Advanced Microeconomic Theory EC104
Answers Problem Set 2

1. In the following two games find the strategy profiles that survive iterated elimination of dominated strategies. (In part b do not forget mixed strategies).

a.

	x_2	y_2	z_2
x_1	4,3	5,1	6,2
y_1	2,1	8,4	3,6
z_1	3,0	9,6	2,8

The game does not admit dominant strategies for any player. Nevertheless, for player 2, y_2 is strictly dominated by strategy z_2. Player 1 can anticipate player 2 will never play y_2, therefore y_1 and z_1 are dominated by x_1. Given that player 2 is never going to play y_2 and player 1 is never going to play y_1 and z_1, then player 2 is never going to play z_2 (dominated by x_2). Therefore the strategy profile that survives IEDS is (x_1, x_2).

b.

	x_2	y_2
x_1	2,0	$-1, 0$
y_1	0,0	0,0
z_1	$-1, 0$	2,0

No dominant strategy for any player. No pure strategy is dominated by any other pure strategy. Nevertheless, notice that the mixed strategy of playing x_1 with probability 0.5 and z_2 with probability 0.5 dominates the pure strategy y_1. Hence, $(x_1, x_2), (x_1, y_2)$ (z_1, x_2) and (z_1, y_2) are the surviving pure strategy profiles.

2. For each of the following two player games, find all equilibria. As usual, player 1 chooses the row and player 2 chooses the column. In part e, player 1 chooses between rows, player 2 between columns and player 3 between the 'boxes'.

a.

	x_2	y_2
x_1	2,1	1,2
y_1	1,5	2,1

It is straightforward to check that this game has no equilibrium in pure strategies. So let's look for a mixed strategy equilibrium. Accordingly, denote p as the probability of x_1 and q the probability of x_2. To find p set $\mathbb{E}_2(x_2) = \mathbb{E}_2(y_2)$ for player 2:

$$\Rightarrow \quad p \times 1 + (1-p) \times 5 = p \times 2 + (1-p) \times 1$$
$$\Rightarrow \quad -4p + 5 = 2p + 1 - p \Rightarrow p = 0.8$$

To find q set $\mathbb{E}_1(x_1) = \mathbb{E}_1(y_1)$ for player 1:

$$\Rightarrow q \times 2 + (1-q) \times 1 = q \times 1 + (1-q) \times 2 \Rightarrow q = 0.5.$$

b.

	x_2	y_2
x_1	3,7	6,6
y_1	2,2	7,3

Check (x_1, x_2) and (y_1, y_2) are the two pure strategy Nash equilibria. We must also look for a mixed strategy equilibria. Let p be the probability of x_1 and q the probability of x_2. Setting $\mathbb{E}_2(x_2) = \mathbb{E}_2(y_2)$ we get

$$p \times 7 + (1-p) \times 2 = p \times 6 + (1-p) \times 3 \Rightarrow p = 0.5.$$

Setting $\mathbb{E}_1(x_1) = \mathbb{E}_1(y_1)$ we get

$$q \times 3 + (1-q) \times 6 = q \times 2 + (1-q) \times 7 \Rightarrow q = 0.5.$$

c.

	x_2	y_2
x_1	7,3	6,6
y_1	2,2	3,7

Notice x_1 dominates y_1. Hence, the only possible equilibrium, in both pure and mixed strategies, is (x_1, y_2).

d.

	x_2	y_2	z_2
x_1	4,2	5,1	0,3
y_1	1,3	0,1	2,2

Clearly y_2 is strictly dominated by x_2 (and also z_2). Hence, we may consider the following "reduced" game

	x_2	z_2
x_1	4,2	0,3
y_1	1,3	2,2

.

In this game the only equilibrium is in mixed strategies. Calculating mixed strategies equilibrium as in parts (a) and (b) we obtain $\Pr(x_1) = \frac{1}{2} = \Pr(y_1)$ and $\Pr(x_2) = \frac{2}{5}, \Pr(z_2) = \frac{3}{5}$.

e.

x_3	x_2	y_2	y_3	x_2	y_2
x_1	5,2,3	6,1,2	x_1	1,2,2	6,1,1
y_1	4,5,1	8,6,4	y_1	9,0,0	3,2,5

In order to derive the best response mapping for each player, proceed as follows. Take player 1's point of view: given player 2 is playing x_2 and player 3 is playing x_3, player 1's best response is to play x_1 because $5 > 4$; if player 2 is still playing x_2, but player 3 is playing $y_{3,}$ player 1's best response is y_1 because $9 > 1$. Proceed in this manner considering all possible strategy profiles and taking each player point of view in turn. You will get the following best response mappings for all players:

	x_3			y_3	
	x_2	y_2		x_2	y_2
x_1	$\underline{5},\underline{2},\underline{3}$	$6,1,\underline{2}$	x_1	$1,\underline{2},2$	$\underline{6},1,1$
y_1	$4,5,\underline{1}$	$\underline{8},\underline{6},4$	y_1	$\underline{9},0,0$	$3,\underline{2},\underline{5}$

Hence, (x_1, x_2, x_3) is the only pure strategy Nash equilibrium. No need to search for mixed strategy equilibria here, as we have a unique equilibrium in pure strategies (search for mixed ones whenever you find more than one Nash equilibrium in pure strategies or you find none).

3. Consider the following normal form game:

1/2	L	R
T	a,b	c,d
B	e,f	g,h

3a. Determine the conditions for (B,R) to be a Nash Equilibrium of the game.

The strategy profile (B,R) is a Nash Equilibrium if no player has an incentive to deviate from his/her strategy given that the other player is playing the strategy specified by the strategy profile (B,R):
1. Given player 2 is playing R, player 1 has no incentive to deviate from playing B only if $g \geq c$.
2. Given player 1 is playing B, player 2 has no incentive to deviate from R only if $h \geq f$.
Therefore, (B,R) is a NE if $g \geq c$ and $h \geq f$.

3b. Assume $a = h = 6, b = g = 1, c = d = 0, e = f = x$. For what values of x does the game has dominant strategy equilibria?

We have:

1/2	L	R
T	6,1	0,0
B	x, x	1,6

For B to be a dominant strategy for player 1 we need to have $x \geq 6$. For L to be a dominant strategy for player 2 we need to have $x \geq 6$. Therefore, (B,L) is the unique dominant strategy equilibrium of the game if $x \geq 6$.

3c. Assume $a = h = 2, b = g = 1$, $c = d = e = f = 0$. Assume player 1 plays the mixed strategy $(r, 1-r)$ and player 2 the mixed strategy $(q, 1-q)$, where $r \in [0,1]$ and $q \in [0,1]$. Derive the best response mapping for both players and find all the Nash Equilibria of the game. Represent both best response mappings and equilibria in an appropriate diagram.

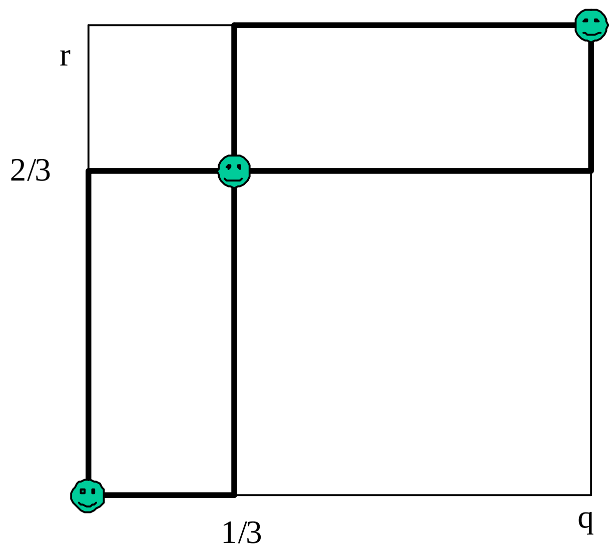
We have:

1/2	L	R
T	2,1	0,0
B	0,0	1,2

The best response mapping for player 1 can be derived by maximizing his expected payoff with respect to r for any given alternative strategy q chosen by player 2:

$$\mathbb{E}_1(r|q) = r(2q) + (1-r)(1-q) = r(3q-1) + 1 - q$$

If $3q - 1 < 0$, i.e. $q < 1/3$, the best response for player 1 is to play $r = 0$; if $q > 1/3$, $\mathbb{E}_1(r|q)$ is maximized at $r = 1$ and if $q = 1/3$ player 1 is indifferent between any $r \in [0,1]$. $\mathbb{E}_2(q|r) = q(r) + 2(1-q)(1-r) = q(3r-2) + 2(1-q)$. Therefore $\mathbb{E}_2(q|r)$ is maximized at $q = 0$ if $r < 2/3$, at $q = 1$ if $r > 2/3$ and $\mathbb{E}_2(q|r) = 2/3$ if $r = 2/3$ regardless of the value taken by q. We can therefore represent the best response mapping using the diagram in the following figure:



There are three Nash Equilibria, 2 in pure strategies and 1 in mixed strategies, identified by the intersection of the best-response mappings of the two players:

$$\{(0,1;0,1),(1,0;1,0),(2/3,1/3;1/3,2/3)\}$$

where each element of this set is a strategy profile $(r, 1-r; q, 1-q)$.

4. Suppose you play in a football team, and you are about to take a penalty kick. You have to decide whether to kick to the left or right corner of the goal. Your opponent team's goalkeeper, in turn, has to decide whether to dive left or right. To put some numbers to this, assume that if the goalkeeper dives left (right) when you kick left (right), then the goalkeeper blocks the kick with probability one. On the other hand, if you kick left (right) and the goalkeeper dives right (left), then you will definitely score a goal with probability one.

4a. Model this story as a normal form game (use a matrix in which the payoffs for the penalty kicker and the goalkeeper are the probabilities of scoring a goal and blocking a kick, respectively, for any combination of strategies).

The payoff matrix is:

K/G	L	R
L	0,1	1,0
R	1,0	0,1

4b. Find all the Nash Equilibria of the game (in pure and/or mixed strategies).

The best-response functions are given by:

K/G	L	R
L	$0,\underline{1}$	$\underline{1},0$
R	$\underline{1},0$	$0,\underline{1}$

Hence, no NE in pure strategies. The game admits 1 NE in mixed strategies given by $(1/2, 1/2; 1/2, 1/2)$.

4c. Find all the Nash Equilibria of the game when the penalty kicker has 2/3 chance of scoring if he kicks left and the goalkeeper dives left, and only 1/3 chance if he kicks right and the goalkeeper dives right.

The payoff matrix is now given by:

K/G	L	R
L	2/3,1/3	1,0
R	1,0	1/3,2/3

Again, no NE in pure strategies. The game admits 1 NE in mixed strategies: $(2/3, 1/3; 2/3, 2/3)$.

5. (Mas-Colell, Whinston and Green)**

Consumers are uniformly distributed along a boardwalk that is 1 mile long. Ice-cream prices are regulated, so consumers go to the nearest vendor because they dislike walking (assume that at the regulated prices all consumers will purchase an ice cream even if they have to walk a full mile). If more than one vendor is at the same location, they split the business evenly.

5a. Consider a game in which two ice-cream vendors pick their locations simultaneously. Model this situation as a strategic game. In particular, give the exact values of the payoff functions of the two vendors as a function of their relative locations.

A strategic game that models this situation is:

Players: The two vendors.

Actions: Let x_i be the location of vendor $i = 1, 2$. The set of actions of each player $i = 1, 2$ is the set of possible locations, which we can take to be the set of numbers x_i for which $x_i \in [0, 1]$.

Preferences: We need to find the payoff function of each vendor. Since the price of the ice cream is regulated, we can identify the profit of each vendor with the number of customers she gets.

Suppose that $x_1 < x_2$. Then vendor 1 will get $x_1 + (x_2 - x_1)/2 = (x_1 + x_2)/2$ customers. Thus all customers located to the left of $(x_1 + x_2)/2$ will purchase from vendor 1 while all consumers located to the right of $(x_1 + x_2)/2$ will purchase from vendor 2. Therefore:

$$u_1(x_1, x_2) = \frac{x_1 + x_2}{2}$$

$$u_2(x_1, x_2) = 1 - \left(\frac{x_1 + x_2}{2}\right)$$

If now $x_2 < x_1$, we have:

$$u_1(x_1, x_2) = 1 - \left(\frac{x_1 + x_2}{2}\right)$$

$$u_2(x_1, x_2) = \frac{x_1 + x_2}{2}$$

If $x_1 = x_2$, the vendors split the business so that $u_1(x_1, x_2) = u_2(x_1, x_2) = 1/2$.

Summarizing, the payoffs are given by:

$$u_1(x_1, x_2) = \begin{cases} (x_1 + x_2)/2 & \text{if } x_1 < x_2 \\ 1/2 & \text{if } x_1 = x_2 \\ 1 - (x_1 + x_2)/2 & \text{if } x_1 > x_2 \end{cases} \tag{1}$$

$$u_2(x_1, x_2) = \begin{cases} 1 - (x_1 + x_2)/2 & \text{if } x_1 < x_2 \\ 1/2 & \text{if } x_1 = x_2 \\ (x_1 + x_2)/2 & \text{if } x_1 > x_2 \end{cases} \tag{2}$$

5b. Show that there exists a unique pure strategy Nash equilibrium and that it involves both vendors locating at the midpoint of the boardwalk.

From the payoffs (1) and (2), it is straightforward to check that

$$x_1^* = x_2^* = 1/2$$

constitutes a Nash equilibrium since no firm can do better by deviating. Indeed, fix $x_2^* = 1/2$ and see if vendor 1 wants to deviate. If vendor 1 deviates by locating slightly on the left of x_2^*, we are in the case when $x_1 < x_2^* = 1/2$, and according to (1),

$$u_1(x_1, x_2^*) = \frac{(x_1 + x_2^*)}{2} = \frac{x_1}{2} + \frac{1}{4}$$

which is strictly less than 1/2 since $x_1 < 1/2$. Similarly, if vendor 1 deviates by locating slightly on the right of x_2^*, we are in the case when $x_1 > x_2^* = 1/2$, and according to (1),

$$u_1(x_1, x_2^*) = 1 - \frac{(x_1 + x_2^*)}{2} = 1 - \frac{x_1}{2} - \frac{1}{4} = \frac{3}{4} - \frac{x_1}{2}$$

which is strictly less than 1/2 since $x_1 > 1/2$. As a result, vendor 1 will not deviate from $x_1^* = 1/2$. Because of the symmetry, we can apply the same reasoning to show that vendor 2 will not deviate from $x_2^* = 1/2$. As a result, $x_1^* = x_2^* = 1/2$ is a Nash equilibrium.

Let us now show that this Nash equilibrium is unique.

Suppose first that $x_1 = x_2 < 1/2$. Then any firm can do better by moving by $\varepsilon > 0$ to the right, since it will get almost $1 - x_1 > 1/2$. By a similar argument, it can be shown that $x_1 = x_2 > 1/2$ cannot be a Nash equilibrium.

Suppose now that $x_1 < x_2$. Then firm 1 can do better by moving to $x_2 - \varepsilon$, with $\varepsilon > 0$, therefore this could not have been a Nash equilibrium. By a similar argument, it can be shown that $x_1 > x_2$ cannot be a Nash equilibrium.

As a result, $x_1^* = x_2^* = 1/2$ is the unique Nash equilibrium of this game.

5c. Show that with three vendors, no pure strategy Nash equilibrium exists.

Suppose that a Nash equilibrium (x_1^*, x_2^*, x_3^*) exists. Suppose first that $x_1^* = x_2^* = x_3^*$. Then each firm makes a profit of 1/3. But any firm can increase its profit by moving to the right (if $x_1^* = x_2^* = x_3^* < 1/2$) or to the left (if $x_1^* = x_2^* = x_3^* \geq 1/2$), a contradiction. Thus, $x_1^* = x_2^* = x_3^*$ cannot be a Nash equilibrium.

Suppose now that two firms locate at the same point, say (without loss of generality) $x_1^* = x_2^*$. If $x_1^* = x_2^* < x_3^*$, then firm 3 can do better by moving to $x_1^* + \varepsilon$, with $\varepsilon > 0$. If $x_1^* = x_2^* > x_3^*$, then firm 3 can do better by moving to $x_1^* - \varepsilon$, with $\varepsilon > 0$, a contradiction. Thus, two firms locate at the same point cannot constitute a Nash equilibrium.

Finally, suppose that all 3 firms are located at different points. But then the firm that is located the farthest on the right will be able to increase its profit by moving to the left by $\varepsilon > 0$, a contradiction.

As a result, there does not exist a pure strategy Nash equilibrium in location of this game with three vendors.

6. *** Consider the Cournot duopoly model in which two firms 1 and 2, simultaneously choose the quantities they will sell on the market, q_1 and q_2. The price each receives for each unit given these quantities is: $P(q_1, q_2) = a - b(q_1 + q_2)$. Their costs are c per unit sold.

6a. Argue that successive elimination of strictly dominated strategies yields a unique prediction in this game, which is exactly the unique Nash equilibrium of this game.

Suppose player 1 produces q_1. Player 2's best response can be calculated by maximizing (this is symmetric for both players):

$$\max_{q_2} \pi_2 = [a - b(q_1 + q_2) - c]\, q_2$$

which yields the following first-order condition:

$$a - b(2q_2 + q_1) - c = 0$$

so the best response is:

$$q_2(q_1) \equiv BR_2(q_1) = \frac{(a-c)}{2b} - \frac{q_1}{2} \tag{3}$$

Now, since $q_1 \geq 0$, then

$$q_2 \leq (a-c)/2b$$

(all other strategies in excess of $(a-c)/2b$ would be strictly dominated by $q_2 = (a-c)/2b$). Therefore, since $q_2 \leq (a-c)/2b$, we have that (by symmetry of (3)):

$$q_1 = \frac{(a-c)}{2b} - \frac{q_2}{2} \geq \frac{(a-c)}{2b} - \frac{(a-c)/2b}{2} = \frac{(a-c)}{4b}$$

or equivalently

$$q_1 \geq \frac{(a-c)}{4b}$$

Now, since $q_1 \geq (a-c)/4b$, then

$$q_2 = \frac{(a-c)}{2b} - \frac{q_1}{2} \leq \frac{(a-c)}{2b} - \frac{(a-c)/4b}{2} = \frac{3(a-c)}{8b}$$

or equivalently

$$q_2 \leq \frac{3(a-c)}{8b}$$

Let continue this process. Since $q_2 \leq \frac{3(a-c)}{8b}$, then

$$q_1 = \frac{(a-c)}{2b} - \frac{q_2}{2} \geq \frac{(a-c)}{2b} - \frac{3(a-c)/8b}{2} = \frac{5(a-c)}{16b}$$

Now since $q_1 \geq \frac{5(a-c)}{16b}$, then

$$q_2 = \frac{(a-c)}{2b} - \frac{q_1}{2} \leq \frac{(a-c)}{2b} - \frac{5(a-c)/16b}{2} = \frac{11(a-c)}{32b}$$

Since $q_2 \leq \frac{11(a-c)}{16b}$, then

$$q_1 = \frac{(a-c)}{2b} - \frac{q_2}{2} \geq \frac{(a-c)}{2b} - \frac{11(a-c)/32b}{2} = \frac{21(a-c)}{64b}$$

Now since $q_1 \geq \frac{21(a-c)}{64b}$, then

$$q_2 = \frac{(a-c)}{2b} - \frac{q_1}{2} \leq \frac{(a-c)}{2b} - \frac{21(a-c)/64b}{2} = \frac{43(a-c)}{128b}$$

We have now that

$$q_1 \geq \frac{21(a-c)}{64b} = 0.328\frac{(a-c)}{b} \text{ and } q_2 \leq \frac{43(a-c)}{128b} = 0.336\frac{(a-c)}{b}$$

Since $q_2 \leq \frac{43(a-c)}{128b}$, then

$$q_1 = \frac{(a-c)}{2b} - \frac{q_2}{2} \geq \frac{(a-c)}{2b} - \frac{43(a-c)/128b}{2} = \frac{85(a-c)}{256b}$$

Now since $q_1 \geq \frac{85(a-c)}{256b}$, then

$$q_2 = \frac{(a-c)}{2b} - \frac{q_1}{2} \leq \frac{(a-c)}{2b} - \frac{85(a-c)/256b}{2} = \frac{171(a-c)}{512b}$$

We have now that

$$q_1 \geq \frac{85(a-c)}{256b} = 0.33203\frac{(a-c)}{b} \text{ and } q_2 \leq \frac{171(a-c)}{512b} = 0.33398\frac{(a-c)}{b}$$

Since $q_2 \leq \frac{171(a-c)}{512b}$, then

$$q_1 = \frac{(a-c)}{2b} - \frac{q_2}{2} \geq \frac{(a-c)}{2b} - \frac{171\,(a-c)\,/512b}{2} = \frac{341\,(a-c)}{1024b}$$

Now since $q_1 \geq \frac{341(a-c)}{1024b}$, then

$$q_2 = \frac{(a-c)}{2b} - \frac{q_1}{2} \leq \frac{(a-c)}{2b} - \frac{341\,(a-c)\,/1024b}{2} = \frac{683\,(a-c)}{2048b}$$

We have now that

$$q_1 \geq \frac{341\,(a-c)}{1024b} = 0.33301\frac{(a-c)}{b} \text{ and } q_2 \leq \frac{683\,(a-c)}{2048b} = 0.33350\frac{(a-c)}{b}$$

Continuing in this fashion, we will obtain:

$$q_1 = q_2 = \frac{(a-c)}{3b} = 0.333\frac{(a-c)}{b}$$

Thus, after successive elimination of strictly dominated strategies, we obtain:

$$q_1 = q_2 = \frac{(a-c)}{3b} = 0.333\frac{(a-c)}{b}$$

which is the unique Nash equilibrium of this game.

6b. Would this be true if there were three firms instead of two?

Suppose that player 2 produce q_2 and player 3 produces q_3. Player 1's best response is given by:

$$\max_{q_1} \pi_1 = [a - b\,(q_1 + q_2 + q_3) - c]\,q_1$$

which yields the following first-order condition:

$$a - b\,(2q_1 + q_2 + q_3) - c = 0$$

so the best response is:

$$q_1(q_2, q_3) \equiv BR_1(q_2, q_3) = \frac{(a-c)}{2b} - \frac{(q_2 + q_3)}{2} \tag{4}$$

Now, since $q_2 \geq 0$ and $q_3 \geq 0$,

$$q_1 \leq \frac{(a-c)}{2b}$$

(all other strategies would be strictly dominated by $q_1 = (a-c)\,/2b$). Therefore, since $q_1 \leq (a-c)\,/2b$, and (by symmetry of (3)) similarly, $q_3 \leq (a-c)\,/2b$, we have:

$$q_2 = \frac{(a-c)}{2b} - \frac{(q_2 + q_3)}{2} \geq \frac{(a-c)}{2b} - \frac{(a-c)\,/2b + (a-c)\,/2b}{2} = 0$$

or equivalently

$$q_2 \geq 0$$

Therefore, successive elimination of strictly dominated strategies, implies that $q_1 \geq 0$, $q_2 \geq 0$, $q_3 \geq 0$ and $q_1 \leq (a-c)\,/2b$, $q_2 \leq (a-c)\,/2b$, $q_3 \leq (a-c)\,/2b$. However, a unique prediction cannot be obtained.

7. (Osborne, Exercise 114.4)*** Swimming with sharks

You and a friend are spending two days at the beach; you both enjoy swimming. Each of you believes that with probability π the water is infested with sharks. If sharks are present, a swimmer will surely be attacked. Each of you has preferences represented by the *expected value* of a payoff function that assigns $-c$ to being attacked by a shark, 0 to sitting on the beach, and 1 to a day's worth of undisturbed swimming (where $c > 0!$).

If a swimmer is attacked by sharks on the first day, then you both deduce that a swimmer will surely be attacked the next day, and hence do not go swimming the next day.

If at least one of you swims on the first day and is not attacked, then you both know that the water is shark-free.

If neither of you swims on the first day, each of you retains the belief that the probability of the water's being infested is π, and hence on the second day swim only if $-\pi c + 1 - \pi > 0$ and sits on the beach if $-\pi c + 1 - \pi < 0$, thus receiving an expected payoff of $\max\{-\pi c + 1 - \pi, 0\}$.

7a. Model this situation as a strategic game in which you and your friend each decides whether to go swimming on your first day at the beach. If, for example, you go swimming on the first day, you (and your friend, if she goes swimming) are attacked with probability π, in which case you stay out of the water on the second day; you (and your friend, if she goes swimming) swim undisturbed with probability $1 - \pi$, in which case you swim on the second day. Thus your expected payoff if you swim on the first day is

$$\pi(-c + 0) + (1 - \pi)(1 + 1) = -\pi c + 2(1 - \pi),$$

independent of your friend's action.

A strategic game that models this situation is:

Players: The two friends.

Actions: The set of actions of each player $i = 1, 2$ is $\{Swim\ today, Wait\}$.

Preferences:

(i) If you swim today, your expected payoff is $-\pi c + 2(1 - \pi)$, regardless of your friend's action.

(ii) If you do not swim today and your friend does, then with probability π your friend is attacked and you do not swim tomorrow, and with probability $1 - \pi$ your friend is not attacked and you do swim tomorrow.

Thus your expected payoff in this case is

$$\pi \times 0 + (1 - \pi) \times 1 = 1 - \pi$$

(iii) If neither of you swims today then your expected payoff is

$$\max\{-\pi c + 1 - \pi, 0\}$$

since you are acting optimally on the second day.

Hence player 1's and 2's payoffs in this game are given by the following matrix:

1/2	*Swim today*	*Wait*
Swim today	$-\pi c + 2(1-\pi)$, $-\pi c + 2(1-\pi)$	$-\pi c + 2(1-\pi)$, $1-\pi$
Wait	$1-\pi$, $-\pi c + 2(1-\pi)$	$\max\{-\pi c + 1 - \pi, 0\}$, $\max\{-\pi c + 1 - \pi, 0\}$

7b. Find the mixed strategy Nash equilibria of the game (depending on c and π).

To find the mixed strategy Nash equilibria, first note that if

$$-\pi c + 1 - \pi > 0$$

or equivalently

$$c < \frac{(1-\pi)}{\pi} \tag{5}$$

then "Swim today" is the best response to both "Swim today" and "Wait". Indeed, if (5) holds, "Swim today" is a dominant strategy for both players since this guarantees that $-\pi c + 2(1-\pi) > 1 - \pi$ and

$$-\pi c + 2(1-\pi) > \max\{-\pi c + 2(1-\pi), 0\} = -\pi c + 2(1-\pi)$$

Thus, if (5) holds, there is a unique mixed strategy Nash equilibrium, in which both players choose "Swim today". In other words,

(i) If $c < (1-\pi)/\pi$, there is a unique mixed strategy Nash equilibrium in which both players choose Swim today.

At the other extreme, if

$$-\pi c + 2(1-\pi) < 0$$

or equivalently

$$c > \frac{2(1-\pi)}{\pi} \tag{6}$$

then "Wait" is the best response to both "Swim today" and "Wait". Indeed, if (6) holds, "Wait" is a dominant strategy for both players since this guarantees that $1 - \pi > -\pi c + 2(1-\pi)$ and $\max\{-\pi c + 1 - \pi, 0\} = 0 > -\pi c + 2(1-\pi)$.

Thus, if (6) holds, there is a unique mixed strategy Nash equilibrium, in which neither of you swims today, and consequently neither of you swims tomorrow. In other words,

(ii) If $c > 2(1-\pi)/\pi$, there is a unique mixed strategy Nash equilibrium in which in which neither of you swims today, and consequently neither of you swims tomorrow.

In the intermediate case in which $0 < -\pi c + 2(1-\pi) < 1-\pi$, or equivalently

$$\frac{(1-\pi)}{\pi} < c < \frac{2\,(1-\pi)}{\pi} \tag{7}$$

the best response to "Swim today" is "Wait" and the best response to "Wait "is "Swim today". Denoting by q the probability that player 2 chooses "Swim today", player 1's expected payoff to "Swim today" is

$$\begin{aligned} EU_1(ST) &= q\,[-\pi c + 2(1-\pi)] + (1-q)\,[-\pi c + 2(1-\pi)] \\ &= -\pi c + 2(1-\pi) \end{aligned}$$

and her expected payoff to "Wait" is

$$\begin{aligned} EU_1(W) &= q(1-\pi) + (1-q)\max\{-\pi c + 1 - \pi, 0\} \\ &= q(1-\pi) + (1-q) \times 0 \\ &= q(1-\pi) \end{aligned}$$

Indeed, because of (7), $\max\{-\pi c + 1 - \pi, 0\} = 0$.

To find the mixed-strategy equilibrium, we have to find the q that equalizes these two expected payoffs. We have:

$$EU_1(ST) = EU_1(W)$$

$$\Leftrightarrow -\pi c + 2(1-\pi) = q(1-\pi)$$

which is equivalent to:

$$q = 2 - \left(\frac{\pi c}{1-\pi}\right)$$

Now, denoting by p the probability that player 1 chooses "Swim today", player 2's expected payoff to "Swim today" is

$$\begin{aligned} EU_2(ST) &= p\,[-\pi c + 2(1-\pi)] + (1-p)\,[-\pi c + 2(1-\pi)] \\ &= -\pi c + 2(1-\pi) \end{aligned}$$

and her expected payoff to "Wait" is

$$\begin{aligned} EU_2(W) &= p(1-\pi) + (1-p)\max\{-\pi c + 1 - \pi, 0\} \\ &= p(1-\pi) + (1-p) \times 0 \\ &= p(1-\pi) \end{aligned}$$

To find the mixed-strategy equilibrium, we have to find the p that equalizes these two expected payoffs. We have:

$$EU_2(ST) = EU_2(W)$$

$$\Leftrightarrow -\pi c + 2(1-\pi) = p(1-\pi)$$

which is equivalent to:

$$p = 2 - \left(\frac{\pi c}{1-\pi}\right)$$

To conclude, if (7) holds, then the game has a unique mixed strategy Nash equilibrium, in which each person swims today with probability $2 - \pi c/(1-\pi)$. In other words,

(iii) If $(1-\pi)/\pi < c < 2(1-\pi)/\pi$, there is a unique mixed strategy Nash equilibrium in which each person swims today with probability $2 - \pi c/(1-\pi)$.

There is another case to study. Assume now that

$$c = \frac{(1-\pi)}{\pi} \tag{8}$$

In that case, the payoff matrix is now given by

1/2	*Swim today*	*Wait*
Swim today	$(1-\pi)$, $(1-\pi)$	$(1-\pi)$, $(1-\pi)$
Wait	$(1-\pi)$, $(1-\pi)$	0 , 0

The set of mixed strategy Nash equilibria in this case is the set of all mixed strategy pairs $((p, 1-p), (q, 1-q))$ for which either $p = 1$ (and q can take any value between 0 and 1) or $q = 1$ (and p can take any value between 0 and 1).

Indeed, if player 1 decides to always swim today (i.e. $p = 1$), then any strategy q of player 2 is a best reply since her expected payoff is always $1-\pi$ (this is true for $q = 0$, and $q = 1$, and $q \in (0,1)$).

Similarly, if player 2 decides to always swim today (i.e. $q = 1$), then any strategy p of player 1 is a best reply since her expected payoff is always $1-\pi$ (this is true for $p = 0$, and $p = 1$, and $p \in (0,1)$).

To conclude, if (8) holds, then the game has an infinity of mixed strategy Nash equilibria for which either $p = 1$ or $q = 1$. In other words,

(iv) If $c = (1-\pi)/\pi$, then the game has an infinity of mixed strategy Nash equilibria for which either $p = 1$ or $q = 1$.

Finally, assume that

$$c = \frac{2\,(1-\pi)}{\pi} \tag{9}$$

In that case, the payoff matrix is now given by:

1/2	*Swim today*	*Wait*
Swim today	0 , 0	0 , $(1-\pi)$
Wait	$(1-\pi)$, 0	0 , 0

The set of mixed strategy Nash equilibria in this case is the set of all mixed strategy pairs $((p, 1-p), (q, 1-q))$ for which either $p = 0$ (and q can take any value between 0 and 1) or $q = 0$ (and p can take any value between 0 and 1).

Indeed, if player 1 decides to always wait (i.e. $p = 0$), then any strategy q of player 2 is a best reply since her expected payoff is always 0 (this is true for $q = 0$, and $q = 1$, and $q \in (0, 1)$).

Similarly, if player 2 decides to always wait (i.e. $q = 0$), then any strategy p of player 1 is a best reply since her expected payoff is always 0 (this is true for $p = 0$, and $p = 1$, and $p \in (0, 1)$).

To conclude, if (9) holds, then the game has an infinity of mixed strategy Nash equilibria for which either $p = 0$ or $q = 0$. In other words,

(*v*) If $c = 2(1-\pi)/\pi$, then the game has an infinity of mixed strategy Nash equilibria for which either $p = 0$ or $q = 0$.

General conclusion

(*i*) If $c < (1-\pi)/\pi$, there is a unique mixed strategy Nash equilibrium in which both players choose to swim today.

(*iv*) If $c = (1-\pi)/\pi$, then the game has an infinity of mixed strategy Nash equilibria for which either $p = 1$ or $q = 1$.

(*iii*) If $(1-\pi)/\pi < c < 2(1-\pi)/\pi$, there is a unique mixed strategy Nash equilibrium in which each person swims today with probability $2 - \pi c/(1-\pi)$.

(*v*) If $c = 2(1-\pi)/\pi$, then the game has an infinity of mixed strategy Nash equilibria for which either $p = 0$ or $q = 0$.

(*ii*) If $c > 2(1-\pi)/\pi$, there is a unique mixed strategy Nash equilibrium in which in which neither of you swims today, and consequently neither of you swims tomorrow.

7c. Does the existence of a friend make it more or less likely that you decide to go swimming on the first day? (Penguins diving into water where seals may lurk are sometimes said to face the same dilemma, though Court (1996) argues that the evidence suggests that they do not.)

If you were *alone* your expected payoff to swimming on the first day would be

$$-\pi c + 2(1-\pi)$$

Your expected payoff to staying out of the water on the first day and acting optimally on the second day would be

$$\max\{0, -\pi c + 1 - \pi\}$$

Thus,

(*i*) if $-\pi c + 2(1-\pi) > 0$, or $c < 2(1-\pi)/\pi$, you swim on the first day (and stay out of the water on the second day if you get attacked on the first day)

(ii) if $c > 2(1-\pi)/\pi$, you stay out of the water on both days.

We have seen that, in the presence of your friend,
(i) if $c < (1-\pi)/\pi$, you swim on the first day.
(ii) if $(1-\pi)/\pi < c < 2(1-\pi)/\pi$, you do not swim for sure on the first day as you would if you were alone, but rather swim with probability less than one.

That is, the presence of your friend decreases the probability of your swimming on the first day when c lies in this range.

For other values of c your decision is the same whether or not you are alone.

8. ***(Mas-Colell, Whinston and Green)

There are n firms in an industry. Each can try to convince Congress to give the industry a subsidy. Let h_i denote the number of hours of effort put in by firm i, and let

$$c_i(h_i) = w_i\,(h_i)^2$$

be the cost of this effort to firm i (w_i is a positive constant).

When the effort levels of the firms are $(h_1, h_2, ..., h_n)$, the value of the subsidy that gets approved for each firm i is:

$$\alpha \sum_{i=1}^{i=n} h_i + \beta \prod_{i=1}^{i=n} h_i = \alpha\,(h_1 + ... + h_n) + \beta\,(h_1 \times ... \times h_n)$$

where α and β are constants. This means that the utility of each firm $i = 1, ..., n$ is given by:

$$U_i(h_1, ...h_n) = \alpha \sum_{j=1}^{j=n} h_j + \beta \prod_{j=1}^{j=n} h_j - w_i\,(h_i)^2$$

8a. Consider a game in which the firms decide simultaneously and independently how many hours they will each devote to this effort. Model this situation as a strategic game.

A strategic game that models this situation is:
Players: The n firms .
Actions: The actions are the hours of effort put in by firm i and the set of actions of each firm $i = 1, ..., n$ is the set of numbers h_i for which $h_i \in [0, +\infty[$.
Preferences: The payoff function of each firm $i = 1, ..., n$ is:

$$U_i(h_1, ...h_n) = \alpha \sum_{j=1}^{j=n} h_j + \beta \prod_{j=1}^{j=n} h_j - w_i\,(h_i)^2$$

8b. Show that each firm has a strictly dominant strategy if and only if $\beta = 0$. What is firm i's strictly dominant strategy when this is so?

Each firm i solves the following program:

$$\max_{h_i} \left\{ \alpha \sum_{j=1}^{j=n} h_j + \beta \prod_{j=1}^{j=n} h_j - w_i (h_i)^2 \right\}$$

The first-order condition is:

$$\alpha + \beta \prod_{j=1, j \neq i}^{i=n} h_i - 2w_i h_i = 0$$

The best-response for firm i is thus:

$$h_i(h_{-i}) \equiv BR_i(h_{-i}) = \frac{1}{2w_i} \left[\alpha + \beta \prod_{j=1, j \neq i}^{i=n} h_i \right]$$

Therefore, firm i has a strictly dominant strategy if and only if $\beta = 0$, i.e. the best response function of i is not dependent on the action of the other firms.

If $\beta = 0$, then firm i's strictly dominant strategy is

$$h_i = \frac{\alpha}{2w_i}$$

9. *** Consider the Bertrand duopoly model in which two firms 1 and 2, simultaneously choose the prices they will sell on the market, p_1 and p_2. Assume that the strategy space S_i of firm i is the interval $[0, 1]$, i.e. $p_1 \in [0, 1]$ and $p_2 \in [0, 1]$. The linear market demand is given by:

$$D = 1 - 2p_i + p_j$$

Assume that there are zero marginal costs so that the profit of firm $i = 1, 2$ is equal to:

$$\pi_i = (1 - 2p_i + p_j)\, p_i$$

This game is called a *supermodular game*. It is important for the answer to 9b to observe that we have imposed the strategy space S_i to be a bounded interval so that prices p_i and p_j cannot be strictly less than zero and strictly greater than one.

9a. Determine the unique Nash equilibrium of this game.

Each firm i chooses p_i that maximizes its profit. First-order condition gives:

$$\frac{\partial \pi_i}{\partial p_i} = 1 - 4p_i + p_j = 0$$

which gives the following best-reply function of firm i:

$$p_i(p_j) \equiv BR_i(p_j) = \frac{1}{4} + \frac{p_j}{4}$$

We need to solving for the two best-reply functions, i.e.

$$\begin{cases} p_1 = \frac{1}{4} + \frac{p_2}{4} \\ p_2 = \frac{1}{4} + \frac{p_1}{4} \end{cases}$$

Solving these two equations, we easily obtain the unique Nash equilibrium of this game, which is given by:

$$p_1^* = p_2^* = \frac{1}{3} \tag{10}$$

9b. Argue that successive elimination of strictly dominated strategies yields a unique prediction in this game, which is exactly the unique Nash equilibrium of this game.

The initial strategy space is $S_i^0 = [0, 1]$ (the superscript 0 indicates that we are in the first step). Suppose player j produces p_j. Player i's best response can be calculated by maximizing:

$$\max_{p_i} \pi_i = (1 - 2p_i + p_j)\, p_i$$

which yields the following best-response function:

$$p_i = \frac{1}{4} + \frac{p_j}{4} \tag{11}$$

Now, since $p_j \geq 0$ (i.e. $p_j \in [0, 1]$), then any strategy $p_i < 1/4$ is strictly dominated by $p_i = 1/4$ and thus it has to be that:

$$p_i \geq \frac{1}{4} \tag{12}$$

We also know by assumption that $p_j \leq 1$ (i.e. $p_j \in [0, 1]$), which using (11) implies

$$p_i = \frac{1}{4} + \frac{p_j}{4} \leq \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

that is

$$p_i \leq \frac{1}{2} \tag{13}$$

By combining (12) and (13), the new strategy space for firm $i = 1, 2$ is now:

$$S_i^1 = \left[\frac{1}{4}, \frac{1}{2}\right] = [0.25 \ ; \ 0.5]$$

(the superscript 1 indicates that we are in the first step).

Let us continue with the elimination of strictly dominated strategies. Consider the best-reply function (11), i.e.

$$p_i = \frac{1}{4} + \frac{p_j}{4}$$

Now, since $p_j \geq 1/4$ (i.e. $p_j \in \left[\frac{1}{4}, \frac{1}{2}\right]$), then any strategy $p_i < 5/16$ is strictly dominated by $p_i = 5/16$ since

$$p_i = \frac{1}{4} + \frac{p_j}{4} \geq \frac{1}{4} + \frac{1/4}{4} = \frac{5}{16}$$

and thus it has to be that:

$$p_i \geq \frac{5}{16} \tag{14}$$

We also know that $p_j \leq 1/2$ (i.e. $p_j \in \left[\frac{1}{4}, \frac{1}{2}\right]$), which using (11) implies

$$p_i = \frac{1}{4} + \frac{p_j}{4} \leq \frac{1}{4} + \frac{1/2}{4} = \frac{3}{8}$$

that is

$$p_i \leq \frac{3}{8} \tag{15}$$

By combining (14) and (15), the new strategy space for firm $i = 1, 2$ is now:

$$S_i^2 = \left[\frac{5}{16}, \frac{3}{8}\right] = [0.3125 \ ; \ 0.375]$$

Let us continue with the elimination of strictly dominated strategies. Consider the best-reply function (11), i.e.

$$p_i = \frac{1}{4} + \frac{p_j}{4}$$

Now, since $p_j \geq 5/16$ (i.e. $p_j \in \left[\frac{5}{16}, \frac{3}{8}\right]$), then any strategy $p_i < 21/64$ is strictly dominated by $p_i = 5/16$ since

$$p_i = \frac{1}{4} + \frac{p_j}{4} \geq \frac{1}{4} + \frac{5/16}{4} = \frac{21}{64}$$

and thus it has to be that:

$$p_i \geq \frac{21}{64} = 0.32813 \tag{16}$$

We also know that $p_j \leq 3/8$ (i.e. $p_j \in \left[\frac{5}{16}, \frac{3}{8}\right]$), which using (11) implies

$$p_i = \frac{1}{4} + \frac{p_j}{4} \leq \frac{1}{4} + \frac{3/8}{4} = \frac{11}{32}$$

that is

$$p_i \leq \frac{11}{32} = 0.34375 \tag{17}$$

By combining (14) and (15), the new strategy space for firm $i = 1, 2$ is now:

$$S_i^2 = \left[\frac{21}{64}, \frac{11}{32}\right] = [0.32813 \ ; \ 0.34375]$$

Let us continue with the elimination of strictly dominated strategies. Consider the best-reply function (11), i.e.

$$p_i = \frac{1}{4} + \frac{p_j}{4}$$

Now, since $p_j \geq 21/64$ (i.e. $p_j \in \left[\frac{21}{64}, \frac{11}{32}\right]$), then any strategy $p_i < 85/256$ is strictly dominated by $p_i = 85/256$ since

$$p_i = \frac{1}{4} + \frac{p_j}{4} \geq \frac{1}{4} + \frac{21/64}{4} = \frac{85}{256} \approx \frac{1}{3}$$

and thus it has to be that:

$$p_i \geq \frac{85}{256} \approx \frac{1}{3} \tag{18}$$

We also know that $p_j \leq 43/128$ (i.e. $p_j \in \left[\frac{21}{64}, \frac{11}{32}\right]$), which using (11) implies

$$p_i = \frac{1}{4} + \frac{p_j}{4} \leq \frac{1}{4} + \frac{11/32}{4} = \frac{43}{128} = \frac{86}{256} \approx \frac{1}{3}$$

$$p_i \leq \frac{86}{256} \approx \frac{1}{3} \tag{19}$$

By combining (18) and (19), the new strategy space for firm $i = 1, 2$ is now:

$$S_i^3 = \left[\frac{85}{256}, \frac{86}{256}\right] \approx \left[\frac{1}{3}, \frac{1}{3}\right]$$

So we have shown that the elimination of strictly dominated strategies converges to $(1/3, 1/3)$, which is the unique Nash equilibrium of this game (see (10)).