“Game Theory"

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Lecture Note 3

- Nash equilibrium
- Mainly pure strategies
- Applications of Nash equilibrium
- Multiple equilibrium
- Focal equilibrium

Nash equilibrium

- Nash equilibrium is an action profile *a** such that;

$$u_i(a^*) \geq u_i(a_i, a_{-i}^*) \ \forall a_i \text{ of player } i, \forall i,$$

where u_i is a payoff function representing player i's preferences.

- In words; Given that all other players play their NE actions, player *i* cannot gain by changing actions. This has to be true for *all* players.
- A NE can be strict or nonstrict.

Nash Equilibrium

- A strategy profile such that
- it's held as a common conjecture (every player thinks that's how the game will be played)
- everyone is playing optimally against what everyone else is doing

How should we think about NE?

- Players’ decisions formed on rationality and beliefs about other players.
- How are beliefs formed?
- Idealized setting (2-player example):
  - Two populations, and each time the game is played one player per population is drawn to play.
  - A NE corresponds to a ”steady state” or ”social norm”.
  - However, each game is considered in isolation…
- Other ways to think about this
  - Rational introspection, learning, … (see section 4.9 in Osborne)
- BUT! The concept of NE tells us nothing about how we attain the particular NE – only that it is not optimal for anyone to deviate given a certain NE action profile.

Solution concept 3: Pure Strategies Nash Equilibrium (NE)

Def. In the N-player normal form game $G = \{I, u_1(\ ),..., u_N(\ ), S_1,...,S_N\}$, the strategy profile $s^*=(s^*_1,...,s^*_N)$ is a NE if for every $i\in I$:

$$u_i(s^*_i, s^*_{-i}) \geq u_i(s_i, s^*_{-i}) \text{ for all } s_i \in S_i$$

Note:

- s^*_i is player i's best response to the strategies s^*_{-i} played by the N-1 other players.
- s^*_i maximises player i's utility *given* that the remaining N-1 players are playing s^*_{-i}.

In equilibrium every player is happy to play his strategy and has no desire to change it in response to the other players' strategic choices (*strategically stable solution*).

The Prisoner's Dilemma game can be solved using a NE reasoning:

Player 1' s *Best-Response Mapping*:

$BR_1(NC) = C$ $(0 > -1)$
$BR_1(C) = C$ $(-6 > -9)$

Player 2' s *Best-Response Mapping*:

$BR_2(NC) = C$ $(0 > -1)$
$BR_2(C) = C$ $(-6 > -9)$

A pure strategy NE is a *fixed point* under the best-response mapping: the unique NE is (C,C).

Features of NE as a solution concept:

1. Each player is playing a best response given his *belief* about what the other players are playing.
2. Each player's beliefs are *correct* (i.e. consistent with the equilibrium actually being played)
3. It does not necessarily imply Pareto optimality
4. NE is a stronger solution concept than <u>strong IEDS</u>
5. Many NE are possible
6. May fail to provide a solution (nonexistence)

Relation between NE and iterated elimination of strictly dominated strategies (strong IEDS):

If s^* is a NE, then it survives strong IEDS. If strong IEDS eliminates *all but* the strategy profile s^*, then s^* is the unique NE of the game (see Prisoner's Dilemma).

Nevertheless there can be strategy profiles that survive strong IEDS, but are not part of any NE (See Battle of the Sexes: (F,B), (B,F) survive strong IEDS - no dominated strategies!!! - but they are not NE!).

$\Rightarrow$ NE is a stronger solution concept than strong IEDS.

A game can have multiple Nash Equilibria:

Example 6: **The Battle of the Sexes**

H/W	Football	Ballet
Football	(2,1)	(0,0)
Ballet	(0,0)	(1,2)

BR_H(Football) = Football (2 > 0)
BR_H(Ballet) = Ballet (1 > 0)

BR_W(Football) = Football (1 > 0)
BR_W(Ballet) = Ballet (2 > 0)

$\Rightarrow$ 2 NE (Football, Football); (Ballet, Ballet)

Application of NE!

PD

		Thief 2	
		Quiet	*Fink*
Thief 1	*Quiet*	-1, -1	-3, 0
	Fink	0, -3	-2,-2

Matching Pennies

		Player 2	
		Head	*Tail*
Player 1	*Head*	1, -1	-1, 1
	Tail	-1, 1	1, -1

Security dilemma

		State 2	
		Disarm	*Arm*
State 1	*Disarm*	3, 3	0, 2
	Arm	2, 0	1, 1

Battle of the sexes

		Partner 2	
		Hockey	*Bach*
Partner 2	*Hockey*	2, 1	0, 0
	Bach	0, 0	1, 2

Application of NE!

Prisoners' Dilemma:
Unique NE (Not optimal)

		Thief 2	
		Quiet	*Fink*
Thief 1	*Quiet*	-1, -1	-3, 0
	Fink	0, -3	**-2,-2**

Application of NE!

Matching Pennies: No Nash equilibrium

		Player 2	
		Head	*Tail*
Player 1	*Head*	$\underline{1}$, -1	-1, $\underline{1}$
	Tail	-1, $\underline{1}$	$\underline{1}$, -1

Application of NE!

Security dilemma: Multiple NE (Coordination games)

		State 1	
		Disarm	*Arm*
State 2	*Disarm*	3, 3	0, 2
	Arm	2, 0	1, 1

Application of NE!

Battle of the sexes: Multiple NE (Coordination games)

		Partner 2	
		Hockey	*Bach*
Partner 2	*Hockey*	2, 1	0, 0
	Bach	0, 0	1, 2

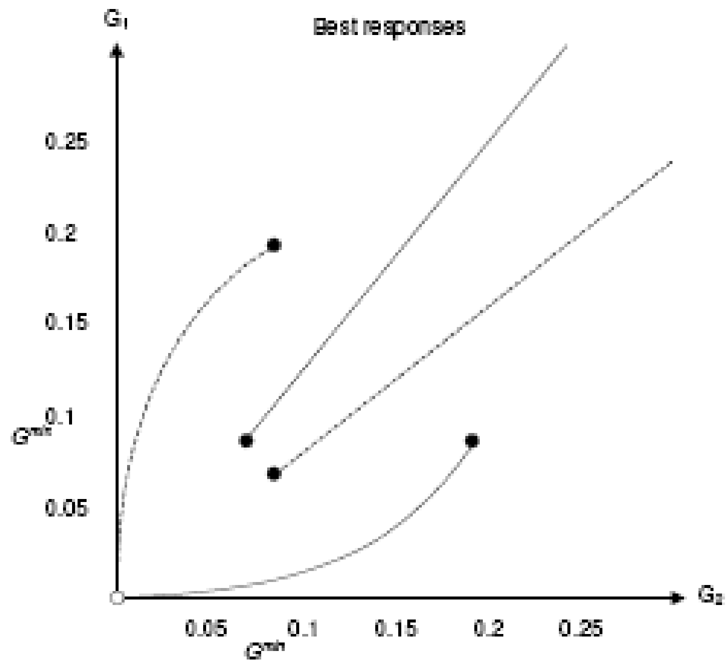
Best response functions

- Useful when there are many actions!

$$B_i(a_{-i}) = \{a_i \text{ in } A_i : u_i(a_i, a_{-i}) \geq u_i(a_i', a_{-i}) \text{ for all } a_i' \text{ in } A_i\}$$

- This is set – valued! There may be many best responses to a given action by your opponents.
- In a NE action profile a^*, a_i^* is in $B_i(a_{-i}^*)$ for every player i.

Sometimes, life is difficult…



Applications of NE: Auctions

Basic setup

- Every buyer knows his/her own valuation of the object being sold.
- These valuations are known by all buyers.
- That is, a game with perfect information.

Second-price sealed-bid auction

- v_i is the valuation of player *i* (=highest price he/she is willing to pay).
- b_i is the (sealed) bid of player *i.*
- *p* = price he/she pays for the object.
- v_i-*p* = payoff of buying.
- Players are numbered according to; $v_1 > v_2 > \ldots > v_n$
- The highest bidder wins the object but pays the second highest bid.
- If there is a tie, the player with the lowest number wins.

The game

- ***Players***: *n* bidders, where n≥2.
- ***Actions***: The set of all possible bids (nonnegative numbers).
- ***Preferences:*** b_i = the bid of player *i* and b^{hat} = is the highest bid submitted by a player other than *i*. If either $b_i > b^{hat}$ *or* $b_i = b^{hat}$ and the number of all other players who bid is greater than *i*, then the payoff is $v_i - b^{hat}$. Otherwise it is 0.

Some NE of Spsba

- $(b_1, \ldots, b_n) = (v_1, \ldots, v_n)$
- $(b_1, \ldots, b_n) = (v_1, 0, \ldots, 0)$
- $(b_1, \ldots, b_n) = (v_2, v_1, 0, \ldots, 0)$
- *Etc…*
- Think about the plausibility of these different equilibria, that is, do they contain weakly dominated actions?
- In spsba, bidding your valuation weakly dominates any other action.

First-price sealed-bid auction

The only difference from the second-price sealed bid auction is that the winner pays his/her own bid.

- ***Players***: *n* bidders, where $n \geq 2$.
- ***Actions***: The set of all possible bids (nonnegative numbers).
- ***Preferences:*** b_i = the bid of player *i* and b^{hat} = is the highest bid submitted by a player other than *i*. If either $b_i > b^{hat}$ *or* $b_i = b^{hat}$ and the number of all other players who bid is greater than *i*, then the payoff is $v_i - b_i$. Otherwise it is 0.

Some NE of Fpsba

- One example is $(b_1, \ldots, b_n) = (v_2, v_2, \ldots, v_n)$
- In every NE, the winner has the highest valuation. Why?
  - *Consider an action profile where player $i \neq 1$, $b_i > b_1$.*
  - *If $b_i > v_2$, then i's payoff is negative. Better to change to $b_i = 0$.*
  - *If $b_i \leq v_2$, player 1 can gain by switching to $b_1 = b_i$.*
  - *Hence, this cannot be a NE.*
- As in the spsba, a bid such that $b_i > v_i$ is weakly dominated by v_i.
- A bid $b_i < v_i$ is not weakly dominated by v_i *(or any other bid).*
- The bid $b_i = v_i$ is weakly dominated by a bid lower than b_i.

Auctions with two players

Basic setup

- There are two buyers, and every buyer knows his/her own valuation of the object being sold.
- These valuations are known by the two buyers.
- That is, a game with perfect information.

Second-price sealed-bid auction

- v_i is the valuation of player i=1,2 (=highest price he/she is willing to pay).
- b_i is the (sealed) bid of player i=1,2.
- p = price he/she pays for the object.
- v_i-p = payoff of buying.
- Players are numbered according to: $v_1 > v_2$
- The highest bidder wins the object but pays the second highest bid.
- If there is a tie, the player with the lowest number (i.e. Player 1) wins.

Question (a)

- Define the game.

The game

- ***Players***: The 2 bidders.
- ***Actions***: The set of all possible bids (nonnegative numbers).
- ***Preferences:*** b_1 = the bid of player *i* and b_2 = is the bid submitted by 2.

If $b_1 \geq b_2$ then the payoff of 1 is $v_1 - b_2$

Otherwise it is 0.

If $b_1 < b_2$ then the payoff of 2 is $v_2 - b_1$

Question (b)

- What are the best reply functions of each player?

Answer Question (b)

- Best-reply function of player 1

- If player 2’s bid b2 is less than v1 then any bid of b2 or more is a best response of player 1 (she wins and pays the price b2).

Answer Question (b)

- If player 2's bid is equal to v1 then every bid of player 1 yields her the payoff zero (either she wins and pays v1, or she loses), so every bid is a best response.

- If player 2's bid b2 exceeds v1 then any bid of less than b2 is a best response of player 1. (If she bids b2 or more she wins, but pays the price b2 > v1, and hence obtains a negative payoff.)

Answer Question (b)

- In summary, player 1's best response function is:

$$BR_1(b_2) = \begin{cases} \{b_1 : b_1 \geq b_2\} & if \quad b_2 < v_1 \\ \{b_1 : b_1 \geq 0\} & if \quad b_2 = v_1 \\ \{b_1 : 0 \leq b_1 < b_2\} & if \quad b_2 > v_1 \end{cases}$$

Figure: BR functions

- Observe

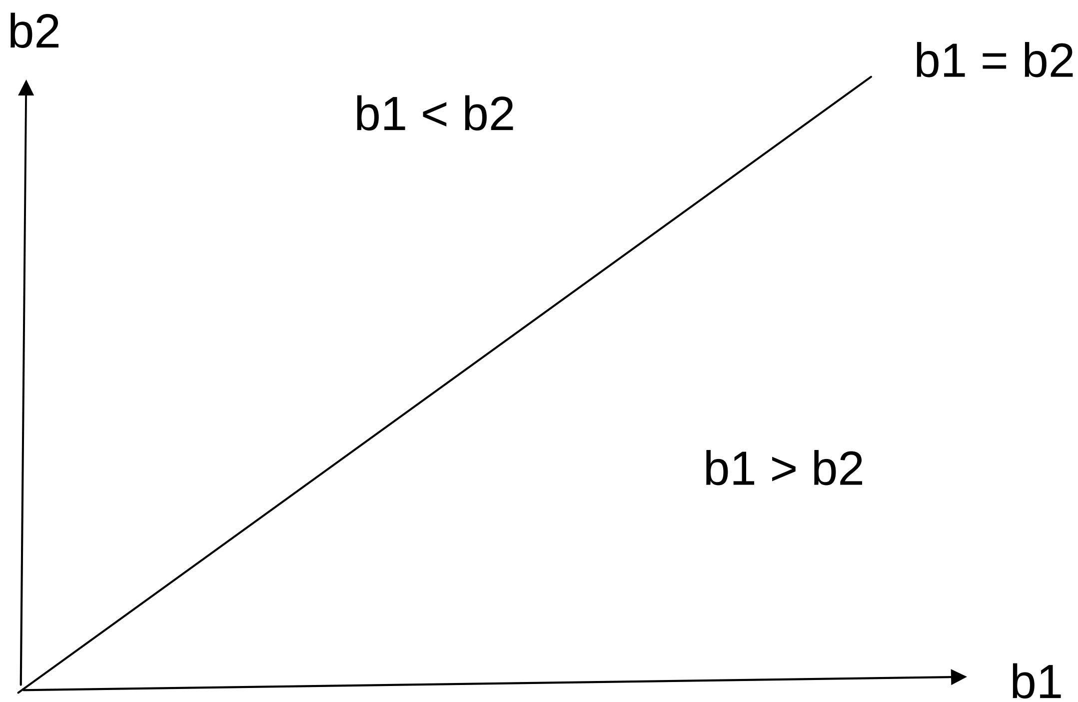
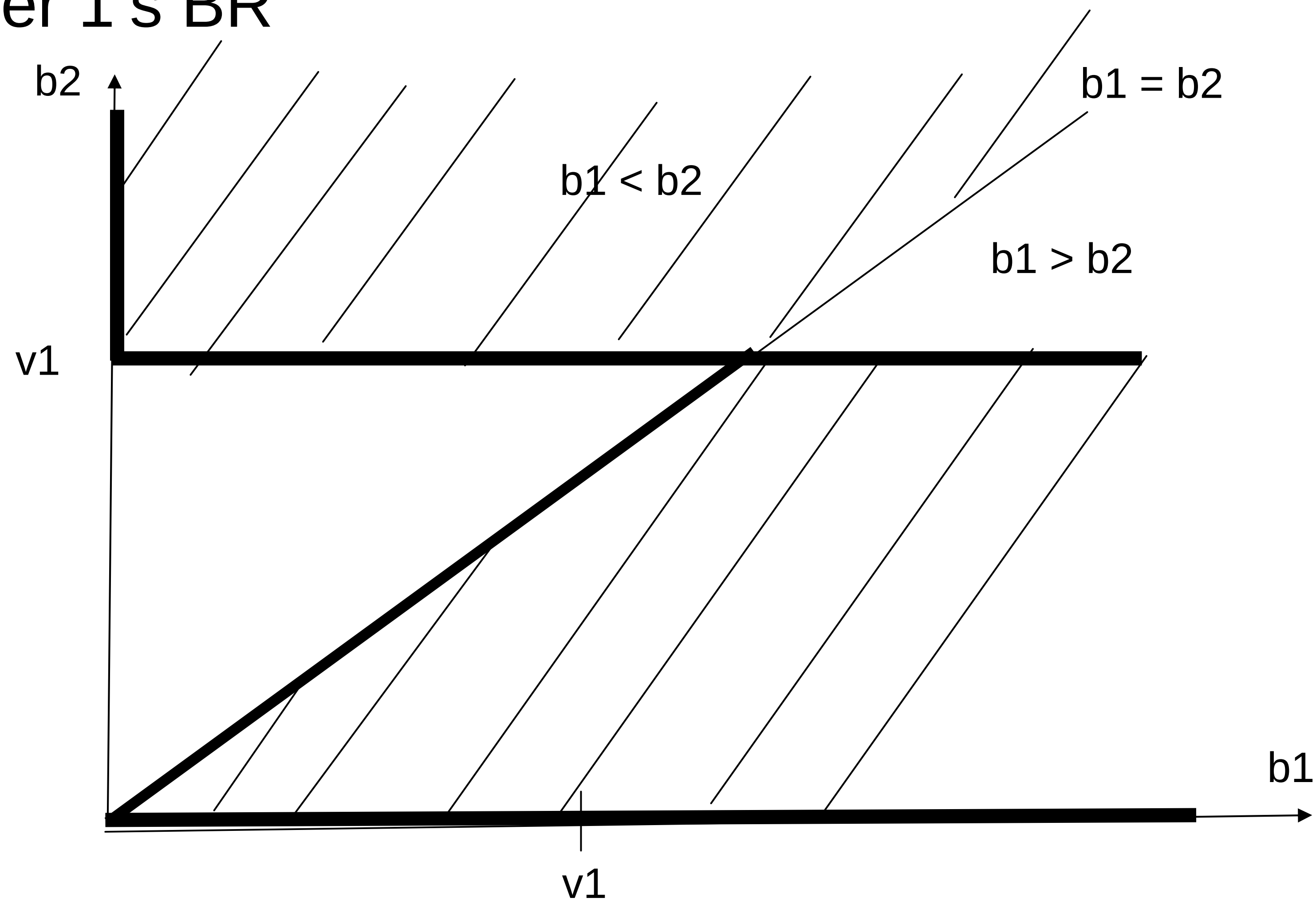


Figure: Player 1's BR function

- Player 1's BR



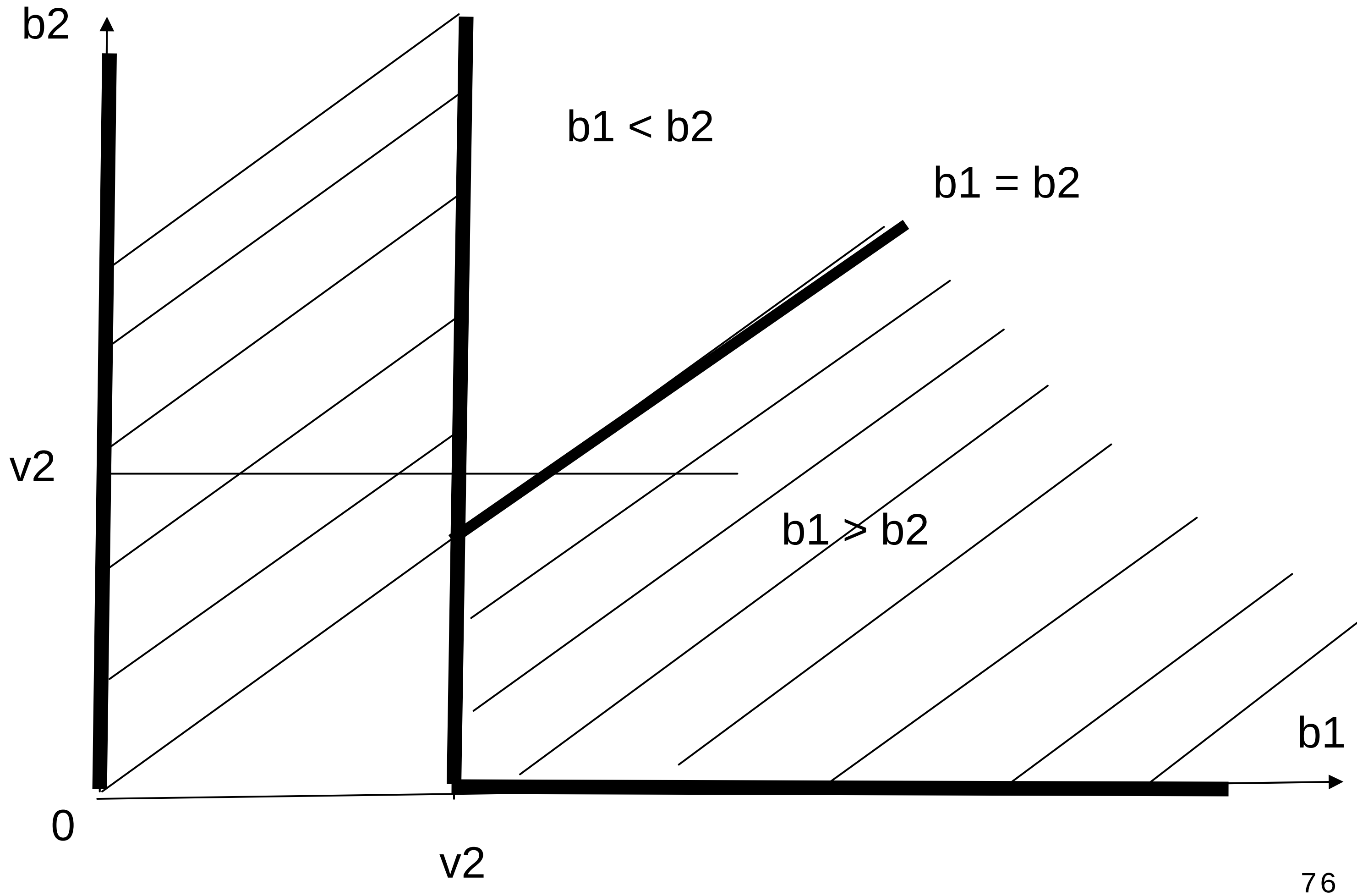
Answer Question (b)

- Similarly, player 2's best response function is:

$$BR_1(b_2) = \begin{cases} \{b_2 : b_2 > b_1\} & if \quad b_1 < v_2 \\ \{b_2 : b_2 \geq 0\} & if \quad b_1 = v_2 \\ \{b_2 : 0 \leq b_2 \leq b_1\} & if \quad b_1 > v_2 \end{cases}$$

Figure: Player 2’s BR function

- Player 2’s BR



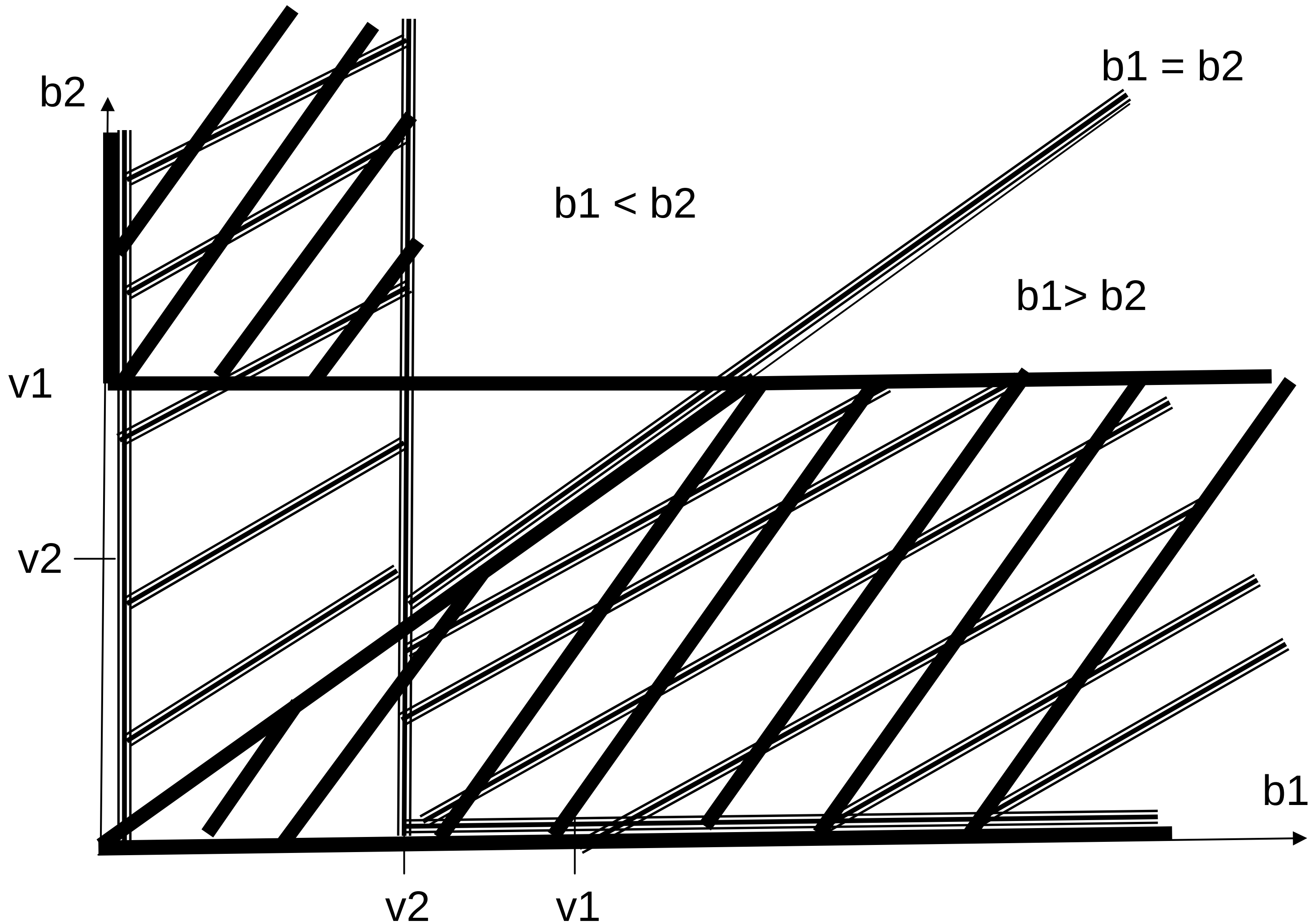
Question (c)

- Calculate all the Nash equilibria in pure strategies?

Answer question (c)

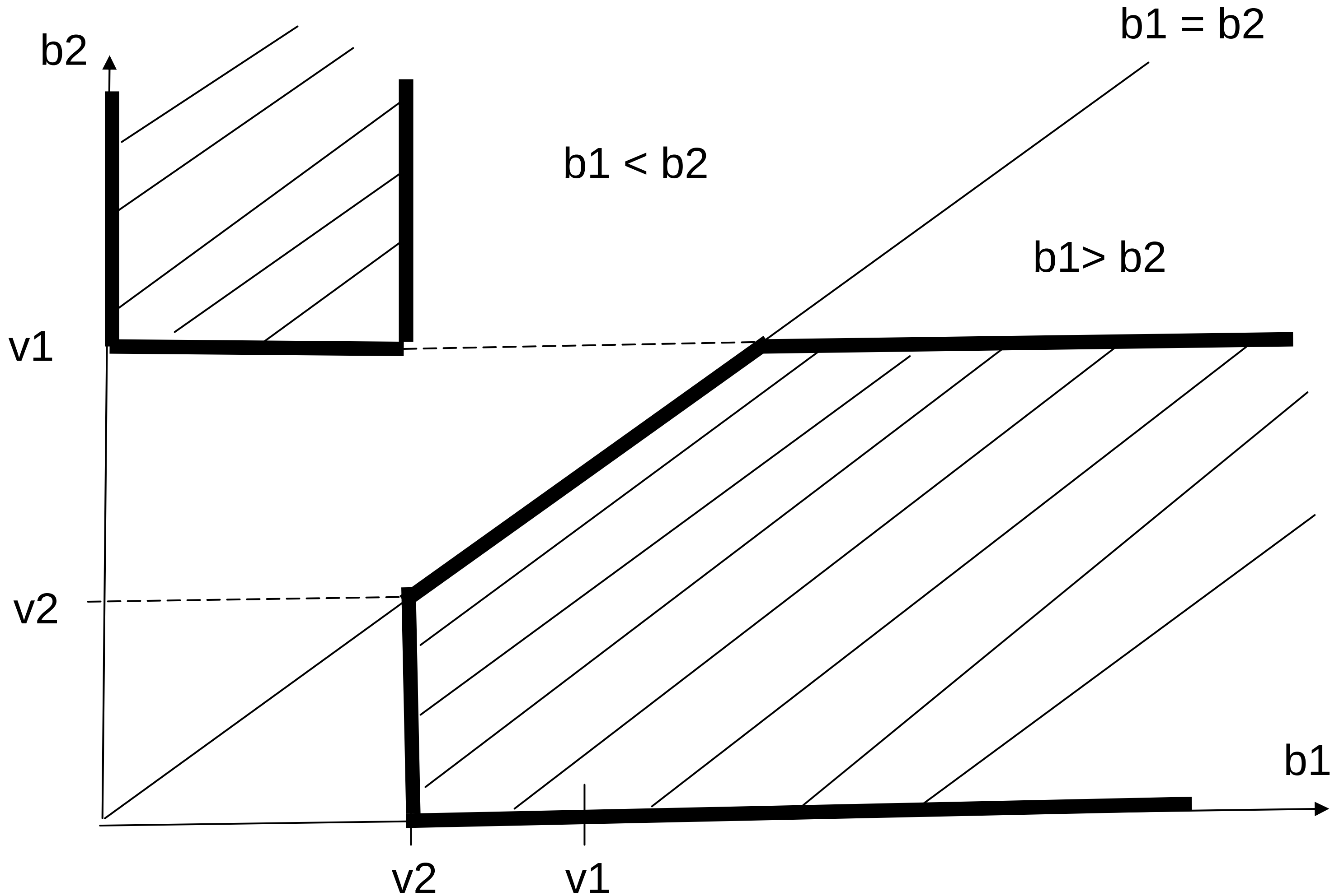
Nash equilibria: Common areas

- NE



Nash equilibria

- NE



First-price sealed-bid auction with 2 players

- v_i is the valuation of player i=1,2 (=highest price he/she is willing to pay) with $v_1 > v_2$
- b_i is the (sealed) bid of player i=1,2.
- p = price he/she pays for the object.
- v_i-p = payoff of buying.
- Players are numbered according to: $v_1 > v_2$
- The highest bidder wins the object and pays her bid.
- If there is a tie, the player with the lowest number (i.e. Player 1) wins.

Question (a)

- Define the game of a first-price sealed-bid auction with two players.

The game

- ***Players***: The 2 bidders.
- ***Actions***: The set of all possible bids (nonnegative numbers).
- ***Preferences:*** b_1 = the bid of player *i* and b_2 = is the bid submitted by 2.

If $b_1 \geq b_2$ then the payoff of 1 is $v_1 - b_1$

Otherwise it is 0.

If $b_1 < b_2$ then the payoff of 2 is $v_2 - b_2$

Question (b)

- What is the Nash equilibrium of this game?

Answer Question (b)

- **First:** A profile of bids in which the two highest bids are not the same is not a Nash equilibrium. Why?

Here is the argument (for *n* players)

- Because the player naming the highest bid can reduce her bid slightly, continue to win, and pay a lower price.

First-price sealed-bid auction

- **First conclusion:** For 2 players, it has to be that, at the NE, $b_1 = b_2$.

Answer Question (b)

- **Second:** In a first-price sealed-bid auction, in all equilibria (even with *n* players), the winner is the player who values the object most highly (player 1). Why?

Here is the argument (for *n* players)

- If there is an equilibrium $(b_1, \ldots, b_n)$ in which some player $i \neq 1$ wins, then $b_i > b_1$.
- But there is *no* equilibrium in which $b_i > b_1$!
- Indeed, if $b_i > v_2$, then *i*'s payoff is negative, so that she can do better by reducing her bid to 0.
- Now, $b_i \leq v_2$, then player 1 can increase her payoff from 0 to $v_1 - b_i$ by bidding b_i, in which case she wins.

First-price sealed-bid auction

- **First and second conclusion (together):** For 2 players, it has to be that, at the NE, $b_1 = b_2$ and player 1 wins.

First-price sealed-bid auction

- What are the values of b_1 and b_2 at the NE?

First-price sealed-bid auction

- **Third:** The highest bid (here b_1) is at least v_2 and at most v_1. Why?

Here is the argument

- If the highest bid is less than v_2, then player 2 can increase her bid to a value between the highest bid and v_2, win, and obtain a positive payoff.
- Thus in an equilibrium the highest bid is at least v_2.

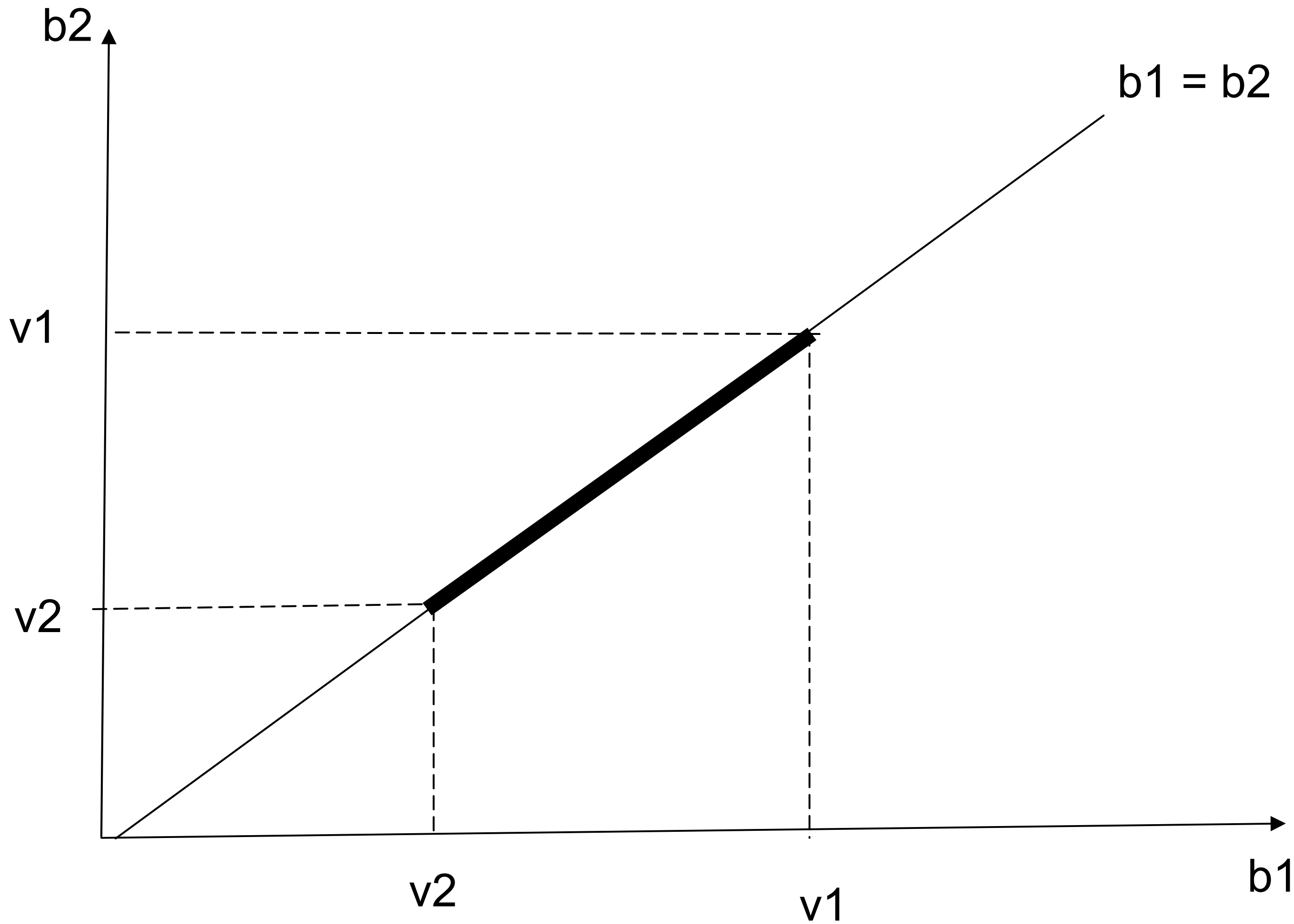
- If the highest bid exceeds v_1, player 1's payoff is negative, and she can increase this payoff by reducing her bid.
- Thus in an equilibrium the highest bid is at most v_1.

Nash Equilibrium in first-price sealed-bid auction

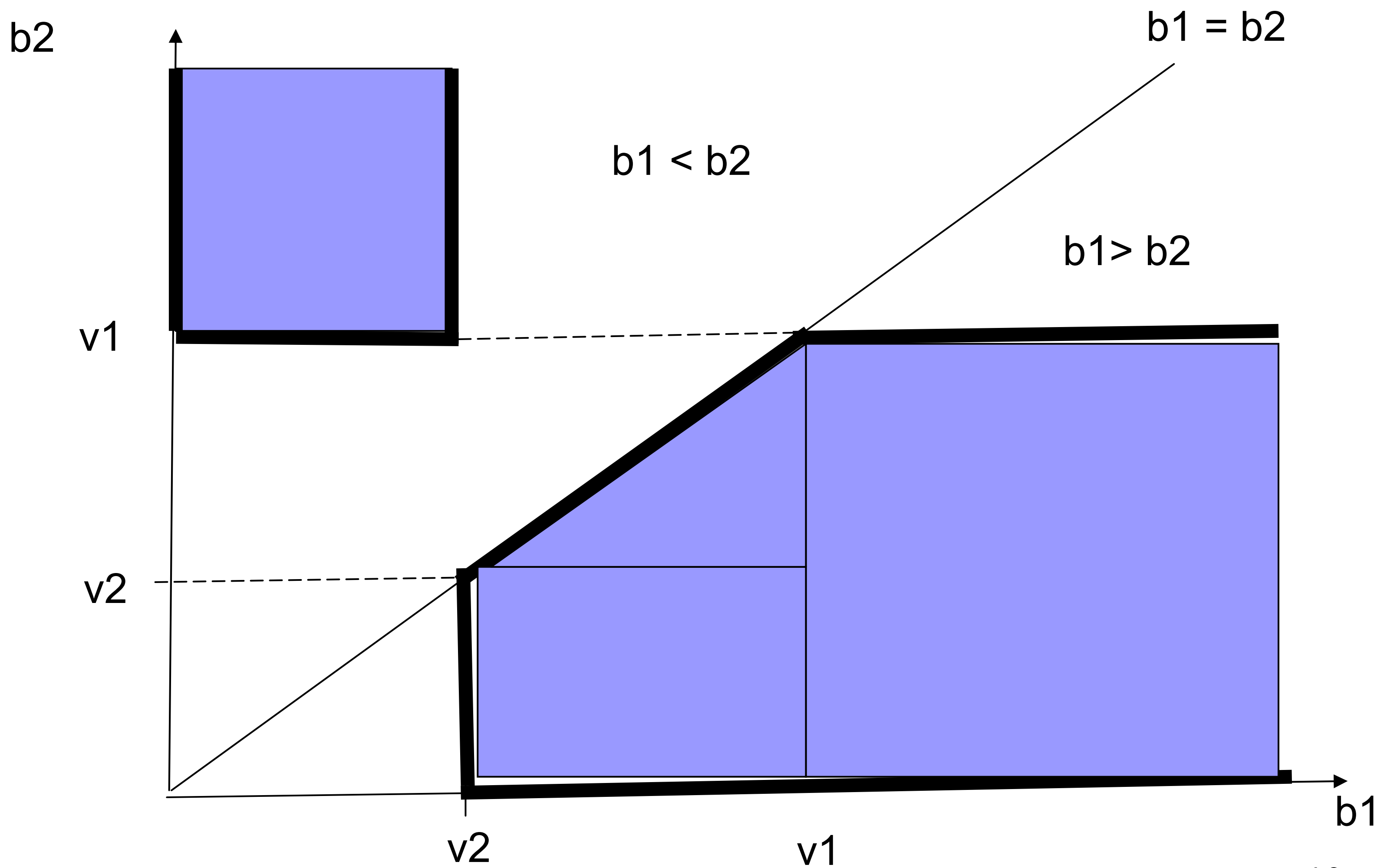
- **Conclusion:**
- In any NE with two bidders, it is has to be that: $v_2 \leq b_1 = b_2 \leq v_1$

Nash equilibria in FPSBA

- NE



Compare Nash equilibria between FPSBA and SPSBA



Nash Equilibrium in first-price sealed-bid auction with *n* players

- In the case of *n* players, in all Nash equilibria, player 1 wins the object and pays the price v_2.

First-price sealed-bid auction with 2 players

- **Exercise:**
- Imagine that we still have $v_1 > v_2$, i.e. player 1 values more the object that player 2 but in case of a tie player 2 wins.
- What are the NE of this game?

- **There is no Nash Equilibrium!**
- Proof: We have seen that:
- In any NE with two bidders, it is has to be that: $v_2 \leq b_1 = b_2 \leq v_1$.
- This cannot be a NE since player 1 loses the auction and will deviate by bidding more than b_2.
- Thus, in any NE, player 1 has to bid more than 2 and it has to be slightly more than v_2, otherwise deviation is profitable from 1.
- But $b_1 = v_2 + \varepsilon$ cannot be a NE since player 1 can always deviate by decreasing b_1.

First-price sealed-bid auction with 2 players

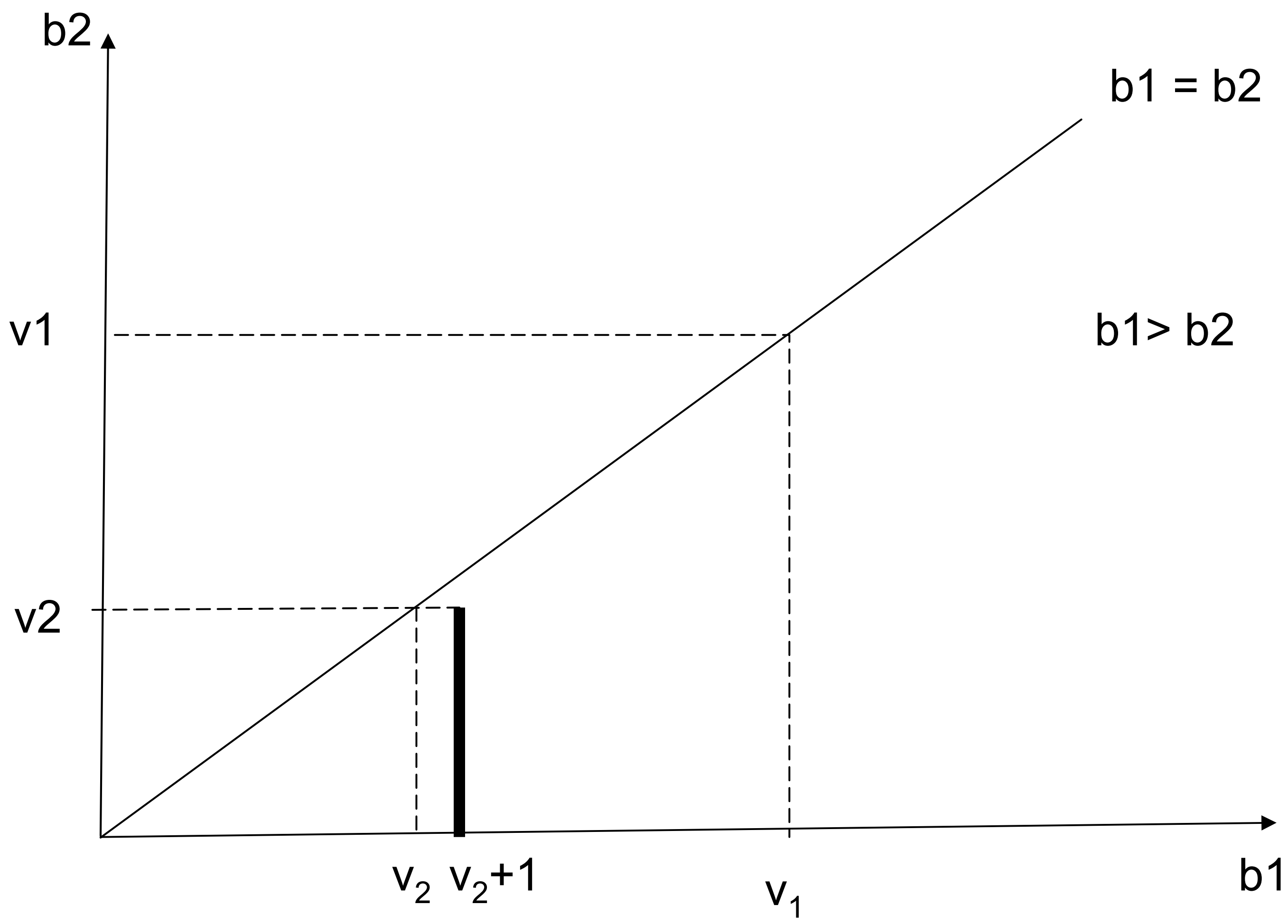
- **Exercise:**
- Imagine that we still have $v_1 > v_2$, i.e. player 1 values more the object that player 2 but in case of a tie player 2 wins.
- Now, bids are discrete numbers 1,2,....,
- What are the NE of this game?

Answer

- There are a lot of NE which are such that:
- $b_1 = v_2 + 1$ and $b_2 \in [0, v_2]$

Nash equilibria in FPSBA

- NE



Example of multiple equilibria (Prisoner's Dilemma)

		FIRM II	
		Invest	Not Invest
FIRM I	Invest	2,2	4,1
	Not Invest	1,4	3,3

- When we solved this by strict dominance, we got the outcome of (Invest, Invest).
- Is this a Nash? Yes—no one can profitably deviate. But is there anything else?
- There is a Pareto optimal outcome at (Not, Not).
- Yes, this is a Pareto improvement, but not a Nash because players can profitably deviate.
- If you solve a game by iterated deletion of **strictly** dominated strategies, the outcome is **always** a Nash Equilibrium.

Example: Multiple equilibria (Game of Chicken)

- Russia and Georgia are deciding if they want to Compromise or Stand Firm.

		Russia	
		Compromise	**Firm**
Georgia	**Compromise**	3,3	2,4
	Firm	4,2	1,1

(Compromise, Firm) and (Firm, Compromise) are both Nash Equilibria. There are two Nash Equilibria.

Example: Divide the dollar

- I have a dollar that we are going to split.
- You write down on a piece of paper how much of the dollar you want.
- I also write down on paper how much of the dollar I want.
- We both look at what we wrote at the same time.
- If the sum is a dollar or less, we each get what we want.
- If sum is more than a dollar, we both get nothing.

Example: Divide the dollar

- What are the possibilities? There are 101 actions.
- You can ask for anything from $0.00 to $1.00.
- Let’s say you can only bid in quarters.
- These are your options: $0.00, 0.25, 0.50, 0.75, 1.00

Example: Divide the dollar

- The strategy space for this game is any amount between $0 and $1.00.
- Is there a Nash Equilibrium for this game?

Example: Divide the dollar

- An **Equilibrium** is a set of strategies.
- An **Equilibrium Outcome** is what would happen if the players follow those strategies.
- An **Equilibrium Payoff** is what each player would get if they followed those strategies.

- For this game, all three happen to be (50,50).

Example: Divide the dollar

- Are there other equilibria in this game?
- Suppose the other player will demand 75 and I know that.
- If I really think he will follow that strategy, I will demand 25.
- So is (75,25) an equilibrium? Yes it is, if I am convinced that the other player will bid 75.

Example: Divide the dollar

- Let's say Player 1 demands x amount. Player 2's best response is 1-x.
- So (x,1-x) is an equilibrium for the game.
- There are infinite many possibilities here.

Conjecture and focal equilibrium

- With each equilibrium, each player is trying to play optimally around a common **conjecture** of the game.
- Difference between dominance arguments and equilibrium strategies.
- Dominance states that players will not play dominated strategies. Equilibrium argues that players will play a strategy following a common conjecture.
- **Focal Equilibrium** is a strategy that draws the attention of all the players, so that it becomes the common conjecture.

Divide the Dollar

- What seems to be the Focal Equilibrium? 50-50?
- What if the game is set up like this: one player is young and one player is old, and the younger person tends to defer to the older person.
- The focal equilibrium in this type of setting may no longer be 50-50.
- Sometimes outside elements such as culture may impact the focal equilibrium of a game.

Example: Paying taxes

- The best outcome is if both players pay taxes and everyone gets the benefit of paid taxes.
- The worst outcome for you is if you paid taxes, but the other person did not.
- The intermediate outcome is if neither pays taxes.

Example: Paying taxes

		Player 2	
		Pay	**Don't Pay**
Player 1	**Pay**	4,4	1,2
	Don't Pay	2,1	3,3

There are two Nash Equilibria in this game.
(Pay, Pay) is a high compliance equilibrium (US, Sweden),
as long as both players believe that the other player is going to pay.

(Don't, Don't) is a low compliance equilibria if both players think that the other will not pay taxes (Italy?).

Sometimes the Nash Equilibria can just depend on what's going on in people's heads.

Example: Paying taxes

- Is one equilibria better than the other?

- We can argue that (Pay, Pay) is a more focal equilibria because it is a Pareto superior outcome—everyone does at least as good or better than the outcome of (Don't, Don't).

Exercise with 3 players: NE?

x3			y3		
	x2	y2		x2	y2
x1	(5,2,3)	(6,1,2)	x1	(1,2,2)	(6,1,1)
y1	(4,5,1)	(8,6,4)	y1	(9,0,0)	(3,2,5)

3 players: Best replies and NE

x3			y3		
	x2	y2		x2	y2
x1	(**5**,**2**,**3**)	(6,1,**2**)	x1	(1,**2**,2)	(**6**,1,1)
y1	(4,5,**1**)	(**8**,**6**,4)	y1	(**9**,0,0)	(3,**2**,**5**)

Exercise with 3 players

- The unique Nash Equilibrium in pure strategies is thus: (x1, x2, x3)

Applications: Strategies as Continuous Variables

Cournot (Duopoly) Competition

The strategies are quantities. Firm 1 and firm 2 *simultaneously* choose their respective output levels, q_i, $i = 1, 2$, from feasible sets $S_i = [0, +\infty)$.

They sell output at the market-clearing price $p(Q)$, where $Q = q_1 + q_2$.

It is assumed that the market-clearing price is linear and given by:

$$p(Q) = \begin{cases} a - b\,Q & \text{if } Q < a/b \\ 0 & \text{otherwise} \end{cases}$$

Firms *simultaneously* choose q_i translate the problem into a normal form game. Let us find the NE.

Definition of the Cournot game:

Players: $I = \{1, 2\}$;

Strategy space for firm $i = 1, 2$: $S_i = [0, +\infty)$;

Strategy for firm $i = 1, 2$: $s_i = q_i$;

Payoff for firm $i = 1, 2$: $u_i(s_i, s_{-i}) = \pi_i(q_i, q_{-i}) = p(Q)q_i - c\, q_i$ (with $c > 0$)

Derive firm i's Best Response (BR) function:

$$\begin{aligned} \max_{q_i} \pi_i(q_i, q_{-i}) &= \max_{q_i} \{p(Q)q_i - c\, q_i\} \\ &= \max_{q_i} \{(a - b\, Q)\, q_i - c\, q_i\} \end{aligned}$$

First-order condition for firm 1 (observe that $q = q_1 + q_2$):

$$\frac{\partial \pi_1(q_1, q_2)}{\partial q_1} = 0$$

We have:

$$\frac{\partial \pi_1(q_1, q_2)}{\partial q_1} = -bq_1 + a - b\ (q_1 + q_2) - c$$

Observe that SOC:

$$\frac{\partial^2 \pi_1(q_1, q_2)}{\partial q_1^2} = -2b < 0$$

Thus BR function for firm 1 is:

$$-bq_1 + a - b\,(q_1 + q_2) - c = 0$$

$$\Leftrightarrow BR_1(q_2) \equiv q_1(q_2) = \frac{(a-c)}{2b} - \frac{q_2}{2}$$

Note: In computing her best response to q_2 , firm 1 ignores the adverse effect that positive variations in q_1 have on firm 2's profits (a larger q_1 lowers the market price $p(q)$).

That is, she ignores the fact that also player 2's profits vary with q_1.

Indeed, $\partial \pi_2(q_1, q_2)/\partial q_1 = -bq_2$!!!

Because of symmetry, we have:

$$BR_2(q_1) \equiv q_2(q_1) = \frac{(a-c)}{2b} - \frac{q_1}{2}$$

If the quantity pair (q_1^*, q_2^*) is to be a NE, then it must satisfy:

$$\begin{cases} q_1^* = BR_1(q_2^*) \\ q_2^* = BR_2(q_1^*) \end{cases}$$

In other words:

$$\begin{cases} q_1^* = \frac{(a-c)}{2b} - \frac{q_2^*}{2} \\ q_2^* = \frac{(a-c)}{2b} - \frac{q_1^*}{2} \end{cases}$$

This is equivalent to:

$$\begin{aligned} q_1^* &= \frac{(a-c)}{2b} - \frac{\left[\frac{(a-c)}{2b} - \frac{q_1^*}{2}\right]}{2} \\ &= \frac{(a-c)}{2b} - \frac{(a-c)}{4b} + \frac{q_1^*}{4} \end{aligned}$$

which means that

$$q_1^* - \frac{q_1^*}{4} = \frac{(a-c)}{4b}$$

$$\Leftrightarrow q_1^* = \frac{(a-c)}{3b}$$

By symmetry:

$$q_2^* = \frac{(a-c)}{3b}$$

This the unique symmetric NE of this game, which is well-defined since:

$$Q^* = q_1^* + q_2^* = \frac{2\,(a-c)}{3b} < \frac{a}{b}$$

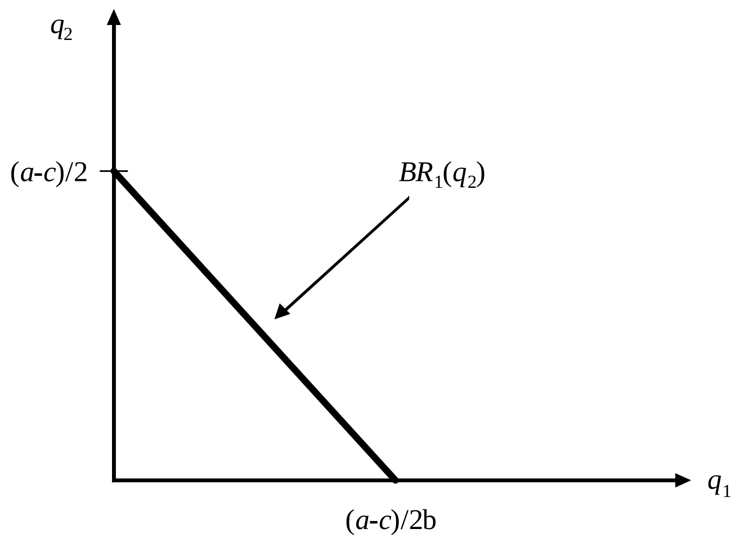
We can plot the BR functions and the NE in the plane (y_1, y_2).

We have for firm 1:

$$BR_1(q_2) \equiv q_1(q_2) = \frac{(a-c)}{2b} - \frac{q_2}{2}$$

which is equivalent to:

$$q_2 = \frac{(a-c)}{2} - 2q_1$$

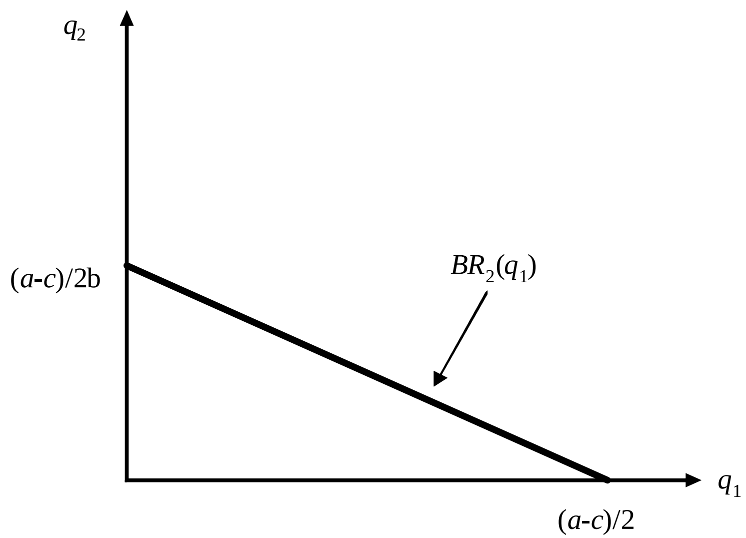


We have for firm 2:

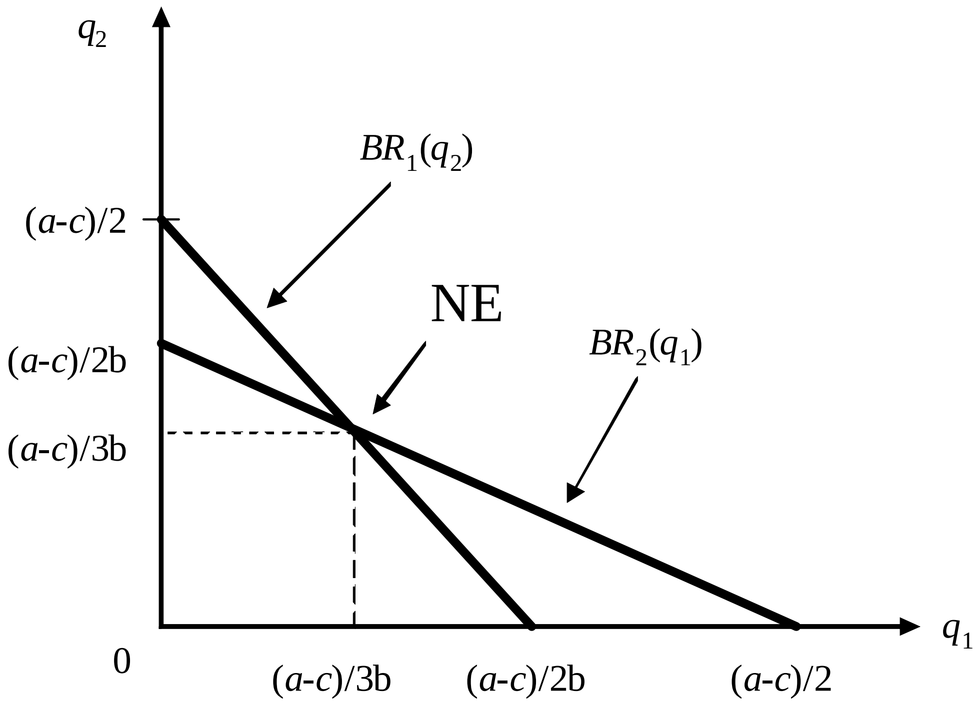
$$BR_2(q_1) \equiv q_2(q_1) = \frac{(a-c)}{2b} - \frac{q_1}{2}$$

that is:

$$q_2 = \frac{(a-c)}{2b} - \frac{q_1}{2}$$



Thus NE is:



q_2
$BR_1(q_2)$
$(a\text{-}c)/2$
NE
$BR_2(q_1)$
$(a\text{-}c)/2b$
$(a\text{-}c)/3b$
q_1
0
$(a\text{-}c)/3b$
$(a\text{-}c)/2b$
$(a\text{-}c)/2$

Equilibrium price and profits:

$$q_1^{NE} = q_2^{NE} = \frac{(a-c)}{3b}$$

$$Q^{NE} = q_1^{NE} + q_2^{NE} = \frac{2\,(a-c)}{3b}$$

$$p(Q^{NE}) = a - bq^{NE} = \frac{a+2c}{3}$$

$$\pi_1^{NE} = \pi_2^{NE} = p(Q^{NE})q_i^{NE} - c\,q_i^{NE} = \frac{(a-c)^2}{9b}$$

Social optimum (Cartel)

$$\max_{q_1,q_2} \{\Pi(q_1, q_2) = \pi_1(q_1, q_2) + \pi_2(q_1, q_2)\}$$

Note: In computing the socially optimum outcome we are taking into account the adverse impact that positive variations in q_1 (q_2), have on the industry profit (i.e. firm 1's + firm 2's profits!!!!). We can anticipate that the socially optimum quantities will be lower than those derived in a Nash Equilibrium!!!!

$$\max_{Q} \{(a - bQ - c)\, Q\}$$

First-order conditions:

$$\frac{\partial \Pi}{\partial Q} = 0$$
$$\frac{\partial \Pi}{\partial Q} = 0$$

That is:

$$-bQ + (a - bQ - c) = 0$$
$$-bQ + (a - bQ - c) = 0$$

which means that

$$Q^{SO} = \frac{(a-c)}{2b}$$

and thus

$$q_1^{SO} = q_2^{SO} = \frac{(a-c)}{4b}$$

Profits:

$$\pi_1^{SO} = \pi_2^{SO} = \left(a - bQ^{SO} - c\right) q_1^{SO} = \frac{(a-c)^2}{8b}$$

Monopoly (one firm):

$$\max_q (a - bq - c)\, q$$

FOC:

$$\frac{\partial \pi(q)}{\partial q} = 0$$

$$\Leftrightarrow -bq + a - bq - c = 0$$

$$\Leftrightarrow q^{MO} = \frac{(a - c)}{2b}$$

Profit:

$$\pi^{MO} = \frac{(a - c)^2}{4b}$$

Thus:

$$Q^{MO} = Q^{SO} < Q^{NE}$$

and

$$\pi^{MO} > \pi_1^{SO} = \pi_2^{SO} > \pi_1^{NE} = \pi_2^{NE}$$

1. $Q^{MO} < Q^{NE}$ ($\pi^{MO} > \pi_1^{NE}$): in a Cournot Nash Equilibrium, each firm ignores the adverse effect its overproduction has on the other firm's profits.

2. The quantity pair (q_1^{SO}, q_2^{SO}) is not a NE: each firm has a private incentive to deviate and produce more since $BR_i(q_1^{SO}) = 3(a-c)/8b > (a-c)/4b = q_1^{SO}$!!!!

Exercise:

Suppose there are n identical firms, which all have a cost function $c_i(q_i) = cq_i$ $(c > 0)$ and compete a la Cournot. The market price is: $p(q) = a - b\,Q$ where $Q = \sum_{j=1}^{j=n} q_j$.

(1) Calculate the symmetric Nash equilibrium of this game (both quantities, price and profit).

Each firm $i = 1, ..., n$ has the following profit

$$\pi_i(q_i, q_{-i}) = p(Q)q_i - c\, q_i$$

FOC:

$$\frac{\partial \pi_i(q_i, q_{-i})}{\partial q_i} = 0 = -bq_i + a - b\, Q - c = 0$$

which is equivalent to:

$$a - b\sum_{j=1}^{j=n} q_j - bq_i - c = 0$$

In a symmetric Nash equilibrium, $q_1^* = q_2^* = ... = q_n^* = q^*$, and we have:

$$a - bnq^* - bq^* - c = 0$$

and thus

$$q^* = \frac{(a-c)}{b\,(n+1)}$$

with

$$\frac{\partial q^*}{\partial n} < 0$$

The market price is thus:

$$p(Q^*) = a - b\,Q^* = a - bn\frac{(a-c)}{b\,(n+1)} = a - \frac{n\,(a-c)}{(n+1)}$$

with

$$\frac{\partial p(Q^*)}{\partial n} = -\frac{(a-c)}{(n+1)^2} < 0$$

The profit of each firm is:

$$\begin{aligned}
\pi^* &= [p(Q^*) - c]\, q^* \\
&= (a - bnq^* - c)\, \frac{(a-c)}{b\,(n+1)} \\
&= \left[a - c - \frac{n\,(a-c)}{(n+1)}\right] \frac{(a-c)}{b\,(n+1)} \\
&= \frac{(a-c)^2}{b\,(n+1)^2}
\end{aligned}$$

with

$$\frac{\partial \pi^*}{\partial n} < 0$$

(2) Compute the limit of the Nash equilibrium (quantities, price and profits) as $n \to +\infty$. Comment.

We have:

$$\lim_{n\to+\infty} q^* = \lim_{n\to+\infty} \frac{(a-c)}{b\,(n+1)} = 0$$

$$\begin{aligned}
\lim_{n\to+\infty} p(Q^*) &= \lim_{n\to+\infty} \left[a - \frac{n\,(a-c)}{(n+1)}\right] \\
&= \lim_{n\to+\infty} \left[a - \frac{(a-c)}{\left(1+\frac{1}{n}\right)}\right] \\
&= \lim_{n\to+\infty} [a-(a-c)] \\
&= c
\end{aligned}$$

$$\lim_{n\to+\infty} \pi^* = \lim_{n\to+\infty} \frac{(a-c)^2}{b\,(n+1)^2} = 0$$

Perfect competition with price equal marginal cost. The Cournot market structure yields approximately the same price, output and profit as the competitive market structure when the number of firms is large.

Exercise.

Suppose now that the n firms are heterogenous in terms of marginal cost so that the total cost of producing q_i for firm i is equal to: $c_i q_i$.

(1) Calculate the *asymmetric* Nash equilibrium of this game (both quantities, price and profit).

Each firm $i = 1, ..., n$ has the following profit

$$\pi_i(q_i, q_{-i}) = p(Q)q_i - c_i\, q_i$$

FOC:

$$\frac{\partial \pi_i(q_i, q_{-i})}{\partial q_i} = 0 = -bq_i + a - b\,Q - c_i = 0$$

which is equivalent to:

$$a - b \sum_{j=1, j\neq i}^{j=n} q_j - 2bq_i - c_i = 0$$

which gives the following BR function for i:

$$BR_i(q_{-i}) = q_i(q_{-i}) = \frac{(a - c_i)}{2b} - \frac{1}{2} \sum_{j=1, j\neq i}^{j=n} q_j$$

which can be written as:

$$a - bq_i - bQ^* - c_i = 0 \quad , \quad i = 1, ..., n$$

In other words:

$$\begin{cases} a - bq_1 - bQ^* - c_1 = 0 \\ a - bq_2 - bQ^* - c_2 = 0 \\ ..... \\ a - bq_{n-1} - bQ^* - c_{n-1} = 0 \\ a - bq_n - bQ^* - c_n = 0 \end{cases}$$

Summing over all q_i, $i = 1, ..., n$, yields:

$$na - bQ^* - nbQ^* - \sum_{j=1}^{j=n} c_j = 0$$

and thus total equilibrium demand is:

$$Q^* = \frac{na}{(n+1)\,b} - \frac{\sum_{j=1}^{j=n} c_j}{(n+1)\,b}$$

Since the FOC for each firm i is:

$$a - bq_i - bQ^* - c_i = 0 \quad , \quad i = 1, ..., n$$

we have for $i = 1, ..., n$:

$$q_i^* = \frac{(a - c_i)}{b} - Q^*$$

$$\Leftrightarrow q_i^* = \frac{(a - c_i)}{b} - \frac{na}{(n+1)\,b} + \frac{\sum_{j=1}^{j=n} c_j}{(n+1)\,b}$$

$$\Leftrightarrow q_i^* = \frac{a - (n+1)\,c_i}{(n+1)\,b} + \frac{\sum_{j=1}^{j=n} c_j}{(n+1)\,b}$$

$$\Leftrightarrow q_i^* = \frac{a - nc_i}{(n+1)\,b} + \frac{\sum_{j=1, j \neq i}^{j=n} c_j}{(n+1)\,b}$$

The equilibrium price is given by:

$$\begin{aligned} p(Q^*) &= a - b\,Q^* \\ &= a - \frac{na}{(n+1)} + \frac{\sum_{j=1}^{j=n} c_j}{(n+1)} \\ &= \frac{a}{(n+1)} + \frac{\sum_{j=1}^{j=n} c_j}{(n+1)} \end{aligned}$$

The equilibrium profit is of firm i:

$$
\begin{aligned}
\pi_i^* &= (p(Q^*) - c_i)\; q_i^* \\
&= \left(\frac{a}{(n+1)} + \frac{\sum\limits_{j=1}^{j=n} c_j}{(n+1)} - c_i \right) \left(\frac{a - nc_i}{(n+1)\,b} + \frac{\sum\limits_{j=1, j\neq i}^{j=n} c_j}{(n+1)\,b} \right) \\
&= \left(\frac{a - nc_i}{(n+1)} + \frac{\sum\limits_{j=1, j\neq i}^{j=n} c_j}{(n+1)} \right) \left(\frac{a - nc_i}{(n+1)\,b} + \frac{\sum\limits_{j=1, j\neq i}^{j=n} c_j}{(n+1)\,b} \right) \\
&= \left(\frac{a - nc_i}{(n+1)} + \frac{\sum\limits_{j=1, j\neq i}^{j=n} c_j}{(n+1)} \right)^2
\end{aligned}
$$

More general results for the non-linear case

Consider the Cournot duopoly (2 firms). Suppose that the market price is now given by

$$p = D(Q)$$

with $p'(Q) \leq 0$ and where $Q = q_1 + q_2$. Suppose that the total cost for i is now: $C(q_i)$, with $C'(q_i) > 0$. Assume that the two firms have the *same cost function*.

Theorem 0.1 *Suppose that $p = D(Q)$ has two continuous derivatives, is nonincreasing, and is concave in the interval $0 \leq Q \leq \widehat{q}$, and suppose that*

$$D(0) > 0 \text{ and } D(Q) = 0 \text{ when } Q \geq \widehat{q}$$

Suppose also that the cost function $C(q_i)$ has two continuous derivatives, is strictly increasing, nonnegative, and convex, and that

$$C'(0) < D(0)$$

There there is one and only one Nash equilibrium given by (q^*, q^*), where $q^* \in [0, \widehat{q}/2]$ is the unique solution of the equation:

$$D(2q^*) + qD'(2q^*) - C'(q^*) = 0$$

Proof of the Theorem

The profit function of each firm i can be written as:

$$\pi_i = D(Q)q_i - C(q_i)$$

First order conditions are:

$$\frac{\partial \pi_1}{\partial q_1} = D(q_1^* + q_2^*) + q_1^* D'(q_1^* + q_2^*) - C'(q_1^*) = 0 \tag{1}$$

$$\frac{\partial \pi_2}{\partial q_2} = D(q_1^* + q_2^*) + q_2^* D'(q_1^* + q_2^*) - C'(q_2^*) = 0 \tag{2}$$

Let's substract these two equations:

$$(q_1^* - q_2^*)\, D'(q_1^* + q_2^*) - \Big[C'(q_1^*) - C'(q_2^*)\Big] = 0 \qquad (3)$$

Let us show that only a symmetric equilibrium can be considered.

We have $C''(0) \geq 0$, meaning that C' is increasing, and $D' < 0$.

If $q_1^* < q_2^*$, then this is impossible since $q_1^* - q_2^* < 0$ and thus total sum in (3) cannot be equal to zero (since, in that case, $C'(q_1^*) - C'(q_2^*) < 0$).

Thus, it has to be that $q_1^* \geq q_2^*$. However, by a similar argument, $\left(q_1^* - q_2^*\right) D'(q_1^* + q_2^*) \leq 0$, and $C'(q_1^*) - C'(q_2^*) > 0$, and the sum in (3) is impossible.

Thus, $q_1^* = q_2^* = q^*$.

Solving (1) and (2) for the symmetric equilibrium leads to:

$$D(2q^*) + q^* D'(2q^*) - C'(q^*) = 0$$

and $0 \leq q^* \leq \widehat{q}$.

Let us show that this NE is unique:

$$\frac{\partial^2 \pi_1}{\partial q_1^2} = 2D'(q_1^* + q_2^*) + q_1^* D''(q_1^* + q_2^*) - C''(q_1^*) < 0$$

since $D' < 0$, $D'' \leq 0$ and $C'' \geq 0$.

Remark: What do we need $C'(0) < D(0)$?

$C'(0) < D(0)$ is called a *boundary condition* and guarantees that an *interior* solution exists:

$$\frac{\partial \pi_1}{\partial q_1} |_{q_1=0, q_2=0} = D(0) - C'(0) > 0$$

In fact, more generally, one needs:

$$D(q) - C'(0) > 0 \text{ for all } q \leq \widehat{q}/2$$

since this also guarantees that:

$$\frac{\partial \pi_1}{\partial q_1} |_{q_1=0, q_2>0} = D(q_2^*) - C'(0) > 0$$

$$\frac{\partial \pi_2}{\partial q_2} |_{q_1>, q_2=0} = D(q_1^*) - C'(0) > 0$$

There is also another boundary condition:

$$\frac{\partial \pi_1}{\partial q_1} \quad | \quad {}_{q_1=\widehat{q}-q_2, 0 \leq q_2 \leq \widehat{q}} = D(\widehat{q}) + (\widehat{q} - q_2^*)\, D'(\widehat{q}) - C'(\widehat{q} - q_2^*)$$
$$= \ (\widehat{q} - q_2^*)\, D'(\widehat{q}) - C'(\widehat{q} - q_2^*)$$

Since $D'(\widehat{q}) \leq 0$ and $\widehat{q} - q_2^* \geq 0$, we need that $C'(\widehat{q} - q_2^*) \geq 0$, which is true by assumption. Thus

$$\frac{\partial \pi_1}{\partial q_1} \,|_{q_1=\widehat{q}-q_2, 0 \leq q_2 \leq \widehat{q}} < 0$$

In the same way, one can show that:

$$\frac{\partial \pi_2}{\partial q_2} \,|_{q_2=\widehat{q}-q_1, 0 \leq q_1 \leq \widehat{q}} < 0$$

To summarize, to have a *unique interior* Nash equilibrium in a Cournot game with two firms and a demand function $D(Q)$ defined in the interval $0 \leq Q \leq \widehat{q}$, with $D(0) > 0$, and $D(Q) = 0$ when $Q \geq \widehat{q}$, we need:

(i) Quasi-concavity of the utility function:

$$\frac{\partial^2 u_i}{q_i^2} \leq 0 \text{ , for } i = 1, 2$$

This guarantees that any interior solution to the first-order solution conditions is a best response.

(ii) Boundary conditions:

$$\frac{\partial \pi_i}{\partial q_i} |_{q_i=0, q_j \geq 0} > 0 \text{ , for } i = 1, 2 \text{ and } i \neq j$$

$$\frac{\partial \pi_i}{\partial q_i} |_{q_i=\widehat{q}-q_j, 0 \leq q_j \leq \widehat{q}} < 0 \text{ , for } i = 1, 2 \text{ and } i \neq j$$

This guarantees that the solution is interior that is player $i = 1, 2$ will always find it optimal to deviate from $q_i = 0$ or $q_i = \widehat{q} - q_j$.

Exercise

Consider a Cournot game where the demand function is:

$$p(Q) = \begin{cases} a - Q & \text{if } Q < a \\ 0 & \text{otherwise} \end{cases}$$

The cost function is $C(q_i)$ for firm $i = 1, 2$. What conditions on $C(.)$ are needed for the second-order conditions and the boundary conditions to be satisfied.

The profit of firm i is

$$\pi_i = p(Q)q_i - C(q_i) = (a - Q)\, q_i - C(q_i)$$

We have for $i \neq j$:

$$\frac{\partial \pi_i(q_i, q_j)}{\partial q_i} = -q_i + a - \left(q_i + q_j\right) - C'(q_i) = 0$$

Thus the second-order condition are satisfied as soon as $C''(q_i) \geq 0$, $\forall q_i \leq a$, $i = 1, 2$ since:

$$\frac{\partial^2 \pi_i(q_i, q_j)}{\partial q_i^2} = -2 - C''(q_i) < 0$$

Boundary conditions:

(1)

$$\frac{\partial \pi_i}{\partial q_i} |_{q_i=0, q_j \geq 0} = a - q_j - C'(0)$$

For this to be strictly positive, it has to be that:

$$a - q_j > C'(0)$$

A sufficient condition is that $C'(0) = 0$.

(2)

$$\frac{\partial \pi_i}{\partial q_i} |_{q_i = a - q_j, 0 \leq q_j \leq \widehat{q}} = q_j - a - C'(a - q_j)$$

For this to be strictly negative, it has to be that:

$$q_j - a < C'(a - q_j)$$

Since $q_j - a \leq 0$, a sufficient condition is that $C'(q) > 0$, $\forall q \leq a$.

To summarize, if the cost function satisfies the following conditions:

$$C'(0) = 0, \ C'(q) > 0, \forall q \leq a, \text{ and } C''(q) \geq 0, \forall q \leq a$$

then there is a unique interior NE.

Observe that for constant marginal cost, $C(q) = cq$, $C'(0) = c > 0$. So the first boundary condition: $a - q_j > C'(0) = c$ needs not hold.

Bertrand game

Imagine now that the firms choose prices instead of quantities.

Two firms simultaneously set a price for a non-differentiated good.

Hence, strategic (continuous) variable is price: p_i.

Cost of production: $c(q_i) = cq_i$.

Define a demand function $D(p)$ rather than an inverse demand function as we did for the Cournot game.

The interpretation of $D(p)$ is that if the good is available at the price p, then the total amount demanded is $D(p)$.

Denote by $P = p_1 + p_2$ the market price. The inverse demand function is:

$$D(p) = \begin{cases} a - p & \text{if } p < a \\ 0 & \text{otherwise} \end{cases}$$

Denote by $\pi_i(p_1, p_2)$ the profit of firm i:

$$\pi_1(p_1, p_2) = \begin{cases} (p_1 - c)(a - p_1) & \text{if} \quad p_1 < p_2 \\ \frac{1}{2}(p_1 - c)(a - p_1) & \text{if} \quad p_1 = p_2 \\ 0 & \text{if} \quad p_1 > p_2 \end{cases}$$

$$\pi_2(p_1, p_2) = \begin{cases} (p_2 - c)(a - p_2) & \text{if} \quad p_2 < p_1 \\ \frac{1}{2}(p_2 - c)(a - p_2) & \text{if} \quad p_1 = p_2 \\ 0 & \text{if} \quad p_2 > p_1 \end{cases}$$

Calculate the unique Nash equilibrium (prices and profits) of this game

Undercutting argument

The unique NE is:

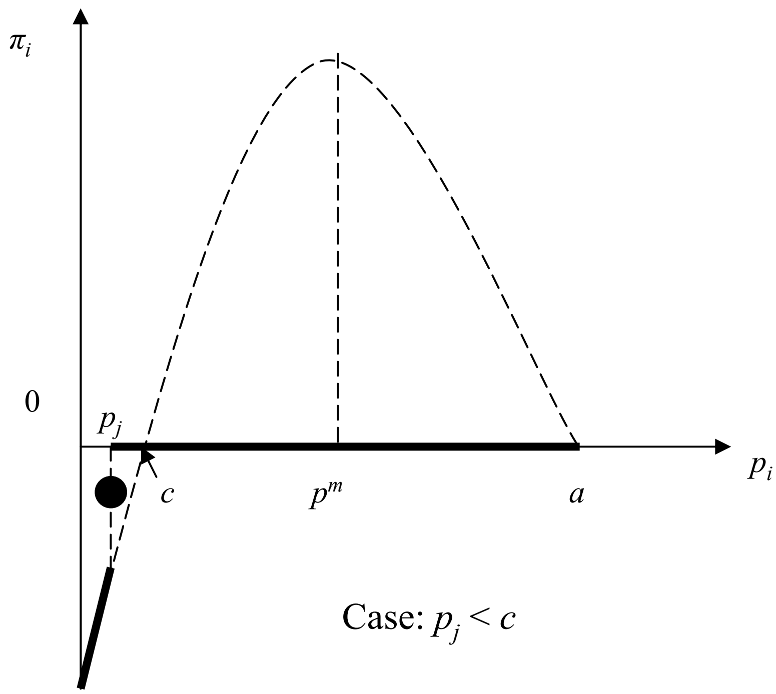
$$p_1^{NE} = p_2^{NE} = c$$

and

$$\pi_1^{NE} = \pi_2^{NE} = 0$$

Let's make these arguments clearer using figures by studying firm i's payoff as of function of its price p_i. Denote by p^m the monopoly price (i.e. the value of price that maximizes $(p-c)\,(a-p)$.

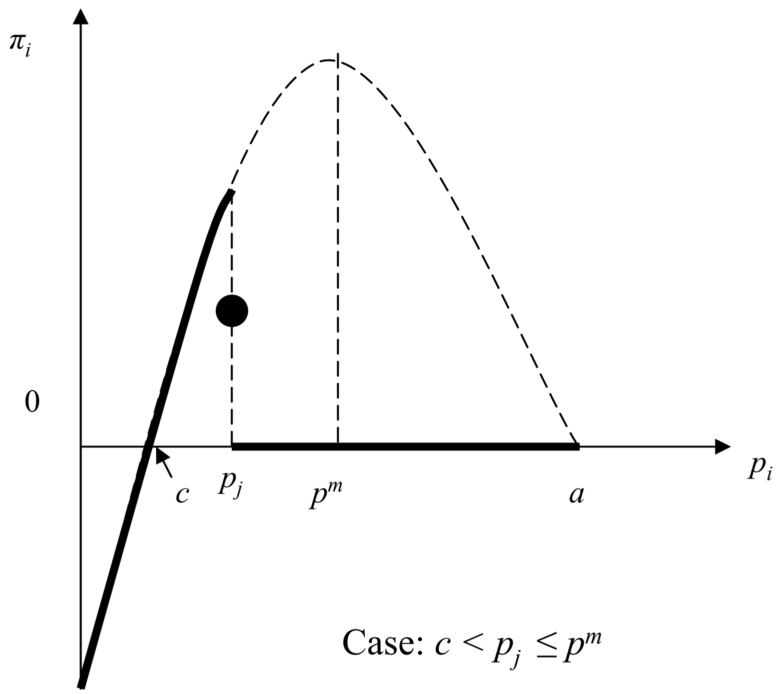
Case 1: $p_j < c$: $BR_i(p_j) = \{p_i : p_i \geq p_j\}$



Case: $p_j < c$

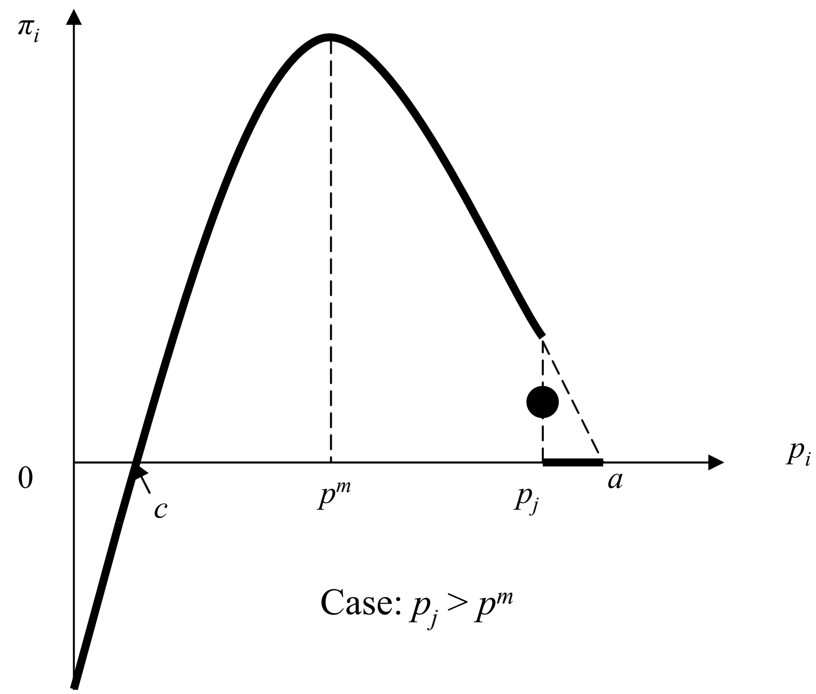
Case 2: $p_j = c$: $BR_i(p_j) = \{p_i : p_i \geq p_j\}$

Case 3: $c < p_j \leq p^m$: $BR_i(p_j) = \varnothing$



Case: $c < p_j \leq p^m$

Case 4: $p_j > p^m$: $BR_i(p_j) = \{p^m\}$



Case: $p_j > p^m$

Differentiated Goods in Bertrand Game

IF firms 1 and 2 choose prices p_1 and p_2, the quantity that consumers demand from firm i is:

$$q_i(p_i, p_j) = a - p_i + bp_j$$

where $b > 0$ reflects the extent to which firm i's product is a *substitute* for firm j's product since

$$\frac{\partial p_i}{\partial p_j} = b > 0$$

The cost of production for firm i is: cq_i.

Determine the NE of this game in prices assuming that $b < 2$ and $c < a$. Determine the equilibrium profits.

Profit of firm i is:

$$\pi_i = (p_i - c)\, q_i(p_i, p_j) = (p_i - c) \Big(a - p_i + bp_j\Big)$$

FOC:

$$\frac{\partial \pi_1}{\partial p_1} = a - p_1 + bp_2 - p_1 + c = 0$$

$$\frac{\partial \pi_2}{\partial p_2} = a - p_2 + bp_1 - p_2 + c = 0$$

and thus the best-reply functions are:

$$p_1^*(p_2) = \frac{a + c + bp_2}{2}$$

$$p_2^*(p_1) = \frac{a + c + bp_1}{2}$$

Solving these two equations lead to:

$$p_1^* = p_2^* = \frac{a + c}{2 - b}$$

which means that $b < 2$ for these prices to be strictly positive.

$$\pi_1^* = \pi_2^* = \left[\frac{a - c(1 - b)}{2 - b}\right]^2$$

Hardin, G. (1968, “The Tragedy of the Commons”, *Science*.

Strategic usage of a commonly owned resource by N (> 1) economic agents.

A “Commons”: a finite, exhaustible resource available to many individuals (environment, international waters. . .).

Tragedy: overuse of the Commons compared to Pareto optimality.

Suppose that we have a common property resource of size $Y > 0$.

Each individual $i = 1, .., N$ can withdraw a nonnegative amount C_i for consumption, provided that $C = \sum_{j=1}^{j=N} C_j \leq Y$.

In the event that they attempt to consume in excess of what is available, we assume that the total amount is simply split between them, that is each individual ends up consuming Y/N.

When total consumption is less than Y, then the leftover amount, $Y - C$, form the resource for the second period.

Period 1: Consumption by agent $i = 1, ..., N$ is: C_i

Thus total consumption in period 1: $C = \sum_{j=1}^{j=N} C_j$

If $C > Y$, then each consumer consumes Y/N;

If $C \leq Y$, then $Y - C$ forms **period 2** resource base

Utility from consumption for individual i: $u_i = \log(C_i) + \log(\text{second period})$

(1) Find the unique symmetric NE of this game.

Assume that player i conjectures that the others consume an amount $\widetilde{C}$ each in period 1, which implies that in period 2, consumption is: $\left[Y - C_j - (N-1)\widetilde{C}\right]/N$

Period 1: Each agent i maximizes:

$$\max_{C_i}\left\{\log C_i + \log\left[\frac{Y - C_i - (N-1)\widetilde{C}}{N}\right]\right\}$$

FOC:

$$\frac{1}{C_i} - \frac{1}{Y - C_i - (N-1)\widetilde{C}} = 0$$

which is equivalent to:

$$C_i = \frac{Y - (N-1)\widetilde{C}}{2}$$

Symmetric equilibrium: $C_1^* = ... = C_n^* = C^{NE}$. Thus,

$$C^{NE} = \frac{Y}{N+1}$$

and utility is for each i:

$$\begin{aligned} u^{NE} &= \log\left(\frac{Y}{N+1}\right) + \log\left(\frac{Y - \frac{Y}{(N+1)}N}{N}\right) \\ &= \log\left(\frac{Y}{N+1}\right) + \log\left[\frac{Y}{N(N+1)}\right] \\ &= \log\left[\frac{Y^2}{N\,(N+1)^2}\right] \end{aligned}$$

(2) Calculate the social optimum of this game.

The planner maximizes the sum of utilities, i.e.

$$\max_{C_1,\ldots,C_n} \left\{ \log C_1 + \ldots + \log C_n + N \log \left(\frac{Y - \sum_{j=1}^{j=N} C_j}{N} \right) \right\}$$

FOC for i:

$$\frac{1}{C_i} - \frac{N}{Y - \sum_{j=1}^{j=N} C_j} = 0$$

that is:

$$Y - \sum_{j=1}^{j=N} C_j = N C_i$$

Symmetry: $C_1^* = ... = C_n^* = C^{SO}$. Thus,

$$Y - NC^{SO} = NC^{SO}$$

Therefore:

$$C^{SO} = \frac{Y}{2N}$$

and for each firm

$$\begin{aligned} u^{SO} &= \log C^{SO} + \log\left(\frac{Y - NC^{SO}}{N}\right) \\ &= \log\left(\frac{Y}{2N}\right) + \log\left(\frac{Y}{2N}\right) \\ &= \log\left(\frac{Y^2}{4N^2}\right) \end{aligned}$$

Tragedy since:

$$Y/(N+1) = C^{NE} > C^{SO} = Y/2N$$

i.e. overuse of Y in the NE.

Also

$$u^{SO} > u^{NE} \text{ for } N \geq 2$$

Remark: As N becomes large, a vanishingly small amount of the resource reaches the second period.