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EX POST EFFICIENCY AND INDIVIDUAL RATIONALITY IN INCENTIVE COMPATIBLE TRADING MECHANISMS

by

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This is a preliminary paper. Comments are welcome.

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Abstract.

This paper concerns the design of a trading mechanism for a group of traders when their valuations of the good are private information and they bargain over who shall consume more than his initial endowment and who shall consume less. It is shown that there generally exists a set of initial endowments of the traded commodity such that it is possible to design a trading mechanism which is incentive compatible, individually rational and ex post efficient.

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1. Introduction.

A trading mechanism is defined as a set of rules which determines the pattern of trade and the associated payments between participating traders as functions of the traders' reported valuations of the traded commodity. A viable trading mechanism must be individually rational, i.e. each trader's expected utility from participation should be greater than the expected utility he can obtain outside the trading mechanism. A desirable property of any trading mechanism is ex post efficiency. A trading mechanism is ex post efficient when the outcome determined by the mechanism is Paretoefficient conditional on the actual valuations of the traders.

A trading mechanism is an example of a revelation game, i.e. a game where each player has to report something from his domain of private information. A simple but important fact, known as the Revelation Principle, is that it suffices to study direct revelation games in order to characterize the outcome of revelation games. In direct revelation games each player accurately reports his private information if he expects each other trader to do the same. A trading mechanism such that each trader's equilibrium strategy is to report accurately is said to be incentive compatible.

Myerson and Satterthwaite (1983) characterized the set of incentive compatible and individually rational trading mechanisms for a seller and a buyer bargaining over a single good. They did in particular show the general impossibility of ex post efficient mechanisms without outside subsidies. Chatterjee (1982) had earlier illustrated this impossibility for certain simple mechanisms. If an incentive compatible trading mechanism is ex post efficient then there are some valuations for which the expected gains from participation in the trading mechanism is negative for at least one of the traders unless the mechanism is subsidized by some outside third party. Without outside subsidies such a mechanism is not individually rational.

The Myerson–Satterthwaite model focuses on traders with very asymmetric initial endowments of the commodity. There is a potential seller with an initial endowment of one unit of the commodity and there is a potential buyer whose initial endowment is zero. But in many trading situations all potential traders have a positive initial endowment of the good and they bargain over who shall consume more than his initial endowment and who shall consume less. One example is electric power companies with a power pool arrangement. The participating companies all have positive reserve capacities and the aim of the pool is to allocate the available capacity between companies on the basis of reported marginal production costs. Thus, depending on its reported marginal cost, a pool member may be a net buyer or a net seller of capacity.

In this paper I shall characterize incentive compatible and individually rational trading mechanisms in a more general context. There is a given but arbitrary number of traders and each trader may have a positive initial endowment of the commodity. The traders are asymmetric, i.e. the probability distributions over each trader's type are not the same for all traders. The main result is that there always exists a set of initial endowments such that if the actual initial endowment allocation is in this set it is possible to design a trading mechanism which is incentive compatible, individually rational and ex post efficient. If a trading mechanism is incentive compatible and ex post efficient, individual rationality requires a positive expected gain from participation for each trader in all possible states. When the traded commodity initially is owned by a single trader Myerson and Satterthwaite (1983) established the impossibility of this. But the expected gains from participation are functions of the initial allocation of the commodity. A trader's expected gain is lower the larger his initial endowment of the commodity is, because a larger initial endowment increases the utility of non–participation and thus the cost of participation. It turns out to be the case that starting from a single ownership initial allocation a decrease of the initial endowment of the single owner increases his expected gain from participation relatively more than it reduces that of those whose initial holdings are increased. Thus the total expected gains from participation increase and there exist initial allocations of the commodity for which the expected gains from participation is positive for all traders.

Cramton, Gibbons and Klemperer (1987) derived this result for the case where all traders are symmetric with respect to probability distributions over types. But the set of initial allocations which allow ex post efficiency depends on the probability distributions over types. It is therefore of interest to investigate how this set is affected by differences in these distributions. This paper shows that their result generalizes to the case with asymmetric traders regardless of how different the traders are.

The next section characterizes the set of incentive compatible and individually rational trading mechanisms. The possibility of achieving ex post efficiency is analyzed in section 3 and section 4 concludes the paper.

2. Incentive compatible and individually rational trading mechanisms.

There are n potential traders, $i \in N = \{1,...,n\}$. They have initial endowments of a commodity and of money income. Units are chosen so that the sum of the traders' endowments of the commodity is unity. Let α_i be the initial commodity endowment of trader i. Thus, $\alpha_i \in [0,1]$, $i = 1,...,n$ and $\sum_{i=1}^{n} \alpha_i = 1$.

Let the traders' valuations of the commodity be independent stochastic variables V_i distributed over the interval [a,b]. Let $f_i(v_i)$ be the continuous and positive probability density function of V_i and let $F_i(v_i)$ be the corresponding distribution function. Each trader is assumed to have an additively separable utility for money income and the commodity and to be risk neutral.

The probability distributions and the form of the utility functions are common knowledge. Each trader knows his own valuation when he submits a bid to the trading institution, but regards the others' valuations as random variables.

A <u>trading mechanism</u> is defined by two outcome functions $p : \mathbb{R}^n \to \mathbb{R}^n$ and $x : \mathbb{R}^n \to \mathbb{R}^n$. Let $v = (v_1,...,v_n)$ be the traders' submitted bids. The function p(v) denotes the allocation of the commodity among the traders determined by the trading mechanism when their submitted bids are v. Thus, $p_i(v) \in [0,1]$, $i = 1,...,n$ and $\sum_{i=1}^{n} p_i(v) = 1$. The function x(v) denotes the payments received by the traders when their submitted bids are v. The sum of payments is required to balance, i.e. $\sum_{i=1}^{n} x_i(v) = 0$.

Let E_{-i} be the expectation operator with respect to $v_{-i} = (v_1,...,v_{i-1},v_{i+1},...,v_n)$ and let $P_i(v_i) = E_{-i}[p_i(v)]$ and $X_i(v_i) = E_{-i}[x_i(v)]$. Trader i's conditional expected net trade of the commodity is $P_i(v_i) - \alpha_i$ and he receives the conditional expected payment $X_i(v_i)$ so his conditional expected gain from trade is

$$U_i(v_i) = X_i(v_i) + v_i[P_i(v_i) - \alpha_i], \qquad i = 1,...,n$$

These definitions make sense only if truthful revelation of v_i is an equilibrium strategy for all traders. If this is the case, the trading mechanism (p,x) is <u>incentive compatible</u>, i.e.

$$U_i(v_i) \geq X_i(u) + v_i[P_i(u) - \alpha_i] \quad \forall\, v_i, u \in [a,b], \ \forall\, i \in N$$

According to the Revelation Principle the analysis may be restricted to incentive compatible trading mechanisms without any loss of generality. For any Bayesian equilibrium in any trading mechanism there is always an equivalent incentive compatible trading mechanism which yields exactly the same outcome. (For a discussion of the Revelation Principle, see Myerson (1979)).

Lemma 1 establishes necessary and sufficient conditions for incentive compatibility. The proof, which is standard, is given in the Appendix.

LEMMA 1: The trading mechanism (p,x) is incentive compatible if and only if for every $i \in N$

(P) $P_i(v_i)$ is increasing

(IC) $X_i(v_i) - X_i(u) = -\int_u^{v_i} t\,dP_i(t) \quad \forall\, v_i, u \in [a,b]$

In a sense (P) is the crucial condition. If (P) holds for all traders it is namely possible to construct a payment function x(v) such that the trading mechanism (p,x) is incentive compatible. This result is given in Lemma 2. The proof is in the Appendix.

LEMMA 2: If $p(v)$ is any function mapping $[a,b]^n$ into the n–1–dimensional simplex, such that $P_i(v_i)$ is increasing for every $i \in N$, then there exists a function $x(v)$, where $\sum_{i=1}^{n} x_i(v) = 0$, such that (p,x) is incentive compatible.

A trading mechanism (p,x) is (interim) individually rational if

$$U_i(v_i) \geq 0 \qquad \forall\, v_i \in [a,b],\ \forall\, i \in N$$

i.e. irrespective of what the actual value of V_i is, the expected utility of participating in the trading mechanism shall be at least as large as the attainable utility outside the mechanism. To ensure individual rationality it is sufficient to show that the lowest possible conditional expected gain from trade is nonnegative. In order to identify this lowest possible expected utility the following two lemmas characterize the properties of $U_i(v_i)$. Proofs are given in the Appendix.

LEMMA 3: If the trading mechanism (p,x) is incentive compatible then $U_i(v_i)$ is a convex and continuous function which attains a minimum at

$$v_i^*(\alpha_i) = \tfrac{1}{2}\inf(\mathcal{V}_i(\alpha_i)) + \tfrac{1}{2}\sup(\mathcal{V}_i(\alpha_i))$$

where

$$\mathcal{V}_i(\alpha_i) = \{v_i\colon P_i(u) \leq \alpha_i\ \forall\, u < v_i \text{ and } P_i(u) \geq \alpha_i\ \forall\, u > v_i\}$$

LEMMA 4: For an incentive compatible trading mechanism (p,x) the worst–off valuation $v_i^*(\alpha_i)$ is an increasing function. The conditional expected gain from trade in the worst outcome, $U_i(v_i^*(\alpha_i))$, is a decreasing and concave function of α_i.

It follows from Lemma 3 that the trading mechanism (p,x) is individually rational if and only if for every $i \in N$

$$U_i(v_i^*(\alpha_i)) = X_i(v_i^*(\alpha_i)) + v_i^*(\alpha_i)\cdot(P_i(v_i^*(\alpha_i))-\alpha_i) \geq 0$$

According to Lemma 3 the worst–off valuation $v_i^*(\alpha_i)$ is the one for which the trader typically expects to be neither a net buyer nor a net seller, i.e. his expected trade $P_i(v_i^*(\alpha_i))-\alpha_i$ is zero. If $v_i < v_i^*(\alpha_i)$ trader i expects to be a net seller in an incentive compatible trading mechanism since $P_i(v_i)-\alpha_i < 0$. A slightly higher valuation for trader i would decrease his net sales. If it shall be an equilibrium strategy for him to report his valuation accurately his expected payment must decrease to exactly match the value of the increase in his expected commodity consumption. But then the only remaining effect of the higher valuation is that the value of his expected net trade increases, which reduces his conditional expected gain from trade. Thus $U_i(v_i)$ is decreasing when $v_i < v_i^*(\alpha_i)$ and by a similar argument increasing when $v_i > v_i^*(\alpha_i)$.

Since $P_i(v_i)$ is increasing the worst–off valuation is higher the higher the trader's initial endowment is. By the envelope theorem an increase in trader i's initial endowment decreases his conditional expected gain from trade in the worst outcome by $v_i^*(\alpha_i)$. Thus, a trader's lowest possible expected gain from trade is decreasing and concave in α_i.

Since the traders' worst–off expected gains from trade depend on the initial endowment allocation the question of whether a particular incentive compatible trading mechanism (p,x) can be individually rational or not cannot be settled independently of the initial allocation of the commodity. To accomplish this we shall introduce the following mapping from the n–1 – dimensional simplex to the real line. Define $G: S^{n-1} \to R$ as

$$G(\alpha\,;p) = \sum_{i=1}^{n}\Big[\int_{v_i^*(\alpha_i)}^{b}(1-F_i(t))t\,dP_i(t) - \int_{a}^{v_i^*(\alpha_i)}F_i(t)t\,dP_i(t) + v_i^*(\alpha_i)(P_i(v_i^*(\alpha_i))-\alpha_i)\Big] \tag{1}$$

where S^{n-1} is the n–1–dimensional simplex. Then we have the following result.

LEMMA 5: If (p,x) is incentive compatible then $G(\alpha\,;p)$ is concave in α and

$$G(\alpha\,;p) = \sum_{i=1}^{n} U_i(v_i^*(\alpha_i))$$

where $G(\alpha\,;p)$ is defined in (1).

Finally, Lemma 6 gives a necessary and sufficient condition for a trading mechanism to be incentive compatible and individually rational.

LEMMA 6: If p(v) is any function mapping $[a,b]^n$ into the n–1–dimensional simplex, then there exists a function x(v), where $\sum_{i=1}^{n} x_i(v) = 0$, such that (p,x) is incentive compatible and individually rational if and only if, for every $i \in N$, $P_i(v_i)$ is increasing and

$$G(\alpha;p) \geq 0.$$

3. Ex post efficiency.

Ex post efficiency is a property of the allocation function p(v). It results in an ex post efficient allocation of the commodity if and only if

$$p_i(v) = 1 \text{ iff } v_i = \max v_j$$
$$p_i(v) = 0 \text{ iff } v_i < \max v_j$$

for all $i \in N$.

If p(v) is ex post efficient

$$P_i(v_i) = E_{-i}[p_i(v)] = \prod_{j \neq i} F_j(v_i)$$

so $P_i(v_i)$ is increasing. By Lemma 2 there exists a payment function x(v) such that the ex post efficient trading mechanism (p,x) is incentive compatible and by Lemma 6 it is individually rational if and only if $G(\alpha;p) \geq 0$. We are thus interested in the set of initial allocations α which satisfy this inequality, i.e. the set

$$A = \{\alpha \in S^{n-1} : G(\alpha;p) \geq 0\}$$

If the set A is nonempty it consists of the initial allocations α for which it is possible to design a trading mechanism which is ex post efficient, incentive compatible and individually rational.

Since $G(\alpha;p)$ is concave and continuous in α it attains a maximum on S^{n-1}. Let

$$\alpha^* \in \underset{\alpha \in S^{n-1}}{\operatorname{argmax}} \, G(\alpha;p)$$

and

$$v^* = v_1^*(\alpha_1^*) = v_2^*(\alpha_2^*) = \dots = v_n^*(\alpha_n^*)$$

The necessary and sufficient condition for maximum is that the worst–off valuations are the same for all traders. The common value is v^*. If they were not equal a reduction of the initial endowment of the trader with the highest worst–off valuation and an increase with the same amount for the trader with the lowest increases the value of $G(\alpha;p)$ since $dU_i(v_i^*(\alpha_i))/d\alpha_i = -v_i^*(\alpha_i)$. When p(v) is ex post efficient $v_i^*(1) = b$ and $v_i^*(0) = a$ for every $i \in N$ and $v_i^*(\alpha_i)$ is strictly increasing. Thus there is a unique α^* in the interior of S^{n-1}.

We shall now show that the set A is always nonempty when p(v) is ex post efficient, i.e. for any set of distribution functions there is always a set of initial allocations such that it is possible to design an ex post efficient, incentive compatible and individually rational trading mechanism.

Consider first the function $G(\alpha;p)$ when p(v) is ex post efficient. Then $P_i(v_i^*(\alpha_i)) = \alpha_i$ for all i so $G(\alpha;p)$ reduces to

$$G(\alpha;p) = \sum_{i=1}^{n}[\int_{v_i^*(\alpha_i)}^{b}(1-F_i(t))tdP_i(t) - \int_{a}^{v_i^*(\alpha_i)}F_i(t)tdP_i(t)]$$

where $P_i(t) = \prod_{j\neq i} F_j(t)$. Define $R(t) = F_i(t)P_i(t) = \prod_{j=1}^{n} F_j(t)$. Using $dR(t) = F_i(t)dP_i(t)+P_i(t)dF_i(t) = \sum_{k=1}^{n} f_k(t)\prod_{j\neq k} F_j(t)dt$ in $G(\alpha;p)$ and integrating by parts results in

$$\begin{aligned} G(\alpha) &= b - \sum_{i=1}^{n} \alpha_i \cdot v_i^*(\alpha_i) + (n-1)\int_a^b R(t)dt - \sum_{i=1}^{n}\int_{v_i^*(\alpha_i)}^{b} P_i(t)dt \\ &= b - \sum_{i=1}^{n} \alpha_i \cdot v_i^*(\alpha_i) + (n-1)\int_a^b \prod_{j=1}^{n} F_j(t)dt - \sum_{i=1}^{n}\int_{v_i^*(\alpha_i)}^{b} \prod_{j\neq i} F_j(t)dt \quad (2) \end{aligned}$$

THEOREM 1: For any set of distribution functions F_i, $i = 1,\dots,n$, there exists a nonempty, convex set $A \subseteq S^{n-1}$ such that it is possible to design a trading mechanism which is ex post efficient, incentive compatible and individually rational if and only if $\alpha \in A$.

PROOF: Let p(v) be ex post efficient. Then $P_i(v_i)$ is increasing for every $i \in N$ and by Lemma 2 there exists a function x(v) such that (p,x) is incentive compatible. Define

$$A = \{\alpha \in S^{n-1}: G(\alpha) \geq 0\}$$

where $G(\alpha)$ is defined as in (2). Then $\alpha^* \in A$ since

$$G(\alpha^*) = b - v^* + (n-1)\int_a^b \prod_{j=1}^n F_j(t)dt - \sum_{i=1}^n \int_{v^*}^b \prod_{j\neq i} F_j(t)dt =$$

$$= b - v^* - \int_{v^*}^b [\sum_{i=1}^n \prod_{j\neq i} F_j(t) - (n-1)\prod_{j=1}^n F_j(t)]dt + (n-1)\int_a^{v^*} \prod_{j=1}^n F_j(t)dt =$$

$$= \int_{v^*}^b [1 - H(t)]dt + (n-1)\int_a^{v^*} \prod_{j=1}^n F_j(t)dt > 0$$

where $H(t) = [\sum_{i=1}^n \prod_{j\neq i} F_j(t) - (n-1)\prod_{j=1}^n F_j(t)]$ and the inequality follows from $H(b) = 1$ and $H(t) < 1$ for all $t \in [v^*,b)$.

Thus α^*, as well as a neighborhood around α^*, is in A.

A is convex since by Lemma 5 $G(\alpha)$ is concave. If $\alpha \notin A$ then $G(\alpha) < 0$ and by Lemma 6 a mechanism (p,x) which is ex post efficient and incentive compatible cannot be made individually rational.

Q.E.D.

Theorem 1 tells us that there always exist a set of initial endowments that makes efficient trading possible regardless of how different the traders are with respect to the probability distributions over types, F_i, at least as long as there is a positive probability density on all possible types. But there are also inital endowments that never allow efficient trading, which Myerson and Satterthwaite (1983) demonstrated for the two trader case. This is established in Proposition 1.

PROPOSITION 1: A is a proper subset of S^{n-1}. In particular, A does not contain any of the vertex points of the simplex.

PROOF: Let without loss of generality $\alpha_1 = 1$ and $\alpha_2 = \ldots = \alpha_n = 0$. Ex post efficiency implies $v_1^*(1) = b$ and $v_2^*(0) = \ldots = v_n^*(0) = a$ and

$$G(\alpha) = b - \sum_{i=1}^{n} \alpha_i \cdot v_i^*(\alpha_i) + (n-1)\int_a^b \prod_{j=1}^{n} F_j(t)dt - \sum_{i=1}^{n} \int_{v_i^*(\alpha_i)}^{b} \prod_{j\neq i} F_j(t)dt =$$

$$= (n-1)\int_a^b \prod_{j=1}^{n} F_j(t)dt - \sum_{i=2}^{n} \int_a^b \prod_{j\neq i} F_j(t)dt =$$

$$= -\sum_{i=2}^{n} \int_a^b [1-F_i(t)] \prod_{j\neq i} F_j(t)dt < 0$$

Q.E.D.

The set A depends on the distribution functions F_i. When they are the same for all traders, $\alpha^* = (1/n,\ldots,1/n)$ and A is a symmetric set centered on α^* (Proposition 1 in Cramton et al. (1987)). To gain some insight into how α^*, and thus indirectly A, is affected by differences in the traders' distribution functions consider a family of such functions $\{F(v_i,\theta_i)\}$. When θ_i is the same for all i the game is symmetric. Suppose that F is differentiable w r t θ_i and that $\partial F/\partial\theta_i > 0$ for all $i \in N$. Thus a trader's expected valuation decreases with a higher θ_i.

Let $\theta_{-i} = (\theta_1,\ldots,\theta_{i-1},\theta_{i+1},\ldots,\theta_n)$. The worst–off valuations are implicitly determined as

$$\prod_{j\neq i} F(v_i^*(\alpha_i,\theta_{-i}),\theta_j) = P_i(v_i^*(\alpha_i,\theta_{-i}),\theta_{-i}) = \alpha_i$$

which implies that

$$\frac{\partial v_i^*}{\partial\theta_i} = 0$$

$$\frac{\partial v_i^*}{\partial\theta_h} = -\frac{\partial P_i/\partial\theta_h}{\partial P_i/\partial v_i} = -\frac{\partial P_i}{\partial\theta_h}\cdot\frac{\partial v_i^*}{\partial\alpha_i} < 0$$

The point α^* is determined by the conditions

$v_1^*(\alpha_1^*,\theta_{-1}) = \ldots = v_n^*(\alpha_n^*,\theta_{-n})$ and $\sum_{i=1}^{n} \alpha_i^* = 1$ so the effect on α^* of a change $d\theta_h$ is given as

$$\frac{\partial v_i^*}{\partial \alpha_i} \cdot \frac{\partial \alpha_i^*}{\partial \theta_h} + \frac{\partial v_i^*}{\partial \theta_i} = k \qquad \forall i \neq h$$

$$\frac{\partial v_h^*}{\partial \alpha_h} \cdot \frac{\partial \alpha_h^*}{\partial \theta_h} = k$$

where k is some constant which must satisfy

$$0 = \sum_{i=1}^{n} \frac{\partial \alpha_i^*}{\partial \theta_h} = k \cdot \sum_{i=1}^{n} \frac{\partial P_i}{\partial v_i} - \sum_{i=1}^{n} \frac{\partial P_i}{\partial v_i} \frac{\partial v_i^*}{\partial \theta_h}$$

i.e.

$$k = \frac{\sum_{i=1}^{n} \frac{\partial P_i}{\partial v_i} \cdot \frac{\partial v_i^*}{\partial \theta_h}}{\sum_{i=1}^{n} \frac{\partial P_i}{\partial v_i}} < 0$$

Thus $\partial \alpha_h^* / \partial \theta_h = k \cdot \partial P_h / \partial v_h < 0$. For the trader whose expected valuation decreases α_h^* should be lower. Since the initial endowments sum to unity $\sum_{i \neq h} \partial \alpha_i^* / \partial \theta_h > 0$. For some traders, other than h, $\partial \alpha_i^* / \partial \theta_h$ is also negative, namely for those where the decrease in the worst–off valuation $\partial v_i^* / \partial \theta_h$ is smaller in absolute value than the average decrease in worst–off valuations measured by k since

$$\frac{\partial \alpha_i^*}{\partial \theta_h} = \frac{\partial P_i}{\partial v_i} \left(k - \frac{\partial v_i^*}{\partial \theta_h}\right)$$

Consider the implications of these results for the two trader case. Suppose to begin with that the traders are symmetric so that A is an interval contained in the unit interval and centered on $\alpha_1^* = 1/2$. Let us now make the traders different by decreasing θ_1 and increasing θ_2, i.e. the expected valuation of trader 1 increases whereas that of trader 2 decreases. As a result of the lower θ_1 α_1^* increases (and $\alpha_2^* = 1 - \alpha_1^*$ decreases) and the higher θ_2 works in the same direction. As α_1^* gets closer to one the interval A must also move closer to one. Thus as the traders become more different efficient trading can be sustained with more unequal initial endowments with a larger initial share for the trader whose expected valuation increases. To illustrate this consider the example $F(v_i,\theta_i) = v_i^{1/\theta_i}$ where [a,b] = [0,1]. Thus, trader i's expected valuation is $\frac{1}{1+\theta_i}$. When $\theta_i = 1$ for all i we have symmetric and uniform distributions over types and with two traders $\alpha^* = 1/2$ and A = [0.21,0.79]. Table 1 illustrates how α^* and A is affected by changes in θ_1 and θ_2. The more different the traders are the smaller is the interval A and it moves in the direction of a larger initial share for the trader with a high expected valuation. With symmetric distributions the set A is larger the lower, or the higher, the common expected valuation is.

θ_1	θ_2	α^*	A
1.0	1.0	0.5	[0.21,0.79]
0.5	2.0	0.72	[0.48,0.94]
0.25	4.0	0.88	[0.75,0.99]
0.1	10.0	0.97	[0.93,0.998]
1.0	2.0	0.62	[0.34,0.88]
1.0	10.0	0.83	[0.63,0.97]
1.0	100.0	0.97	[0.88,0.996]
10.0	10.0	0.5	[0.15,0.85]
40.0	40.0	0.5	[0.06,0.94]
0.02	0.02	0.5	[0.19,0.81]

Table 1 The effects on α^* and A of different values for the distribution parameters θ_i.

A higher expected valuation for one of the traders implies lower expected consumption for the other traders. Thus their worst–off valuations increase, whereas it remains constant for the trader with the higher expected valuation. The total expected gains from trade would then be larger if the initial commodity holding of the latter were larger, while it, on average, were lower for the other traders, which explains the direction in which α^* moves.

4. Conclusions.

As an economic institution trade should serve the purpose of allocating goods to people who value them the most, i.e. to allocate goods ex post efficiently. When there is incomplete information Myerson and Satterthwaite (1983) demonstrated that there does not exist any trading mechanism which always yield an outcome which is ex post efficient if trade is voluntary and the good initially is owned by a single trader. Cramton, Gibbons and Klemperer (1987) did however show that for symmetric traders there exists a set of initial allocations of the commodity such that there exists mechanisms which ensures ex post efficient and voluntary trade. In this paper it has been shown that this is true also when traders are different, and possibly very different, with respect to expected valuations.

The key to these results is that individual rationality requires the expected gain from participation to be positive in all possible states. In particular, the expected gain at the worst–off valuation must be positive. But this expected gain is a decreasing and concave function of the trader's initial holding of the commodity. Myerson and Satterthwaite (1983) showed that the sum of expected gains at the worst–off valuations must be negative if the commodity initially is owned by a single trader. The concavity property implies however that a reduction of the endowment of the single owner increases his expected gain by more than it reduces that of those whose initial holdings increase. Thus the total gains from participation must increase and there exist a reallocation of the initial allocation such that the total expected gain becomes positive.

APPENDIX

This Appendix contains the proofs of Lemmas 1 – 6.

LEMMA 1: The trading mechanism (p,x) is incentive compatible if and only if for every $i \in N$

(P) $P_i(v_i)$ is increasing

(IC) $X_i(v_i) - X_i(u) = -\int_u^{v_i} t dP_i(t) \quad \forall\, v_i, u \in [a,b]$

PROOF: Suppose (p,x) is incentive compatible. The incentive compatibility definition implies

$$U_i(v_i) = X_i(v_i) + v_i(P_i(v_i)-\alpha_i) \geq X_i(u) + v_i(P_i(u)-\alpha_i) =$$
$$= U_i(u) + (v_i-u)(P_i(u)-\alpha_i)$$

and

$$U_i(u) \geq U_i(v_i) + (u-v_i)(P_i(v_i)-\alpha_i)$$

Combining these inequalities yields

$$(v_i-u)(P_i(v_i)-\alpha_i) \geq U_i(v_i) - U_i(u) \geq (v_i-u)(P_i(u)-\alpha_i)$$

or

$$(v_i-u)(P_i(v_i)-P_i(u)) \geq 0$$

so $P_i(v_i)$ is increasing (P).

The inequality $U_i(v_i) \geq U_i(u) + (v_i-u)(P_i(u)-\alpha_i)$ implies that U_i has a supporting hyperplane at u with slope $P_i(u)-\alpha_i$. Thus, U_i is convex and has derivative $dU_i/dv_i = P_i(v_i)-\alpha_i$ almost everywhere so

$$U_i(v_i) - U_i(u) = \int_u^{v_i}[P_i(t) - \alpha_i]dt =$$
$$= v_iP_i(v_i) - uP_i(u) - \int_u^{v_i} t dP_i(t) - \alpha_i(v_i-u)$$

which together with the definition of U_i yields (IC).

To prove sufficiency, suppose conditions (P) and (IC) hold. Then adding $v_i[P_i(v_i)-P_i(u)] = v_i\int_u^{v_i} dP_i(t)$ to (IC) yields

$$X_i(v_i) + v_iP_i(v_i) - [X_i(u) + v_iP_i(u)] = \int_u^{v_i}(v_i-t)dP_i(t) \geq 0$$

where the inequality follows from (P). Thus,

$$U_i(v_i) = X_i(v_i) + v_i(P_i(v_i)-\alpha_i) \geq X_i(u) + v_i(P_i(u)-\alpha_i)$$

which is incentive compatibility.

Q.E.D.

LEMMA 2: If p(v) is any function mapping $[a,b]^n$ into the n−1–dimensional simplex, such that $P_i(v_i)$ is increasing for every $i \in N$, then there exists a function x(v), where $\sum_{i=1}^{n} x_i(v) = 0$, such that (p,x) is incentive compatible.

PROOF: The proof is by construction. Let

$$x_i(v) = c_i - \int_a^{v_i} t dP_i(t) + \frac{1}{n-1}\sum_{j\neq i}\int_a^{v_j} t dP_j(t)$$

where c_i, $i = 1,\ldots,n$ are constants such that $\sum_{i=1}^{n} c_i = 0$. Then

$$X_i(v_i) - X_i(u) = -\int_u^{v_i} t dP_i(t)$$

so by Lemma 1 (p,x) is incentive compatible.

The balance condition $\sum_{i=1}^{n} x_i(v) = 0$ is also satisfied since

$$\sum_{i=1}^{n} x_i(v) = \sum_{i=1}^{n} c_i - \sum_{i=1}^{n}\int_a^{v_i} t dP_i(t) + \frac{1}{n-1}\sum_{i=1}^{n}\sum_{j\neq i}\int_a^{v_j} t dP_j(t) =$$

$$= \sum_{i=1}^{n} c_i = 0$$

Q.E.D.

LEMMA 3: If the trading mechanism (p,x) is incentive compatible then $U_i(v_i)$ is a convex and continuous function which attains a minimum at

$$v_i^*(\alpha_i) = \tfrac{1}{2}\inf(\mathcal{V}_i(\alpha_i)) + \tfrac{1}{2}\sup(\mathcal{V}_i(\alpha_i))$$

where

$$\mathcal{V}_i(\alpha_i) = \{v_i\colon P_i(u) \le \alpha_i \ \forall\, u < v_i \text{ and } P_i(u) \ge \alpha_i \ \forall\, u > v_i\}$$

PROOF: If (p,x) is incentive compatible then

$$U_i(v_i) \ge U_i(u) + (v_i - u)(P_i(u) - \alpha_i)$$

so U_i is everywhere above its supporting hyperplane at u. Thus U_i is convex and has derivative $P_i(v_i) - \alpha_i$ almost everywhere and U_i is also continuous.

Since U_i is convex and continuous it attains a minimum on the interval [a,b] and since $dU_i/dv_i = P_i(v_i) - \alpha_i$ there are five possible situations:

i) $P_i(v_i) - \alpha_i \ge 0 \quad \forall\, v_i \in [a,b]$. Then $v^*(\alpha_i) = a$.

ii) $P_i(v_i) - \alpha_i \le 0 \quad \forall\, v_i \in [a,b]$. Then $v^*(\alpha_i) = b$.

iii) P_i is continuous and strictly increasing and $P_i(a) \le \alpha_i \le P_i(b)$. Then

$$v_i^*(\alpha_i) = P_i^{-1}(\alpha_i).$$

iv) $P_i(v_i) = \alpha_i \quad \forall\, v_i$ in an intervall $I \subseteq [a,b]$. The any v_i in I minimizes U_i and we can choose $v_i^*(\alpha_i) = \tfrac{1}{2}[\inf(I) + \sup(I)]$.

v) P_i is not continuous and jumps past α_i at some $\tilde{v} \in [a,b]$. Then $v_i^*(\alpha_i) = \tilde{v}$.

Q.E.D.

LEMMA 4: For an incentive compatible trading mechanism (p,x) the worst–off valuation $v_i^*(\alpha_i)$ is an increasing function. The conditional expected gain from trade in the worst outcome, $U_i(v_i^*(\alpha_i))$, is a decreasing and concave function of α_i.

PROOF: Since P_i is an increasing function it follows that $\inf(\mathcal{V}(\alpha_i))$ and $\sup(\mathcal{V}(\alpha_i))$ both are increasing w r t α_i, which establishes that v_i^* is increasing.

Consider next the effect on $U_i(v_i^*(\alpha_i)) = X_i(v_i^*(\alpha_i)) + v_i^*(\alpha_i)[P_i(v_i^*(\alpha_i)) - \alpha_i]$ of a change in α_i. Since $v_i^*(\alpha_i)$ minimizes U_i the envelope theorem implies that

$$dU_i(v_i^*(\alpha_i))/d\alpha_i = -v_i^*(\alpha_i)$$

Since v_i^* is increasing, $U_i(v_i^*(\alpha_i))$ is a concave, decreasing function w r t α_i.

Q.E.D.

LEMMA 5: If (p,x) is incentive compatible then $G(\alpha\,;p)$ is concave in α and

$$G(\alpha\,;p) = \sum_{i=1}^{n} U_i(v_i^*(\alpha_i))$$

where $G(\alpha\,;p)$ is defined in (1).

PROOF: If (p,x) is incentive compatible

$$X_i(v_i) = X_i(v_i^*(\alpha_i)) - \int_{v_i^*(\alpha_i)}^{v_i} t\,dP_i(t)$$

Taking the expectation w r t v_i results in

$$E_i[X_i(v_i)] = X_i(v_i^*(\alpha_i)) - \int_a^b \int_{v_i^*(\alpha_i)}^{v_i} t\,dP_i(t)\,dF_i(t) =$$

$$= X_i(v_i^*(\alpha_i)) - \int_{v_i^*(\alpha_i)}^{b} [1-F_i(t)]t\,dP_i(t) + \int_a^{v_i^*(\alpha_i)} F_i(t)t\,dP_i(t)$$

where the second row is obtained by changing the order of integration. Adding over all traders yields

$$\sum_{i=1}^{n} E_i[X_i(v_i)] = E_N[\sum_{i=1}^{n} x_i(v)] = 0$$

so that

$$0 = \sum_{i=1}^{n} E_i[X_i(v_i)] = \sum_{i=1}^{n} X_i(v_i^*(\alpha_i)) - \sum_{i=1}^{n} [\int_{v_i^*(\alpha_i)}^{b} [1-F_i(t)]t\,dP_i(t) +$$

$$+ \int_a^{v_i^*(\alpha_i)} F_i(t)t\,dP_i(t)]$$

From the definition of U_i we have

$$\sum_{i=1}^{n} X_i(v_i^*(\alpha_i)) = \sum_{i=1}^{n} [U_i(v_i^*(\alpha_i)) - v_i^*(\alpha_i)(P_i(v_i^*(\alpha_i)) - \alpha_i)]$$

so that

$$\sum_{i=1}^{n} U_i(v_i^*(\alpha_i)) = \sum_{i=1}^{n} [\int_{v_i^*(\alpha_i)}^{b} [1-F_i(t)]tdP_i(t) - \int_a^{v_i^*(\alpha_i)} F_i(t)tdP_i(t) + v_i^*(\alpha_i)(P_i(v_i^*(\alpha_i))-\alpha_i)] = G(\alpha;p)$$

By Lemma 4 $U_i(v_i^*(\alpha_i))$ is concave if (p,x) is incentive compatible which establishes that $G(\alpha;p)$ is concave.

Q.E.D.

LEMMA 6: If p(v) is any function mapping $[a,b]^n$ into the n–1–dimensional simplex, then there exists a function x(v), where $\sum_{i=1}^{n} x_i(v) = 0$, such that (p,x) is incentive compatible and individually rational if and only if, for every $i \in N$, $P_i(v_i)$ is increasing and

$$G(\alpha;p) \geq 0.$$

PROOF: Only if: Let (p,x) be incentive compatible and individually rational. Then $P_i(v_i)$ is increasing for every $i \in N$ and

$$G(\alpha;p) = \sum_{i=1}^{n} U_i(v_i^*(\alpha_i)) \geq 0$$

If: By hypothesis $P_i(v_i)$ is increasing for every $i \in N$. By Lemma 2 there exists a payment function x(v) such that (p,x) is incentive compatible. Then

$$G(\alpha;p) = \sum_{i=1}^{n} U_i(v_i^*(\alpha_i)) = \sum_{i=1}^{n} [c_i - \int_a^{v_i^*(\alpha_i)} tdP_i(t) + \frac{1}{n-1}\sum_{j\neq i}\int_a^b\int_a^{v_j} tdP_j(t)dF_j(t) + v_i^*(\alpha_i)(P_i(v_i^*(\alpha_i))-\alpha_i)] \geq 0$$

Thus by a proper choice of the constant terms c_i, i = 1,...,n, such that $\sum_{i=1}^{n} c_i = 0$, one can always ensure that $U_i(v_i^*(\alpha_i)) \geq 0$ for every $i \in N$, i.e. (p,x) is individually rational.

Q.E.D.

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