Game Theory

The study of multi-person decision problems (few agents).
The study of rational behaviour in situations involving interdependency.
A set of tools to formally describe situations of strategic interaction.

We can use Game Theory to model:

- Trading processes (auctions, bargaining)
- Strategic voting in committees (Bank of England)
- Competition in oligopolistic markets
- Competition/collusion among countries in choosing tariffs/trade policies/environmental standards
- Interaction between monetary authority and unions

What is a Game?

A set of rules specifying:

- Players
- Alternative choices/actions players choose from
- Order of play
- Outcomes and payoffs

Four basic classes of games:

- Non-cooperative vs. cooperative games
- Strategic (or normal form) games vs. extensive form games
- Games with complete vs. incomplete information

- Static vs. dynamic games

Game Theory provides **solution concepts** (notions of equilibrium):

- Dominance
- Nash Equilibrium
- Subgame-perfect Nash Equilibrium
- Bayesian Nash Equilibrium
- Perfect Bayesian Nash Equilibrium

Normal Form Representation of Games

- Players: $I=\{1,2,\ldots,N\}$
- (pure) strategies: $\forall i \in I,\ s_i \in S_i$ (strategy space)
- Players choose their strategies simultaneously (or without knowledge of the others' strategies)
- Payoff functions (vNM utility functions)

$$u_i : S_1 \times S_2 \times \ldots \times S_N \to \Re$$

In complete information games these are *common knowledge* among players.

Def.: ***Strategy profile***: $s=(s_1,\ldots, s_N) \in S_1 \times S_2 \times \ldots \times S_N$

Or alternatively…. $s=(s_i, s_{-i}) \in (S_i , S_{-i})$

…every s induces an **outcome** in the game ($u_i(s)$ for each i).

Example 1: **The Prisonner's Dilemma**

	Not confess	Confess
Not Confess	(-1,-1)	(-9,0)
Confess	(0,-9)	(-6,-6)

Identify the primitives of the game…

$I=\{1,2\}$; $S_1 \times S_2 = \{(NC,NC), (C,C), (C,NC), (NC,C)\}$

$u_i(NC,NC) = -1 \quad i = 1,2$ $\qquad u_i(C,C) = -6 \quad i = 1,2$
$u_1(C,NC) = 0 = u_2(NC,C)$ $\qquad u_1(NC,C) = -9 = u_2(C,NC)$

Solution Concept 1: Dominance

Def. A *strictly dominant strategy* is the best choice for a player *regardless* of what the others are doing.

s_i is a *strictly dominant* strategy for i if for all $s'_i \neq s_i$ and all $s_{-i} \in S_{-i}$:

$$u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i})$$

Def. A strategy is weakly dominant if does as least as well as any other of my strategies against *all* of my opponents' strategies, and it does strictly better for *some* of them.

s_i is a *weakly dominant* strategy for i if for all $s'_i \neq s_i$:

$$u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i}) \text{ for all } s_{-i} \in S_{-i}:$$
$$u_i(s_i, s'_{-i}) > u_i(s'_i, s'_{-i}) \text{ for some } s'_{-i} \in S_{-i}:$$

Dominant Strategy Equilibrium (DSE): a strategy profile is A DSE if each player's strategy is a dominant strategy.

Prisoner's Dilemma: (C,C) is the unique DSE
Pareto efficient outcome: (NC,NC)

Applications of the Prisoner's Dilemma Framework:

Cooperation issues in environmental economics
Price-setting in oligopoly
Free-riding in the provision of public goods
Arms races
Theoretical sociology
…

Solution concept 2: Iterated elimination of <u>strictly</u> dominated strategies

Example 2:

	$L^{(3)}$	M	$R^{(1)}$
U	(1,0)	(1,2)	(0,1)
$D^{(2)}$	(0,3)	(0,1)	(2,0)

Def. A *strategy* is *strictly dominated* for player i if there is at least another strategy he can play that does strictly better *regardless* of what the others are doing.

s_i is *strictly dominated* if there exist an $s'_i \neq s_i$ such that for all $s_{-i} \in S_{-i}$:

$$u_i(s_i, s_{-i}) < u_i(s'_i, s_{-i})$$

A rational player will not play strictly dominated strategies…(U,M) unique equilibrium.

Iterated elimination of weakly dominated strategies…?
…very imprecise solution concept

a) layers of rationality need to be assumed
b) order/speed of deletion matters (multiple outcomes possible)

Example 3 : iterated elimination of weakly dominated strategies
Layers of rationality (Dutta p. 57)

(4,5)	(1,6)	(5,6)
(3,5)	(2,5)	(5,4)
(2,5)	(2,0)	(7,0)

Example 4: iterated elimination of weakly dominated strategies
Order of deletion matters…multiple outcomes (Dutta p.58)

1/2	L	R
T	(0,0)	(0,1)
B	(1,0)	(0,0)

Features of **dominance** as a solution concept:

a) Knowledge of actions taken by players not required.
b) Need to assume that it is common knowledge that players are rational
c) It does not imply Pareto optimality
d) May fail to provide a solution (nonexistence):

Example 6: **The battle of the Sexes**

H/W	Football	Ballet
Football	(2,1)	(0,0)
Ballet	(0,0)	(1,2)

Solution concept 3: Pure Strategies Nash Equilibrium (NE)

Def. In the N-player normal form game $G = \{I, u_1( ),..., u_N( ), S_1,...,S_N\}$, the strategy profile $s^*=(s^*_1,...,s^*_N)$ is a NE if for every $i \in I$:

$$u_i(s^*_i, s^*_{-i}) \geq u_i(s_i, s^*_{-i}) \text{ for all } s_i \in S_i$$

Note:

- s^*_i is player i's best response to the strategies s^*_{-i} played by the N-1 other players.
- s^*_i maximises player i's utility *given* that the remaining N-1 players are playing s^*_{-i}.

In equilibrium every player is happy to play his strategy and has no desire to change it in response to the other players' strategic choices (*strategically stable solution*).

The Prisoner's Dilemma game can be solved using a NE reasoning:

Player 1' s *Best-Response Mapping*:

$BR_1(NC) = C$ $(0 > -1)$
$BR_1(C) = C$ $(-6 > -9)$

Player 2' s *Best-Response Mapping*:

$BR_2(NC) = C$ $(0 > -1)$
$BR_2(C) = C$ $(-6 > -9)$

A pure strategy NE is a *fixed point* under the best-response mapping: the unique NE is (C,C).

Features of NE as a solution concept:

1. Each player is playing a best response given his *belief* about what the other players are playing.
2. Each player's beliefs are *correct* (i.e. consistent with the equilibrium actually being played)
3. It does not necessarily imply Pareto optimality
4. NE is a stronger solution concept than <u>strong IEDS</u>
5. Many NE are possible
6. May fail to provide a solution (nonexistence)

Relation between NE and iterated elimination of strictly dominated strategies (strong IEDS):

If s^* is a NE, then it survives strong IEDS. If strong IEDS eliminates *all but* the strategy profile s^*, then s^* is the unique NE of the game (see Prisoner's Dilemma).

Nevertheless there can be strategy profiles that survive strong IEDS, but are not part of any NE (See Battle of the Sexes: (F,B), (B,F) survive strong IEDS - no dominated strategies!!! - but they are not NE!).

$\Rightarrow$ NE is a stronger solution concept than strong IEDS.

A game can have multiple Nash Equilibria:

Example 6: **The Battle of the Sexes**

H/W	Football	Ballet
Football	(2,1)	(0,0)
Ballet	(0,0)	(1,2)

BR_H(Football) = Football (2 > 0)
BR_H(Ballet) = Ballet (1 > 0)

BR_W(Football) = Football (1 > 0)
BR_W(Ballet) = Ballet (2 > 0)

$\Rightarrow$ 2 NE (Football, Football); (Ballet, Ballet)