“Game Theory"

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Lecture Note 5

- What is an extensive game?
- Strategies
- Backward induction
- NE and SPNE

What is an extensive game?

- In a game on strategic form, players picked their actions simultaneously and ”forever”.
- Now we consider the timing of decisions.
- We assume perfect information, that is, players know the components of the game, including its history (that is, previous actions).

Definition

- To describe an extensive game with perfect information, we need (as for a strategic game) to specify:
- The set of players
- Their preferences.
- **In addition**, we need to specify:
- The **order** of the players' moves and the **actions** each player may take at each point.

- How do we do that?

- Need to specify the set of all **sequences** of actions that can possibly occur, together with the player who moves at each point in each sequence.
- We refer to each possible sequence of actions as a ***terminal history***
- and to the function that gives the player who moves at each point in each terminal history as the ***player function***.

Some definitions

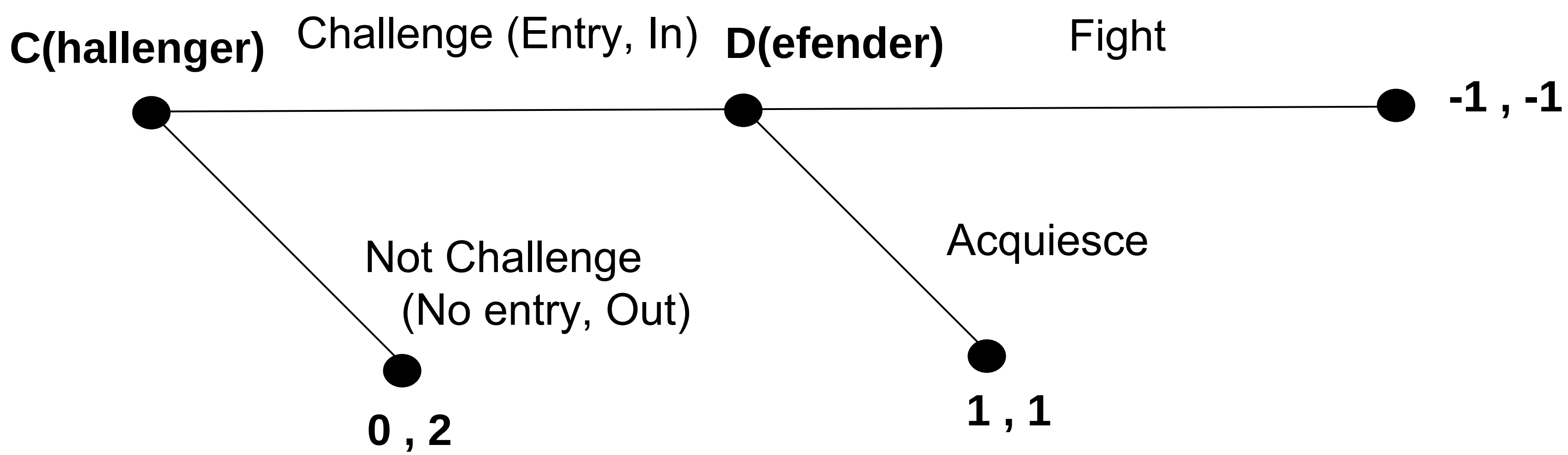
- The **subhistories** of a finite sequence $(a^1, a^2, \ldots, a^k)$ of actions are the empty sequence consisting of no actions, denoted $\varnothing$ (the **empty history**, representing the start of the game), and all sequences of the form $(a^1, a^2, \ldots, a^m)$ where $1 \le m \le k$.

- Similarly, the **subhistories** of an infinite sequence $(a^1, a^2, \ldots)$ of actions are the empty sequence $\varnothing$, every sequence of the form $(a^1, a^2, \ldots, a^m)$ where m is a positive integer, and the entire sequence $(a^1, a^2, \ldots)$.

- A subhistory not equal to the entire sequence is called a **proper subhistory**.
- A sequence of actions that is a subhistory of some terminal history is called simply a **history**.

Some definitions: An example

- <u>Example: Basic Threat Game</u>



Example: Basic Threat Game

- Subhistories of (*In*, *Acquiece*) are:
- (i) Empty history ∅ and
- (ii) the sequences *In* and (*In*, *Acquiece*).

- The proper subhistories are the empty history ∅ and the sequence *In*.

Definition (Extensive game with perfect information)

- An extensive game with perfect information consists of:
- A set of **players;**
- A set of sequences (**terminal histories**) with the property that no sequence is a proper subhistory of any other sequence;
- A function (the **player function)** assigning a player to each sequence that is a proper subhistory of a terminal history
- For each player, **preferences** over the set of terminal histories.

Definition: Basic Threat Game

- ***Players***: The challenger and the incumbent (defender).
- ***Terminal histories***: (*In, Acquiesce*), (*In, Fight*), and *Out*.
- ***Player function***: $P(\varnothing)$ = *Challenger* and $P(In)$ = *Incumbent*.
- ***Preferences***: The challenger's preferences are:

  $u_1(In, Acquiesce) = 1$, $u_1(Out) = 0$, and

  $u_1(In, Fight) = -1$.

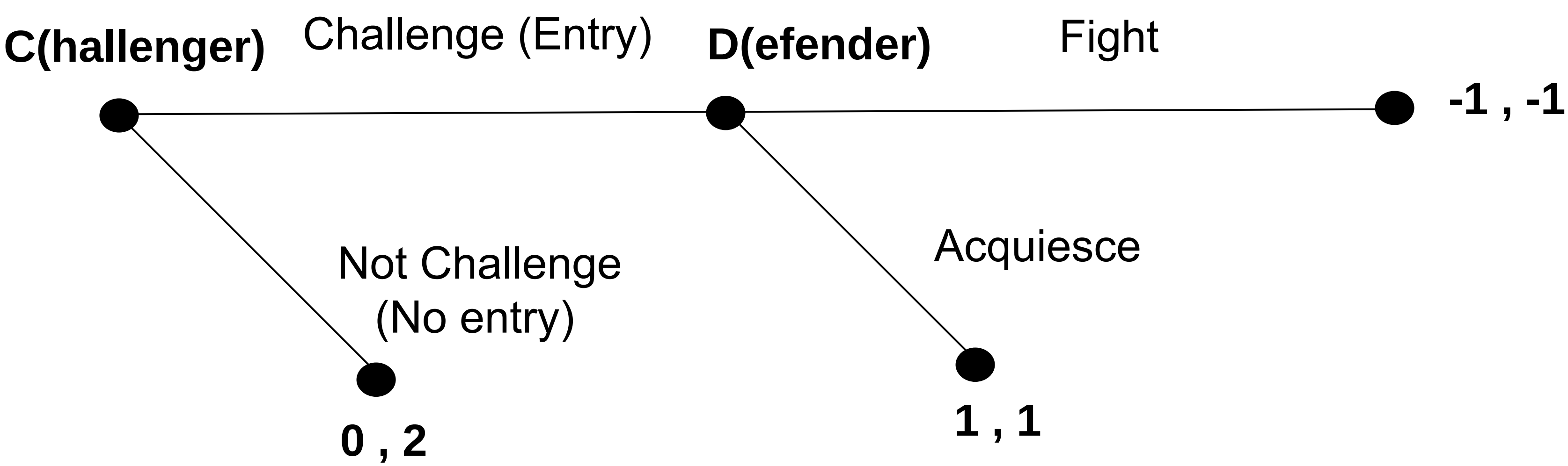
  The incumbent's preferences are: $u_2(Out) = 2$, $u_2(In, Acquiesce) = 1$, and $u_2(In, Fight) = -1$.

- We also need to specify the sets of actions available to the players at their various moves.
- If, for some non-terminal history h, the sequence (h, a) is a history, then a is one of the actions available to the player who moves after h.
- Thus the set of all actions available to the player who moves after h is:

$$A(h) = \{a: (h,a) \text{ is a history}\}$$

Example: Basic Threat Game

- The histories are $\emptyset$, *In*, *Out*, (*In, Acquiesce*), and (*In, Fight*).
- Thus, $A(\emptyset) = \{In, Out\}$ and $A(In) = \{Acquiesce, Fight\}$



Strategies

- Definition : A **strategy** of player *i* in an extensive game with perfect information is a function that assigns to each history *h* after which it is player *i*'s turn to move

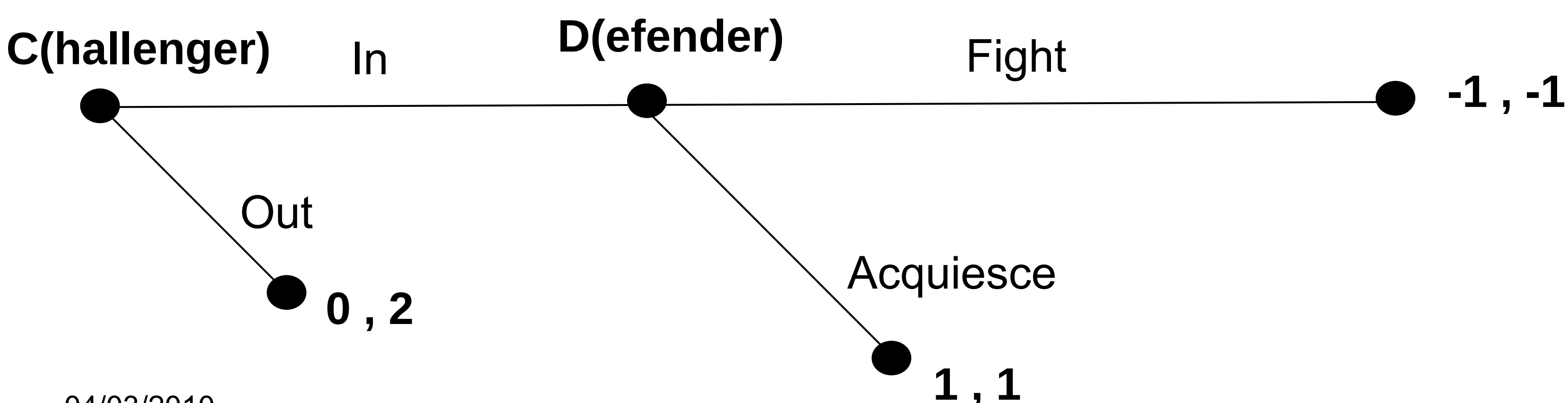
  (i.e. $P(h) = i$, where P is the player function) an action in $A(h)$ (the set of actions available after *h*).

Strategies

- That is, a strategy is a ***complete*** plan that tells the player what to do at ***all*** of the points in the game where he/she might be called upon to make a decision.

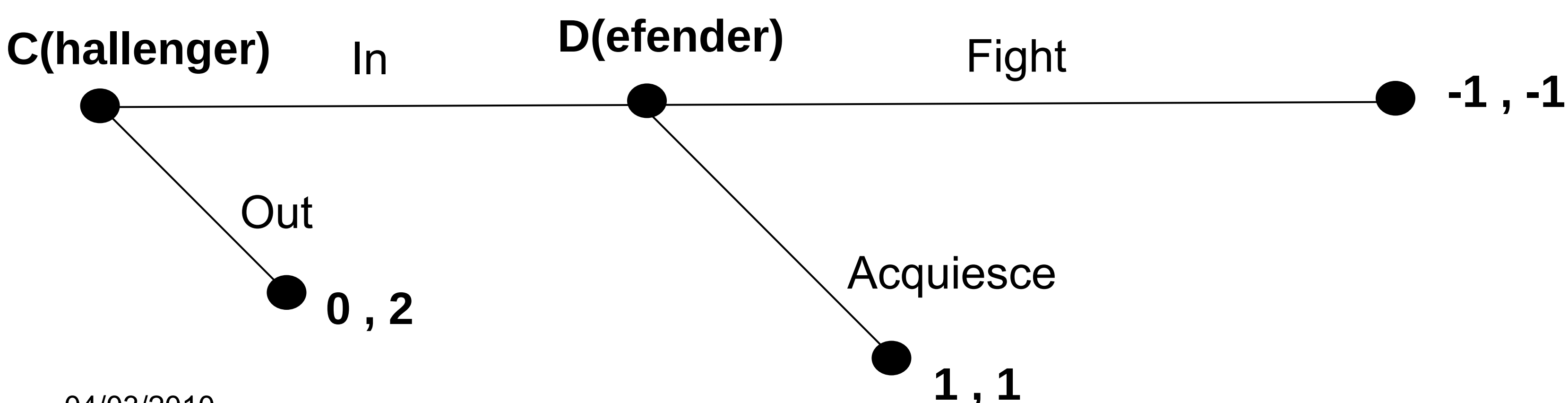
Example: Basic Threat Game

- Player 1 (Challenger) moves only at the start of the game and the **actions** available are *In* and *Out*.
- Thus, player 1 has two **strategies**: one that assigns *In* to the empty history and one that assigns *Out* to the empty history.



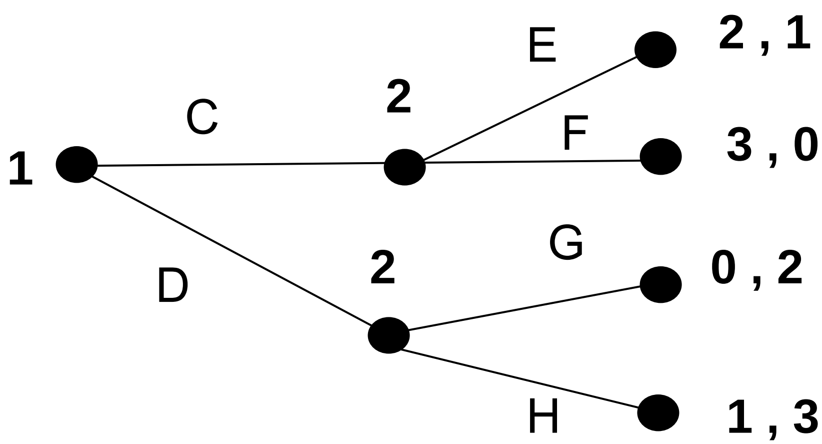
Example: Basic Threat Game

- Player 2 (Defender) moves only after the history *In*. After history *In*, the **actions** available to her are *Fight* or *Acquiesce*.
- Thus, player 1 has two strategies: one that assigns *Fight* to history *In* and one that assigns *Acquiesce* to history *In*.



Extended Threat Game

- Player 1 moves only at the start of the game and the **actions** available are *C* and *D*.
- Thus, player 1 has two **strategies** available to her: one that assigns *C* to the empty history and one that assigns *D* to the empty history.



Extended Threat Game

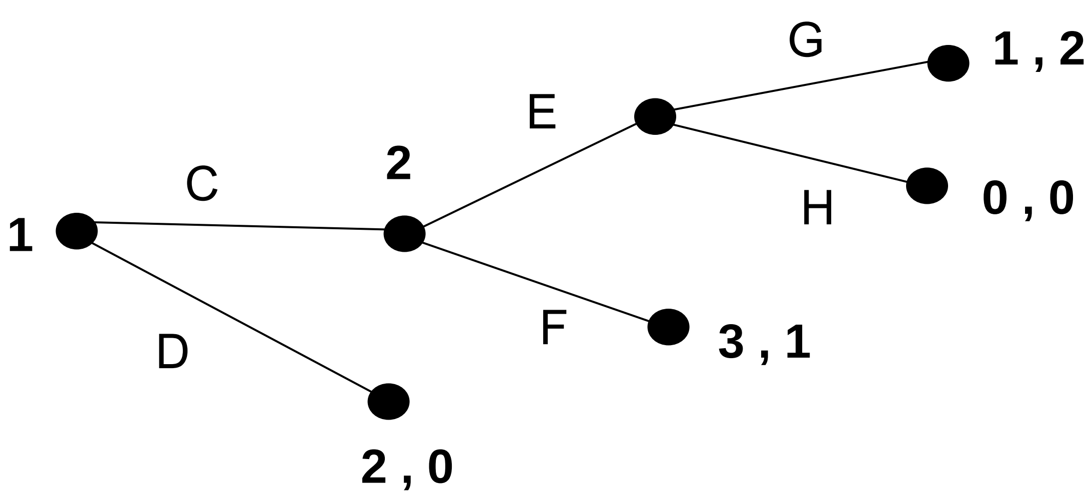
- Player 2 moves after both the histories *C* and *D*. After history *C*, the **actions** available to her are *E* and *F* and after history *D*, *G* and *H*.
- Thus a **strategy** of player 2 is a function that assigns either *E* or *F* to the history *C*, and either *G* or *H* to history *D*.
- That is, player 2 has *four strategies*, which are: *EG*, *EH*, *FG*, *FH*.

Extended Threat Game

	Action assigned to history *C*	Action assigned to history *D*
Strategy 1	*E*	*G*
Strategy 2	*E*	*H*
Strategy 3	*F*	*G*
Strategy 4	*F*	*H*

New Game

- What are the **actions** and the **strategies** available to each player?



New game

- Player 1 moves both at the start of the game and after the history (*C*,*E*).
- In each case, she has **two actions** but has **four strategies**: *CG*, *CH*, *DG*, *DH*.
- The definition of a strategy of player *i* requires that it specifies an action for *every* history after which it is player *i*'s turn to move, *even for histories that, if the strategy is followed, do not occur (e.g. DG* or *DH*).

Nash equilibrium of extensive game with perfect information

- The strategy profile s^* in an extensive game with perfect information is a **Nash equilibrium** if, for each player i and every strategy r_i of player i, the terminal history $O(s^*)$ generated by s^* is at least as good according to player i's preferences as the terminal history $O(r_i, s^*_{-i})$ generated by the strategy profile (r_i, s^*_{-i}) in which player i chooses r_i while every other player j chooses s^*_j.
- Equivalently, for each player i,

$$u_i(O(s^*)) \geq u_i(O(r_i, s_{-i}^*))$$

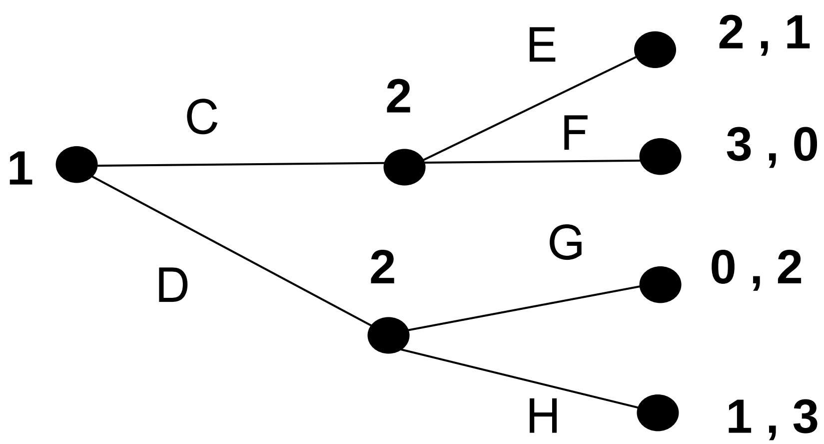
for every strategy r_i *of player i, where O(.)* is the outcome function of the game

Nash equilibrium of extensive game with perfect information

- Note! Here there is no mention of histories! Still simultaneous choices.
- The Nash equilibrium disregards the sequential structure.

Nash equilibrium of extensive game with perfect information

- Consider the following extensive game:



Normal form representation of this game

1/2	EG	EH	FG	FH
C	2 , 1	2 , 1	3 , 0	3 , 0
D	0 , 2	1 , 3	0 , 2	1 , 3

BR functions

1/2	EG	EH	FG	FH
C	**2** , **1**	**2** , **1**	**3** , 0	**3** , 0
D	0 , 2	1 , **3**	0 , 2	1 , **3**

Nash Equilibria

1/2	EG	EH	FG	FH
C	**2** , **1**	**2** , **1**	**3** , 0	**3** , 0
D	0 , 2	1 , **3**	0 , 2	1 , **3**

Some more definitions

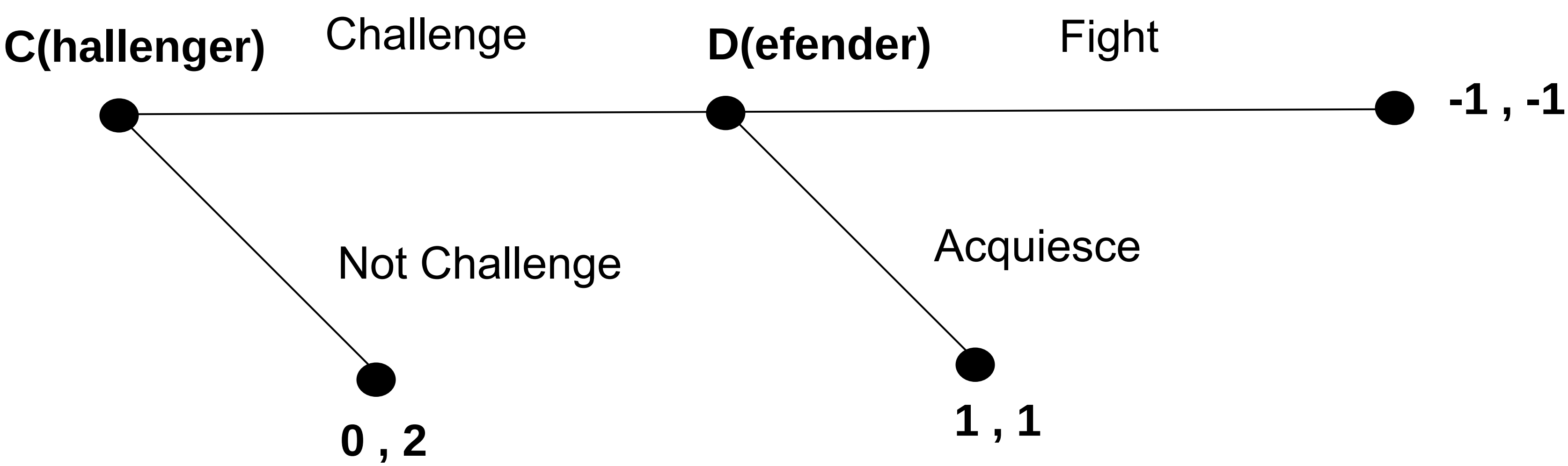
- **Game Tree**: plots the extensive form a game.
- **Games in Extensive Form** include:
- Who the players are
- What is the order in which players make decisions
- What alternatives does a player have when making a decision
- What does a player know when making a decision
- How do players rank the possible outcomes
- Probabilities of events outside the control of the players

Some more definitions

- **Decision node**: Point where only one player has to make a decision.
- **Branches**: Alternative choices available to that player.
- **Root**: Initial decision node.
- **Terminal nodes**: No branches emanate from them.

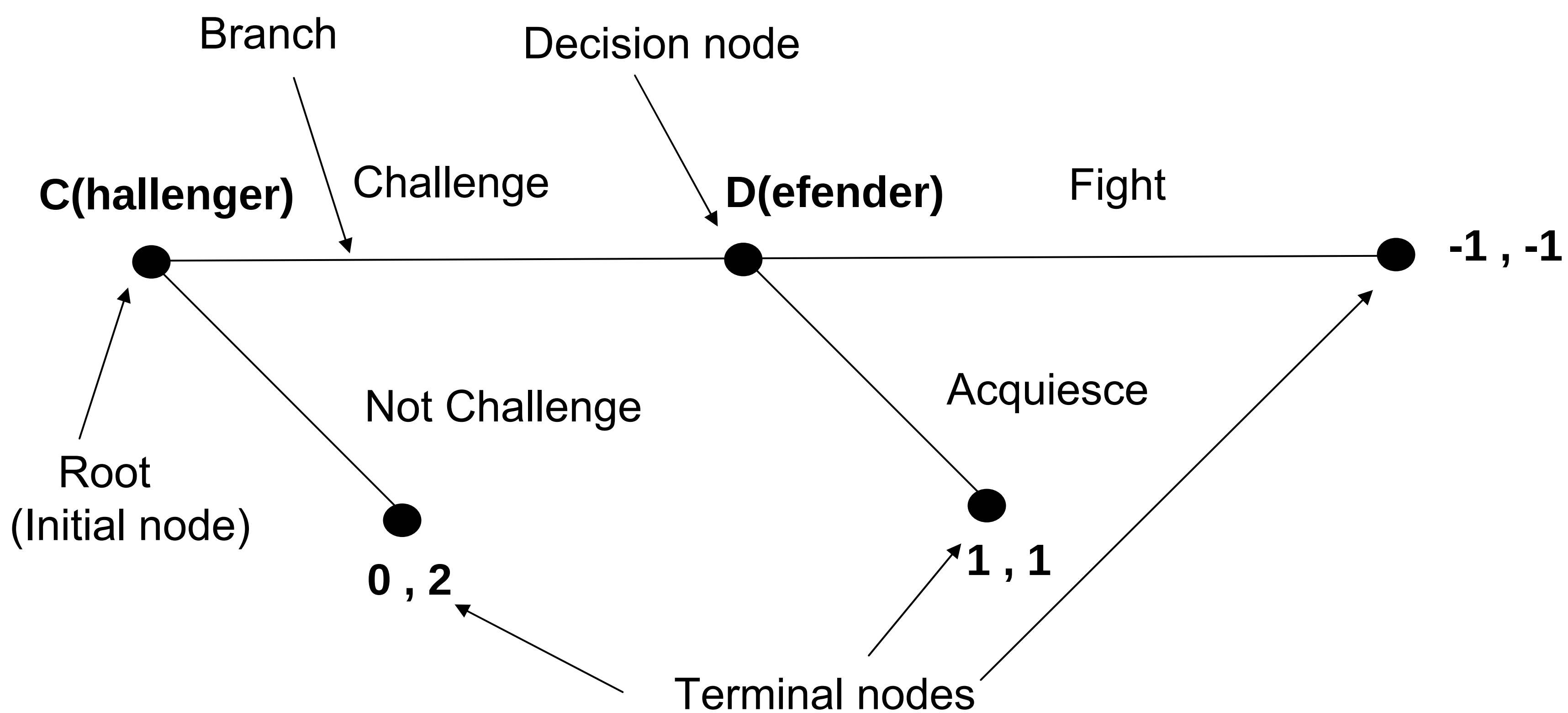
Some more definitions

- Example: Basic Threat Game



Some more definitions

- <u>Example: Basic Threat Game</u>



Some more definitions

- **Root**: The Challenger has a decision to make between 2 alternatives: Challenge or Not Challenge.
- The Challenger has a decision to make between 2 alternatives, so s/he has 2 **branches**.
- Where the tree ends at a decision is called the **terminal node**. Node: Defender is the terminal node.

Perfect versus imperfect information

- A game of **perfect information** is one where all players know all the preceding moves that all the players before him/her have taken.
- Games of **imperfect information** involve at least one player who is unsure about what preceding players have done (what moves they made).

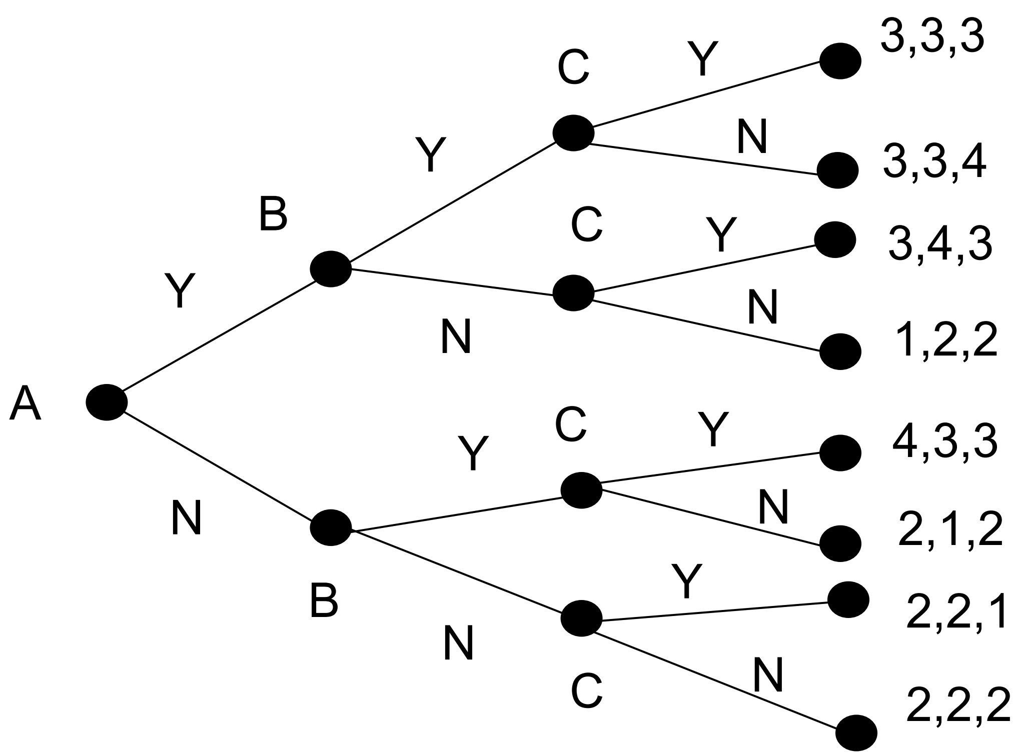
Perfect versus imperfect information

- <u>Example: Congressional Committee</u>
- A Committee can vote Yes or No whether to give themselves a pay raise.
- Player A votes first, Player B votes second, Player C votes third. Everyone has to vote so there can be no tie.

Congressional Committee

- Assume all the Players want a raise. Here are the Player's Preferences:
- Raise passes, Player did not vote for it. (best)
- Raise passes, Player voted for it.
- Raise did not pass, Player did not vote for it
- Raise did not pass, Player voted for it. (worst)

Congressional Committee



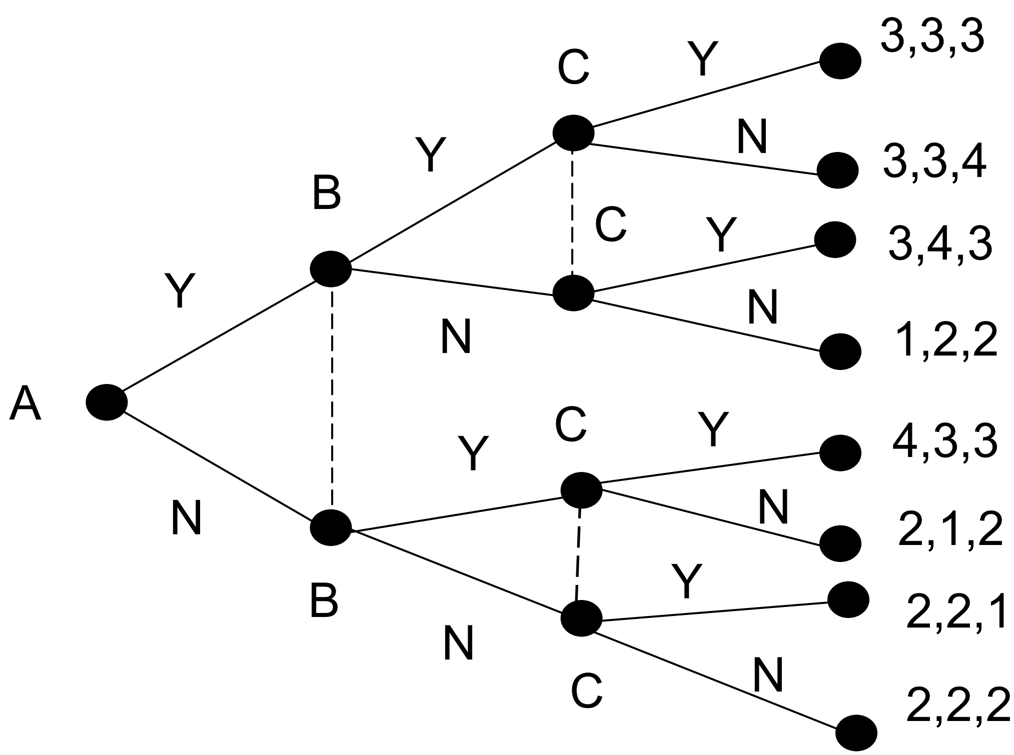
Congressional Committee

- If this had been a secret ballot game, then nobody would know how anyone voted (i.e: at which decision node they are at), therefore would be a game of imperfect information.

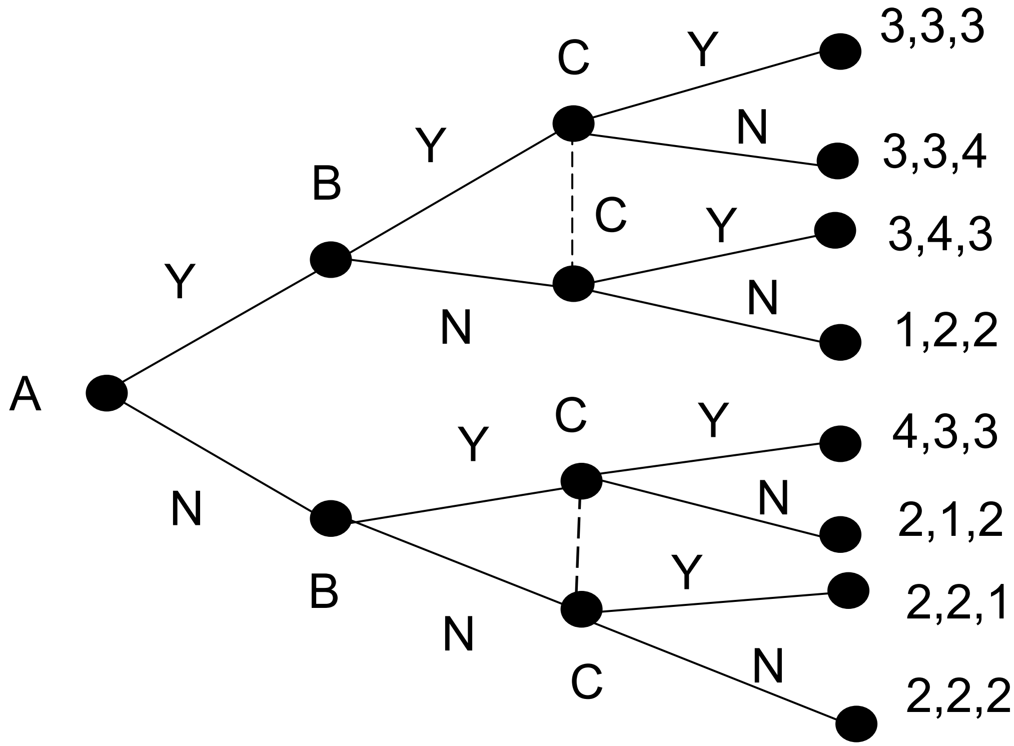
Perfect versus imperfect information

- In situations where a player does not know what decision a player before her made, we represent this by drawing a dashed line on the game tree.

B does not know what A voted and C does not know what B and A voted



B knows what A voted, C knows what A voted but does not know what B voted



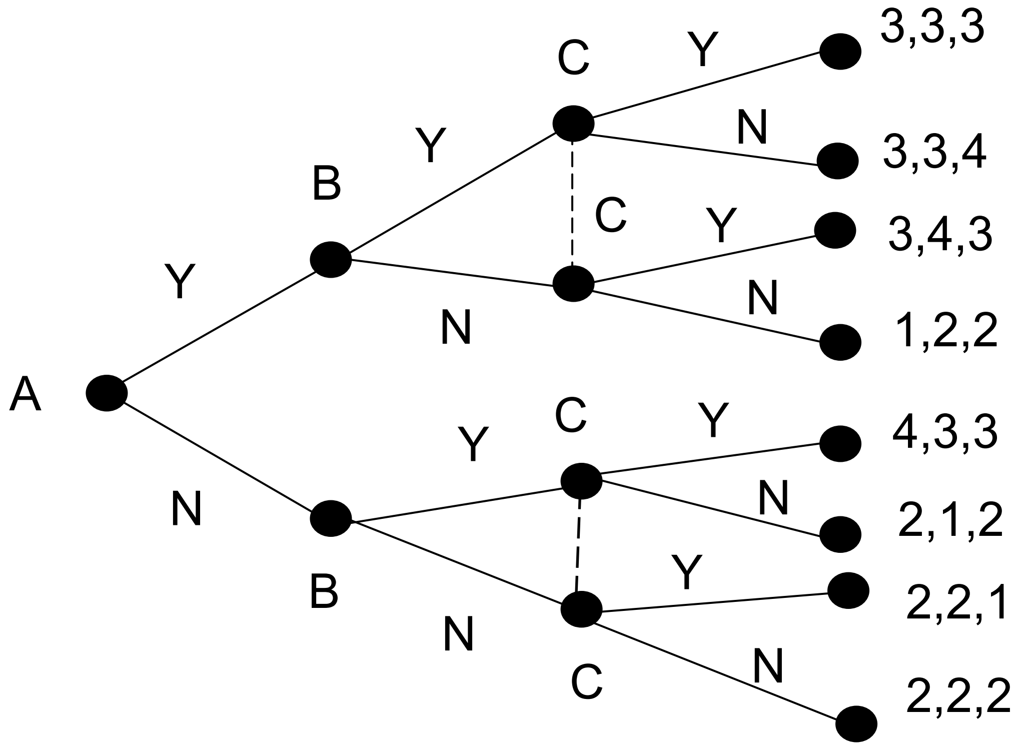
Information sets

- In this last example, there is no dashed line at B’s decision nodes because everyone can see how A voted.
- However there are dashed lines at C’s decision nodes because she does not know how B voted.
- C’s decision nodes are **enclosed in the same information set** because they all look the same to her (she does not know if she is at the Yes or the No node for Player B).

Information sets

- **Information sets**: Sets of nodes that are indistinguishable from the decision maker's standpoint.
- If they contain:
- 1 decision node: perfect information.
- Multiple nodes: imperfect information.

B knows what A voted, C knows what A voted but does not know what B voted



Information sets

- Player B has **two information sets**:
- **IS1**: Player A plays Yes
- **IS2**: Player A plays No.
- S/he has complete information and has no uncertainty about what Player A has done.

Information sets

- Player C has **two information sets**:
- **IS1**: Player A plays Yes and Player B plays Yes OR Player B plays No
- **IS2**: Player A plays No and Player B plays Yes OR Player B No.
- Player C does not have complete information because she does not know what Player B has done, therefore she has multiple nodes at her information sets.

Difference between static/normal and extensive games

- In a static game, **strategies** and **alternatives** are synonymous.
- In a dynamic game there is a difference between strategy and alternative. In a dynamic game, a **strategy** is a complete plan for how to play the game.

Static vs dynamic games

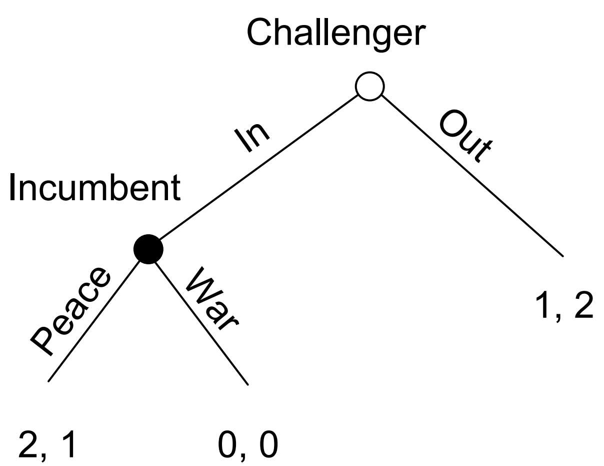
- **Static game**: you only have one decision to make and you only have *one* information set.
- A strategy for a player in a **dynamic game** is a complete plan for how to play the game in **any** contingency: what a player will do in any information set.

- A strategy can be considered a **function**: you take an action, given a specific action, written as an "if-then" statement. You need to specify every strategy in a game, even if most of them won't be actuated.

Definition of an extensive game

- **Players**
- **Terminal histories**
  - I.e. the set of all possible sequences of actions where no sequence is a proper subhistory of any other sequence.
- **Player function**
  - A function that tells us which player moves at a particular point in the game. That is, assigns a player to every sequence that is a proper subhistory.
- **Preferences for the players**
  - Over the set of possible terminal histories.

Example



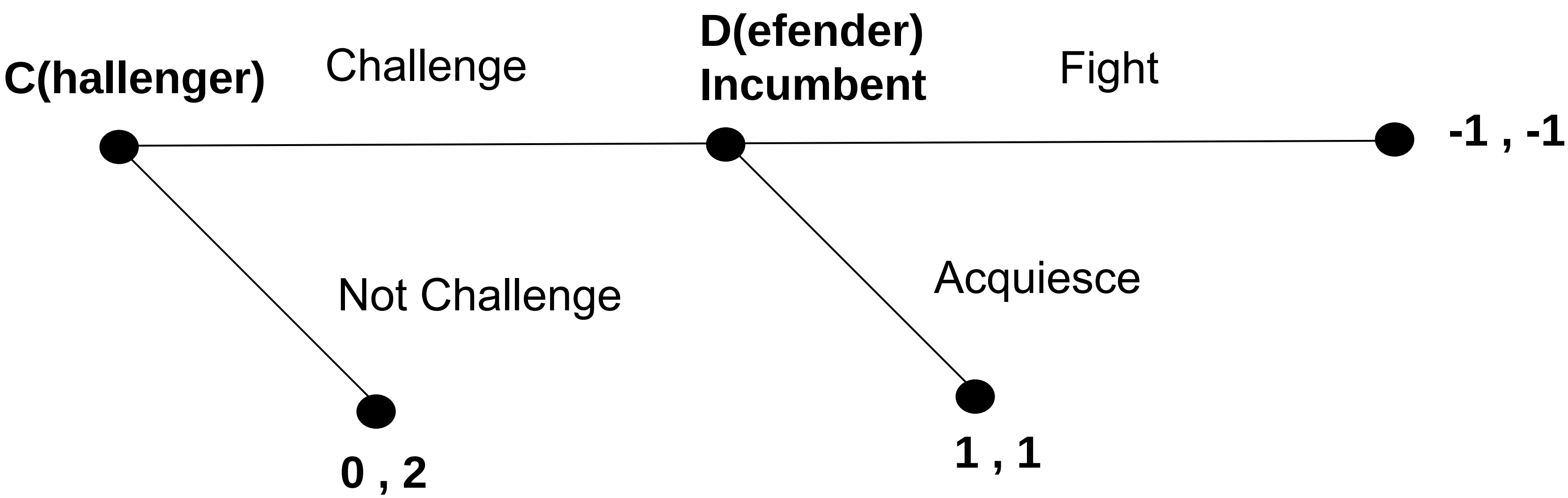
Action set available at the history *h* = *In*:

A(In)={Peace, War}

- **Players:** challenger and incumbent
- **Terminal histories:** *(In, Peace), (In, War), (Out).*
- **Player function:** *P(In)*=Incumbent, *P(empty set)*= challenger.
- **Example of prefs:** u_1*(In, Peace)*=2 etc...

Static vs dynamic games

- <u>Example: Basic Threat Game</u>



Basic Threat Game

- Players: I={1,2}
- Strategy spaces:

S(E)= {In,Out}, S(I)= {T,A}

- Strategy profiles: {S(E), S(I)}={in,T}, {in,A}, {Out,T}, {Out,A}.
- Payoffs at terminal nodes:

e.g. u(In,T)=(-2,-1)

Basic Threat Game

- What is a reasonable prediction of the play?
- In dynamic games, the central issue is **credibility**.
- Is the threat of fighting by the incumbent if C challenges her a credible one?

Basic Threat Game

- What is a reasonable prediction of the play?
- In dynamic games, the central issue is **credibility**.
- Is the threat of fighting by the incumbent if C challenges her a credible one?

Backward induction

- Solution concept: Backward induction
- Key idea: principle of sequential rationality: at each decision node, a player picks the best action given what she thinks is going to be the future play of the game.

Basic Threat Game

- **Step 1**: Consider the Defender (Incumbent) decision node (final decision node).
- If the C challenges D, what action maximizes D’s payoff?
- Answer: Acquiesce.

Basic Threat Game

- **Step 2**: Consider the Challenger decision node (penultimate node and root at the same time).
- Given that D would Acquiesce if C challenges her, what action maximizes C’s payoff?
- Answer: Challenge

Basic Threat Game

- The Backward Induction (sequential rational outcome) is thus:

  (Challenge, Acquiesce)

Backward induction

- In general:
- Step 1: Final decision nodes: decision maker picks the action that maximizes her payoff.
- Step 2: Penultimate decision nodes: decision maker knows the exact consequences (final payoffs) of her moves.
- Step 3: Fold back the game tree one step at a time until the root of the game is reached.

From extensive to normal form game

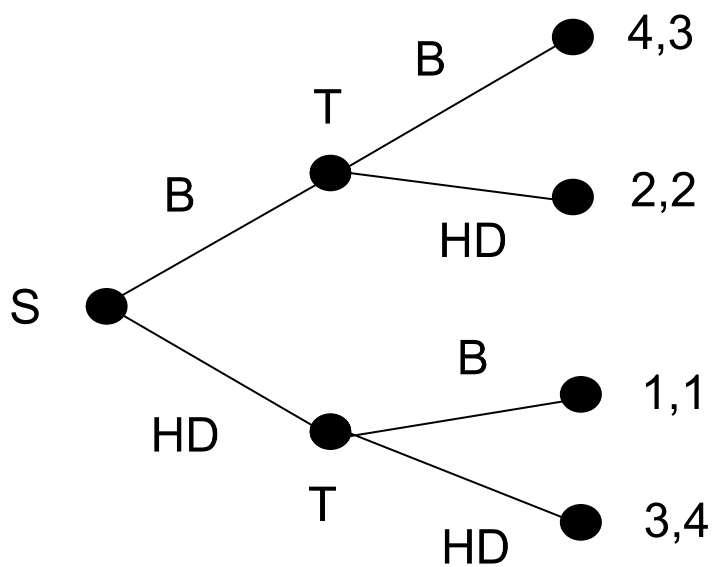
		Incumbent	
		Acq.	**Fight**
Challenger	**Ch.**	1,1	-1,-1
	Out	0,2	0,2

Two Nash Equilibria:
(Challenge, Acquiesce) (Out, Fight).
NE: not good concept, keeps non-credible threats.

Example: HD vs Blue Ray

- Sony and Toshiba are trying to decide which standard system they want to adopt: HD vs Blue Ray.
- Sony prefers Blue Ray and Toshiba prefers HD, but they both prefer that they are using the same system.
- This is an example of **sequential decision making**.

HD vs Blue Ray: Complete information



Backward induction

- Backward induction solution: (B,B)
- First-move advantage.

From extensive to normal form game

- One can collapse certain types of extensive form games (not with continuous actions) into normal form by simply defining an action to be a **complete specification of how an agent would act in all possible contingencies.**
- Agents then choose these actions **simultaneously** at the beginning of the game.

From extensive to normal form game

- In the present example, Sony has only two strategies: ”B” or ”HD”.
- However, for Toshiba, we need to completely specify how it would act in **all possible contingencies**.
- For Toshiba, we need to define 2 x 2 = 4 strategies.

From extensive to normal form game

- For Toshiba, each strategy must have 2 components.
- The **first component** indicates whether Toshiba chooses ”B” or ”HD” given that Sony has chosen ”B”
- The **second component** indicates whether Toshiba chooses ”B” or ”HD” given that Sony has chosen ”HD”.

- As a result, Toshiba has the 4 following strategies: BB, BHD, HDB, HDHD, where for example HDB means that Toshiba chooses ”HD” if Sony has chosen ”B” and chooses ”B” if Sony has chosen ”HD”.

From extensive to normal form game

S / T	BB	BHD	HDB	HDHD
B	(4,3)	(4,3)	(2,2)	(2,2)
HD	(1,1)	(3,4)	(1,1)	(3,4)

From extensive to normal form game

S / T	BB	BHD	HDB	HDHD
B	(**4**,**3**)	(**4**,**3**)	(**2**,2)	(2,2)
HD	(1,1)	(3,**4**)	(1,1)	(**3**,**4**)

NE in normal form game

S / T	BB	BHD	HDB	HDHD
B	(**4**,**3**)	(**4**,**3**)	(**2**,2)	(2,2)
HD	(1,1)	(3,**4**)	(1,1)	(**3**,**4**)

Two Nash Equilibria: (B, B) (HD, HD).
Indeed (B,BB) and (HD,BHD) are equivalent since if Sony plays ”B” then the second component of the strategy of Toshiba becomes irrelevant.

IEWDS

- It is well-known that by solving the **normal-form game** by IEWDS (Iteration of Elimination of Weakly Dominated Strategies) gives the same solution as in the SPNE.
- Let’s show this result.

IEWDS in normal form game

S / T	BB	BHD	HDB	HDHD
B	(4,3)	(4,3)	(2,2)	(2,2)
HD	(1,1)	(3,4)	(1,1)	(3,4)

IEWDS in normal form game

S / T	BB	BHD	HDB	HDHD
B	(4,3)	(4,3)	(2,2)	(2,2)
HD	(1,1)	(3,4)	(1,1)	(3,4)

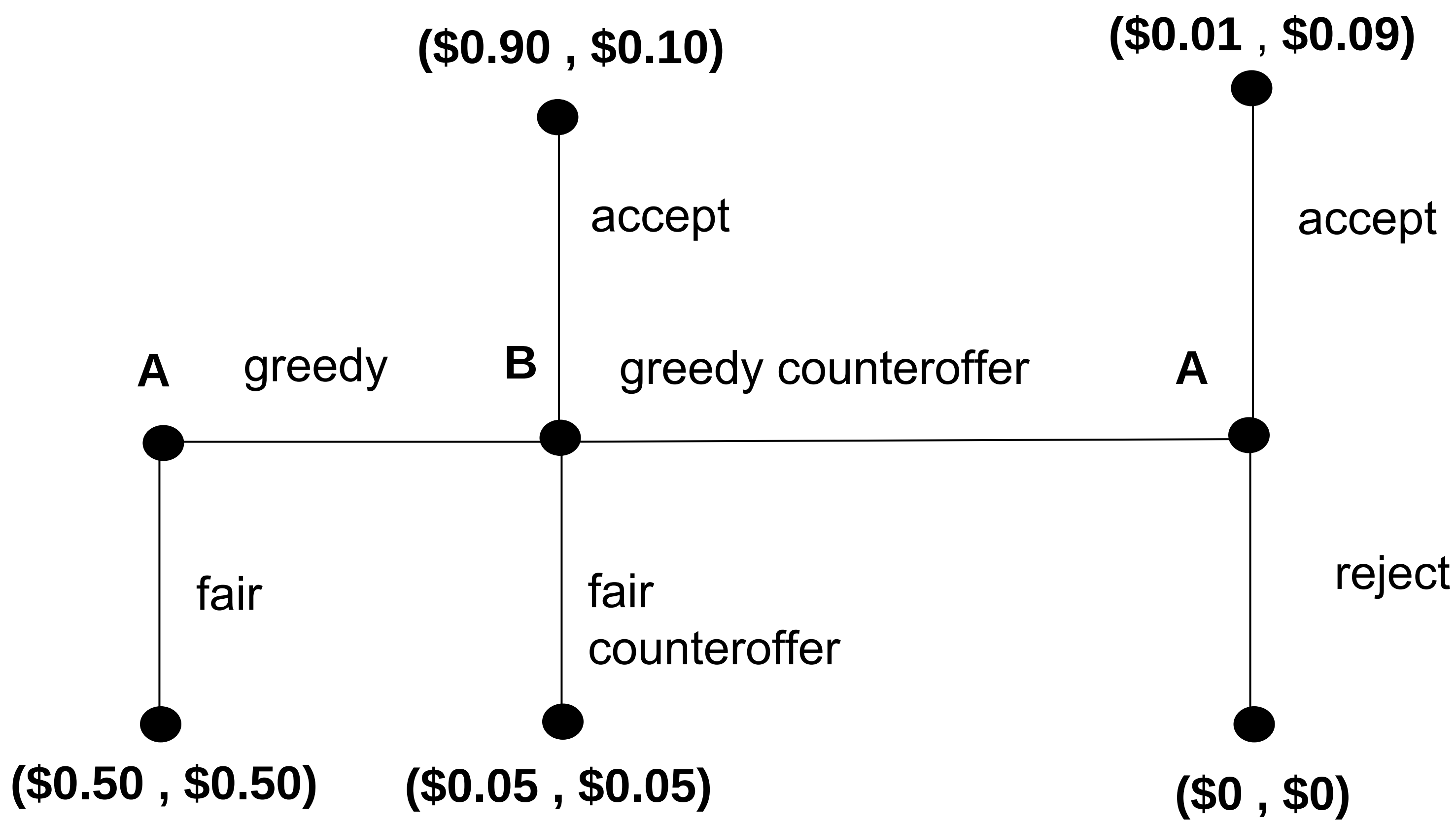
IEWDS in normal form game
Unique SPNE

S / T	BB	BHD	HDB	HDHD
B	(4,3)	(4,3)	(2,2)	(2,2)
HD	(1,1)	(3,4)	(1,1)	(3,4)

Limitations of Backward induction

- Example: **The ultimatum game**
- Two players A and B has to divide a dollar.
- Player A plays first and can be greedy (i.e. give \$0.90 to herself and \$0.10 to B) or fair (\$0.50 each). If A is greedy, B must accept the offer Player or make a fair or greedy counteroffer. Finally, A has to accept or reject B's offer.

The ultimatum game: payoffs are monetary prizes



Money is not payoffs

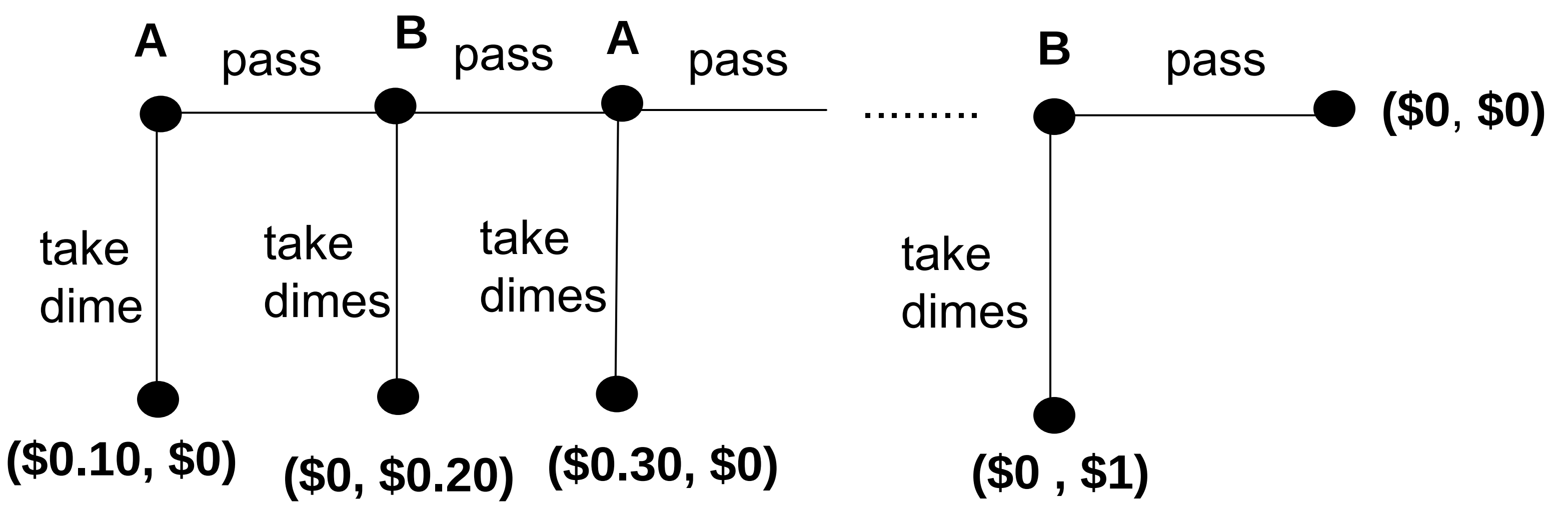
- Backward induction: A is greedy and B accepts the greedy offer.
- Payoff: ($0.90 , $0.10)
- Not true in the real world: often A is fair at the start.
- Money is not everything to the players.

The centripede game

- Two players A and B are chosen. The experimenter puts a dime on a table. Player A can take it or pass. If A takes the dime, the game is over and she gets 10 cents. If A passes, the experimenter adds a dime, and now B has the choice of taking 20 cents or passing, and so on.

The centripede game

- B is sure to take the dollar at the last stage; so A should take the $0.90 at the penultimate stage, and so on.
- Backward induction: A takes the dime ($0.10) and the game ends.



The centripede game

- Not true in the classroom.
- Such games go on for at least a few rounds.

Backward induction: Too many iterations

- The predictions do poorly when the number of nodes along branches through the tree gets very large.

Backward Induction and SPNE

- A solution concept that formalizes the backward induction solution and applies to more general games is known as subgame perfect equilibrium or subgame perfect Nash equilibrium (SPNE).
- This is due to Selten (1975).

Subgame perfect Nash equilibrium (SPNE)

- A good idea that got Prof. Selten Nobel prize in 1994.

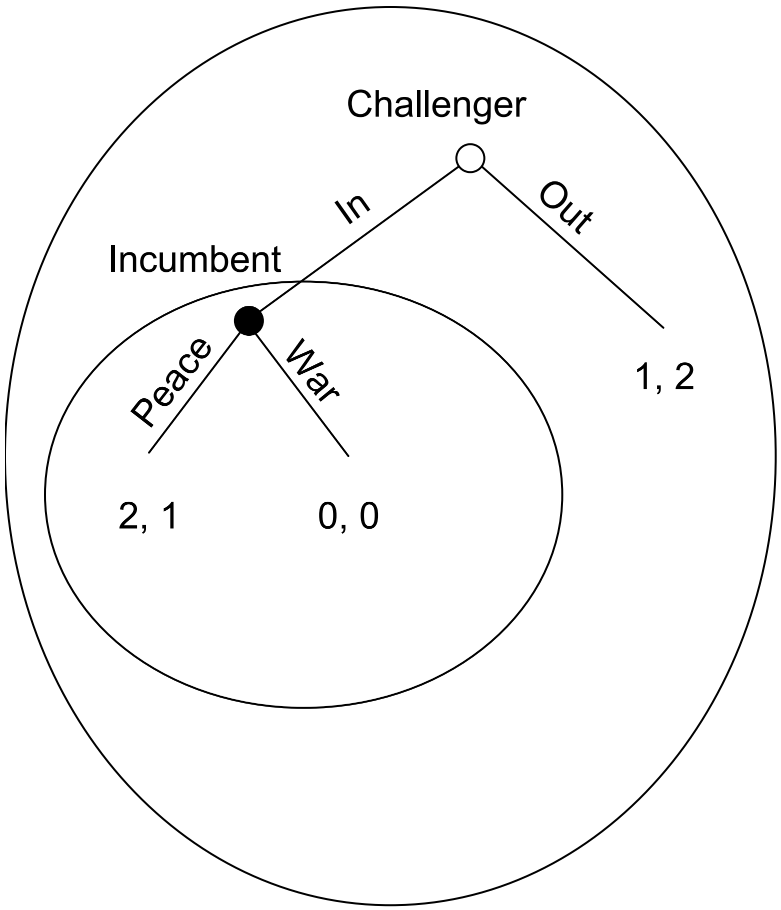
(check out http://nobelprize.org/nobel_prizes/economics/laureates/1994/)

- A refinement of NE.
- The notion of SPNE requires a player's strategy to prescribe a best response at all points in the extensive game where the player might be called upon to play, i.e. in every subgame.

Some definitions

- **Successor** to node x: all of those nodes that can be reached by following some sequence of branches originating from x.
- **Subgame**: part of the extensive form of a game satisfying:
- (i) Starts at a single decision node;
- (ii) It contains every successor to this node;
- (iii) If it contained any part of an information set, then it contains all the nodes in that information set.

Subgames



- Consider an extensive game,*Γ*, with a player function *P*.
- For any non terminal history *h* of *Γ*, there is a subgame *Γ(h)* following the history *h.*
- This incudes the entire game, i.e. when *h*=the empty history.
- A proper subgame is any other subgame other than the one following the empty history.

SPNE

- A SPNE is a strategy profile s^* such that it is not optimal for any player i to unilaterally deviate in any subgame given that the other players adhere to s_{-i}^*.
- This has to be true in ***ALL*** subgames!!!
- Put another way: for every player i and after every history h after which it is player i's turn to move;

$$u_i(O_h(s^*)) \geq u_i(O_h(r_i, s_{-i}^*)),$$

for all r_i of player i, where:

  - $u_i(O_h(s))$ is the utility of player i in the outcome that is generated by s after the history h.

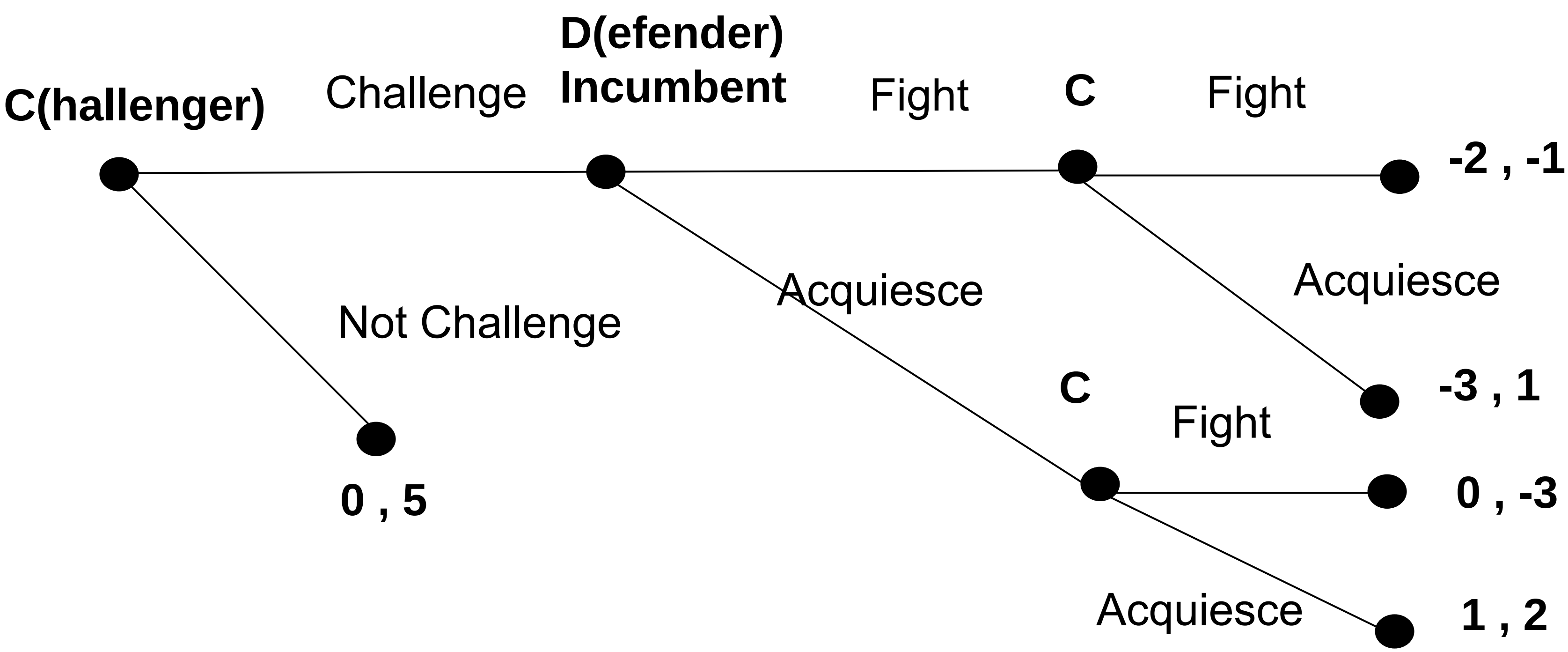
SPNE

- To find a SPNE of a game, use the backward induction procedure:
- **Step 1**: Start from a final subgame and find all the NE.
- **Step 2**: Replace each final subgame with one of its NE payoffs.
- **Step 3**: Consider the penultimate subgame and find all the NE.
- **Step 4**: Fold back the game until you reach the subgame that starts with the root of the game.

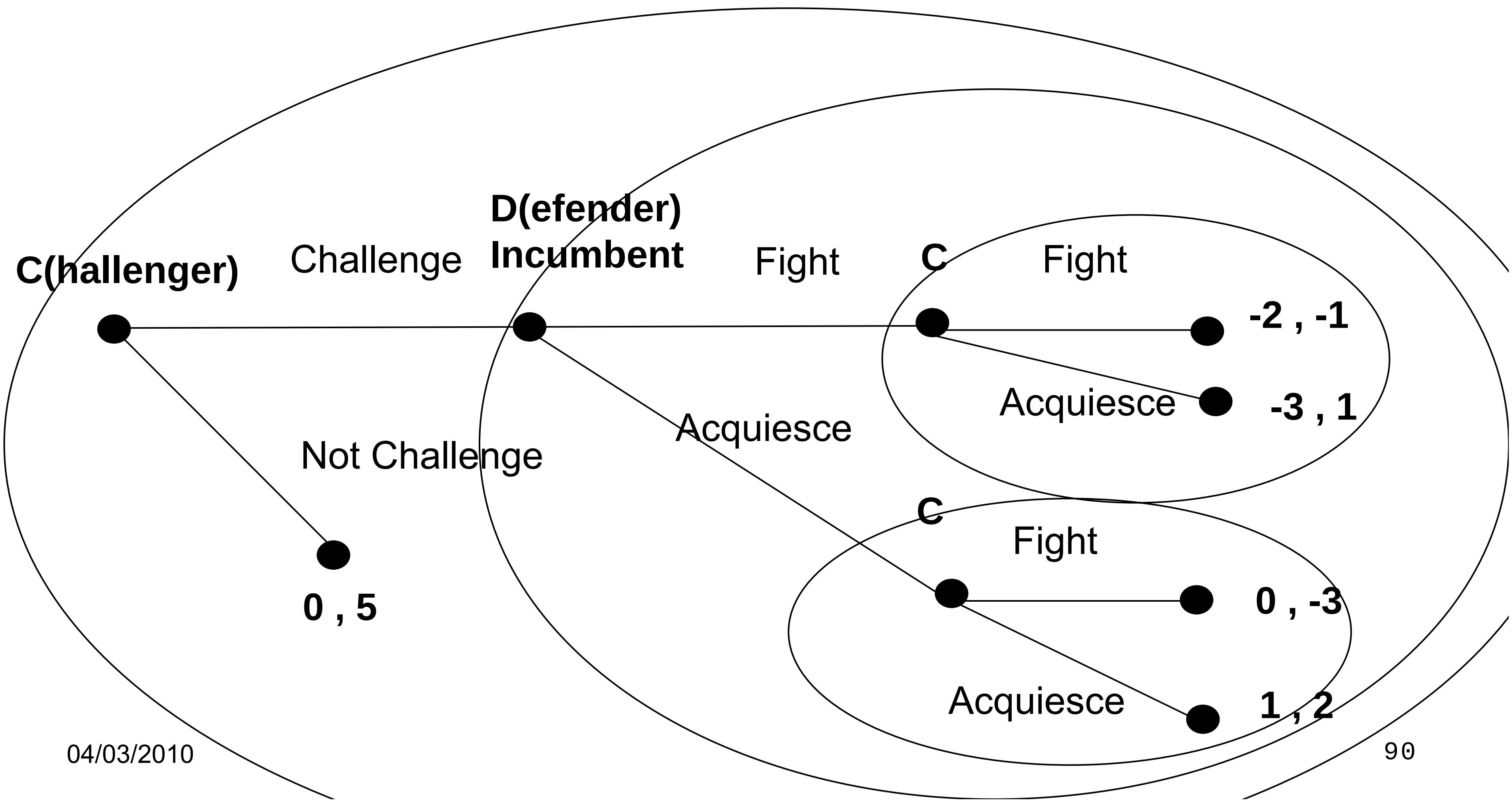
SPNE

- Multiple NE in subgames lead to multiple SPNE.

Example: Extended Threat Game



Example: Extended Threat Game



Basic Threat Game

- Final subgame:
- C chooses Fight if D fights and Acquiece if D acquieces

Basic Threat Game

- Penultimate subgame:
- The defender Acquiece since it gives her a higher payoff: 2 > -1.

Basic Threat Game

- Root game
- If C challenges, she gets 1 (A,A)
- If does not challenge, she gets 0.
- 1 SPNE: (C,A,A)
- Payoff: (1,2)

Normal form of this game

- Let us define the strategies of each player.
- For player 1(Challenger), every strategy must have 3 components.
- Here ”not challenge” is denoted by ”O”
- The first component tells C whether or not to challenge (takes value C or O), the second tells her whether or not to fight if D acts ”fight” (takes value F or A) and the third one tells her whether or not to fight if D acts ”acquiesce” (takes value F or A)

Normal form of this game

- For example, CAF means (1) challenge, (2) acquiesce against a defender that fights, (3) fight against a defender that acquieces.
- As a result, player 1 (Challenger) has 8 strategies.
- Player 2 (Defender) has only two strategies: F or A.

Normal form

C / D	F	A
CFF	(-2 , -1)	(0 , -3)
CFA	(-2 , -1)	(1 , 2)
CAF	(-3 , 1)	(0 , -3)
CAA	(-3 , 1)	(1 , 2)
OFF	(0 , 5)	(0 , 5)
OFA	(0 , 5)	(0 , 5)
OAF	(0 , 5)	(0 , 5)
OAA	(0 , 5)	(0 , 5)

BR functions

1/2	F	A
CFF	(-2 , **-1**)	(0 , -3)
CFA	(-2 , -1)	(**1** , **2**)
CAF	(-3 , **1**)	(0 , -3)
CAA	(-3 , 1)	(**1** , **2**)
OFF	(**0** , **5**)	(0 , **5**)
OFA	(**0** , **5**)	(0 , **5**)
OAF	(**0** , **5**)	(0 , **5**)
OAA	(0 , **5**)	(0 , **5**)

Nash equilibria

1/2	F	A
CFF	(-2 , **-1**)	(0 , -3)
CFA	(-2 , -1)	(**1** , **2**)
CAF	(-3 , **1**)	(0 , -3)
CAA	(-3 , 1)	(**1** , **2**)
OFF	(**0** , **5**)	(0 , **5**)
OFA	(**0** , **5**)	(0 , **5**)
OAF	(**0** , **5**)	(0 , **5**)
OAA	(0 , **5**)	(0 , **5**)

IEWDS Normal form = SPNE

C / D	F	A
~~CFF~~	~~(-2 , -1)~~	~~(0 , -3)~~
CFA	(-2 , -1)	(1 , 2)
~~CAF~~	~~(-3 , 1)~~	~~(0 , -3)~~
~~CAA~~	~~(-3 , 1)~~	~~(1 , 2)~~
OFF	(0 , 5)	(0 , 5)
OFA	(0 , 5)	(0 , 5)
OAF	(0 , 5)	(0 , 5)
OAA	(0 , 5)	(0 , 5)

IEWDS Normal form = SPNE

C / D	F	A
~~CFF~~	~~(-2 , -1)~~	~~(0 , -3)~~
CFA	(-2 , -1)	(1 , 2)
~~CAF~~	~~(-3 , 1)~~	~~(0 , -3)~~
~~CAA~~	~~(-3 , 1)~~	~~(1 , 2)~~
OFF	(0 , 5)	(0 , 5)
OFA	(0 , 5)	(0 , 5)
OAF	(0 , 5)	(0 , 5)
OAA	(0 , 5)	(0 , 5)

IEWDS Normal form = SPNE

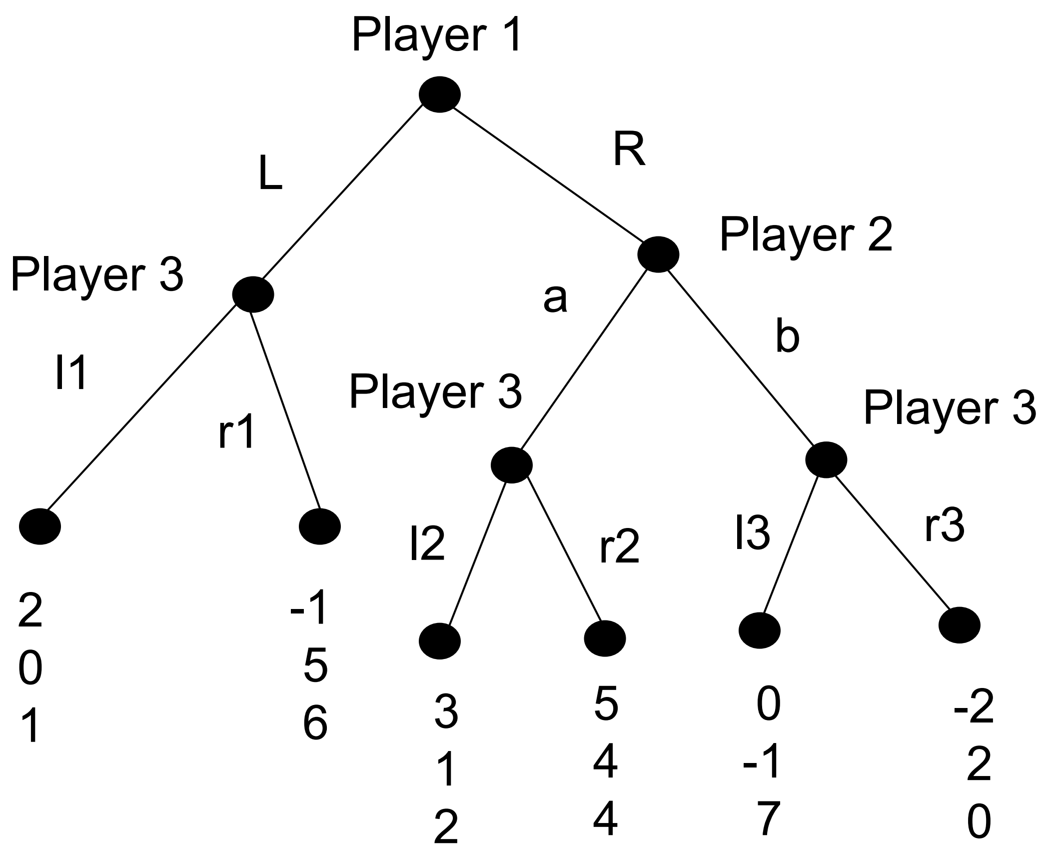
C / D	F	A
~~CFF~~	~~(-2 , -1)~~	~~(0 , -3)~~
CFA	(-2 , -1)	**(1 , 2)**
~~CAF~~	~~(-3 , 1)~~	~~(0 , -3)~~
~~CAA~~	~~(-3 , 1)~~	~~(1 , 2)~~
~~OFF~~	~~(0 , 5)~~	~~(0 , 5)~~
~~OFA~~	~~(0 , 5)~~	~~(0 , 5)~~
~~OAF~~	~~(0 , 5)~~	~~(0 , 5)~~
~~OAA~~	~~(0 , 5)~~	~~(0 , 5)~~

SPNE

- Useful in finite horizon games for finding SPNEs.
- Assumes all players are fully informed and use rational calculations.
- Method: solve every subgame starting at the back.
- Every finite extensive game with perfect information has at least one SPNE

Exercise: SPNE

- Consider the game depicted in the following figure:

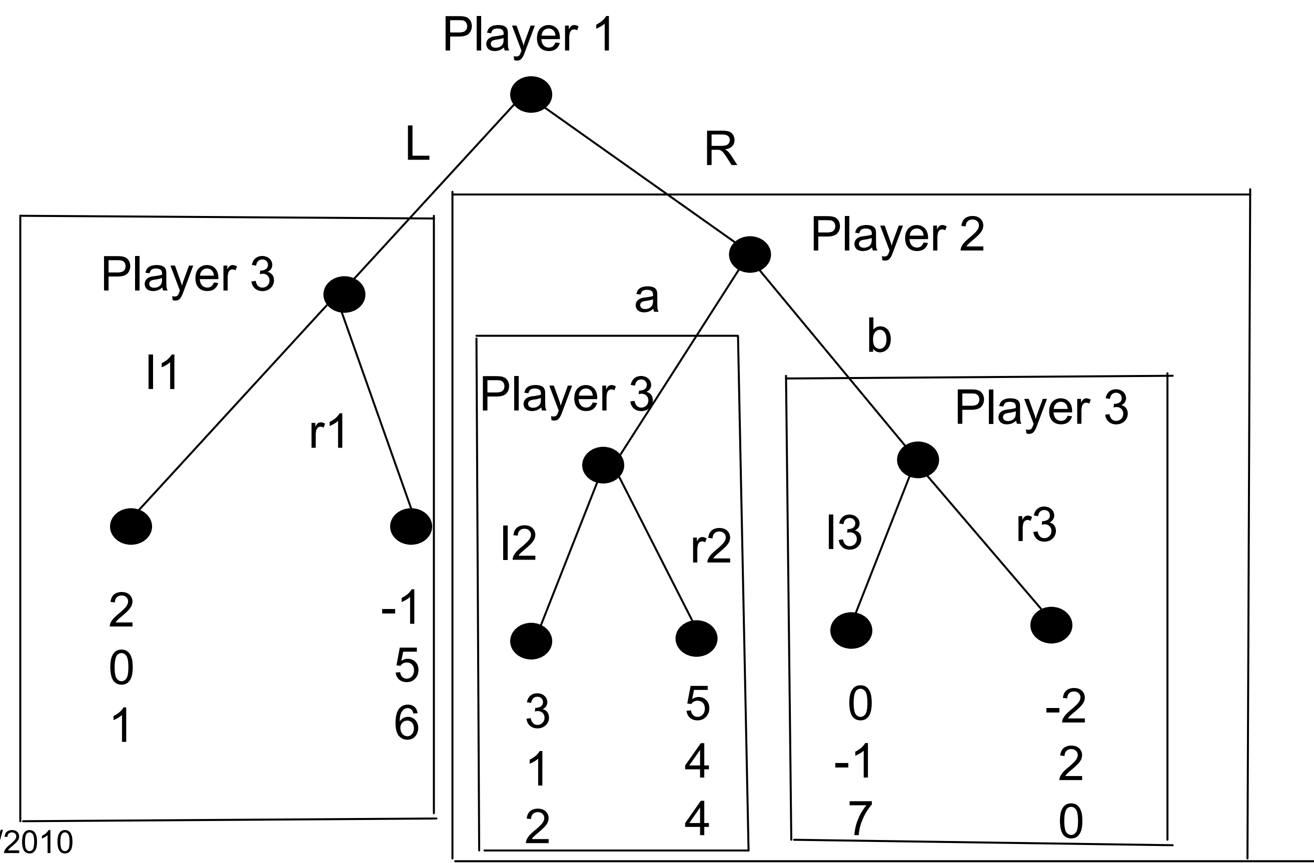


Question (a)

- (a) How many subgames does this game have?

Answer to question (a)

- This game admits 4 proper subgames plus an improper one (the game itself).



Question (b)

- (b) Find the unique subgame perfect Nash equilibrium of this game.

Answer to question (b)

- **Final subgames**: Player 3 plays *l*3 since 7 > 0 and plays r2 since 4 > 2.
- **Penultimate subgames**: Player 3 plays *r*1 since 6 > 1
- and player 2 plays a since 4 >-1 because she anticipates that player 3 will plays *l*3 and *r*2.
- **Initial (improper) subgame**:
- Player 1 plays *R* since 5 > -1 because she anticipates player 3 plays *r*1 if she plays *L* and player 2 plays *a* and then player 3 plays *r*2 if she plays *R*.

- The unique SPNE of the game is the strategy profile: (*s*1; *s*2; *s*3) = (*R*; a; *r*1, *r*2, *l*3). The subgame perfect outcome is therefore (*R*; *a*; *r*2) with payoffs (5,4,4)

Question (c)

- Identify all other pure strategy Nash equilibria of this game. Take any one of these other equilibria and argue that it is not sub-game perfect

Answer to question (c)

- In order to compute all the NE of the game we need to write the **normal form** representation of the game:

Normal form of the game

	L		*R*	
	a	*b*	*a*	*b*
*r*1,*r*2,*r*3	(-1,5,6)	(-1,5,6)	(5,4,4)	(-2,2,0)
*l*1,*r*2,*r*3	(2,0,1)	(2,0,1)	(5,4,4)	(-2,2,0)
*r*1,*l*2,*r*3	(-1,5,6)	(-1,5,6)	(3,1,2)	(-2,2,0)
*l*1,*l*2,*r*3	(2,0,1)	(2,0,1)	(3,1,2)	(-2,2,0)
*r*1,*r*2,*l*3	(-1,5,6)	(-1,5,6)	(5,4,4)	(0,-1,7)
*l*1,*r*2,*l*3	(2,0,1)	(2,0,1)	(5,4,4)	(0,-1,7)
*r*1,*l*2,*l*3	(-1,5,6)	(-1,5,6)	(3,1,2)	(0,-1,7)
*l*1,*l*2,*l*3	(2,0,1)	(2,0,1)	(3,1,2)	(0,-1,7)

Best-reply functions (bold)

	L		*R*	
	a	*b*	*a*	*b*
*r*1,*r*2,*r*3	(-1,**5**,**6**)	(**-1**,**5**,**6**)	(**5**,**4**,**4**)	(-2,2,0)
*l*1,*r*2,*r*3	(2,**0**,1)	(**2**,**0**,1)	(**5**,**4**,**4**)	(-2,2,0)
*r*1,*l*2,*r*3	(-1,**5**,**6**)	(**-1**,**5**,**6**)	(**3**,1,2)	(-2,**2**,0)
*l*1,*l*2,*r*3	(2,**0**,1)	(**2**,**0**,1)	(**3**,1,2)	(-2,**2**,0)
*r*1,*r*2,*l*3	(-1,**5**,**6**)	(-1,**5**,**6**)	(**5**,**4**,**4**)	(**0**,-1,**7**)
*l*1,*r*2,*l*3	(2,**0**,1)	(**2**,**0**,1)	(**5**,**4**,**4**)	(0,-1,**7**)
*r*1,*l*2,*l*3	(-1,**5**,**6**)	(-1,**5**,**6**)	(**3**,**1**,2)	(**0**,-1,**7**)
*l*1,*l*2,*l*3	(2,**0**,1)	(**2**,**0**,1)	(**3**,**1**,2)	(0,-1,**7**)

6 Nash equilibria (framed)

	L		*R*	
	a	*b*	*a*	*b*
*r*1,*r*2,*r*3	(-1,**5**,**6**)	(**-1**,**5**,**6**)	(**5**,**4**,**4**)	(-2,2,0)
*l*1,*r*2,*r*3	(2,**0**,1)	(**2**,**0**,1)	(**5**,**4**,**4**)	(-2,2,0)
*r*1,*l*2,*r*3	(-1,**5**,**6**)	(**-1**,**5**,**6**)	(**3**,1,2)	(-2,**2**,0)
*l*1,*l*2,*r*3	(2,**0**,1)	(**2**,**0**,1)	(**3**,1,2)	(-2,**2**,0)
*r*1,*r*2,*l*3	(-1,**5**,**6**)	(-1,**5**,**6**)	(**5**,**4**,**4**)	(**0**,-1,**7**)
*l*1,*r*2,*l*3	(2,**0**,1)	(**2**,**0**,1)	(**5**,**4**,**4**)	(0,-1,**7**)
*r*1,*l*2,*l*3	(-1,**5**,**6**)	(-1,**5**,**6**)	(**3**,**1**,2)	(**0**,-1,**7**)
*l*1,*l*2,*l*3	(2,**0**,1)	(**2**,**0**,1)	(**3**,**1**,2)	(0,-1,**7**)

Answer to question (c)

- The game admits 6 pure strategies Nash Equilibria:
- (*L*;*b*;*r*1, *r*2, *r*3), (*L*;*b*;*r*1, *l*2, *l*3), (*R*;*a*;*r*1, *r*2, *r*3), (*R*;*a*;*l*1, *r*2, *r*3), (*R*;*a*;*r*1, *r*2, *l*3), and (*R*;*a*;*l*1, *r*2, *l*3).

- Only (*R*;*a;r*1, *r*2, *l*3) is a SPNE.

Answer to question (c)

- Consider for instance the NE (*L*;*b*;*r*1, *l*2, *l*3).
- It is not a SPNE because it contains a strategy by player 1 which is not sequentially rational.
- Given that player 3 plays (*r*1, *l*2, *l*3) and player 2 plays "*b*", player 1's best move is to play *R* rather than *L* since 0 > -1.

The ultimatum game (Exercises 183.1 and 183.2 in Osborne)

Two people use the following procedure to split c dollars. Person 1 offers person 2 an amount of money up to c dollars. If 2 accepts this offer, then 1 receives the remainder of the c dollars. If 2 rejects the offer, then neither person receives any payoff. Each person cares only about the amount of money he receives, and prefers to receive as much as possible.

Assume that the amount person 1 offers can be any number, not necessarily an integral number of cents.

1a) Formulate this situation as an extensive game with perfect information.

Players: The two people

Terminal histories: The set of sequences (x, Z), where x is a number with $0 \leq x \leq c$ (the amount of money that person 1 offers to person 2) and Z is either Y ("yes, I accept") or N ("no, I reject").

Player function: $P(\emptyset) = 1$ and $P(x) = 2$ for all x.

Preferences: Each person's preferences are represented by a payoff equal to the amount of money she receives. For the terminal history (x, Y) person 1 receives $c - x$ and person 2 receives x; for the terminal history (x, N) both people receive 0.

1b) Find the subgame perfect equilibrium (equilibria?) of the game.

To find the subgame perfect equilibria of this game, we use backward induction.

First consider the subgame that follows the history x in which 1 offers the amount x to 2.

If $x > 0$, then 2's optimal action is to accept (since otherwise she gets nothing);

If $x = 0$, then 2 is indifferent between accepting and rejecting.

Thus in a subgame perfect equilibrium person 2's strategy either accepts all offers (including 0), or accepts all offers $x > 0$ and rejects the offer $x = 0$.

Now consider the whole game.

For each possible subgame perfect equilibrium strategy of person 2, we need to find the optimal strategy of person 1.

• If person 2 accepts all offers (including 0), then person 1's optimal offer is 0 (which yields her the payoff \$$c$).

• If person 2 accepts all offers but zero, then no offer of person 1 is optimal!

If she offers a positive amount then she is better off offering half as much, because person 2 accept both offers.

If she offers 0 then person 2 rejects the offer and she receives 0; she is better off offering any positive amount less than \$$c$.

We conclude that only the strategy of person 2 that accepts all offers is part of a subgame perfect equilibrium. In this equilibrium person 1 offers 0.

That is, the game has a unique subgame perfect equilibrium, in which person 1 offers nothing, and player 2 accepts all offers; person 1's payoff is \$c and person 2's payoff is zero.

This one-sided outcome is a consequence of the *one-sided* structure of the game.

If we allow person 2 to make a counteroffer after rejecting person 1's opening offer (and possibly further responses by both players), so that the model corresponds more closely to a "bargaining" situation, then under some circumstances the outcome is less one-sided.

1c) Find the values of x for which there is a Nash equilibrium of the ultimatum game in which person 1 offers x.

For every amount x there are Nash equilibria in which person 1 offers x.

For example, for any value of x there is a Nash equilibrium in which person 1's strategy is to offer x and person 2's strategy is to accept x and any offer more favorable, and reject every other offer.

Given person 2's strategy, person 1 can do no better than offer x. Given person 1's strategy, person 2 should accept x; whether person 2 accepts or rejects any other offer makes no difference to her payoff, so that rejecting all less favorable offers is, in particular, optimal.

1d) Find the subgame perfect equilibrium (equilibria?) of the variant of the ultimatum game in which the amount of money is available only in multiple of a cent (indivisible units).

In this case each player has finitely many actions, and for both possible subgame perfect equilibrium strategies of player 2 there is an optimal strategy for player 1.

If player 2 accepts all offers then player 1's best strategy is to offer 0, as before.

If player 2 accepts all offers except 0 then player 1's best strategy is to offer one cent (which player 2 accepts).

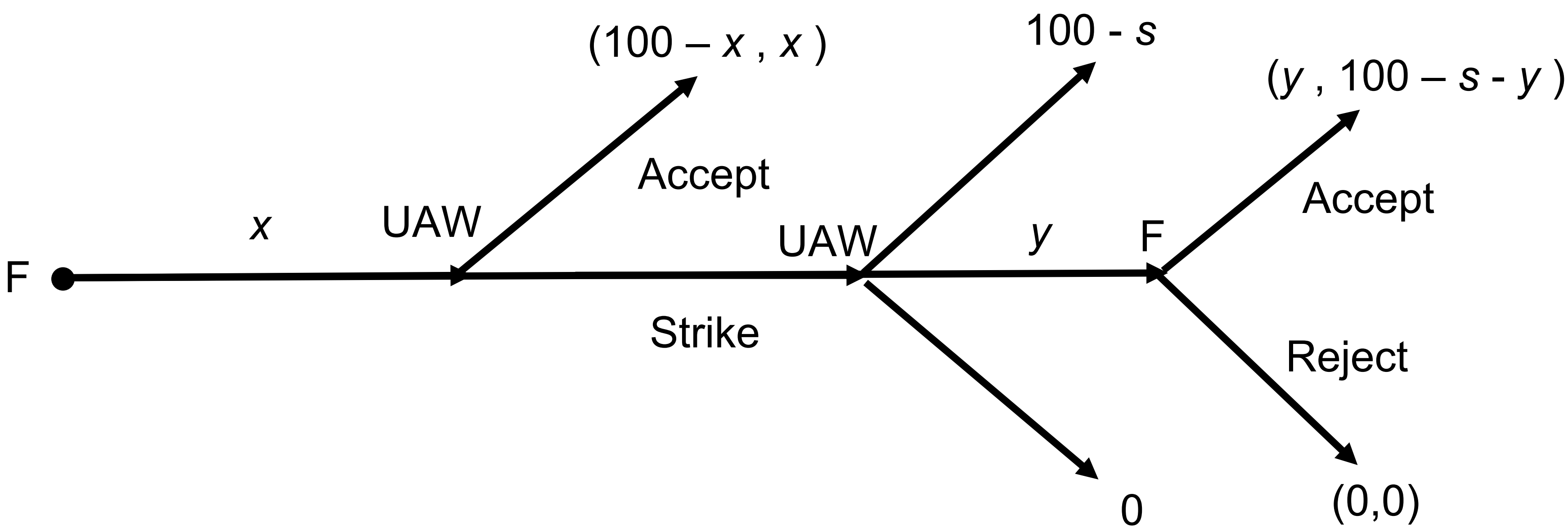
Thus the game has **two subgame perfect equilibria**: one in which player 1 offers 0 and player 2 accepts all offers, and one in which player 1 offers one cent and player 2 accepts all offers except 0.

Exercise

- General Motors and Chrysler have already negotiated and settled with the United Auto Workers (UAW).
- Now it is Ford's turn to negotiate. Ford is in a weak position but there appears to be about a $100 million dollars to be divided between the company and the union over the life of the contract, and the issue is how much each will get.

The game tree (extensive form)

- Tree



Rules of the game

- In the game described in the tree, Ford (F), begins by making an offer of x to the UAW.
- The UAW can accept this offer x or go on strike. If the UAW accepts, the $100 million is divided as agreed, that is 100 – x for Ford and x for UAW.

- If the UAW rejects and goes on strike, production will stop, revenues will be lost and the amount to be divided will drop to 100 – s, where s is the cost of the strike.
- The UAW can then make an offer to Ford which can accept or reject the offer. Accepting ends the game in the agreed division, that is y for Ford an 100 – s – y for UAW.

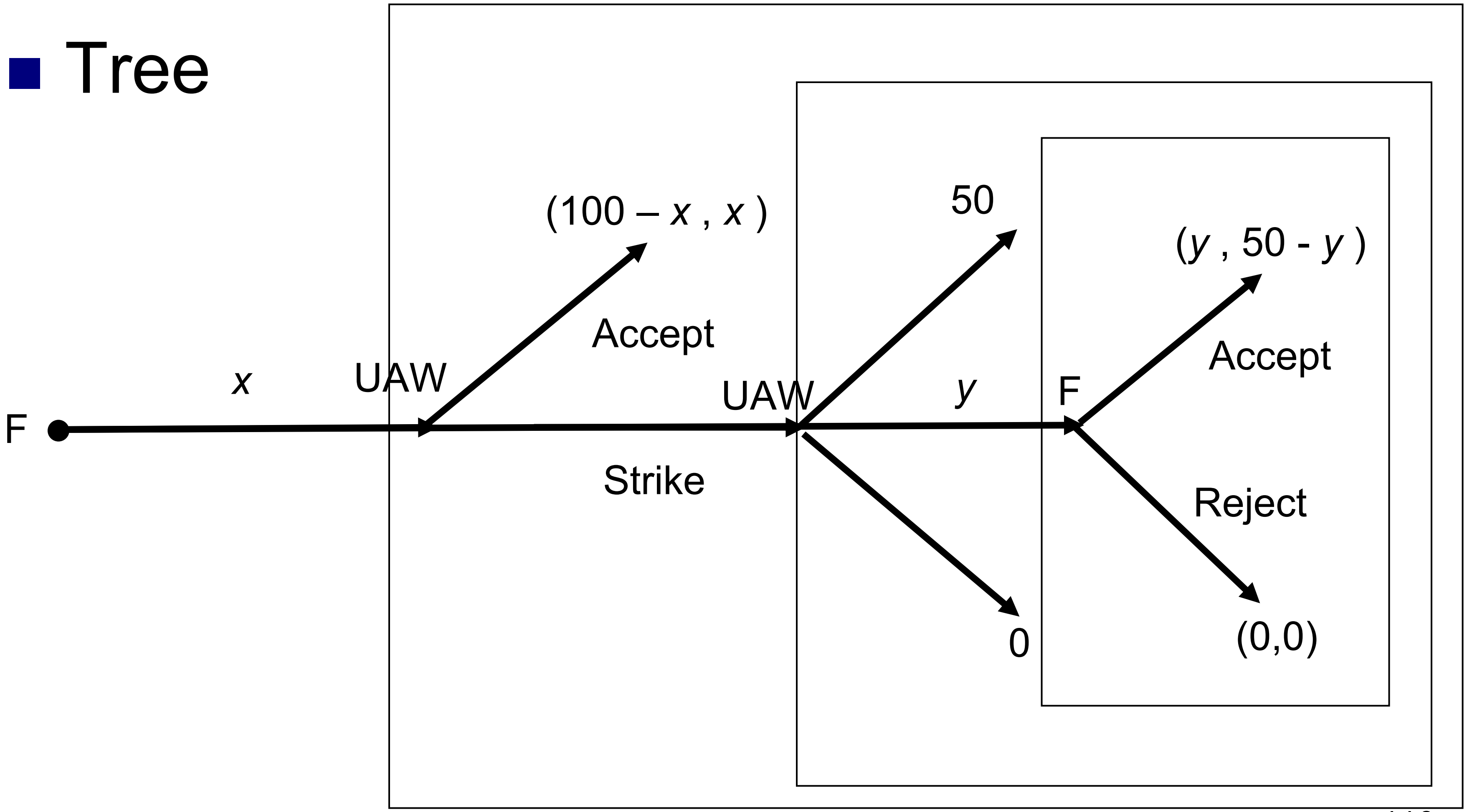
- If Ford rejects the offer, the remaining value is lost and both Ford and the UAW end up with nothing.

Question (a)

- Assume that the cost of a strike is \$50 million, i.e., $s = 50$.
- Find the backwards-induction equilibrium (SPNE) and specify the equilibrium path and outcome.

The game tree when s = 50

- Tree



Answer Question (a)

- Starting from the last node, F will accept any offer that makes him weakly better off than rejecting, i.e. accept $y \geq 0$
- At UAW's offer node, they buy the "cheapest yes" by offering y = 0.
- The outcome of this subgame is thus (0,50).

Answer Question (a)

- At node 2, UAW will accept any offer that makes them better off than rejecting continuing to the aforementioned subgame.
- Thus, they will accept any $x \geq 50$ and reject x < 50.

Answer Question (a)

- At node 1, if Ford offers $x < 50$ then they will receive a payoff of 0, so they choose to buy the ”cheapest yes”, which is $x = 50$.

Answer Question (a)

- Backwards induction equilibrium:

  {(**offer** x = 50, **accept** $y \geq 0$;

  (**accept** if $x \geq 50$, **offer** y = 0)}

- Equilibrium Path: Ford offers x = 50, UAW accepts.
- Outcome: (50,50)

Question (b)

- Assume that the cost of a strike is $50 million, i.e., $s = 50$.
- Find a Nash equilibrium in which the *UAW* gets 75.

Answer Question (b)

- With *s* = 50 there is no way that the UAW can get 75 in the last round.

- So the UAW must threaten to reject anything less than 75 at the second node.

Answer Question (b)

- Ford's best response to this threat is to offer *x* = 75. Since our Nash does not have to be subgame perfect (and is not), Ford can also make threats to reject some value of y (for example accept $y \geq 25$ and reject y < 25).

- Catering to this threat, UAW would buy the cheapest possible yes by offering y = 25.

Answer Question (b)

For simplicity, suppose Ford's strategy at node 4 remains subgame perfect so that F accepts any $y \geq 0$

- UAW's best response to this is to offer y = 0.
- The Nash Equilibrium described above is formalized as {(**offer** x = 75, **accept** $y \geq 0$); (**accept** if $x \geq 75$, **offer** y = 0)}

Answer Question (b)

- Note that this (and other variations), also works: {(offer x = 75, accept $y \geq 25$);
  (accept if $x \geq 75$, offer y = 25)}

Question (c)

- Is the equilibrium in (b) subgame perfect? Explain why or why not.

Answer Question (c)

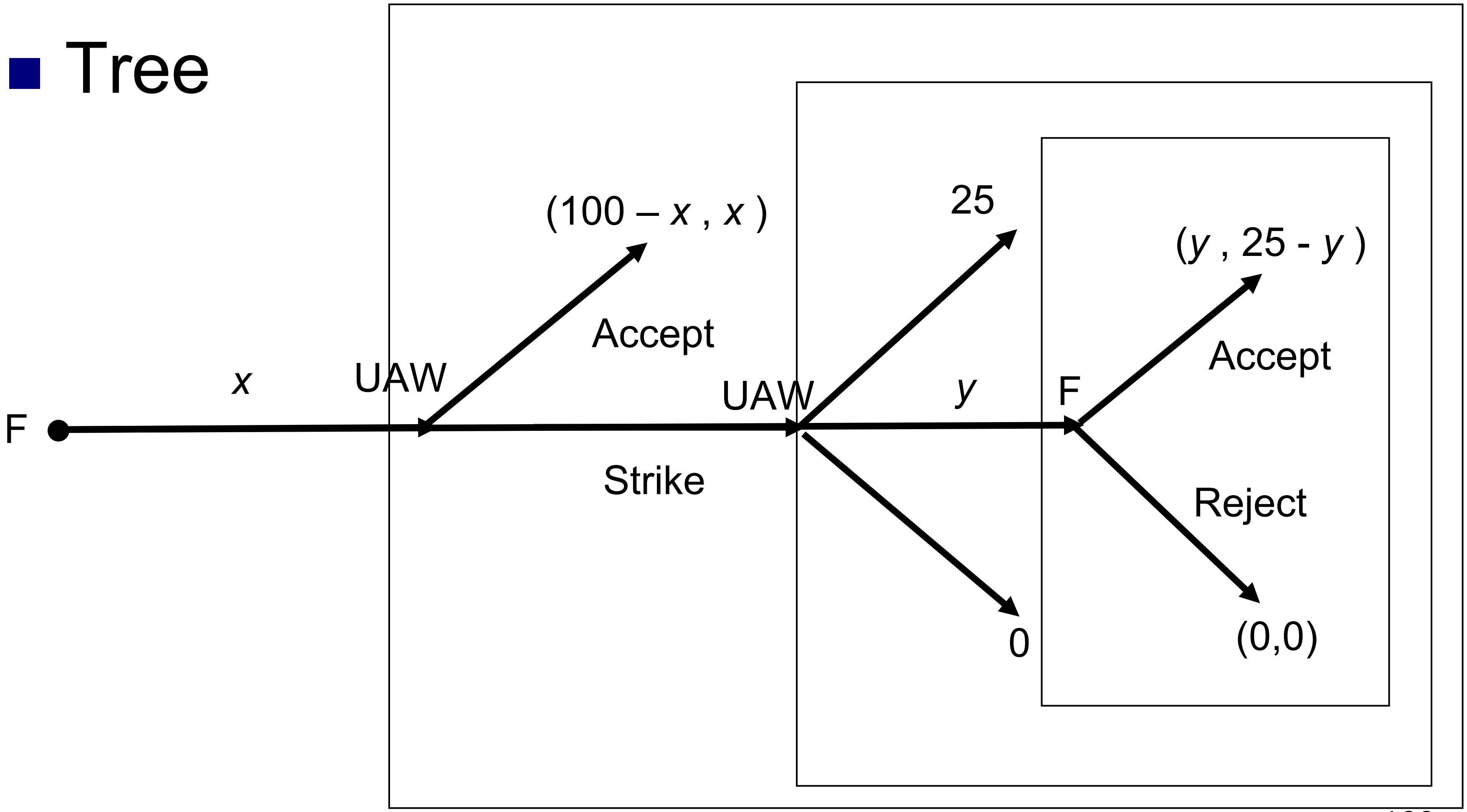
- No. UAW's threat to reject $x < 75$ is not credible.
- UAW's payoff for accepting 70, for example, is still higher than if UAW rejects and gets a payoff of 50.
- If in (b) a Nash was created where Ford also threatened to reject $y < 25$ (for example), then this threat is also not credible, since we know that accepting 10, for example, gives Ford a higher payoff than rejecting and getting 0.

Question (d)

- Now suppose the cost of a strike increases to s = \$75. What is the *UAW*'s backward-induction equilibrium payoff? .

The game tree when s = 75

- Tree



Answer Question (d)

- Starting from the last node, F will accept any offer that makes him weakly better off than rejecting, i.e. accept $y \geq 0$

- At UAW's offer node, they buy the ”cheapest yes” by offering y = 0.
- The outcome of this subgame is (0,25).

Answer Question (d)

- At node 2, UAW will accept any offer that makes them better off than rejecting continuing to the aforementioned subgame.
- They will accept any $x \geq 25$ and reject x < 25.

Answer Question (d)

- At node 1, if Ford offers $x < 25$ then they will receive a payoff of 0, so they choose to buy the ”cheapest yes” which is $x = 25$.
- So, Ford will offer 25 and UAW will accept.
- UAW’s backward induction equilibrium payoff is 25.

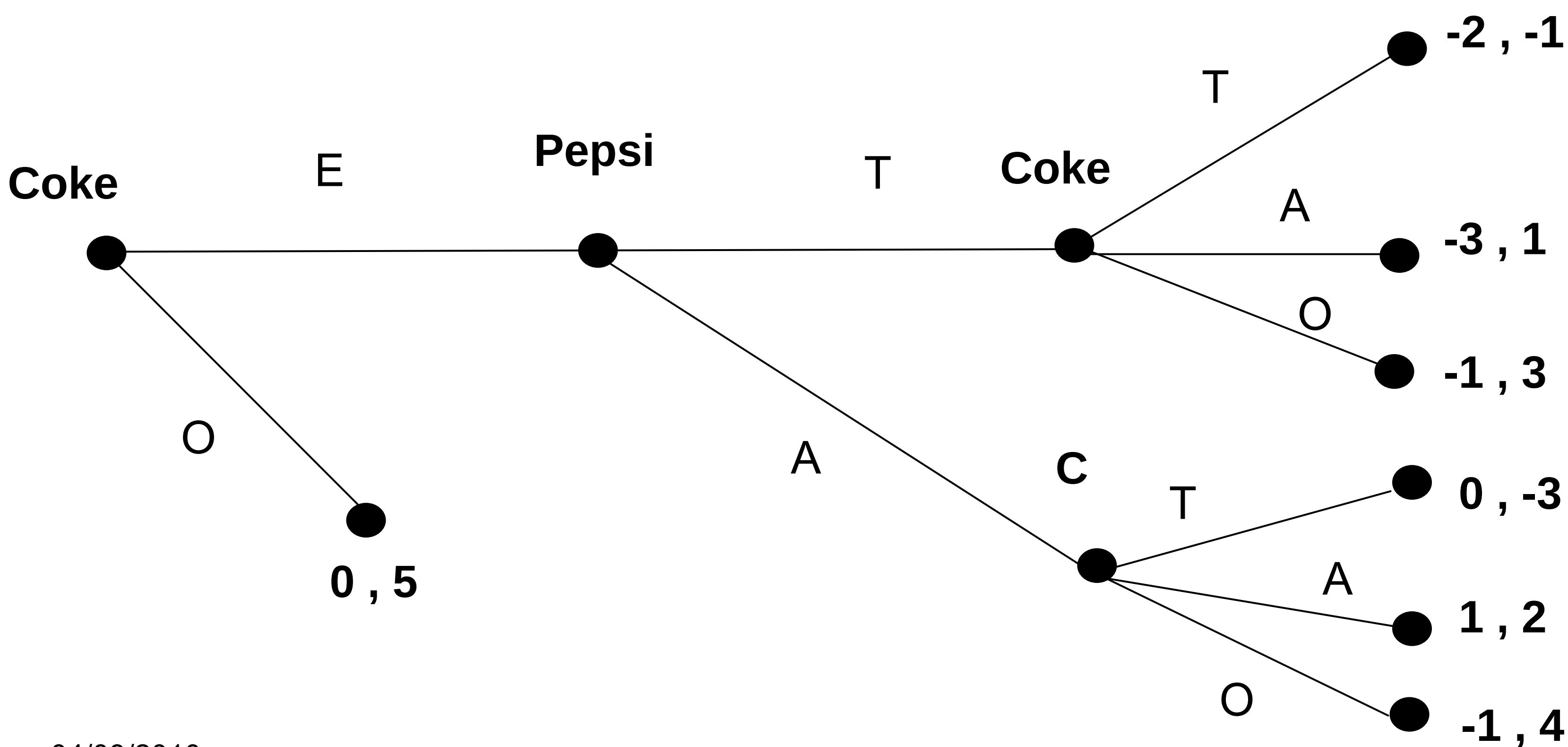
Exercise (Dutta)

- Suppose that Coke's (Player 1) decision on the FSU market is reversible in the following sense.
- After it has entered and after Pepsi (Player 2) has chosen T(ough) or A(ccommodated), Coke has any one of three options to choose from: T, A, O(ut).

Exercise

- Suppose that exiting at that point nets Coke a payoff of -1 and Pepsi a payoff of 3 if it had been *Tough* and 4 had it *Accommodated*.

Extensive form of this game



Question (a)

- Calculate the SPNE of this game

Answer Question (a)

- Backward induction solution:
- In the last stage, Coke exits (plays O) if Pepsi is *Tough* and accommodates (plays A) if Pepsi accommodates.
- In the stage before, Pepsi chooses to be tough (plays T) because it gives a payoff of 3, which is greater than 2 (plays A).

Answer Question (a)

- In the first stage, Coke anticipating the decision of Pepsi will not enter the market (plays O) since her payoff is 0, which is greater than -1.
- As a result, the SPNE is that Coke plays (**O, O, A**) and Pepsi plays **T**.
- Outcome: (0,5)

Question (b)

- Write down the strategic form of the game and calculate all the Nash equilibria.

Normal form of this game

- Let us define the strategies of each player.
- For player 1(Coke), every strategy must have 3 components. She has 3 x 3 x 2 = 18 possible strategies.
- The first component tells C whether or not to enter (takes value E or O), the second tells her whether to play T,A or O if Pepsi acts ”tough” and the third one tells her whether to play T,A or O if Pepsi acts ”accommodate”.

Normal form of this game

- For example, EOT means (1) enter, (2) out against a defender that is tough, (3) tough against a defender that accommodate.
- As a result, player 1 (Challenger) has 18 strategies.
- Player 2 (Pepsi) has only two strategies: T or A.

	T	A
ETT	(-2 , -1)	(0 , -3)
ETA	(-2 , -1)	(1 , 2)
EAT	(-3 , 1)	(0 , -3)
EAA	(-3 , 1)	(1 , 2)
ETO	(-2 , -1)	(-1 , 4)
EOT	(-1 , 3)	(0 , -3)
EOO	(-1 , 3)	(-1 , 4)
EAO	(-3 , 1)	(-1 , 4)
EOA	(-1 , 3)	(1 , 2)
OTT	(0 , 5)	(0 , 5)
OTA	(0 , 5)	(0 , 5)
OAT	(0 , 5)	(0 , 5)
OAA	(0 , 5)	(0 , 5)
OTO	(0 , 5)	(0 , 5)
OOT	(0 , 5)	(0 , 5)
OOO	(0 , 5)	(0 , 5)
OAO	(0 , 5)	(0 , 5)
OOA	(0 , 5)	(0 , 5)

	T	A
ETT	(-2 , **-1**)	(0 , -3)
ETA	(-2 , -1)	(**1** , **2**)
EAT	(-3 , **1**)	(0 , -3)
EAA	(-3 , 1)	(**1** , **2**)
ETO	(-2 , -1)	(-1 , **4**)
EOT	(-1 , **3**)	(0 , -3)
EOO	(-1 , 3)	(-1 , **4**)
EAO	(-3 , 1)	(-1 , **4**)
EOA	(-1 , **3**)	(**1** , 2)
OTT	(**0** , **5**)	(0 , **5**)
OTA	(**0** , **5**)	(0 , **5**)
OAT	(**0** , **5**)	(0 , **5**)
OAA	(**0** , **5**)	(0 , **5**)
OTO	(**0**, **5**)	(0 , **5**)
OOT	(**0** , **5**)	(0 , **5**)
OOO	(**0** , **5**)	(0 , **5**)
OAO	(**0** , **5**)	(0 , **5**)
OOA	(**0** , **5**)	(0 , **5**)

	T	A
ETT	(-2 , **-1**)	(0 , -3)
ETA	(-2 , -1)	(**1** , **2**)
EAT	(-3 , **1**)	(0 , -3)
EAA	(-3 , 1)	(**1** , **2**)
ETO	(-2 , -1)	(-1 , **4**)
EOT	(-1 , **3**)	(0 , -3)
EOO	(-1 , 3)	(-1 , **4**)
EAO	(-3 , 1)	(-1 , **4**)
EOA	(-1 , **3**)	(**1** , 2)
OTT	(**0** , **5**)	(0 , **5**)
OTA	(**0** , **5**)	(0 , **5**)
OAT	(**0** , **5**)	(0 , **5**)
OAA	(**0** , **5**)	(0 , **5**)
OTO	(**0**, **5**)	(0 , **5**)
OOT	(**0** , **5**)	(0 , **5**)
OOO	(**0** , **5**)	(0 , **5**)
OAO	(**0** , **5**)	(0 , **5**)
OOA	(**0** , **5**)	(0 , **5**)

Question (c)

- Solve the normal-form game by IEWDS (Iteration of Elimination of Weakly Dominated Strategies) and show that you obtain the same solution as in the SPNE.

	T	A
ETT	(-2 , -1)	(0 , -3)
ETA	(-2 , -1)	(1 , 2)
EAT	(-3 , 1)	(0 , -3)
EAA	(-3 , 1)	(1 , 2)
ETO	(-2 , -1)	(-1 , 4)
EOT	(-1 , 3)	(0 , -3)
EOO	(-1 , 3)	(-1 , 4)
EAO	(-3 , 1)	(-1 , 4)
EOA	(-1 , 3)	(1 , 2)
OTT	(0 , 5)	(0 , 5)
OTA	(0 , 5)	(0 , 5)
OAT	(0 , 5)	(0 , 5)
OAA	(0 , 5)	(0 , 5)
OTO	(0 , 5)	(0 , 5)
OOT	(0 , 5)	(0 , 5)
OOO	(0 , 5)	(0 , 5)
OAO	(0 , 5)	(0 , 5)
OOA	(0 , 5)	(0 , 5)

	T	A
~~ETT~~	~~(-2 , -1)~~	~~(0 , -3)~~
ETA	(-2 , -1)	(1 , 2)
~~EAT~~	~~(-3 , 1)~~	~~(0 , -3)~~
~~EAA~~	~~(-3 , 1)~~	~~(1 , 2)~~
~~ETO~~	~~(-2 , -1)~~	~~(-1 , 4)~~
EOT	(-1 , 3)	(0 , -3)
EOO	(-1 , 3)	(-1 , 4)
~~EAO~~	~~(-3 , 1)~~	~~(-1 , 4)~~
EOA	(-1 , 3)	(1 , 2)
OTT	(0 , 5)	(0 , 5)
OTA	(0 , 5)	(0 , 5)
OAT	(0 , 5)	(0 , 5)
OAA	(0 , 5)	(0 , 5)
OTO	(0 , 5)	(0 , 5)
OOT	(0 , 5)	(0 , 5)
OOO	(0 , 5)	(0 , 5)
OAO	(0 , 5)	(0 , 5)
OOA	(0 , 5)	(0 , 5)

	T	A
ETT	(-2 , -1)	(0 , -3)
ETA	(-2 , -1)	(1 , 2)
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EAA	(-3 , 1)	(1 , 2)
ETO	(-2 , -1)	(-1 , 4)
EOT	(-1 , 3)	(0 , -3)
EOO	(-1 , 3)	(-1 , 4)
EAO	(-3 , 1)	(-1 , 4)
EOA	(-1 , 3)	(1 , 2)
OTT	(0 , 5)	(0 , 5)
OTA	(0 , 5)	(0 , 5)
OAT	(0 , 5)	(0 , 5)
OAA	(0 , 5)	(0 , 5)
OTO	(0 , 5)	(0 , 5)
OOT	(0 , 5)	(0 , 5)
OOO	(0 , 5)	(0 , 5)
OAO	(0 , 5)	(0 , 5)
OOA	(0 , 5)	(0 , 5)

	T	A
~~ETT~~	~~(-2 , -1)~~	~~(0 , -3)~~
~~ETA~~	~~(-2 , -1)~~	~~(1 , 2)~~
~~EAT~~	~~(-3 , 1)~~	~~(0 , -3)~~
~~EAA~~	~~(-3 , 1)~~	~~(1 , 2)~~
~~ETO~~	~~(-2 , -1)~~	~~(-1 , 4)~~
~~EOT~~	~~(-1 , 3)~~	~~(0 , -3)~~
~~EOO~~	~~(-1 , 3)~~	~~(-1 , 4)~~
~~EAO~~	~~(-3 , 1)~~	~~(-1 , 4)~~
EOA	(-1 , 3)	(1 , 2)
OTT	(0 , 5)	(0 , 5)
OTA	(0 , 5)	(0 , 5)
OAT	(0 , 5)	(0 , 5)
OAA	(0 , 5)	(0 , 5)
OTO	(0 , 5)	(0 , 5)
OOT	(0 , 5)	(0 , 5)
OOO	(0 , 5)	(0 , 5)
OAO	(0 , 5)	(0 , 5)
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	T	A
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OAT	(0 , 5)	(0 , 5)
OAA	(0 , 5)	(0 , 5)
OTO	(0 , 5)	(0 , 5)
OOT	(0 , 5)	(0 , 5)
OOO	(0 , 5)	(0 , 5)
OAO	(0 , 5)	(0 , 5)
OOA	(0 , 5)	(0 , 5)

Matching pennies

- Moving first is not always advantageous!
- For ex, when playing the matching pennies game sequentially, it is clearly not good for a player to move first.

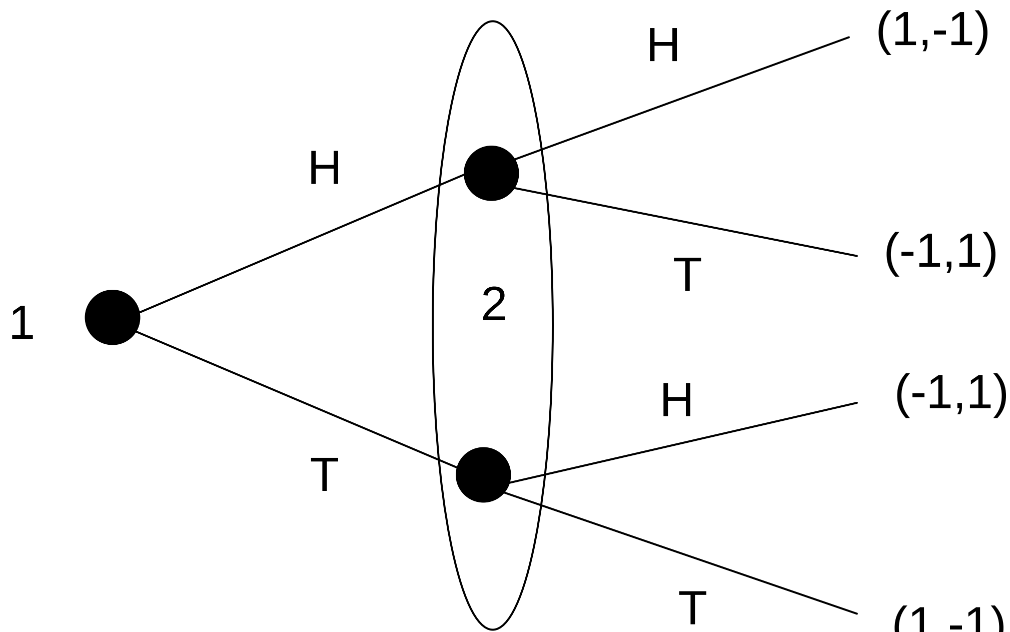
Matching Pennies

Matching Pennies: No Nash equilibrium in pure strategies

		Player 2	
		Head	*Tail*
Player 1	*Head*	1, -1	-1, 1
	Tail	-1, 1	1, -1

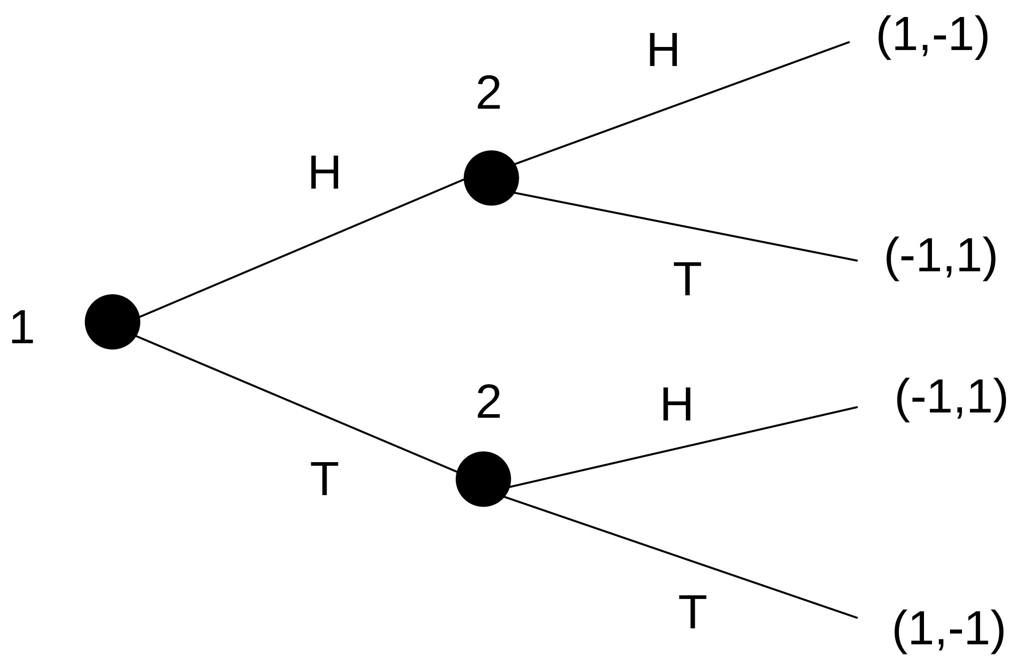
Simmultanous Matching Pennies

- Extensive-form game:



Sequential Matching Pennies

- Extensive-form game:



SPNE: Matching Pennies

- In the last stage, player 2 chooses T if player 1 chooses H and H if player 1 chooses T.
- In the first stage, player 1 will be indifferent between choosing between H and T since she will always get -1.
- There are two SPNE (H,T) and (T,H) which both give payoffs (-1,1).
- Second-mover advantage!

Normal representation of this game

- Player 1 has two strategies: H and T.
- Player 2 has 4 strategies: HH, HT, TH, TT.
- For ex, HT means that player 2 plays H when player 1 plays H and plays T when player 1 plays T.

Normal representation of this game

1/2	HH	HT	TH	TT
H	(1,-1)	(1,-1)	(-1, 1)	(-1, 1)
T	(-1, 1)	(1,-1)	(-1, 1)	(1,-1)

Normal representation of this game

1/2	HH	HT	TH	TT
H	(**1**,-1)	(**1**,-1)	(**-1**, **1**)	(-1, **1**)
T	(-1, **1**)	(**1**,-1)	(**-1**, **1**)	(**1**,-1)

Normal representation of this game 2 NE (H,T) and (T,H)

1/2	HH	HT	TH	TT
H	(**1**,-1)	(**1**,-1)	(**-1**, **1**)	(-1, **1**)
T	(-1, **1**)	(**1**,-1)	(**-1**, **1**)	(**1**,-1)

IEWD in Normal form =SPNE

1/2	HH	HT	TH	TT
H	(1,-1)	(1,-1)	(-1, 1)	(-1, 1)
T	(-1, 1)	(1,-1)	(-1, 1)	(1,-1)

IEWD in Normal form =SPNE

1/2	HH	HT	TH	TT
H	(1,-1)	(1,-1)	(-1, 1)	(-1, 1)
T	(-1, 1)	(1,-1)	(-1, 1)	(1,-1)

IEWD in Normal form = SPNE

1/2	HH	HT	TH	TT
H	(1,-1)	(1,-1)	(-1, 1)	(-1, 1)
T	(-1, 1)	(1,-1)	(-1, 1)	(1,-1)

Exercise

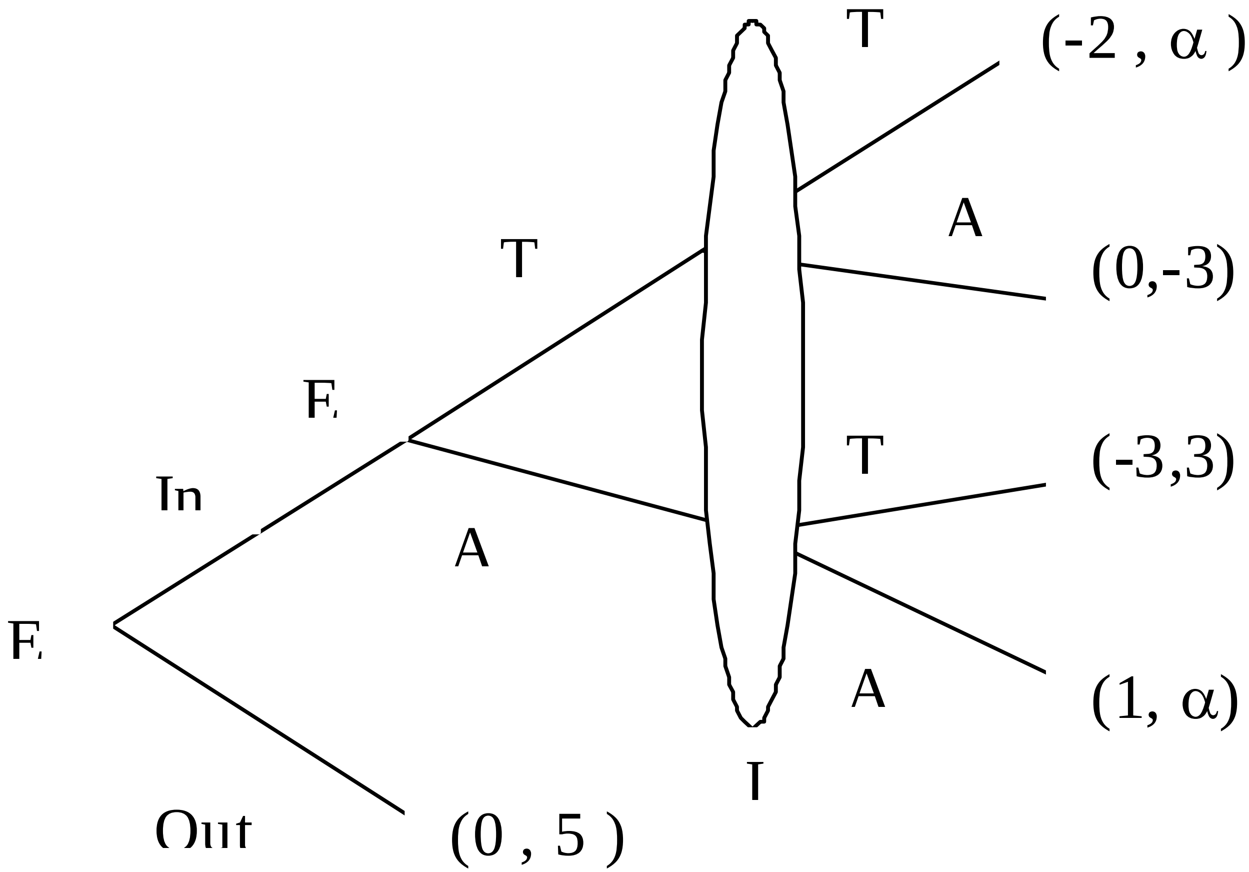
- Firm *I (incumbent)* is alone in a market and has thus monopoly power and earns a profit equals to 5.
- A potential competitor, firm *E* (entrant), considers to enter in this market.
- If *E* does not enter, its profit is 0 and firm *I* has still monopoly power and earns a profit of 5.
- If firm *E* decides to enter, then firms *I* and *E* compete and make decisions *simultaneously* after entry. That is firm *I* can either choose be tough (*T*) or accomodating (*A*).
- Firm E can make the same type of choices: *T* or *A*.
- In that case, post-entry profits are given by: (player 1 is firm *E* and player 2 is firm *I* and α is a constant):

E/I	T	A
T	-2,-α	0,-3
A	-3,3	1, α

Question (a)

- Draw the tree of this game.

Answer Question (a)



Question (b)

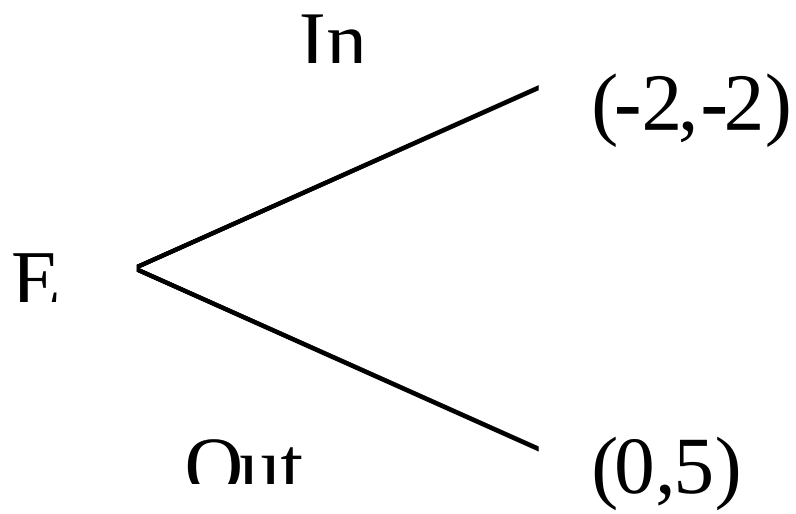
- Assume that $\alpha = 2$. Calculate the subgame perfect Nash Equilibria in pure strategies of this game.

- Backward induction :
- The last subgame is:

E/I	T	A
T	-2,-2	0,-3
A	-3,3	1,2

- Thus, there is a unique Nash equilibrium which is: (T,T)

- The penultimate subgame is now:



Thus, the unique SPNE of this game is to not enter.

Question (c)

- We now suppose that $\alpha > 0$.
- In that case, for what values of α this game has a Subgame Perfect Nash Equilibrium in pure strategies where firm E enters?
- For what values of α this equilibrium is the *unique* Subgame Perfect Nash Equilibrium in pure strategies of this game?

- For firm E to enter, it must have a profit which is either positive or equal to zero.
- This is possible only for strategies (T,A) and (A,A).
- Let us show that (T,A) can never be a NE of the last subgame since it is not a BR for player 1.

- Indeed, firm E's (i.e. player 1) strategy does not depend on α and her BR are given by:

E/I	T	A
T	$\underline{-2}$,$-\alpha$	0,-3
A	-3,3	$\underline{1}$,α

- If $\alpha \geq 3$, then in this last subgame (T,T) and (A,A) are NE.

Conclusion

- If α **= 3**, there are two SPNE: (Out, T, T) and (In, A, A).
- Indeed, last subgame (two NE (T,T) et (A,A))

E/I	T	A
T	-2,-3	0,-3
A	-3,3	1,3

- First stage, (Out, T, T) and (In, A, A) are SPNE.

- If α **> 3**, there is a unique SPNE (In, A, A)
- Indeed, last subgame:

E/I	T	A
T	$\underline{-2}, -\alpha$	$0, \underline{-3}$
A	$-3, 3$	$\underline{1}, \underline{\alpha}$

- One unique NE (A,A).
- First stage:
- (In, A, A): Unique SPNE

Question (d)

- Assume that $\alpha \leq 0$. Determine the SPNE in pure strategies of this game.

- First, observe that, for the last subgame, firm E's (player 1) strategy does not depend on α.
- Furthermore, for player 2 (Firm *I*), when $\alpha \leq 0$, it is easily verified strategy T is a (strictly) dominant strategy and thus will never play strategy A.

- When $\alpha \leq 0$, for the last subgame, we have:

E/I	T	A
T	<u>-2</u>,<u>-α</u>	0,-3
A	-3,<u>3</u>	<u>1</u>,α

- Thus, there is a unique NE which is (T,T).
- Solving the first stage, it is easily seen that it is optimal not to enter for firm E (-2 > 0).

Question (e)

- Give a general conclusion of this game in terms of SPNE in pure strategies for all possible values of α (positive, equal to zero, and negative).
- What is the economic meaning of α ?

- We need first to analyze the case $0 < \alpha < 2$.
- In the last subgame, the BR are:

E/I	T	A
T	$\underline{-2},\underline{-\alpha}$	0,-3
A	$-3,\underline{3}$	$\underline{1},\alpha$

- Thus unique NE (T,T).
- First stage: No enter
- Unique SPNE: (Out, T,T).

General Conclusion

- (*i*) $\alpha \leq 2$. There exists a unique SPNE, which is: (Out, T, T).
- (*ii*) $\alpha = 3$. There are two SPNE: (Out, T, T) and (In, A, A).
- (*iii*) $\alpha > 3$. There exists a unique SPNE: (In, A, A).

- As a result, $|\alpha|$ represents both the price/gain to pay for firm *I* (the incumbent) for being tough against a tough firm *E* and the price/gain for being nice (accomodating) against a « nice » firm E.
- When α is positive, then it pays to be tough for firm I against a tough firm E while it costs to be nice against a nice firm E.
- When α is negative, we have the reverse (i.e. it is better for firm I to be nice upon entry).
- This is why when α is negative, it is better for firm E not to enter because upon entry firm E is better off being agressive.
- When α becomes positive and has a high value, then it is of the interest of firm I to be nice and firm E will enter.

The Stackelberg duopoly model

(example with constant unit cost, linear inverse demand)

- Cost function of firm i= $C_i(q_i)=cq_i$ for all q_i.
- The inverse demand function:

$$P_d(Q) = \begin{cases} \alpha - Q & \text{if } Q < \alpha \\ 0 & \text{if } Q \geq \alpha \end{cases}$$

where $\alpha > 0$ and $c > 0$ and assume that $\alpha > c$.

- Firm *1*'s profits;

$$\pi_1(q_1, q_2) = \begin{cases} q_1(\alpha - c - q_1 - q_2) & \text{if } q_1 + q_2 \leq \alpha \\ -cq_1 & \text{if } q_1 + q_2 > \alpha \end{cases}$$

- Firm *1* sets q_1 before firm 2 sets q_2.

The Stackelberg duopoly game

- **Players:** Firms *1* and *2*
- **Terminal histories:** *The set of all sequences* (q_1, q_2).
- **Player function:** *P(empty history)*=1, $P(q_1)=2$ *for all* q_1.
- **Preferences:**
  - *The payoff to firm 1 after a sequence* (q_1, q_2) *is its profits:* $q_1(\alpha\text{-}c\text{-}q_1\text{-}q_2)$.
  - Permute indices to get firm *2*'s payoff.

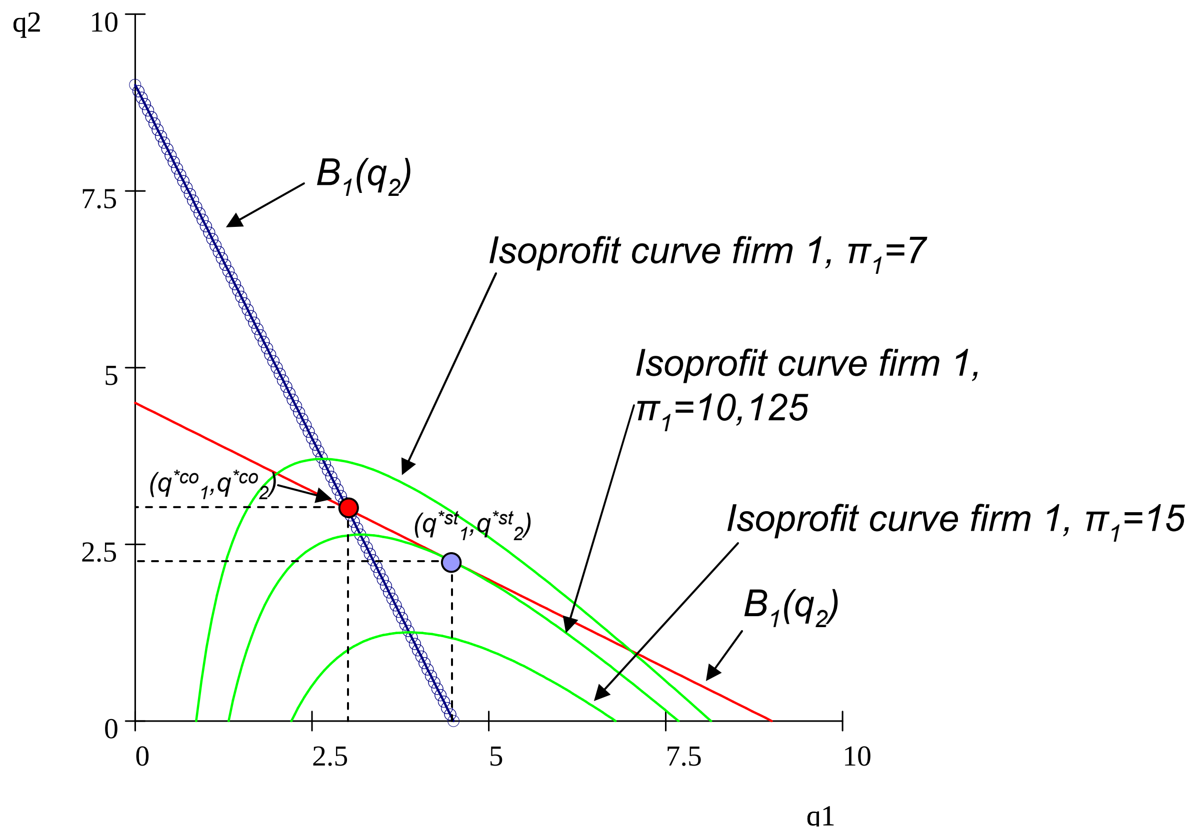
Stackelberg continued…

- Finite horizon game so we can use backward induction.
- Given firm *1*'s choice of q_1, what is the best response of firm *2*?

$$BR_2(q_1) = \begin{cases} \frac{1}{2}(\alpha - c - q_1) & \text{if } q_1 \leq \alpha - c \\ 0 & \text{if } q_1 > \alpha - c \end{cases}$$

- Hence, firm *1* needs to pick the most profitable level of q_1 on firm *2*'s best response function.

Stackelberg continued



Stackelberg continued

- First stage:

$$\pi_1(q_1,q_2) = \begin{cases} q_1\left[\frac{(\alpha-c)}{2} - \frac{q_1}{2}\right] & \text{if } q_1+q_2 \le \alpha \\ -cq_1 & \text{if } q_1+q_2 > \alpha \end{cases}$$

- NE:

$$q_1{}^* = \frac{\alpha - c}{2}$$

$$q_2{}^* = \frac{1}{2}(\alpha - c - q_1{}^*) = \frac{\alpha - c}{4}$$

$$q_1{}^* + q_2{}^* = \frac{3(\alpha - c)}{4} < \alpha \quad \text{since} \quad -3c < \alpha$$

Exercise (with calculus)

- A worker produces the output $q = \sqrt{e}$ when she puts in the amount of effort *e.*
- Her employer pays her a fraction *α* (*α* < 1) of the output she produces;
- if she puts in effort e; the worker's net payoff is: $\alpha\sqrt{e} - e$
- while the employer's payoff is : $(1-\alpha)\sqrt{e}$

Question (a)

- Find all the Nash Equilibria of the game (list the equilibrium strategy profiles and the equilibrium payoffs).

Answer Question (a)

In order to find all the Nash Equilibria of the game, we need to think of it as a two-player simultaneous moves games.

The strategy spaces for the two players would be $[0, 1)$ for the employer (strategy: $\alpha \in [0, 1)$) and $[0,\infty)$ for the worker (strategy: $e \in [0,\infty)$).

The payoff functions for the players would be for the employer:

$$u_E(\alpha, e) = (1 - \alpha)\, e^{1/2}$$

and for the worker:

$$u_W(\alpha, e) = \alpha e^{1/2} - e$$

The best response function for the worker is:

$$\frac{du_W(\alpha, e)}{de} = \frac{\alpha}{2e^{1/2}} - 1 = 0$$

i.e.

$$e = BR_W(\alpha) = \frac{\alpha^2}{4}$$

The reaction function of the employer is given by:

$$\frac{du_E(\alpha, e)}{d\alpha} = -e^{1/2} < 0$$

which implies that the BR function for the employer is:

$$\alpha = BR_E(e) = 0$$

The unique NE of this game is therefore $(\alpha = 0, e = 0)$ and the associated payoffs are $(u_E(0, 0) = 0, u_E(0, 0) = 0)$.

Question (b)

- Assume that first the employer chooses *α* the fraction of output to pay the worker, then the worker chooses her effort level *e*, knowing what the employer has chosen.
- Find the subgame perfect equilibrium choices of *α* and *e* (list the equilibrium strategy profiles, the equilibrium outcome and the resulting equilibrium payoffs)

Answer Question (b)

We need to solve the game using a backward induction approach.

The final subgame is one in which the worker chooses her best effort level given the choice of α made by the employer in the initial stage.

We have seen that the worker will respond to a given α by putting in an effort

$$e = BR_W(\alpha) = \frac{\alpha^2}{4}$$

The initial subgame involves the employer deciding on the size of α given she can anticipate the future play of the game for any given α (i.e. $BR_W(\alpha) = \frac{\alpha^2}{4}$).

Therefore the employer chooses α to maximize

$$\begin{aligned} u_E(\alpha, BR_W(\alpha)) &= (1-\alpha)\,[BR_W(\alpha)]^{1/2} \\ &= (1-\alpha)\left[\frac{\alpha^2}{4}\right]^{1/2} \\ &= \frac{(1-\alpha)\alpha}{2} \end{aligned}$$

The FOC is:

$$\frac{du_E(\alpha, BR_W(\alpha))}{d\alpha} = \frac{d\left[\frac{(1-\alpha)\alpha}{2}\right]}{d\alpha} = 0$$

i.e.

$$\alpha^* = 1/2$$

The unique subgame perfect Nash equilibrium of the game is therefore

$$\alpha^* = 1/2,\ BR_W(\alpha) = \frac{\alpha^2}{4}$$

The unique subgame perfect outcome is

$$\alpha^* = 1/2,\ BR_W(1/2) = \frac{1}{16}$$

and the resulting payoffs are

$$u_E^*(1/2, 1/16) = 1/8$$

$$u_E^*(1/2, 1/16) = 1/16$$

Firm-union bargaining (Exercise 177.1 in Osborne)

A firm's output Q is given by

$$Q = \begin{cases} L(100 - L) & \text{when} \quad L \leq 50 \\ 2500 & \text{when} \quad L > 50 \end{cases}$$

where L is the number of units of labor that the firm uses to produce Q. The price of output is normalized to 1.

A union that represents workers presents a wage demand (a nonnegative number w), which the firm either accepts or rejects. If the firm accepts the demand, it chooses the number L of workers to employ (that maximizes its profits). If the firm rejects the demand, no production takes place and $L = 0$.

The firm's preferences are represented by its profit

$$\Pi = \begin{cases} L(100 - L) - wL & \text{when} \quad L \leq 50 \\ 2500 - wL & \text{when} \quad L > 50 \end{cases}$$

and the union's preferences are represented by

$$U = wL$$

1a) Formulate this situation as an extensive game with perfect information.

Players: The firm and the union

Terminal histories: All sequences of the form (w, Y, L) and (w, N) for nonnegative numbers w and L, where w is the wage, Y means accept, N means reject, and L is the number of workers hired.

Player function: $P(\emptyset)$ is the union, and for any nonnegative number w, $P(w)$ and $P(w, Y)$ are the firm.

Preferences: Profit Π for the firm and U for the union.

1b) Find the subgame perfect equilibrium (equilibria?) of the game.

To find the subgame perfect equilibrium of this game (Nash equilibrium in each subgame), we have to use a backward induction to solve this problem.

Let us thus first consider the subgame following a history (w, Y), i.e. the union demands w and the firm has accepted the demand. In a subgame perfect equilibrium, the firm chooses L to maximize its profit, given w.

For $L \leq 50$, $\Pi = L(100 - L) - wL = (100 - w)L - L^2$. This is a quadratic function in L that is concave and that is equal to zero when $L = 0$ and when $L = 100 - w$, and reached a maximum L^* in between these two values.

This maximum is

$$\frac{\partial \Pi}{\partial L} = 100 - w - 2L^* = 0$$

which implies that

$$L^* = \begin{cases} (100 - w)/2 & \text{if} \quad w \leq 100 \\ 0 & \text{if} \quad w > 100 \end{cases}$$

Given the firm's optimal action in such a subgame, consider now the subgame following a history w, in which the firm has to decide whether to accept or reject w. For any w, the firm's profit, given its subsequent optimal choice of L, is such that:

(i) if $w < 100$, this profit is positive;

(ii) if $w \geq 100$, this profit is zero.

Thus, in a subgame perfect equilibrium, the firm accepts any demand $w < 100$ and either accepts or rejects any demand $w \geq 100$.

Finally, consider the union's choice at the beginning of the game. If it chooses $w < 100$, then the firm accepts and chooses $L - (100 - w)/2$, yielding the union a payoff of $w(100 - w)/2$. If it chooses $w > 100$, then the firm either accepts and chooses $L = 0$ or rejects; in both cases, the union's payoff is 0.

Thus the best value of w for the union is the number that maximizes $U = w(100 - w)/2$. This function U is quadratic in w and concave, is zero when $w = 0$ and $w = 100$ and reaches a maximum w^* in between.

This value is

$$\frac{\partial U}{\partial w} = \frac{100 - 2w^*}{2} = 0$$

which implies that

$$w^* = 50$$

As a result, there is a unique subgame perfect equilibrium in which the union's strategy is $w^* = 50$ and the firm's strategy is to accept this demand and chooses $L^* = 25$.

In this case,

$$\Pi^* = (100 - w^*)L^* - L^{*2} = 625$$

$$U^* = w^* L^* = 1250$$

1c) Is there an outcome of the game that both parties prefer to any subgame perfect equilibrium outcome?

Yes. In any subgame perfect equilibrium $\Pi^* = 625$ and $U^* = 1250$. Thus both parties are better off at the outcome (w, L) that they are in the unique subgame perfect equilibrium if and only if

$$L \leq 50$$

and

$$U = wL > 1250$$

and

$$\Pi = L(100 - L) - wL > 625$$

or

$$L \geq 50$$

and

$$wL > 1250$$

and

$$2500 - wL > 625$$

These conditions are satisfied for a nonempty set of pairs (w, L). For example, if $L = 50$, the conditions are satisfied by $25 < w < 37.5$. If $L = 100$, they are satisfied by $12.5 < w < 18.75$.

1d) Find a Nash equilibrium for which the outcome differs from any subgame perfect equilibrium outcome.

There are many Nash equilibria in which the firm "threatens" to reject high wage demands. In one such Nash equilibrium the firm threatens to reject any positive wage demand. In this equilibrium, the union's strategy is $w = 0$, and the firm's strategy rejects any demand $w > 0$ and accepts the demand $w = 0$ and chooses $L = 50$.