

Urban Labor Economics

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Appendix 1: Basic Urban Economics

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In this appendix, we give the basic ingredients of the standard urban model. For a deeper understanding of these types of models, the reader is strongly advised to read Fujita (1989).

1. The basic model with identical agents

We assume that the city is *linear* and *monocentric*. This means that the city is described by a line in which all jobs and all firms (which are assumed to be identical) are located in the Central Business District (CBD hereafter), which is normalized to zero for simplicity, and all workers/consumers endogenously decide their residential location between 0 and the city fringe x_f . Landlords allocate the land to the highest bids in the city. All workers/consumers are employed and are identical in all respect. There are exactly N identical workers. There are neither mobility costs within the city nor migration costs between outside the city and the city. However, individuals do incur commuting costs to go to work.

1.1. The individual location choice

The question we have to solved is the following: What is the optimal residential choice of each individual? In fact, each individual is a worker and thus works every day in the CBD. He/she obtains a monthly wage of w_L and incurs a monthly pecuniary commuting cost at distance x from the CBD equal to τx , where τ is the commuting cost per unit of distance.

Each individual is also a consumer. He/she consumes housing (or land)¹ and a non-spatial composite good (it is a mixture of all goods that are not related to location). More precisely, he/she optimally determines the lot size of the house h_L and the optimal consumption of the non-spatial composite good z_L .

In order to solve this question we have to determine the preferences of this worker/consumer. For simplicity, we use preferences that are well-behaved, i.e. preferences are continuous and increasing at all $h_L > 0$ and $z_L > 0$ and all indifference curves are strictly convex and smooth and do not cut axes. The utility function (that represent these preferences) is given by:

$$\Gamma(z_L, h_L) \tag{1.1}$$

¹Land and housing is the same here as we do not modeled the optimal choice of land developers (see e.g. Brueckner, 1987, for such an analysis).

with

$$\frac{\partial \Gamma(z_L, h_L)}{\partial h_L} > 0 \quad \frac{\partial \Gamma(z_L, h_L)}{\partial z_L} > 0$$

and the function $\Gamma(z_L, h_L)$ is strictly quasi-concave.

We can also write the budget constraint. Each worker has a net monthly wage of $w_L - \tau x$ and spend money on the composite good z_L plus housing $h_L R(x)$, where $R(x)$ is the land price at a distance x from the CBD. The budget constraint is thus given by:

$$w_L - \tau x = h_L R(x) + z_L \quad (1.2)$$

Each individual chooses h_L and z_L that maximize $U(z_L, h_L)$ under the budget constraint (1.2). We need to calculate the bid rent of each individual, that is how much he/she is ready to pay for land at each location x in order to reach a utility level $\Gamma(z_L, h_L)$. There are three different approaches that are all equivalent (Fujita, 1989, chap. 2).

1.1.1. The indirect Marshallian approach

In this approach, each individual solves the following program:

$$\max_{h_L, z_L} \Gamma(z_L, h_L) \text{ s.t. } w_L - \tau x = h_L R(x) + z_L \quad (1.3)$$

Solving $w_L - \tau x = h_L R(x) + z_L$ in z_L , this is equivalent to:

$$\max_{h_L} U(w_L - \tau x - h_L R(x), h_L) \quad (1.4)$$

where $z_L = w_L - \tau x - h_L R(x)$. First and second order conditions of (1.4) give a unique Marshallian demand for housing (lot size) $h_L^M(R(x), w_L)$, which is implicitly defined as:

$$-\frac{\partial U}{\partial z_L} R(x) + \frac{\partial U}{\partial h_L} = 0 \quad (1.5)$$

By using the budget constraint, we obtain the Marshallian demand for the composite good:

$$z_L^M(R(x), w_L) = w_L - \tau x - h_L^M(x, w_L) R(x) \quad (1.6)$$

The indirect utility function can be written as:

$$W(w_L, \tau, x, R) = \Gamma(w_L - \tau x - h_L^M(R(x), w_L) R(x), h_L^M(R(x), w_L)) \equiv W_L \quad (1.7)$$

where W_L is the equilibrium utility level in the city. Finally, we can solve equation (1.7) to obtain the so-called bid rent (see below for an exact definition) for specific preferences:

$$\Psi_L(x, W_L) \tag{1.8}$$

Let us explain the Marshallian approach using Figure A1.1. The aim of each individual is to solve (1.3), that is to find the highest utility level compatible with the budget constraint (1.2). Thus, by fixing the budget constraint as in Figure A1.1, we vary the utility functions W_L and the solution of (1.3) is given by (z_L^*, h_L^*) , which is precisely the highest utility level compatible with the budget constraint; in Figure A1.1, it corresponds to $W_{L,2}$.

[Insert Figure A1.1 here]

We would like to know where this representative individual choose his/her residential location in the city. For that, he/she solves the following program:

$$\max_x \Gamma(w_L - \tau x - h_L R(x), h_L)$$

It is easily checked that this leads to:

$$\tau + h_L R'(x) = 0 \tag{1.9}$$

which is the so-called Alonso-Muth condition. It says that, by optimally choosing his/her location, each individual face the following trade off. If the individual decides to reside closer to the city-center (here 0), then he/she pays a higher marginal land rent ($R'(x) < 0$) but incurs a lower marginal commuting cost ($\tau > 0$). On the contrary, if he/she locates closer to the periphery of the city, then the marginal land rent is lower but the marginal commuting cost is greater. Of course, in equilibrium, all (identical) individuals are indifferent between all location since they reach the same utility level W_L (there is no mobility costs).

1.1.2. The direct approach

Let us define more in a more direct way the concept of bid rent. This is referred to as the *direct* approach of calculating the bid rent.

Definition 1. *The bid rent $\Psi_L(x, W_L)$ is the maximum rent per unit of land that an individual can pay for residing at a distance x from the CBD while enjoying a fixed utility level W_L .*

Using the budget constraint (1.2), this bid rent can be written as:

$$\Psi_L(x, W_L) = \max_{z_L, h_L} \left\{ \frac{w_L - \tau x - z_L}{h_L} \mid \Gamma(z_L, h_L) = W_L \right\} \quad (1.10)$$

where, for an individual residing at x , $w_L - \tau x - z_L$ is the money available for land rent and where $(w_L - \tau x - z_L)/h_L$ represents the land rent per unit of land. By solving the equation $\Gamma(z_L, h_L) = W_L$ we obtain a composite good consumption which is a function of h_L and W_L , that is $Z_L(h_L, W_L)$ and thus we can write (1.10) as:

$$\Psi_L(x, W_L) = \max_{h_L} \left\{ \frac{w_L - \tau x - Z_L(h_L, W_L)}{h_L} \right\} \quad (1.11)$$

Solving (1.11) leads to an the optimal lot size which is a function of x and W_L , that is $h_L^d(x, W_L)$, which is implicitly defined by:

$$\frac{\partial Z_L(h_L^d(x, W_L), W_L)}{\partial h_L} h_L + w_L - \tau x - Z_L(h_L^d(x, W_L), W_L) = 0 \quad (1.12)$$

Plugging $h_L^d(x, W_L)$ back to (1.11), we obtain:

$$\Psi_L(x, W_L) = \frac{w_L - \tau x - Z_L(h_L^d(x, W_L), W_L)}{h_L^d(x, W_L)} \quad (1.13)$$

We can illustrate the direct approach using Figure A1.2. To solve program (1.10), one has to fix the utility level to W_L and then find the highest bid rent $\Psi_L(x, W_L)$ compatible with W_L . Since in the plane (h_L, z_L) , $\Psi_L(x, W_L) = R(x)$ is the slope of the budget constraint, i.e. $z_L = w_L - \tau x - R(x) h_L$, then graphically solving program (1.10) means to find the optimal slope of the budget constraint that is compatible with W_L . In Figure A1.2, it is $\Psi_{L,2}(x, W_L)$.

[Insert Figure A1.2 here]

We can again calculate the optimal location for each individual. By differentiating (1.13) with respect to x , we obtain:

$$\begin{aligned} & \frac{\partial \Psi_L(x, W_L)}{\partial x} \\ &= \frac{1}{h_L^2} \left\{ \left[-\tau - \frac{\partial Z_L(h_L, W_L)}{\partial h_L} \frac{\partial h_L}{\partial x} \right] h_L - [w_L - \tau x - Z_L(h_L, W_L)] \frac{\partial h_L}{\partial x} \right\} \\ &= \frac{1}{h_L^2} \left\{ -\tau h_L - \frac{\partial h_L}{\partial x} \left[\frac{\partial Z_L(h_L, W_L)}{\partial h_L} h_L + [w_L - \tau x - Z_L(h_L, W_L)] \right] \right\} \end{aligned}$$

where $h_L \equiv h_L^d(x, W_L)$. Now observing from (1.12) that $\frac{\partial Z_L(h_L, W_L)}{\partial h_L} h_L - [w_L - \tau x - Z_L(h_L, W_L)] = 0$, we obtain (this is known as the Envelope Theorem):

$$\frac{\partial \Psi_L(x, W_L)}{\partial x} = -\frac{\tau}{h_L^d(x, W_L)}$$

In equilibrium (see below), $\Psi_L(x, W_L) = R(x)$ and thus this equation is exactly equivalent to (1.9).

1.1.3. The indirect Hicksian approach

The last approach consists in using the Hicksian compensated demand for land. In that case, the consumer minimizes his/her expenses in order to reach a certain level of utility (this is referred to as the expenditure-minimization problem), that is:

$$\mathbf{E}(x, W_L) \equiv \min_{z_L, h_L} \{z_L + h_L R(x) + \tau x \mid \Gamma(z_L, h_L) = W_L\} \quad (1.14)$$

where $\mathbf{E}(x, W_L)$ is the expenditure function. Again solving $\Gamma(z_L, h_L) = W_L$, we obtain $Z_L(h_L, W_L)$. Thus this program can be written as:

$$\mathbf{E}(x, W_L) \equiv \min_{z_L, h_L} \{z_L + h_L R(x) + \tau x \mid \Gamma(z_L, h_L) = W_L\}$$

where $\mathbf{E}(x, W_L)$ is the expenditure function. Again solving $\Gamma(z_L, h_L) = W_L$, we obtain $Z_L(h_L, W_L)$. Thus this program can be written as:

$$\mathbf{E}(x, W_L) \equiv \min_{h_L} \{Z_L(h_L, W_L) + h_L R(x) + \tau x\}$$

which defines implicitly $h_L^H(R(x), W_L)$, the (compensated) Hicksian demand for land, as follows:

$$\frac{\partial Z_L(h_L^H(R(x), W_L), W_L)}{\partial h_L} + R(x) = 0 \quad (1.15)$$

Plugging $h_L^H(R(x), W_L)$ back to the expenditure function, we easily obtain:

$$\mathbf{E}(x, W_L) \equiv Z_L(h_L^H(R(x), W_L), W_L) + h_L^H(R(x), W_L) R(x) + \tau x \quad (1.16)$$

From the budget constraint (1.2), we have

$$\mathbf{E}(x, W_L) = w_L$$

which using (1.16) is equivalent to:

$$Z_L(h_L^H(R(x), W_L), W_L) + h_L^H(R(x), W_L) R(x) = w_L - \tau x \quad (1.17)$$

Inverting this equation and observing that $\Psi_L(x, W_L) = R(x)$, we obtain the bid rent function.

Now, by differentiating equation (1.17), we obtain:

$$\begin{aligned} & \frac{\partial \Psi_L(x, W_L)}{\partial x} \\ &= \frac{1}{h_L^2} \left\{ \left[-\tau - \frac{\partial Z_L(h_L, W_L)}{\partial h_L} \frac{\partial h_L}{\partial x} \right] h_L - [w_L - \tau x - Z_L(h_L, W_L)] \frac{\partial h_L}{\partial x} \right\} \\ &= \frac{1}{h_L^2} \left\{ -\tau h_L - \frac{\partial h_L}{\partial x} \left[\frac{\partial Z_L(h_L, W_L)}{\partial h_L} h_L + w_L - \tau x - Z_L(h_L, W_L) \right] \right\} \end{aligned}$$

where $h_L \equiv h_L^H(R(x), W_L)$. Using the budget constraint (1.2), we have

$$w_L - \tau x - Z_L(h_L, W_L) = h_L \Psi_L(x, W_L)$$

and thus

$$\begin{aligned} & \frac{\partial Z_L(h_L, W_L)}{\partial h_L} h_L + [w_L - \tau x - Z_L(h_L, W_L)] \\ &= \left[\frac{\partial Z_L(h_L, W_L)}{\partial h_L} + \Psi_L(x, W_L) \right] h_L \end{aligned}$$

Observing that in equilibrium $\Psi_L(x, W_L) = R(x)$ and using (1.16), it is easy to see that (Envelope Theorem):

$$\frac{\partial Z_L(h_L, W_L)}{\partial h_L} h_L + w_L - \tau x - Z_L(h_L, W_L) = 0$$

and thus

$$\frac{\partial \Psi_L(x, W_L)}{\partial x} = -\frac{\tau}{h_L^H(x, W_L)} < 0$$

Again, we can explain the Hicksian approach using Figure A1.3. It is easy to see that this approach is the dual of the Marshallian approach since we are fixing the utility function to some level, here W_L , and we vary the budget constraint. Indeed, in order to solve program (1.14), one has to find the *lowest* budget constraint compatible with utility W_L and in Figure A1.3 it is BC_2 .

[Insert Figure A1.3 here]

Finally, if we compare the different approaches, it is easy to verify that:

$$h_L^*(x, W_L) \equiv h_L^M(R(x), w_L) = h_L^d(x, W_L) = h_L^H(R(x), W_L) \quad (1.18)$$

when $h_L^M(R(x), w_L)$ and $h_L^H(R(x), W_L)$ are evaluated at $R(x) = \Psi_L(x, W_L)$. Therefore the three approaches are totally equivalent. By differentiating equation (1.18), it is easy to show that:

$$\frac{\partial h_L^*(x, W_L)}{\partial x} < 0 \quad , \quad \frac{\partial h_L^*(x, W_L)}{\partial W_L} \quad , \quad \frac{\partial h_L^*(x, W_L)}{\partial w_L}$$

Indeed, people living further away from the CBD will consume more land because it is cheaper there (remember that $\frac{\partial \Psi_L(x, W_L)}{\partial x} < 0$). When utility W_L increases, bid rent decreases and thus individuals increase their housing consumption because land is cheaper. The same reasoning applies to w_L since when it increases, they are able to pay more for land and thus reduce their housing consumption.

Also, in all these approaches, it is easy to show that:

$$\frac{\partial \Psi_L(x, W_L)}{\partial w_L}, \quad \frac{\partial \Psi_L(x, W_L)}{\partial W_L}, \quad \frac{\partial \Psi_L(x, W_L)}{\partial \tau}$$

Indeed, the bid rent increases (decreases) with the wage w_L (the commuting cost τ) because, individuals being richer (poorer), they are able to offer more (less) for land at each location x . On the other hand, the bid rent decreases with utility W_L because the higher the utility level to be reached the lower the bid rent individuals can afford.

1.2. The urban-land use equilibrium

We have now to determine the equilibrium of this city. We will have different cases depending on whether the city is closed or open and the landlords are absent or not.

1.2.1. The closed-city with absentee landlords

We have now to determine the equilibrium of this city. The fact that the city is closed means that the utility level W_L is endogenous while the size of the population N is given. Also, landlords do not reside in the city and thus the revenue for land does not appear in the income of any resident. We have the following definition:

Definition 2. *A closed-city urban equilibrium where landlords are absent and individuals are all employed is a vector $(W_L, x_f, R(x))$ such that:*

$$\Psi_L(x_f, W_L) = R_A \tag{1.19}$$

$$\int_0^{x_f} \frac{1}{h_L^*(x, W_L)} dx = N \tag{1.20}$$

$$R^*(x) = \begin{cases} \max \{ \Psi_L(x_f, W_L), R_A \} & \text{for } x \leq x_f \\ 0 & \text{for } x > x_f \end{cases} \tag{1.21}$$

where x_f is the city fringe and R_A agricultural land rent outside the city and $h_L^*(x, W_L)$ is defined by (1.18).

Equation (1.21) means that the land is offered to the highest bids in the city. In equation (1.19), the bid rent at the city fringe is equal to agricultural rent (normalized to zero for simplicity). Equation (1.20) is the population constraint condition. By solving the two last equations, we obtain the equilibrium values of the two unknowns W_L^* and x_f^* as functions of the exogenous variables w_L, τ .

Of course these conditions are written for a linear city. In order to better understand them, let us explain the case of a circular city. Let there be $n(x)dx$ individuals residing in a circular ring whose inner radius is x and outer radius is $x + dx$. Their total *demand for land* in that ring is thus $h_L^*(x, W_L)n(x)dx$. On the other hand, the total supply for land in that ring is $2\pi x dx$. Then in equilibrium, we must have supply equals demand for land, that is:

$$h_L^*(x, W_L)n(x)dx = 2\pi x dx$$

which can be written as

$$n(x)dx = \frac{2\pi x}{h_L^*(x, W_L)}$$

As a result, the equilibrium condition for space is given by:

$$N = \int_0^{x_f} n(x)dx = \int_0^{x_f} \frac{2\pi x}{h_L^*(x, W_L)} dx$$

which is exactly the population constraint (1.20) for the case of a circular city since $2\pi x$ is the circumference of a circle of radius x . Indeed, at a distance x from the CBD, there is 1 piece of land available in a linear city and $2\pi x$ in a circular city. The second equilibrium condition (1.19) is independent of the type of city (linear or circular) and is mainly the result of perfect competition in the land market.

1.2.2. The open-city with absentee landlords

We can easily extend this model to the case of an open city. In that case, where mobility is free between cities, the utility obtained by city residents W_L becomes exogenous (it is just their outside option) but the number of people N living in the city is endogenous. The definition of equilibrium is exactly as in definition 2 but one has to solve (1.19), (1.20) and (1.21) in terms of N, x_f and $R(x)$. It can be shown (see Fujita, 1989, Proposition 3.5, pages 62-63) that the closed- and open-city models are identical if one uses the value of equilibrium utility W_L^* of the closed-city model in the open-city case.

1.2.3. The case of resident landlords

In both closed and open cities, landlords can reside in the city. This is referred to as the public land-ownership model. To be more precise, the city residents are now assumed to form a government, which rents the land for the city from rural landlords at agricultural rent R_A . The city government, in turn, subleases the land to city residents at the competitive rent $R(x)$ at each location x . We can define the total differential rent (TDR) from the city as:

$$\begin{aligned} TDR &= \int_0^{x_f} [R(x) - R_A] dx \\ &= \int_0^{x_f} R(x) dx - R_A x_f \end{aligned} \quad (1.22)$$

This will lead us to study , where urban land is rented from absentee landlords at a price equaling the agricultural rent (see, for example, Pines and Sadka, 1986, for the full closed city model, i.e. a closed city model with public land-ownership).

The analysis of this case in both closed and open cities is quite straightforward. The only difference is that the income of each individual is now given by $w_L + TDR/N$. Since TDR/N is taken as given by each individual, the analysis is straightforward and follows closely that of the benchmark case. Again, Fujita (1989, Proposition 3.5, pages 62-63) has shown that the absentee landlord and public land-ownership models are identical if they are evaluated with the same values.

1.3. Example 1. Cobb-Douglas utility function with endogenous housing consumption

The individual location choice We will now illustrate all the results of the previous section using a Cobb-Douglas utility function. Let us first focus on the individual problem. We assume that all individuals have the following preferences:

$$\Gamma(z_L, h_L) = z_L^\alpha h_L^\omega$$

with $\alpha + \omega \leq 1$. We will first determine the bid rent using the Marshallian *indirect* approach. From the budget constraint (1.2), we have: $z_L = w_L - \tau x - h_L R(x)$ and thus the Cobb-Douglas utility function can be written as:

$$\Gamma(h_L) = [w_L - \tau x - h_L R(x)]^\alpha h_L^\omega \quad (1.23)$$

Solving (1.4) leads to the following Marshallian demand for housing:

$$h_L^M(R(x), w_L) = \left(\frac{\omega}{\alpha + \omega} \right) \frac{(w_L - \tau x)}{R(x)} \quad (1.24)$$

and, using the budget constraint, we obtain:

$$z_L^M(R(x), w_L) = \left(\frac{\alpha}{\alpha + \omega} \right) (w_L - \tau x) \quad (1.25)$$

We can compute the indirect utility function by plugging (1.24) into (1.23):

$$\Gamma(z_L^M(R(x), w_L), h_L^M(R(x), w_L)) = \frac{\alpha^\alpha \omega^\omega}{(\alpha + \omega)^{\alpha + \omega}} \frac{(w_L - \tau x)^{\alpha + \omega}}{R(x)^\omega} \equiv W_L$$

By inverting this function, we obtain the following bid rent:

$$\Psi_L(x, W_L) = \frac{\omega \alpha^{\alpha/\omega}}{(\alpha + \omega)^{(\alpha + \omega)/\omega}} (w_L - \tau x)^{(\alpha + \omega)/\omega} W_L^{-1/\omega} \quad (1.26)$$

Let us now determine the bid rent by the *direct* approach. Using the definition given by (1.10), we have:

$$\Psi_L(x, W_L) = \max_{z_L, h_L} \left\{ \frac{w_L - \tau x - z_L}{h_L} \mid z_L^\alpha h_L^\omega = W_L \right\} \quad (1.27)$$

Solving $z_L^\alpha h_L^\omega = W_L$, we obtain: $Z_L(h_L, W_L) = W_L^{1/\alpha} h_L^{-\omega/\alpha}$ and thus (1.27) can be written as:

$$\Psi_L(x, W_L) = \max_{h_L} \left\{ \frac{w_L - \tau x - W_L^{1/\alpha} h_L^{-\omega/\alpha}}{h_L} \right\} \quad (1.28)$$

Solving this program leads to:

$$h_L^d(x, W_L) = \left(\frac{\alpha + \omega}{\alpha} \right)^{\alpha/\omega} \frac{W_L^{1/\omega}}{(w_L - \tau x)^{\alpha/\omega}} \quad (1.29)$$

Thus, we have:

$$\begin{aligned} z_L^d(x, W_L) &= W_L^{1/\alpha} h_L^{-\omega/\alpha} \\ &= \left(\frac{\alpha}{\alpha + \omega} \right) (w_L - \tau x) \end{aligned} \quad (1.30)$$

Now, by plugging the value of h_L^d from (1.29) into (1.28), we obtain exactly (1.26), demonstrating that the direct and indirect approaches are totally identical.

Finally, let us calculate the bid rent under the Hicksian *indirect* approach. By observing that $Z_L(h_L, W_L) = W_L^{1/\alpha} h_L^{-\omega/\alpha}$, the program (1.16) can be written as:

$$\mathbf{E}(x, W_L) \equiv \min_{h_L} \left\{ W_L^{1/\alpha} h_L^{-\omega/\alpha} + h_L R(x) + \tau x \right\}$$

First order condition gives:

$$h_L^H(R(x), W_L) = \left(\frac{\alpha}{\omega} \right)^{-\alpha/(\alpha+\omega)} W_L^{1/(\alpha+\omega)} R(x)^{-\alpha/(\alpha+\omega)} \quad (1.31)$$

Plugging $h_L^H(R(x), W_L)$ back to the expenditure function, we easily obtain:

$$\begin{aligned} \mathbf{E}(x, W_L) &\equiv \left(\frac{\alpha + \omega}{\omega} \right) h_L^H(R(x), W_L) R(x) + \tau x \\ &= \frac{(\alpha + \omega)}{\omega^{\omega/(\alpha+\omega)} \alpha^{\alpha/(\alpha+\omega)}} W_L^{1/(\alpha+\omega)} R(x)^{\omega/(\alpha+\omega)} + \tau x \end{aligned}$$

By the budget constraint, we know that $\mathbf{E}(x, W_L) = w_L$ and thus we can obtain the bid rent function by solving the following equation:

$$\mathbf{E}(x, W_L) = \frac{(\alpha + \omega)}{\omega^{\omega/(\alpha+\omega)} \alpha^{\alpha/(\alpha+\omega)}} W_L^{1/(\alpha+\omega)} R(x)^{\omega/(\alpha+\omega)} + \tau x = w_L$$

This gives:

$$\Psi_L(x, W_L) = \frac{\omega \alpha^{\alpha/\omega}}{(\alpha + \omega)^{(\alpha+\omega)/\omega}} (w_L - \tau x)^{(\alpha+\omega)/\omega} W_L^{-1/\omega}$$

which is identical to (1.26).

In all these approaches, by differentiating (1.26), we have:

$$\begin{aligned} \frac{\partial \Psi_L(x, W_L)}{\partial x} &= -\tau \left(\frac{\alpha}{\alpha + \omega} \right)^{\alpha/\omega} (w_L - \tau x)^{\alpha/\omega} W_L^{-1/\omega} < 0 \\ \frac{\partial^2 \Psi_L(x, W_L)}{\partial x^2} &= \tau^2 \frac{\alpha^{(\alpha+\omega)/\omega}}{\omega (\alpha + \omega)^{\alpha/\omega}} W_L^{(\alpha-\omega)/\omega} (w_L - \tau x) W_L^{-1/\omega} > 0 \end{aligned}$$

The intuition is as follows. In order to guarantee that all individuals have the same utility level W_L , the bid rent decreases with distance to the city-center (CBD) because it compensates individuals for their increasing commuting costs. This decrease is marginally increasing (convex function) because the further individuals reside from the CBD, the higher the compensation to ensure the same utility. The following figure (Figure A1.4) displays the bid rent.

[Insert Figure A1.4 here]

Finally, let us verify (1.18) to check that the three different approaches yield the same result.

First, for the Marshallian direct approach, let us plug the value of the bid rent function $\Psi_L(x, W_L) = R(x)$ given by (1.26) into (1.24). We obtain:

$$\begin{aligned} h_L^M(R(x), w_L) &= \left(\frac{\omega}{\alpha + \omega} \right) \frac{(w_L - \tau x)}{R(x)} \\ &= \left(\frac{\alpha + \omega}{\alpha} \right)^{\alpha/\omega} \frac{W_L^{1/\omega}}{(w_L - \tau x)^{\alpha/\omega}} \end{aligned}$$

which is exactly the value of $h_L^d(x, W_L)$ given in (1.29). Second, for the Hicksian indirect approach, let us plug the value of the bid rent function $\Psi_L(x, W_L) = R(x)$ given by (1.26) into (1.31). We obtain:

$$\begin{aligned} h_L^H(R(x), W_L) &= \left(\frac{\alpha}{\omega} \right)^{-\alpha/(\alpha+\omega)} W_L^{1/(\alpha+\omega)} R(x)^{-\alpha/(\alpha+\omega)} \\ &= \left(\frac{\alpha + \omega}{\alpha} \right)^{\alpha/\omega} \frac{W_L^{1/\omega}}{(w_L - \tau x)^{\alpha/\omega}} \end{aligned} \quad (1.32)$$

which is exactly the value of h_L^d given in (1.29). As a result, we have shown that

$$\begin{aligned} h_L^*(x, W_L) &\equiv h_L^M(R(x), w_L) = h_L^d(x, W_L) = h_L^H(R(x), W_L) \\ &= \left(\frac{\alpha + \omega}{\alpha} \right)^{\alpha/\omega} \frac{W_L^{1/\omega}}{(w_L - \tau x)^{\alpha/\omega}} \end{aligned}$$

where $R(x) = \Psi_L(x, W_L)$.

We are now able to determine the Alonso-Muth condition (1.9). By differentiating the utility function (1.23) with respect to x , we have:

$$\frac{\partial \Gamma(h_L)}{\partial x} = -\alpha [\tau + h_L R'(x)] [w_L - \tau x - h_L R(x)]^{\alpha-1} h_L^\omega = 0$$

Since $w_L - \tau x - h_L R(x) \neq 0$, $h_L \neq 0$ and $\alpha \neq 0$, this implies that $t+q R'(x) = 0$ or equivalently

$$R'(x) = -\frac{\tau}{h_L^*(x, W_L)}$$

which is the Alonso-Muth condition.

The urban land use equilibrium Using (1.26) and (1.32), the equilibrium conditions (1.19) and (1.20) can be written as:

$$\omega \alpha^{\alpha/\omega} (w_L - \tau x_f^*)^{(\alpha+\omega)/\omega} = (\alpha + \omega)^{(\alpha+\omega)/\omega} W_L^{1/\omega} R_A \quad (1.33)$$

$$w_L^{(\alpha+\omega)/\omega} - (w_L - \tau x_f^*)^{(\alpha+\omega)/\omega} = \omega^{-1} \alpha^{-\alpha/\omega} (\alpha + \omega)^{(\alpha+\omega)/\omega} \tau N W_L^{1/\omega} \quad (1.34)$$

Furthermore, from equation (1.33), we obtain:

$$(w_L - \tau x_f^*)^{(\alpha+\omega)/\omega} = \omega^{-1} \alpha^{-\alpha/\omega} (\alpha + \omega)^{(\alpha+\omega)/\omega} W_L^{1/\omega} R_A \quad (1.35)$$

By plugging this value in (1.34), we get:

$$w_L^{(\alpha+\omega)/\omega} = \omega^{-1} \alpha^{-\alpha/\omega} (\alpha + \omega)^{(\alpha+\omega)/\omega} W_L^{1/\omega} (\tau N + R_A) \quad (1.36)$$

First, let us solve the case of *closed-city with absentee landlords*. From (1.36), we obtain:

$$W_L^* = \frac{\omega^\omega \alpha^\alpha}{(\alpha + \omega)^{(\alpha+\omega)}} \frac{w_L^{(\alpha+\omega)}}{(\tau N + R_A)^\omega} \quad (1.37)$$

Now plugging this value of W_L^* into (1.35), we have:

$$x_f^* = \frac{w_L}{\tau} \left[1 - \left(\frac{R_A}{\tau N + R_A} \right)^{\omega/(\alpha+\omega)} \right] \quad (1.38)$$

which always strictly positive since $1 > \left(\frac{R_A}{\tau N + R_A} \right)^{\omega/(\alpha+\omega)}$. We can now calculate the equilibrium land rent (1.21) by plugging the value of W_L^* in (1.37) into (1.26). We obtain:

$$R^*(x) = \begin{cases} (w_L - \tau x)^{(\alpha+\omega)/\omega} w_L^{-(\alpha+\omega)/\omega} (\tau N + R_A) & \text{for } x \leq x_f^* \\ R_A & \text{for } x > x_f^* \end{cases} \quad (1.39)$$

where x_f^* is defined by (1.38).

Second, let us solve the case of *open-city with absentee landlords*. The great advantage over the closed-city case is that the two equations (1.33) and (1.34) can be solved independently for x_f^* and N^* . Indeed, from (1.36), we obtain:

$$N^* = \frac{\omega \alpha^{\alpha/\omega} (\alpha + \omega)^{-(\alpha+\omega)/\omega} W_L^{-1/\omega} w_L^{(\alpha+\omega)/\omega} - R_A}{\tau} \quad (1.40)$$

while from (1.35), we have:

$$x_f^* = \frac{w_L - \omega^{-\omega/(\alpha+\omega)} \alpha^{-\alpha/(\alpha+\omega)} (\alpha + \omega) W_L^{1/(\alpha+\omega)} R_A^{\omega/(\alpha+\omega)}}{\tau} \quad (1.41)$$

For x_f^* and N^* to be strictly positive, it has to be assumed that R_A is not too high, i.e.

$$R_A < \frac{\omega \alpha^{\alpha/\omega}}{(\alpha + \omega)^{(\alpha+\omega)/\omega}} W_L^{-1/\omega} \min \left\{ w_L^{(\alpha+\omega)/\omega}, w_L \right\}$$

Moreover, we can determine the equilibrium land rent (1.21) by directly using the value of the bid rent in (1.26) (since it is expressed in terms of exogenous values only). We obtain:

$$R^*(x) = \begin{cases} \omega \alpha^{\alpha/\omega} (\alpha + \omega)^{-(\alpha+\omega)/\omega} (w_L - \tau x)^{(\alpha+\omega)/\omega} W_L^{-1/\omega} & \text{for } x \leq x_f^* \\ R_A & \text{for } x > x_f^* \end{cases} \quad (1.42)$$

where x_f^* is defined by (1.41).

Third, we can analyze the case of a *closed-city with resident landlords*. Using (1.38) and (1.39), the total differential rent, TDR^* , (1.22) is equal to:

$$\begin{aligned} TDR^* &= \int_0^{x_f} R^*(x) dx - R_A x_f^* \\ &= \int_0^{x_f} \left[\left(w_L + \frac{TDR}{N} - \tau x \right)^{(\alpha+\omega)/\omega} \left(w_L + \frac{TDR}{N} \right)^{-(\alpha+\omega)/\omega} (\tau N + R_A) \right] dx \\ &\quad - R_A \left(\frac{w_L}{\tau} + \frac{TDR}{\tau N} \right) \left[1 - \left(\frac{R_A}{\tau N + R_A} \right)^{\omega/(\alpha+\omega)} \right] \end{aligned}$$

which is equivalent to:

$$(\alpha + 2\omega) TDR = \left(w_L + \frac{TDR}{N} \right) \left[\omega N - \frac{(\alpha + \omega)}{\tau} R_A \left[1 + \left(\frac{R_A}{\tau N + R_A} \right)^{\omega/(\alpha+\omega)} \right] \right]$$

By solving this equation, we finally obtain:

$$TDR^* = \frac{\left[\omega - \frac{(\alpha+\omega)}{\tau N} R_A \left[1 + \left(\frac{R_A}{\tau N + R_A} \right)^{\omega/(\alpha+\omega)} \right] \right]}{(\alpha + \omega) \left[1 + \frac{R_A}{\tau N} \left[1 + \left(\frac{R_A}{\tau N + R_A} \right)^{\omega/(\alpha+\omega)} \right] \right]} w_L N \quad (1.43)$$

Now we can solve the whole model since it suffices to replace w_L by $w_L + TDR^*/N$ in (1.37), (1.38) and (1.39) to obtain W_L^* , x_f^* and $R^*(x)$.

Fourth and lastly, we can analyze the case of an *open-city with resident landlords*. Using (1.41) and (1.42), the total differential rent, TDR^* , (1.22) is equal to:

$$\begin{aligned} TDR^* &= \int_0^{x_f} R^*(x) dx - R_A x_f^* \\ &= \omega \alpha^{\alpha/\omega} (\alpha + \omega)^{-(\alpha+\omega)/\omega} W_L^{-1/\omega} \int_0^{x_f} (w_L + TDR/N - \tau x)^{(\alpha+\omega)/\omega} dx \\ &\quad - R_A \frac{w_L + TDR/N - \omega^{-\omega/(\alpha+\omega)} \alpha^{-\alpha/(\alpha+\omega)} (\alpha + \omega) W_L^{1/(\alpha+\omega)} R_A^{\omega/(\alpha+\omega)}}{\tau} \end{aligned}$$

which is equivalent to:

$$TDR^* \left(\frac{\tau N + R_A}{N} \right) = (w_L + TDR/N)^{(\alpha+2\omega)/\omega} \frac{\alpha^{\alpha/\omega} \omega^2}{(\alpha + 2\omega) (\alpha + \omega)^{(\alpha+\omega)/\omega}} W_L^{-1/\omega} \\ + \frac{(\alpha + \omega)}{\alpha^{\alpha/(\alpha+\omega)}} W_L^{1/(\alpha+\omega)} R_A^{(\alpha+2\omega)/(\alpha+\omega)} \omega^{-\omega/(\alpha+\omega)} \left[1 - \frac{\omega^{(\alpha-\omega)/(\alpha+\omega)}}{(\alpha + 2\omega)} \right] - w_L R_A$$

It can be shown, under some condition that there is a unique solution in TDR^* to this equation. Again, we can solve the whole model since it suffices to replace w_L by $w_L + TDR^*/N$ in (1.40), (1.41) and (1.42) to obtain N^* , x_f^* and $R^*(x)$.

1.4. Example 2. Exogenous housing consumption

In order to make the algebra simpler, we make the following assumption. All individuals consume the same amount of land, which is normalized to 1. This means that $h_L = 1$.

This assumption implies that the utility function of each individual (1.1) can be rewritten as:

$$\Gamma(z_L, 1) \tag{1.44}$$

and the budget constraint (1.2) as

$$w_L - \tau x = R(x) + z_L \tag{1.45}$$

Now solving (1.45) and replacing z_L into (1.44) yields

$$\Gamma(w_L - \tau x - R(x), 1)$$

For simplicity, we use the following indirect utility function:

$$\Gamma(w_L - \tau x - R(x), 1) \equiv W_L = w_L - \tau x - R(x) \tag{1.46}$$

which is quite intuitive since it expresses a net income, i.e. wage minus commuting costs minus housing cost. In this context, the bid rent (1.10) or (1.11) can be written as:

$$\Psi_L(x, W_L) = w_L - \tau x - W_L \tag{1.47}$$

It is easily checked that

$$\frac{\partial \Psi_L(x, W_L)}{\partial x} < 0 \quad \frac{\partial \Psi_L(x, W_L)}{\partial w_L} > 0 \quad \frac{\partial \Psi_L(x, W_L)}{\partial \tau} < 0 \quad \frac{\partial \Psi_L(x, W_L)}{\partial W_L} < 0$$

Let us determine the urban-land use equilibrium in the *closed-city case with absentee landlords*. Solving (1.19) and (1.20) yields:

$$x_f^* = N \tag{1.48}$$

$$W_L^* = w_L - \tau N - R_A \quad (1.49)$$

with

$$\frac{\partial x_f^*}{\partial N} > 0$$

$$\frac{\partial W_L^*}{\partial w_L} > 0 \quad \frac{\partial W_L^*}{\partial \tau} < 0 \quad \frac{\partial W_L^*}{\partial N} < 0 \quad \frac{\partial W_L^*}{\partial R_A} < 0$$

When wages increase or commuting costs decrease, workers are richer and their utility level increases. Concerning N , the effect is less obvious. When N rises, the city becomes larger (since the city-fringe is equal to N) and workers are, on average, further away from jobs. This means that their commuting costs increase, implying a reduction in their utility level. In this very simple model, larger cities imply lower levels of utility compared to smaller cities. This is because only commuting costs are taken into account. However, it is well known that large cities offer more diversity and more amenities than smaller cities. For example, one can think of the variety of restaurants, the nice theatres, the fine architecture of monuments that is offered by big cities. If we introduce these elements in the model, then there will obviously be a trade-off between commuting costs and amenities so that big cities do not always imply lower utility levels.

We can now calculate the equilibrium land rent in the city. For that we plug the value of (1.49) into (1.47) and easily obtain:

$$R^*(x) = \begin{cases} \tau(N - x) + R_A & \text{if } x \leq N \\ R_A & \text{if } x > N \end{cases} \quad (1.50)$$

A comparative statics analysis shows that, within the city (i.e. $x \leq N$), land rent linearly decreases from the city center ($x = 0$) to the city-fringe ($x_f = N$) at a rate of τ . The interesting result here is that N is positively correlated with $R(x)$, which means that in big cities land prices are higher than in small cities. In this model, the intuition runs as follows. When N increases, the size of the city increases so that everybody is further away from jobs and thus incurs more commuting costs (even though the commuting per unit of distance τ stays the same). Now, in order for all workers to obtain the same utility level, it must be that the land rent has to decrease.

Let us now study the *open-city case with absentee landlords*, that is utility W_L is now given by N is endogenous. Solving equations (1.19) and (1.20) leads to:

$$x_f^* = N^* = \frac{w_L - W_L - R_A}{\tau} \quad (1.51)$$

with

$$\frac{\partial x_f^*}{\partial w_L} = \frac{\partial N^*}{\partial w_L} > 0 \quad \frac{\partial x_f^*}{\partial W_L} = \frac{\partial N^*}{\partial W_L} < 0 \quad \frac{\partial x_f^*}{\partial \tau} = \frac{\partial N^*}{\partial \tau} < 0$$

Therefore, when wages increase and/or commuting costs decrease, the city become larger because more individuals are attracted to the city. However, when the (exogenous) utility level W_L outside the city increases, more workers are induced to stay outside the city and thus the city-size is reduced.

Using (1.47), we can calculate the equilibrium land rent in the open-city case:

$$R^*(x) = \begin{cases} w_L - \tau x - W_L & \text{if } x \leq N \\ R_A & \text{if } x > N \end{cases} \quad (1.52)$$

We have the same type of figure as for the closed city case (linearly decreasing in x with a slope of τ). However, different parameters affect the equilibrium land rent. Higher wages imply higher land rents (the willingness to pay is higher) and higher outside utility implies lower land rents (this is a direct consequence of the definition of the bid rent function where bid rents and utility are negatively correlated).

We now study the case of a *closed-city with resident landlords*. Using (1.48) and (1.50), the total differential rent, TDR^* , (1.22) is equal to:

$$\begin{aligned} TDR^* &= \int_0^{x_f} R^*(x) dx - R_A x_f^* \\ &= \int_0^{x_f} [\tau(N - x) + R_A] dx - R_A N \end{aligned}$$

and thus

$$TDR^* = \frac{\tau N^2}{2} \quad (1.53)$$

In this equilibrium, we thus have:

$$\begin{aligned} W_L^* &= w_L + \frac{TDR^*}{N} - \tau N - R_A \\ &= w_L - \frac{\tau N}{2} - R_A \end{aligned} \quad (1.54)$$

while $x_f^* = N$ and the equilibrium land rent $R^*(x)$ is still given by (1.50).

Finally, in the case of an *open-city with resident landlords*, by using (1.51) and (1.52), we have:

$$\begin{aligned} TDR^* &= \int_0^{x_f} R^*(x) dx - R_A x_f^* \\ &= \int_0^{x_f} [w_L - \tau x - W_L] dx - R_A \frac{w_L - W_L - R_A}{\tau} \end{aligned}$$

and thus

$$TDR^* = \frac{(w_L - W_L - R_A)^2}{2\tau} \quad (1.55)$$

In that case, we have the following equilibrium values:

$$\begin{aligned} x_f^* &= N^* = \frac{w_L + TDR^*/N - W_L - R_A}{\tau} \\ &= (w_L - W_L - R_A) \left(\frac{1}{\tau} + \frac{w_L - W_L - R_A}{2N} \right) \end{aligned} \quad (1.56)$$

$$R^*(x) = \begin{cases} w_L - W_L + \frac{(w_L - W_L - R_A)^2}{2\tau N} - \tau x & \text{if } x \leq N \\ R_A & \text{if } x > N \end{cases} \quad (1.57)$$

It is easy to see that the four different cases are totally equivalent.

2. The basic model with heterogenous agents

Let us relax the assumption that all workers are identical and let us assume that there are two types of workers: Employed workers who earn w_L and unemployed workers whose wage is w_U , with $w_L > w_U$. Furthermore, the employed commute more often to the CBD than the unemployed so that their total costs is τx at distance x from the CBD while for the unemployed they are equal to $s\tau x$, with $0 < s < 1$. There are L employed workers and U unemployed workers. We need to determine where they reside in the city. A standard result (Fujita, 1989) is that steeper bid rents imply locations closer to the CBD. This is because landlords allocate land to the highest bids so that workers with steeper bid rents will bid away the workers with flatter bid rents.

Let us check this rule in our model. Let us assume that land is a normal good, that is

$$\frac{\partial H_k(x, W_k)}{\partial w_k} > 0 \quad (2.1)$$

In other words, when households become richer, they increase their housing consumption. Let us assume that the bid rent of each household intersects at some distance x_b . In this context, if land is a normal good, it must be the case that

$$H_U(x_b, W_U) < H_k(x_L, W_L)$$

This is equivalent to:

$$\frac{s\tau}{H_U(x_b, W_U)} \stackrel{\leq}{\geq} \frac{\tau}{H_L(x_b, W_L)}$$

and thus, using (2.1), this is equivalent to:

$$-\frac{\partial \Psi_U(x_b, W_U)}{\partial x} \begin{matrix} \leq \\ \geq \end{matrix} -\frac{\partial \Psi_L(x_b, W_L)}{\partial x}$$

The intuition of this ambiguity is quite simple. For employed households, there exists *an attraction force* to the CBD because of higher pecuniary commuting costs than the unemployed as well as a *repulsion force* from the CBD because of housing consumption (or lot size) since land is normal good and land price is cheaper at the periphery. In other words, when deciding to locate, employed and unemployed households trade off commuting costs and housing consumption.

Proposition 1. *If land is a normal good and employed workers have higher commuting costs than the unemployed, then the location pattern is ambiguous.*

Imagine that the commuting cost effect dominates lot-size effect, that is for employed workers what matters most is to reduce their commuting costs, then they will locate close to jobs while the unemployed will reside at the periphery of the city. In this case, we have the following definition for a closed-city model.

Definition 3. *An urban equilibrium with full employment and two types of workers is a vector $(W_L^*, W_U^*, x_b^*, x_f^*, R^*(x))$ such that:*

$$\Psi_L(x_b^*, W_L^*) = \Psi_U(x_b^*, W_U^*) \quad (2.2)$$

$$\Psi_U(x_f^*, W_U^*) = R_A \quad (2.3)$$

$$\int_0^{x_b^*} \frac{1}{H_L(x, W_L^*)} dx = L \quad (2.4)$$

$$\int_{x_b^*}^{x_f^*} \frac{1}{H_U(x, W_U^*)} dx = U \quad (2.5)$$

$$R^*(x) = \max \{ \Psi_L(x, W_L^*), \Psi_U(x, W_U^*), 0 \} \quad \text{at each } x \in (0, x_f^*] \quad (2.6)$$

Equation (2.2) says that, in the land market, at the frontier x_b^* , the bid rent offered by the employed is equal to the bid rent offered by the unemployed. Equation (2.3) in turn says that the bid rent of the unemployed must be equal to the agricultural land at the city fringe. Equations (2.4) and (2.5) give the two population constraints. Finally, equation (2.6) defines the equilibrium land rent as the upper envelope of the equilibrium bid rent curves of all workers' types and the agricultural rent line.

2.1. Example 1. Quasi-linear utility function with endogenous housing consumption

Let us now assume quasi-linear preferences. For worker with employment status $k = L, U$, we have therefore:

$$\Omega(h_k, z_k) = z_k + g(h_k) \quad (2.7)$$

where h_k is the *housing* consumption for a worker with employment status $k = L, U$ and $g(\cdot)$ is any increasing function with $g''(\cdot) \leq 0$. For simplicity, we take

$$g(h_k) = \sqrt{h_k}$$

The budget constraints for employed and unemployed workers are respectively given by:

$$h_L R(x) + \tau x + z_L = w_L \quad (2.8)$$

$$h_U R(x) + s\tau x + z_U = w_U \quad (2.9)$$

where, as above, the composite good is taken as the numeraire good with unit price.

Maximizing utility (2.7) subject to (2.8) for the employed and subject to (2.9) yield the following Marshallian *housing (land) demand* for workers of employment status $k = L, U$:

$$h_k^M = \frac{1}{[2R(x)]^2} \quad (2.10)$$

which implies that

$$h_L(x) = h_U(x) = h(x) = \frac{1}{[2R(x)]^2} \quad (2.11)$$

This result (2.11) is due to the nature of the quasi-linear preferences since housing consumption is independent of income and thus employment status. Using (2.7) and (2.10), we can now derive the following indirect utility

$$W_L = w_L - \tau x + \frac{1}{4R(x)} \quad (2.12)$$

for each *employed* worker at x ,

$$W_U = w_U - s\tau x + \frac{1}{4R(x)} \quad (2.13)$$

for each *unemployed* worker at x .

The bid rents of the employed and the unemployed are thus respectively equal to:

$$\Psi_L(x, W_L) = \frac{1}{4(W_L - w_L + \tau x)} \quad (2.14)$$

$$\Psi_U(x, W_U) = \frac{1}{4(W_U - w_U + s\tau x)} \quad (2.15)$$

Plugging (2.14) in (2.10) and (2.15) in (2.10), we obtain the housing consumptions of the employed and unemployed. There are respectively given by:

$$H_L(x, W_L) = 4(W_L - w_L + \tau x)^2 \quad (2.16)$$

$$H_U(x, W_U) = 4(W_U - w_U + s\tau x)^2 \quad (2.17)$$

We have the following straightforward result:

Proposition 2. *With quasi-linear preferences and endogenous housing consumption, the employed reside close to jobs whereas the unemployed live at the periphery of the city.*

Because we assume quasi-linear preferences, as in the previous section, there is only the commuting cost effect and thus the employed locate close to jobs. Interestingly, this result is robust if one uses any quasi-linear utility function in which the non-linearity is on h .

Because housing consumption is endogenous, we cannot as before equate x_b to L and x_f to N , but we have to determine them. We focus on a *closed-city with absentee landlords*. Since density is 1 at each location, we have exactly the same definition as definition 3. Solving the equations (2.2)–(2.6) and, without loss of generality, normalizing R_A to $1/4$, we obtain:

$$\begin{aligned} x_b^* &= \frac{4L}{[1 + 4s\tau(N - L)](1 + 4L\tau)} \\ x_f^* &= 4 \frac{N + 4L\tau(N - L)}{[1 + 4s\tau(N - L)](1 + 4L\tau)} \\ W_L^* &= w_L + \frac{1}{1 + 4\tau[sN + (1 - s)L]} \\ W_U^* &= w_U + 1 - \frac{4s\tau}{1 + 4s\tau(N - L)} \left[\frac{L}{1 + 4\tau[sN + (1 - s)L]} + N - L \right] \\ R^*(x) &= \begin{cases} \frac{1}{4} \left(\frac{1}{1 + 4s\tau(N - L) + 4\tau L} + \tau x \right)^{-1} & \text{for } x \leq x_b^* \\ \frac{1}{4} \left(1 - \frac{4s\tau}{1 + 4s\tau(N - L)} \left[\frac{L}{1 + 4s\tau(N - L) + 4\tau L} + N - L \right] + s\tau x \right)^{-1} & \text{for } x_b^* < x \leq x_f^* \\ \frac{1}{4} & \text{for } x > x_f^* \end{cases} \end{aligned}$$

The same type of computations can be done for an open-city with absentee landlords, a closed-city with resident landlords and an open-city with resident landlords. We leave this as an exercise. Of course, all the results should be equivalent if properly evaluated.

2.2. Example 2. Exogenous housing consumption

Let us assume now that all individuals consume the same amount of land, which is normalized to 1, that is $h_L = h_U = 1$. Assume the same preferences as in (1.46), then the bid rents for the employed and unemployed are respectively given by:

$$\Psi_L(x, W_L) = w_L - \tau x - W_L \quad (2.18)$$

$$\Psi_U(x, W_U) = w_U - s\tau x - W_U \quad (2.19)$$

Then, solving (2.2)-(2.6) for the *closed-city model with absentee landlords*, gives the following equilibrium values:

$$x_b^* = L \quad (2.20)$$

$$x_f^* = N \quad (2.21)$$

$$W_L^* = w_L - \tau L - s\tau (N - L) - R_A \quad (2.22)$$

$$W_U^* = w_U - s\tau N - R_A \quad (2.23)$$

The employment zone (i.e. the residential zone for the employed workers) is thus $(0, L]$ and the unemployment zone (i.e. the residential zone for the unemployed workers) is thus $[L, N]$. By plugging (2.22) and (2.23) into (2.18) and (2.19), we easily obtain the land rent equilibrium $R(x)$. It is given by:

$$R^*(x) = \begin{cases} \tau (L - x) + s\tau (N - L) + R_A & \text{for } 0 \leq x \leq L \\ s\tau (N - x) + R_A & \text{for } L < x \leq N \\ R_A & \text{for } x > N \end{cases} \quad (2.24)$$

Let us analyze the *closed-city with resident landlords case* (the computations of the equilibrium values of an open-city with absentee landlords and with resident landlords are straightforward and are left as an exercise). Using (2.20), (2.21) and (2.24), the total equilibrium land rent TDR^* is given by:

$$\begin{aligned} TDR^* &= \int_0^{x_f^*} R^*(x)dx - R_A x_f^* \\ &= \int_0^L [\tau (L - x) + s\tau (N - L) + R_A] dx \\ &\quad + \int_L^N [s\tau (N - x) + R_A] dx - R_A N \end{aligned}$$

which is equivalent to:

$$\begin{aligned} TDR^* &= \frac{\tau L^2}{2} + \frac{s\tau}{2} (N^2 - L^2) \\ &= \frac{\tau}{2} [(1-s)L^2 + sN^2] \end{aligned} \quad (2.25)$$

The equilibrium values of W_U^* and $R^*(x)$ are still given by (2.23) and (2.24), respectively, while W_L^* is now equal to (the revenue of each employed individual is now $w_L + TDR/N$):

$$W_L^* = w_L - \tau L \left(\frac{2N - L}{2N} \right) - \frac{s\tau}{2N} (N - L)^2 - R_A \quad (2.26)$$

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Figure A1.1. The Marshallian Approach

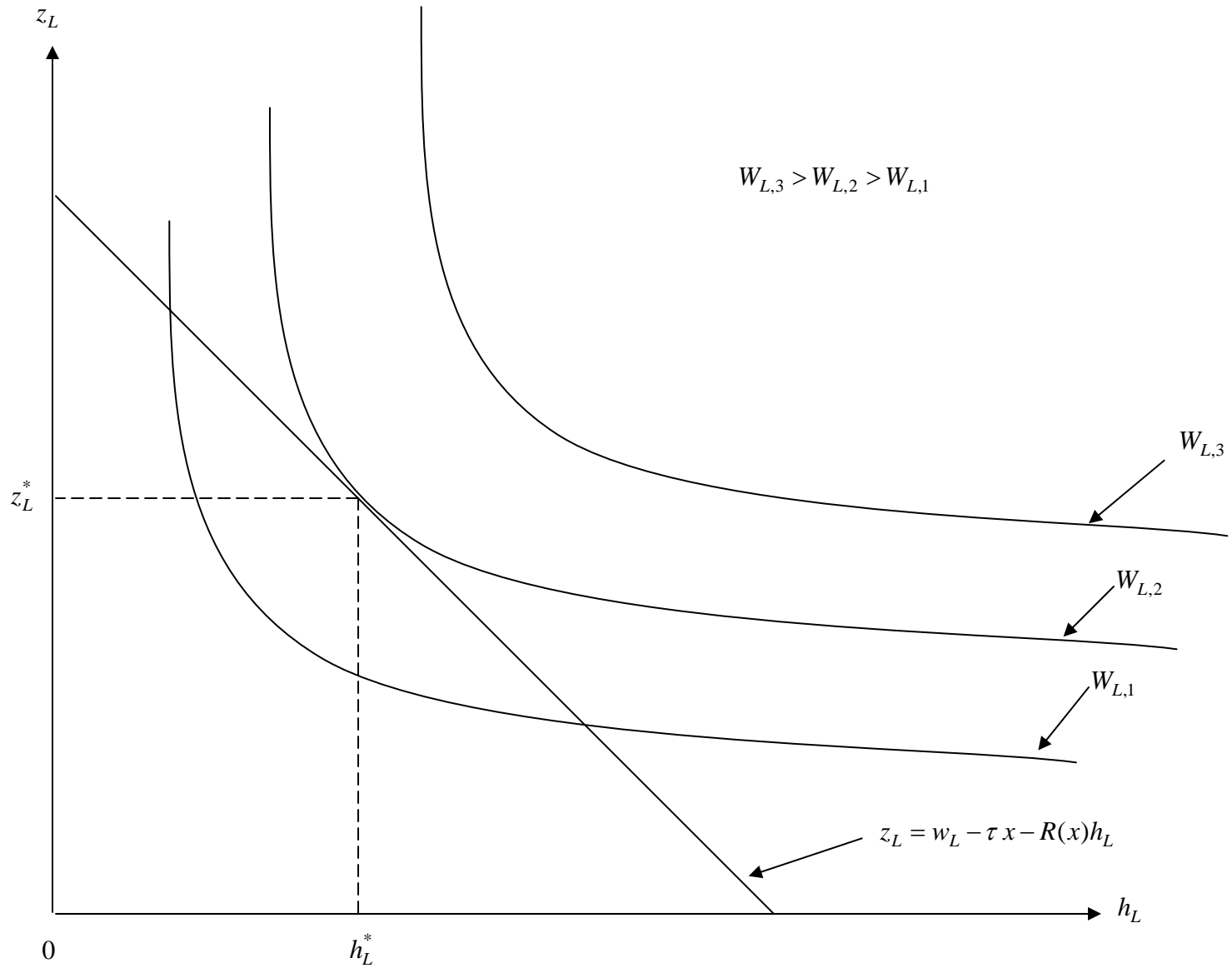


Figure A1.2. The Direct Approach

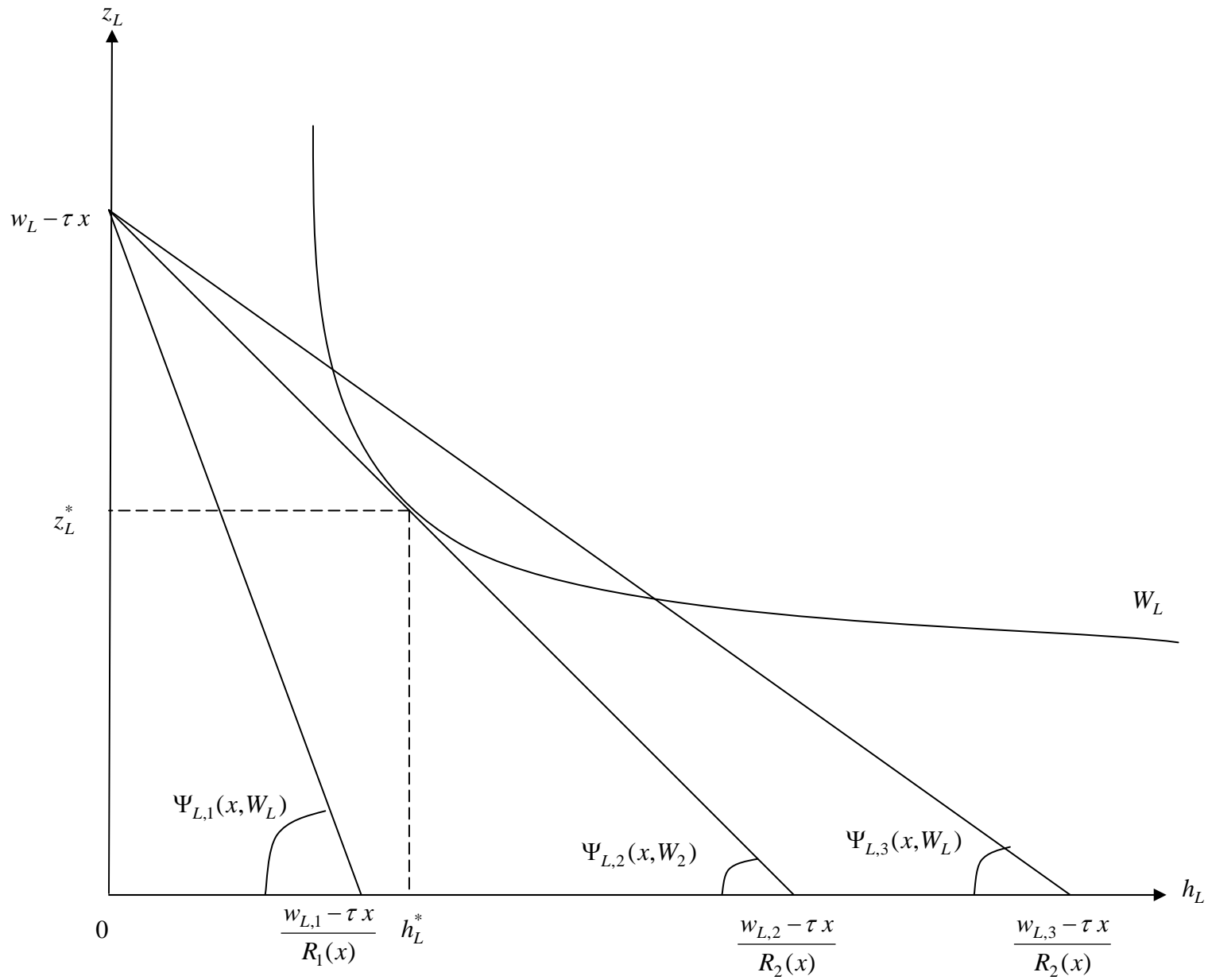


Figure A1.3. The Hicksian Approach

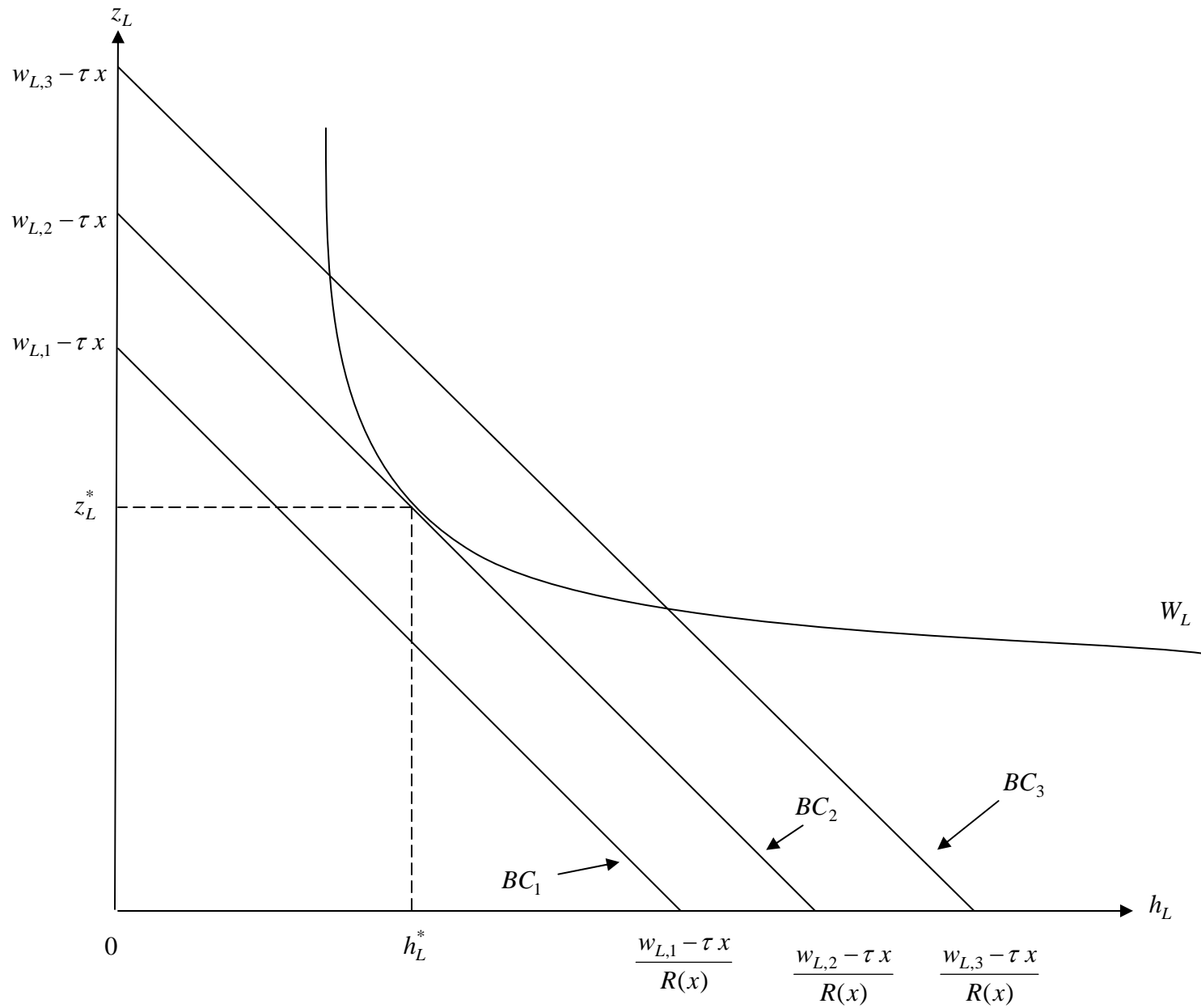


Figure A1.4. The bid rent function in the case of Cobb-Douglas preferences

