Similar to proof of Proposition 5—see Appendix.

We have stated Propositions 5, 6, and 7 for the case where the changes occur over a bounded domain (S_1, p_1, and ε_1 are finite). In this case, the new SFE is unique and coincides with the original SFE beyond a certain point. If S_1, p_1, or ε_1 is infinite, then the SFE in the perturbed model may not be unique; however, except at the origin, all of the equilibria lie everywhere strictly above the original SFE, for an increase in $C'(S)$ or $D_p(p, \varepsilon)$, or everywhere strictly below, for a decrease in $C'(S)$ or $D_p(p, \varepsilon)$.

PROPOSITION 8a: *Suppose that the number, n, of symmetric firms, each with marginal cost curve $C'(S)$, increases. A SFE, which must be symmetric across firms and have strictly positive slope $S_n'(p)$ for all $p \geqslant 0$, continues to exist. All SFE's have a common slope at the origin, $S_n'(0)$, which is increasing in n and tends to the slope of the marginal cost curve there, $1/C''(0)$, as $n \to \infty$. Except at the origin, each firm's supply function $S_n(p)$ in any new equilibrium lies strictly below the original SFE. Industry supply at any price therefore also increases with n, so for any $\varepsilon > e(0,0)$, the market-clearing price falls and industry output increases. (See Figure 3d.)*

PROOF: Similar to proof of Proposition 5—see Appendix.

The results of this proposition apply to the change from monopoly to duopoly as well as to changes to larger numbers of firms: a monopolist's optimal supply function can be shown to coincide with the $f=0$ locus for the duopoly model, so the comparison follows from Claim 1 and Proposition 1.

Adding firms with the same marginal cost curve lowers industry marginal cost (the marginal cost of the industry producing one extra unit) for every level of industry output, and so biases our results towards larger industry outputs and lower prices as the number of firms increases. An alternative approach gives each firm marginal cost curve $C_n'(S) = C'(nS)$ when the industry contains n firms, so that the industry marginal cost curve is independent of n (it is $C'(S) = C_n'(S/n)$).

PROPOSITION 8b: *Suppose that the number, n, of symmetric firms, each with marginal cost curve $C_n'(S) = C'(nS)$, increases. A SFE, which must be symmetric across firms and have strictly positive slope $\tilde{S}_n'(p)$ for all $p \geqslant 0$, continues to exist. All SFE's have a common slope at the origin, $\tilde{S}_n'(0)$, with $n\tilde{S}_n'(0)$ increasing in n and tending to the slope of the industry marginal cost curve at the origin, $1/C''(0)$, as $n \to \infty$. Except at the origin, the industry supply curve $n\tilde{S}_n(p)$ in any new equilibrium lies strictly below the industry supply curve in the original SFE, so for any $\varepsilon > e(0,0)$, the market-clearing price falls and industry output increases.* (*See Figure* 3*e*.)

PROOF: Similar to proof of Proposition 5—see Appendix.

Like Proposition 8a, Proposition 8b applies to the change from monopoly to duopoly as well as to changes to larger numbers of firms.

Under the assumptions of Proposition 8b, entry into the industry increases industry supply, though not necessarily each firm's output, at any given price. Industry profits decrease monotonically and social welfare (the sum of profits and consumer surplus) increases monotonically as n increases, and price falls. When, however, as in Proposition 8a, increasing n reduces industry marginal costs, industry profits need not monotonically decrease in n, although social welfare must increase in n. In this case, increasing n increases *each firm's* supply at each price.

For the linear model analyzed above

$$S'(p) = \frac{-m + \dfrac{n-2}{c} + \sqrt{\left(-m + \dfrac{n-2}{c}\right)^2 + \dfrac{4m(n-1)}{c}}}{2(n-1)}.$$

(See (A5) in the Appendix for the derivation of this slope.) Thus (as shown in Propositions 5, 6, and 8a), increasing c, decreasing m, or decreasing n (keeping the slope of each firm's marginal cost curve constant at c) makes the (linear) equilibrium supply functions steeper. As $n \to \infty$ or $c \to 0$, moreover, the equilibrium supply functions converge to the marginal cost curve. To interpret these results, note that a monopolist's supply function in this market is

$$S = \left(\frac{m}{1+cm}\right)p.$$

This function is steeper, the steeper is marginal cost (higher c) and the steeper is demand (lower m). Now suppose that either c increases or m decreases in an oligopoly. Since each firm is a monopolist with respect to its residual demand, its optimal supply function becomes steeper, for given supply function choices by its rivals. This change in turn makes residual demand steeper for other firms, so their optimal supply functions become steeper. Similarly, a decrease in n, for given choices of supply functions, makes total market supply steeper, so each firm's residual demand, and hence optimal supply function, becomes steeper. Hence increasing c, decreasing m, or decreasing n makes the equilibrium supply functions steeper.

Increasing c or decreasing n raises industry price and lowers industry output for any given realization of ε. However, decreasing m raises price by a smaller proportion than the fall in m and so raises industry output. (Multiplying m by a factor $(1/\lambda) < 1$ multiplies the market-clearing price for any given output by a factor λ, by rotating the demand curve upwards about a fixed horizontal intercept for given ε. Therefore, we are interested in whether the equilibrium price increases by more or less than a factor λ.) To understand this result, observe that making both marginal cost and demand steeper by a factor λ (i.e. multiplying c by λ and m by $1/\lambda$) also makes the equilibrium supply functions steeper, and so raises price, by the same factor and leaves output unchanged. Therefore, decreasing m has the opposite effect on the level of price relative to demand and on the level of output as increasing c. To summarize, industry price is higher and industry output lower the smaller is the number of firms (lower n) and the steeper is the marginal cost curve relative to the demand curve (larger cm).[17]

These comparative statics effects are shared by other oligopoly models. Perhaps more interesting is that the magnitudes of the effects are larger for changes in c and n, but smaller for changes in m, than if firms competed by choosing from a set of supply functions of exogenously fixed slope. In our model, changes in these parameters, by affecting the slopes of the supply functions that firms adopt, indirectly affect market outcomes, in addition to directly affecting outcomes in a manner analogous to that in the standard models of Cournot and Bertrand and in Vives (1986), where supply function slopes are exogenously fixed. To confirm that changes in supply function slopes affect equilibrium outputs, observe that any linear SFE consists of the supply functions that firms would choose in the Nash equilibrium of a game in which they were restricted to supply functions of this slope, and as this slope increases, the Nash equilibrium outputs are reduced.

[17] The generalization of the latter result to the nonlinear case is the following: If $D(p, \varepsilon)$ is changed to $(1/\lambda)D(p, \varepsilon)$, $(1/\lambda) < 1$, with the marginal cost curve remaining fixed, each firm's supply at each price is reduced by a factor between $(1/\lambda)$ and 1. (The proof follows that of Proposition 8b.) Consequently, the market-clearing price falls, and a larger fraction of the market is served.

(Each point $(p_0, 2S_0)$ on a demand curve $D(p, \varepsilon)$ is the unique Nash equilibrium in linear supply functions of slope $f(p_0, S_0)$; Claim 3 thus implies that smaller outputs correspond to steeper supply functions.) Hence, since an increase in c or a decrease in n lowers total equilibrium output when the slope of supply functions is exogenously fixed, these comparative statics effects are reinforced by the increase in the slope of the supply functions adopted.[18] On the other hand, a decrease in m *reduces* output through its effect of increasing the slope of equilibrium supply functions, but its overall effect is nevertheless to increase output.

4. DIFFERENTIATED PRODUCTS

This section sketches the extension of the model of supply function competition to the case of a differentiated products duopoly, in which firms' demands are subject to a common, ex ante unobservable shock. We assume that the firms are symmetric and that the shock translates the demand curves horizontally, so the demand system is

$$q_i = g(p_i, p_j) + \varepsilon,$$

$$q_j = g(p_j, p_i) + \varepsilon,$$

where $g_1 < 0$ and $g_2 > 0$. To derive i's residual demand, we note that if j chooses the supply function $q_j = S^j(p_j)$, then market-clearing in j's market implies $S^j(p_j) = g(p_j, p_i) + \varepsilon$, which determines p_j as a function ϕ^j of p_i and ε: $p_j = \phi^j(p_i, \varepsilon)$. Implicitly differentiating $S^j(\phi^j(p_i, \varepsilon)) = g(\phi^j(p_i, \varepsilon), p_i) + \varepsilon$ with respect to p_i gives

$$\phi_1^j(p_i, \varepsilon) = \frac{g_2(\phi^j(p_i, \varepsilon), p_i)}{S^{j\prime}(\phi^j(p_i, \varepsilon)) - g_1(\phi^j(p_i, \varepsilon), p_i)}.$$

Substituting $\phi^j(p_i, \varepsilon)$ for p_j in i's demand curve yields i's residual demand: $q_i = g(p_i, \phi^j(p_i, \varepsilon)) + \varepsilon$.

For a given ε, i's profit-maximizing price $p_i^0(\varepsilon)$ solves

$$\max_{p_i} p_i\left[g(p_i, \phi^j(p_i, \varepsilon)) + \varepsilon\right] - C\left(g(p_i, \phi^j(p_i, \varepsilon)) + \varepsilon\right). \tag{12}$$

Differentiating with respect to p_i and substituting for ϕ_1^j from above, the

[18] In the linear SFE, the difference between equilibrium price and marginal cost converges to 0 at rate $1/n^2$, for given ε. In Cournot competition, the order of magnitude is $1/n$, while in Vives (1986), it is $1/n^2$. As in Vives (1986), when $n \to \infty$, there are two distinct effects on the SFE, each of which drives the price-cost margin to 0 at rate $1/n$. First, residual demand for each firm shifts inward, and the perceived elasticity of residual demand goes to infinity. Second, residual demand becomes flatter because, with each firm choosing an upward-sloping supply function, the total supply curve of the rest of the industry becomes flatter. (Only the first of these effects is present in the Cournot model.) In the SFE, in contrast to Vives (1986), the increase in the slope of the equilibrium supply functions as n increases also shrinks the price-cost margin, but because this slope remains finite, this third effect does not change the order of magnitude of the price-cost margin.

first-order condition becomes

$$g(p_i, \phi^j(p_i, \varepsilon)) + \varepsilon + [p_i - C'(g(p_i, \phi^j(p_i, \varepsilon)) + \varepsilon)] \cdot \left[g_1(p_i, \phi^j(p_i, \varepsilon)) + \frac{g_2(p_i, \phi^j(p_i, \varepsilon)) g_2(\phi^j(p_i, \varepsilon), p_i)}{S^{j\prime}(\phi^j(p_i, \varepsilon)) - g_1(\phi^j(p_i, \varepsilon), p_i)} \right] = 0. \tag{13}$$

If (12) is globally strictly concave in p_i, (13) implicitly determines a unique $p_i^0(\varepsilon)$ for each ε, and the corresponding optimal quantity is $q_i^0(\varepsilon) = g(p_i^0(\varepsilon), \phi^j(p_i^0(\varepsilon), \varepsilon)) + \varepsilon$. If $p_i^0(\varepsilon)$ is invertible, so the locus of points described by $p_i^0(\varepsilon)$ and $q_i^0(\varepsilon)$ can be written as a function $q_i = S^i(p_i)$, and if $S^i(p_i)$ equilibrates supply and demand in i's market at exactly one point for each ε, $S^i(p_i)$ is i's optimal supply function in response to $S^j(p_j)$. In a symmetric equilibrium, $S^i(\cdot) = S^j(\cdot) \equiv S(\cdot)$, and for each ε, $p_i = \phi^j(p_i, \varepsilon) = p_j \equiv p$. Substituting in (13) and solving for $S'(p)$ yields

$$S'(p) = g_1(p, p) - \frac{(g_2(p, p))^2 (p - C'(S))}{S + g_1(p, p)(p - C'(S))}. \tag{14}$$

The differential equation (14) is a necessary condition for a SFE tracing through ex post optimal points. (In the limit as the goods become undifferentiated ($g_1 \to -\infty$ and $g_2 \to \infty$, with $(g_2/g_1) \to -1$), (14) reduces to (5).)

Existence and characterization results for SFE's could be developed along the lines of Section 3, by examining which of the solutions to (14) satisfy the second-order conditions. Here, we restrict ourselves to the linear case, for which we show existence of a unique SFE and provide comparative statics predictions, when ε has full support.

Let the demand system be

$$q_i = -b_1 p_i + b_2 p_j + \varepsilon,$$

$$q_j = -b_1 p_j + b_2 p_i + \varepsilon,$$

where $b_1 > b_2 > 0$ and $\varepsilon \in [0, \infty)$, and let each firm's cost curve be $C(q) = cq^2/2$, where $c > 0$. The differential equation (14) can be expressed, after some rearrangement, in parametric form as

$$\begin{pmatrix} \dfrac{dS}{dt} \\ \dfrac{dp}{dt} \end{pmatrix} = M \begin{pmatrix} S \\ p \end{pmatrix}, \quad \text{where} \quad M = \begin{pmatrix} 1 + \dfrac{(b_1^2 - b_2^2)c}{b_1} & -\dfrac{(b_1^2 - b_2^2)}{b_1} \\ -\dfrac{1}{b_1} - c & 1 \end{pmatrix},$$

which has solutions

$$\begin{pmatrix} S \\ p \end{pmatrix} = A_1 e^{\lambda_1 t} \begin{pmatrix} v_1 \\ w_1 \end{pmatrix} + A_2 e^{\lambda_2 t} \begin{pmatrix} v_2 \\ w_2 \end{pmatrix},$$

where A_1, A_2 are constants,

$$\lambda_i = \frac{2 + \dfrac{(b_1^2 - b_2^2)c}{b_1} \pm \sqrt{\left(\dfrac{b_1^2 - b_2^2}{b_1}\right)^2 c^2 + 4\dfrac{(b_1^2 - b_2^2)c}{b_1} + 4\left(1 - \dfrac{b_2^2}{b_1^2}\right)}}{2}$$

are the eigenvalues (which are real and unequal), and

$$\frac{v_i}{w_i} = \frac{1 - \lambda_i}{c + \dfrac{1}{b_1}}$$

are the slopes of the corresponding eigenvectors of M ($i = 1, 2$). Letting λ_1 denote the larger eigenvalue, we have $\lambda_1 > 1$, $0 < \lambda_2 < 1$, and $(v_1/w_1) < 0$, $(v_2/w_2) > 0$. As in the case of perfect substitutes, all solutions pass through the origin, and all but one have a common positive slope there. As $t \to \infty$, if $A_1 \neq 0$, $(S/p) \to (v_1/w_1) < 0$, so all solutions with $A_1 \neq 0$ eventually leave the positive quadrant and cannot therefore be SFE's when ε has full support. The single solution with $A_1 = 0$ is the unique linear solution to (14) with positive slope at the origin:

$$S(p) = \left(\frac{v_2}{w_2}\right)p = \frac{p}{2\left(1 + \dfrac{1}{b_1 c}\right)} \cdot \left[-\frac{(b_1^2 - b_2^2)}{b_1} + \sqrt{\left(\frac{b_1^2 - b_2^2}{b_1}\right)^2 + \frac{4}{c}\left(\frac{b_1^2 - b_2^2}{b_1}\right)\left(1 + \frac{1}{b_1 c}\right)} \right]. \tag{15}$$

This function can be shown directly to satisfy the second-order conditions for profit maximization for all $p \geqslant 0$. Furthermore, arguments analogous to those in the proof of Proposition 3 show that no asymmetric equilibrium exists. Therefore, the function $S(p)$ in (15) is the unique SFE.

We note that the analysis of the linear case gives the behavior near the origin of the solutions to (14) in general. All solutions to (14) pass through the origin, and all but one have a common positive slope there, given by the expression in (15) for v_2/w_2, with b_1 replaced by $-g_1(0,0)$, b_2 by $g_2(0,0)$, and c by $C''(0)$.

For the linear case, we now analyze the effect on the unique SFE of changes in the slope of marginal cost, the slope of demands, and the degree of product differentiation.

As the marginal cost curve becomes steeper (higher c), the equilibrium supply functions also become steeper, so the market-clearing outputs fall. However, with differentiated products, even in the limit as marginal cost becomes constant ($c \to 0$), the equilibrium supply functions remain upward-sloping: (S/p)

$\to \sqrt{b_1^2 - b_2^2} < \infty$. (With perfect substitutes, the equilibrium supply functions approach the marginal cost curve as $c \to 0$: $(S/p) \to \infty$.)

As with perfect substitutes, increasing the slope of demand (by proportionately increasing b_1 and b_2, leaving the horizontal intercept unchanged for given ε) raises equilibrium outputs.

We can examine the effect of reducing the degree of product differentiation, while keeping the market size constant, by increasing b_1 and b_2 while holding $b_1 - b_2$ constant.[19] This change makes the equilibrium supply functions flatter and increases outputs. In contrast to the case when firms compete with supply functions of exogenously fixed slope, competition is intensified both directly, by the greater similarity of the products, and also indirectly, by the fact that firms choose flatter supply functions.

5. CONCLUSION

Our objectives in this paper have been to develop a richer model of competition in oligopoly, and at the same time to resolve the competing predictions of different models (Cournot, Bertrand, etc.), in which firms are restricted to using particular, exogenously determined strategic variables (fixed prices, fixed quantities, etc.).

In the presence of uncertainty, firms will wish to adopt supply functions as strategic variables. We have presented conditions under which a Nash equilibrium in supply functions (SFE) exists for a symmetric n-firm oligopoly producing a homogeneous good, have shown that all equilibria are symmetric, and have found (stronger) sufficient conditions for uniqueness. When the demand uncertainty has unbounded support, then even if the SFE is not unique, for any given realization of the market demand curve, the set of equilibrium outcomes along it is a connected set, and each point in the set is supported by a unique supply function. Furthermore, under this assumption on the demand uncertainty, any SFE outcome corresponding to a given value of the demand shock is intermediate in terms of price, output, and profits between the outcomes that would result from Cournot and Bertrand competition if the value of the shock were known.

Recognizing that firms will adapt to exogenous uncertainty by choosing supply functions helps to resolve the indeterminacy of equilibrium in oligopoly models. Under uncertainty, firms have strict preferences over the set of possible strategic variables, because their strategic variable (their supply function) must function well in many possible environments. This rules out almost all of the superabundance of Nash equilibria in supply functions under certainty, because the supply functions in these equilibria are not optimal except at a single point. In addition to determining how a firm's behavior will change in equilibrium with the exogenous demand shock, the chosen supply function also determines how the firm would respond out of equilibrium to a change in a rival's behavior. From this perspective, firms' selection of supply functions in an environment of

[19] Total market demand is $q_i + q_j = (b_2 - b_1)(p_i + p_j) + 2\varepsilon$, but the sensitivity of i's demand to a change in j's price increases as b_2 (and b_1) increase.

uncertainty narrows down the set of possible residual demand curves that their rivals could face in equilibrium, and hence dramatically reduces the set of equilibria.

While our approach has been to model a game in which strategic variables (that is, the form of firms' supply functions) are determined endogenously in Nash equilibrium, rather than to propose a new equilibrium concept, our objective of resolving the indeterminacy of equilibrium in oligopoly is shared by the literature on consistent conjectures equilibria (see, for example, Bresnahan (1981, 1983)). That literature view this indeterminacy as arising from the arbitrariness of firms' conjectures about their rivals' responses to deviations and attempts to remove this arbitrariness through imposition of some form of consistency condition on conjectures. However, while the equilibrium concept has been expressed in game-theoretic terms (Bresnahan (1983), Robson (1983)), the approach relies implicitly on firms' being uncertain, even in equilibrium, about their rivals' behavior, and this uncertainty is neither explained nor explicitly modeled. As a result the justification for the consistency condition is unclear.[20] In contrast, our approach of motivating supply function strategies by adaptation to exogenous uncertainty and focusing on Nash equilibria in these strategies provides, in our view, a sounder game-theoretic model of oligopoly.

The predictions of our model could be tested by adapting the methodology developed by Bresnahan (1982) and Lau (1982) for empirically estimating the degree of market power in an oligopolistic industry, even when the demand and marginal cost curves must be estimated as well. (By the "degree of market power," Bresnahan and Lau mean the slope of a firm's perceived marginal revenue curve, which in our model is endogenously determined by the competitors' equilibrium supply functions.)

We have modeled the endogenous determination of strategic variables in a very general way, allowing firms to choose any function relating their output to their price: we believe that competition in oligopoly is through choices that are better approximated by supply functions than by fixing either price or quantity at a given level and letting the other variable absorb all of the adjustment required for market clearing, as in standard Bertrand and Cournot models. Nevertheless, models in which firms fix prices or quantities are convenient analytically. Our work can be interpreted as suggesting under what conditions a Bertrand or a Cournot model is a better approximation to oligopolistic competition.[21] With a

[20] Both the stronger and the weaker versions of Bresnahan's consistent conjectures equilibrium concept have other drawbacks: Robson (1983) showed that the stronger equilibrium, proposed by Bresnahan (1981), may not exist, and Klemperer and Meyer (1988) showed that any point at which firms earn nonnegative profits can be supported by an infinity of consistent conjectures equilibria of the weaker form defined in Bresnahan (1983).

[21] Viewed in this light, our analysis generalizes Klemperer and Meyer (1986), in which firms were restricted to choosing either a fixed price (horizontal supply function) or a fixed quantity (vertical supply function) and results similar to those here were obtained: there an increase in the slope of the marginal cost curve, a convexification of the demand curve, and an increase in the effect of the demand shock at higher, relative to lower, prices all made firms more likely to select quantities as equilibrium strategic variables.

small number of firms, differentiated products, additive demand uncertainty, and a marginal cost curve that is steep relative to demand, quantity-setting models may be better approximations than price-setting models. With a larger number of firms, more homogeneous products, relatively greater demand uncertainty at lower prices, and a marginal cost curve that is flatter relative to demand, price-setting models may be closer to the truth.

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APPENDIX

PROOF OF CLAIM 1: Differentiation yields

$$f_S(p,S) = \frac{p - C'(S) + SC''(S)}{(p - C'(S))^2},$$

so for all $(p_0, S_0) \neq (0,0)$ such that $f(p_0, S_0) = 0$, $f_S(p_0, S_0) \neq 0$. Therefore, by the Implicit Function Theorem, $f(p,S) = 0$ implicitly defines, in the neighborhood of any such (p_0, S_0), a unique function $S = S^0(p)$, which is continuous and differentiable.

To prove (i) and (ii), observe that since $-\infty < D_p < 0$, either $S^0(p)$ and $p - C'(S^0(p))$ are both positive or they are both 0. (We are ignoring the possibility of negative outputs.) Hence $S^0(0) = 0$ is the unique solution to $f(0,S) = 0$. Furthermore, $p > C'(S^0(p))$ whenever $S^0(p) > 0$. Therefore, since $C'' > 0$ and since $S^0(p) > 0$ if and only if $p > 0$, $(C')^{-1}(p) > S^0(p)$ for all $p > 0$. To prove (iii) and (iv), differentiate $f(p, S^0(p)) = 0$ totally with respect to p and substitute using this equation to get

$$S^{0\prime}(p) = \frac{-D_p(p) - D_{pp}(p)\left[p - C'(S^0(p))\right]}{1 - D_p(p)C''(S^0(p))} > 0 \quad \text{for} \quad p > 0.$$

Now $\lim_{p \to 0} S^{0\prime}(p)$ exists and equals

$$\frac{-D_p(0)}{1 - D_p(0)C''(0)} \equiv S^{0\prime}(0),$$

where $0 < S^{0\prime}(0) < (1/C''(0))$, so $S^0(p)$ is continuous and differentiable at $p = 0$. *Q.E.D.*

PROOF OF CLAIM 2: From (5), $S^\infty(p)$ solves $f(p, S^\infty(p)) = \infty$ implies $S^\infty(p)$ solves $p - C'(S^\infty(p)) = 0$, so since $C'' > 0$, $S^\infty(p) = (C')^{-1}(p)$ for all p. The stated properties of $S^\infty(p)$ follow from the assumptions on $C'(S)$. *Q.E.D.*

PROOF OF CLAIM 3: Since $S/(p - C'(S))$ is finite and increasing in S as long as $p > C'(S)$, for a given $p_0 > 0$, $f(p_0, S)$ is finite and monotonically increasing in S for $S \in [0, (C')^{-1}(p_0))$. Below the $f = \infty$ locus, $0 > S/(p - C'(S)) > -\infty$, so since $0 > D_p > -\infty, 0 > f(p,S) > -\infty$. *Q.E.D.*

PROOF OF CLAIM 4: Consider first any (p_0, S_0) such that $p_0 \neq C'(S_0)$. Then f and f_S are continuous at (p_0, S_0), so f is locally lipschitzian in S at (p_0, S_0). Therefore, by a standard theorem in differential equations (see, e.g. Hale, Chapter 1, Theorem 3.1), there exists a unique trajectory solving (5) and passing through (p_0, S_0), and this trajectory can be expressed as a continuous function $S(p)$ in an open neighborhood of (p_0, S_0).

Now suppose $p_0 = C'(S_0)$ but $(p_0, S_0) \neq (0,0)$. For the differential equation in the form (6), r and r_p are continuous at (p_0, S_0), so r is locally lipschitzian in p at (p_0, S_0). Therefore, there exists a unique trajectory solving (6), and hence (5), and passing through (p_0, S_0), and this trajectory can be expressed as a continuous function $p(S)$ in an open neighborhood of (p_0, S_0). *Q.E.D.*

PROOF OF CLAIM 5: By Claims 1 and 3, (i) any trajectory that crosses the $f = 0$ locus from below can never reenter the region between the $f = 0$ and the $f = \infty$ loci. By Claims 2 and 3, (ii) any trajectory that crosses the $f = \infty$ locus from the left can never reenter the region between the $f = 0$ and the $f = \infty$ loci. (i) and (ii) imply that the unique trajectory through any point between the $f = 0$ and the $f = \infty$ loci, or on one of them, must emerge from (0,0). It remains to show that the unique trajectory through any point in the positive quadrant above the $f = 0$ locus or below the $f = \infty$ locus must emerge from the region between these two loci. For any (p, S) above the $f = 0$ locus, Claim 3 implies that $S'(p) < 0$ and differentiation of (5) implies that $S''(p) < 0$, so as p decreases, the trajectory through a given (p_0, S_0) must eventually cross $f = 0$. For any (p_0, S_0) below the $f = \infty$ locus, Claim 3 implies that as p increases, the trajectory through the point must eventually cross $f = \infty$. *Q.E.D.*

PROOF OF CLAIM 6: Suppose that a trajectory emerges from the origin with slope $S'(0)$. Taking the limit of both sides of (5) as $p \to 0$, we have

$$\lim_{p\to 0} S'(p) = \lim_{p\to 0} \frac{S}{p - C'(S)} + \lim_{p\to 0} D_p(p).$$

Using L'Hôpital's Rule to evaluate the first term on the right-hand side, we get

$$S'(0) = \frac{S'(0)}{1 - C''(0)S'(0)} + D_p(0),$$

which when solved for $S'(0)$ yields a quadratic with solutions

$$\frac{1}{2}\left[D_p(0) \pm \sqrt{\left(D_p(0)\right)^2 - \frac{4D_p(0)}{C''(0)}}\,\right].$$

Let $\bar{S}^+ > 0$ correspond to the positive root and $\bar{S}^- < 0$ correspond to the negative root. $\bar{S}^+$ is easily seen to satisfy $S^{0\prime}(0) < \bar{S}^+ < S^{\infty\prime}(0)$. Since the trajectory through any (p_0, S_0) between the $f = 0$ and the $f = \infty$ loci passes through the origin and must lie between these loci for all $p \in (0, p_0]$ (by the proof of Claim 5), this trajectory must have the positive slope $\bar{S}^+$ at (0,0). *Q.E.D.*

PROOF OF CLAIM 7: Given that j chooses $S^j(p)$, the second derivative of i's profit with respect to p for a given ε is

$$\text{(A1)} \qquad \pi^i_{pp}(p, \varepsilon; S^j(\cdot)) = 2\left[D_p(p) - S^{j\prime}(p)\right] - C''\left(D(p) + \varepsilon - S^j(p)\right)\left[D_p(p) - S^{j\prime}(p)\right]^2 + \left[p - C'\left(D(p) + \varepsilon - S^j(p)\right)\right]\left[D_{pp}(p) - S^{j\prime\prime}(p)\right].$$

If $S^j(p)$ solves (5), we can differentiate (5) totally with respect to p to obtain an expression for $S^{j\prime\prime}(p)$:

$$\text{(A2)} \qquad S^{j\prime\prime}(p) = \frac{S^{j\prime}(p)\left[p - C'\left(S^j(p)\right)\right] - S^j(p)\left[1 - C''\left(S^j(p)\right)S^{j\prime}(p)\right]}{\left[p - C'\left(S^j(p)\right)\right]^2} + D_{pp}(p).$$

Using (5) to substitute for $S^j(p)$ in (A2) gives

$$S^{j\prime\prime}(p) = D_{pp}(p) + \frac{S^{j\prime}(p) + \left[D_p(p) - S^{j\prime}(p)\right]\left[1 - C''\left(S^j(p)\right)S^{j\prime}(p)\right]}{p - C'\left(S^j(p)\right)},$$

so when $S^j(p)$ solves (5), (A1) becomes

$$\pi^i_{pp}(p,\varepsilon;S^j(\cdot)) = \left[D_p(p) - S^{j\prime}(p)\right]\left[1 + C''(D(p)+\varepsilon - S^j(p))S^{j\prime}(p)\right] - C''(D(p)+\varepsilon - S^j(p))\left[D_p(p) - S^{j\prime}(p)\right]^2 - S^{j\prime}(p),$$

which is equation (8) in the text. *Q.E.D.*

Proof of Proposition 4: Define $\hat{p} = 1/p$ and $\hat{S} = 1/S$. We will write the differential equation (5) in terms of $\hat{p}$ and $\hat{S}$ and examine the behavior of its solutions when $\hat{p}$ and $\hat{S}$ are small, i.e. $\hat{p} < (1/p_1)$ and $\hat{S} < (1/S_1)$, so the demand and marginal cost curves are linear in this region. We will show that all solutions approach $(\hat{S} = 0, \hat{p} = 0)$, all but one with a common negative slope $(d\hat{S}/d\hat{p}) = r_2 < 0$ and the remaining one with a positive slope $(d\hat{S}/d\hat{p}) = r_1 > 0$. At $(\hat{S} = 0, \hat{p} = 0)$, $(d\hat{S}/d\hat{p}) = r_i$ implies $(dS/dp) = 1/r_i$, so by Claims 1, 2, and 3, the unique solution with $(d\hat{S}/d\hat{p}) = r_1 > 0$ at $(\hat{S} = 0, \hat{p} = 0)$ is the unique trajectory which remains between the $f = 0$ and the $f = \infty$ loci for all p (or $\hat{p}$). By Propositions 1 and 3, this trajectory is the unique SFE tracing through ex post optimal points.

For $\hat{p} < (1/p_1)$ and $\hat{S} < (1/S_1)$, (5) becomes

$$\text{(A3)} \qquad \frac{d\hat{S}}{d\hat{p}} = \frac{d\hat{S}}{dS}\frac{dS}{dp}\frac{dp}{d\hat{p}} = \frac{\hat{S}^2}{\hat{p}^2}\left[\frac{\frac{1}{\hat{S}}}{\frac{1}{\hat{p}} - \frac{b}{\hat{S}}} - \mu\right].$$

(The constant a in the marginal cost function is negligible when $\hat{S}$ is small.) The substitution $\hat{S} = v(\hat{p})\hat{p}$, after some algebra, yields a separable differential equation for $v(\hat{p})$:

$$\frac{(v-b)\,dv}{v(-\mu v^2 + b\mu v + b)} = \frac{d\hat{p}}{\hat{p}},$$

to which the solution is

$$\frac{(v-r_1)^{-A+\frac{1}{2}}(v-r_2)^{A+1/2}}{v} = \gamma\hat{p},$$

where

$$r_1 = \frac{b}{2} + \frac{1}{2}\sqrt{b^2 + \frac{4b}{\mu}} > 0,$$

$$r_2 = \frac{b}{2} - \frac{1}{2}\sqrt{b^2 + \frac{4b}{\mu}} < 0,$$

$$A = \left(\frac{b}{2} + \frac{1}{\mu}\right)\Big/\sqrt{b^2 + \frac{4b}{\mu}} > \frac{1}{2},$$

and γ is a constant of integration. Substituting $\hat{S}/\hat{p}$ for v and inverting both sides of the equation, we get

$$\left(\frac{\hat{S}}{\hat{p}} - r_1\right)^{A-1/2}\left(\frac{\hat{S}}{\hat{p}} - r_2\right)^{-A-1/2} = \frac{\gamma'}{\hat{S}},$$

where $\gamma' = 1/\gamma$. It is easy to check that for all solutions, as $\hat{p} \to 0$, $\hat{S} \to 0$. To examine the slopes $d\hat{S}/d\hat{p}$ of the solutions as $\hat{p} \to 0$ and $\hat{S} \to 0$, we use the linear approximation $\hat{S} = \theta\hat{p}$ and show how θ varies with γ'. Substituting $\hat{S} = \theta\hat{p}$ gives

$$(\theta - r_1)^{A-1/2}(\theta - r_2)^{-A-1/2} = \frac{\gamma'}{\hat{S}}.$$

Since $A > \frac{1}{2}$, for the solution corresponding to $\gamma' = 0$, $\theta = r_1 > 0$. For any $\gamma' \neq 0$, letting $\hat{S} \to 0$ implies $\theta = r_2 < 0$. Therefore, all solutions to (A3) but one approach ($\hat{S} = 0$, $\hat{p} = 0$) with negative slope r_2, and the remaining solution approaches with positive slope r_1. *Q.E.D.*

PROOF OF PROPOSITION 6: The proofs of the claims for $p_0 > 0$ straightforwardly parallel the arguments used to prove Proposition 5. Even if $p_0 = 0$, since $D_p(0)$ is unchanged, Claim 6 implies that the slope of the SFE at the origin is unchanged. *Q.E.D.*

PROOF OF PROPOSITION 7: If $|D_{p\varepsilon}|/D_\varepsilon$ is sufficiently small, the $f = 0$ locus remains upward-sloping everywhere and, furthermore, any solution to (4) that never crosses either the $f = 0$ or the $f = \infty$ locus satisfies the second-order conditions for all $p \geqslant 0$, so is a SFE. Also, $|D_{p\varepsilon}|/D_\varepsilon$ sufficiently small ensures that any trajectory that crosses the vertical axis violates the second-order conditions where it crosses, so cannot be a SFE.

Now since ε_1 is finite, for $\varepsilon \geqslant \varepsilon_1$ the evolution of trajectories solving (4) is unaffected by the change in $D_p(p, \varepsilon)$ for $\varepsilon < \varepsilon_1$, so there is a unique trajectory which does not cross either the $f = 0$ or the $f = \infty$ locus for (S, p) such that $e(2S, p) \geqslant \varepsilon_1$: this trajectory coincides with the original SFE in this region. This trajectory must emerge from the origin and be everywhere between the $f = 0$ and the $f = \infty$ loci. The arguments above now imply that this trajectory is the unique SFE in the perturbed model.

Suppose that $D_p(p, \varepsilon)$ is increased on $(\varepsilon_0, \varepsilon_1)$. For (S, p) such that $e(2S, p) \in (\varepsilon_0, \varepsilon_1)$, the new SFE must lie everywhere strictly above the original SFE: if at some $(\hat{S}, \hat{p})$ such that $e(2\hat{S}, \hat{p}) = \hat{\varepsilon} \in (\varepsilon_0, \varepsilon_1)$, it were on or below the original equilibrium trajectory then, since $D_p(\hat{p}, \hat{\varepsilon})$ is now higher, we know from (4) that the new trajectory would be flatter at $(\hat{S}, \hat{p})$ than the old one. It follows from (4) that the new trajectory would be strictly below the old one at (S, p) such that $e(2S, p) = \varepsilon_1$, and since the original SFE is unique, the new trajectory would eventually cross the $f = \infty$ locus, thus invalidating it as an equilibrium.

Since for $\varepsilon \leqslant \varepsilon_0$, the trajectories solving (4) are unaffected by the change in $D_p(p, \varepsilon)$ for $\varepsilon > \varepsilon_0$, the new equilibrium must also lie strictly above the old one for (S, p) such that $e(2S, p) \in (e(0,0), \varepsilon_0]$. Since $\varepsilon_0 > e(0,0)$, $D_p(0, e(0,0))$ is unchanged, and application of L'Hôpital's Rule as in the proof of Claim 6 implies that the slope of the SFE at the origin is unchanged.

The arguments for the case where $D_p(p, \varepsilon)$ is decreased on $(\varepsilon_0, \varepsilon_1)$ are parallel. *Q.E.D.*

PROOF OF PROPOSITION 8a: It is easy to show that for $n \geqslant 2$, (5) becomes

$$\text{(A4)} \qquad f_n(p, S) = S'(p) = \frac{1}{n-1}\left[\frac{S}{p - C'(S)} + D_p(p)\right].$$

The $f_n = 0$ and $f_n = \infty$ loci are independent of n (since $D_{p\varepsilon} \equiv 0$), and it is straightforward to confirm the validity, for $n \geqslant 2$, of Claims 1 through 5, as well as of Propositions 1 through 3. Application of L'Hôpital's Rule as in the proof of Claim 6 shows that for all solutions to (A4) that emerge from the origin with a positive slope, the common value of the slope is

$$\text{(A5)} \qquad S_n'(0) = \frac{D_p(0) + \dfrac{n-2}{C''(0)} + \sqrt{\left(D_p(0) + \dfrac{n-2}{C''(0)}\right)^2 - \dfrac{4D_p(0)(n-1)}{C''(0)}}}{2(n-1)},$$

which is increasing in n and tends to $1/C''(0)$ as $n \to \infty$.

At any point between the $f_n = 0$ and $f_n = \infty$ loci, f_n is decreasing in n. Therefore, when n increases, any trajectory lying between these loci that is anywhere on or above the original unique SFE (except at the origin) will move and remain strictly above the original equilibrium. It follows, from uniqueness of the original equilibrium, that any such trajectory will eventually cross the $f_n = 0$ locus, so cannot be a SFE. Thus, when the number of firms increases, the new SFE's must, except at the origin, lie strictly below the original SFE. It is immediate that the price falls and output increases for any $\varepsilon > e(0,0)$. *Q.E.D.*

PROOF OF PROPOSITION 8b: The proof of the existence, symmetry across firms, and strictly positive slope of an SFE for $n \geqslant 2$ parallels the argument in the proof of Proposition 8a, as does the proof that the common slope of all SFE's at the origin, $\tilde{S}_n'(0)$, satisfies $\lim_{n\to\infty} n\tilde{S}_n'(0) = 1/C''(0)$, once we note that $C_n'(S/n) = C'(S)$ implies $C_n''(0) = nC''(0)$.

We proceed in three steps:

Step 1: Fix n at n_0 but change firms' marginal cost curves from $C_{n_0}'(\cdot)$ to $C_{n_1}'(\cdot)$, $n_1 > n_0$, *and* change industry demand from $D(p)+\varepsilon$ to $(n_0/n_1)(D(p)+\varepsilon)$. Before the change, (5) takes the form

$$S'(p) = \frac{1}{n_0 - 1}\left[\frac{S}{p - C_{n_0}'(S)} + D_p(p)\right] \equiv \tilde{f}_{n_0}(p, S).$$

After this change, let (5) take the form $S'(p) = \hat{f}(p, S)$. Now

$$\hat{f}\left(p, \frac{S}{n_1}\right) = \frac{1}{n_0 - 1}\left[\frac{\frac{S}{n_1}}{p - C_{n_1}'\left(\frac{S}{n_1}\right)} + \frac{n_0}{n_1} D_p(p)\right]$$

$$= \frac{n_0}{n_1}\tilde{f}_{n_0}\left(p, \frac{S}{n_0}\right).$$

The change thus rescales f horizontally. Therefore the $f = 0$ and the $f = \infty$ loci, the solutions to the differential equation, and hence also the unique SFE are all rescaled horizontally but retain the same relative positions. Letting $\tilde{S}_{n_0}^*(p)$ and $\hat{S}(p)$ be the unique SFE's before and after the change, we have $\hat{S}(p) = (n_0/n_1)\tilde{S}_{n_0}^*(p)$.

Step 2: Keep n fixed at n_0 and firms' marginal cost curves fixed at $C_{n_1}'(\cdot)$ but change industry demand from $(n_0/n_1)(D(p)+\varepsilon)$ back to $D(p)+\varepsilon$. From Proposition 6, the uppermost new SFE, $\check{S}(p)$, must, except at the origin, lie everywhere strictly below $\hat{S}(p)$.

Step 3: Keep firms' marginal cost curves fixed and industry demand fixed at $D(p)+\varepsilon$, but change n from n_0 to n_1. By a simple extension of Proposition 8a (to the case where the original SFE is not unique), any new SFE $\tilde{S}_{n_1}^*(p)$ must, except at the origin, lie everywhere strictly below $\check{S}(p)$.

Combining Steps 1, 2, and 3, we have that for all $p > 0$

$$n_1\tilde{S}_{n_1}^*(p) > n_1\check{S}(p) > n_1\hat{S}(p) = n_0\tilde{S}_{n_0}^*(p).$$

Since industry supply increases, the market-clearing price falls and industry output increases, for any $\varepsilon > e(0,0)$. *Q.E.D.*

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