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Irregular Fluctuations in Competitive Markets with Production Lags

by

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Abstract

This paper shows how changing patterns of change and irregular or chaotic fluctuations arise in deterministic, competitive markets with production lags using (1) the standard cobweb model with a backward bending supply function and (2) the Robertson-Williams cobweb model with financially constrained supply.

> The question arises whether there are any fluctuations at all which arise out of the behavior of business communities as such and would be observable even if the institutional and natural framework of society remained absolutely invariable.
>
> J.A. Schumpeter

> ... production and prices fluctuate greatly from year to year and there is a good deal of irregularity in these fluctuations too ...
>
> G.S. Shepherd

1 INTRINSIC DYNAMICS

In this paper we show how changing patterns of change and irregular or chaotic fluctuations can arise in deterministic, competitive markets with production lags. Two forms of the cobweb model are used for this purpose, one with a backward bending supply function and one in which supply depends on "working capital" or "reinvestment income". As in other examples of economic chaos investigated by various authors, complicated dynamics do not occur because of random, exogenous shocks, which are not present, but because the feedback effect of past output on current supply and demand exhibits non-linearities that bring about a causal effect reversal.[1]

Of course economic history involves a host of variables that come from outside the strictly economic domain and whose more or less random impulses can be propagated when the deterministic system upon which they impinge is stable, as shown in the classic works of Frisch (1933) or Lucas (1975). But it is also of interest to inquire under what conditions irregular fluctuations of the kind we experience in daily economic life could also be generated in isolation of exogenous perturbations. The present paper is concerned with such intrinsic mechanisms in still another classic or standard context.

The cobweb model presents, of course, a highly simplified picture of supply, demand and of price expectations. In the case where periodic fluctuations occur and suppliers are consistently wrong in their forecasts, it has long been argued that more rational expectations would evolve. If, however, intrinsic instability exists it is not clear that the usual methods of statistical inference, let alone typically used rules of thumb and learning procedures, could help producers avoid ceaseless error. That is an intriguing question for which the present analysis begs an answer, and which may stand as a challenge for further theoretical research. A probe in this direction has recently been made by Brock and Chamberlin (1984). See also Tokens (undated).

2 PRODUCTION LAGS AND MARKET FEEDBACK: THE BASIC COBWEB MODEL

Consider a set of micro supply functions that describe the dependence of supply of a given firm, say y^i_{t+1}, on that firm's expected price, say $\hat{p}^i_{t+1}$,

$$y^i_{t+1} = S^i(\hat{p}_{t+1}) \tag{1}$$

Each firm is assumed to have exactly the same price expectations, and these have the naive, adaptive form

$$\hat{p}^i_{t+1} = p_t \tag{2}$$

where p_t is the market price of period t.[2] Aggregate (macro) market supply is then defined to be

$$y_{t+1} = S(p_t) = \Sigma_i y^i_{t+1} = \Sigma_i S^i(p_t) \tag{3}$$

The model is completed by adding the inverse demand function representing temporary market clearing prices,

$$p_t = D^-(y_t) \tag{4}$$

The micro-macro linkage occurs through market feedback when (4) is substituted into (2). Substituting in turn into (1) the behavior of the ith firm is given by

$$y^i_{t+1} = S^i\left(D^-(y_t)\right) \tag{5}$$

so that what each firm does in a given period turns out to depend on what the aggregate of firms did in the previous period.

If each firm satisfies a proportionality condition then there exists a representative supply function $S(\cdot)$ and constants λ_i for each firm such that

$$S^i(p) = \lambda_i S(p), \ \Sigma_i \lambda_i = 1 \tag{6}$$

Equations (2) and (6) are aggregation conditions which simplify the micro-macro linkages implicit in the aggregate model. With this assumption aggregate supply is simply

$$y_{t+1} = S\left(D^-(y_t)\right) := \Theta(y_t) \tag{7}$$

Moreover, each side of (7) can be divided by 1/n to get the micro behavior of the "average" firm

$$\bar{y}_{t+1} = (1/n)S\left(D^-(n\bar{y}_t)\right) \tag{8}$$

Now consider a situation in which, if the price is low enough, production will be negligible, but as price rises supply increases rapidly, reaching a maximum. Beyond this point output decreases as price increases. This could happen if in this range farmers (say) would prefer to sustain, or modestly improve their material standard of living while at the same time, by substituting leisure for work, increase the time they have to enjoy it.

Given this "backward bending" supply curve, unstable oscillations can occur if the local stability condition at equilibrium y^e is violated, that is, if $S'(pY^e)D^{-\prime}(y^e) < -1$. However, unlike the linear case usually used to illustrate the cobweb model these oscillations do not "explode" but are bounded because of the nonlinearity in the supply function. They may converge to periodic cycles - or they may wander in a pattern that can have irregular sequences of turning points and with erratic amplitudes, a by now well known phenomenon called "chaos" by the mathematicians Li and Yorke (1975). A sufficient condition for the existence of such market solutions can be derived directly from Li, Misiurewicz, Pianigiani and Yorke (1982). It is the existence of a point, say y^c, such that either

$$(a)\quad y_n < y_0 < y_1 \quad \text{or} \quad (b)\quad y_n > y_0 > y_1 \tag{9}$$

for some odd n where $y_0 = y^c$, $y_1 = S[D^-(y_0)],\ldots,$ $y_{t+1} = S[D^-(y_t)]$.

These sufficient conditions do not depend on any particular functional form for the basic difference equation (7) so it is not very restrictive so far as the underlying conditions of demand and supply are concerned.

As an example consider the supply function

$$y = S(\hat{p}) = \begin{cases} 0 & , \hat{p} < c \\ A(\hat{p}-c)^{\alpha}e^{-\beta\hat{p}} & , \hat{p} \geqslant c \end{cases} \tag{10}$$

(where for simplicity we will set c = o) and let demand be

$$p = D^-(y) = a(1-b)y. \tag{11}$$

The adjustment equation for output is then

$$y_{t+1} = \Theta(y_t) = Aa^{\alpha}(1-by_t)^{\alpha}\, e^{-\beta a(1-by_t)} \qquad (12)$$

Figure 1 presents a simulation of this equation.[3] Supply increases at a decreasing, then increasing rate until it switches to an oscillating mode.[4] The underlying supply and demand curves are shown in 1a, the phase diagram in 1b and a specific solution in 1c. A further analysis of this case shows that once the oscillatory mode is entered price and quantity changes reverse sign every period, in this sense following a two period oscillation. But the amplitudes are highly irregular as can be seen in Figure 2a where no periodic motion is evident even after 100 iterations.

Consider the behavior of supply <u>every second period</u>. If we begin for example with the point labelled A on Figure 1c and let it be y_0 then we find that

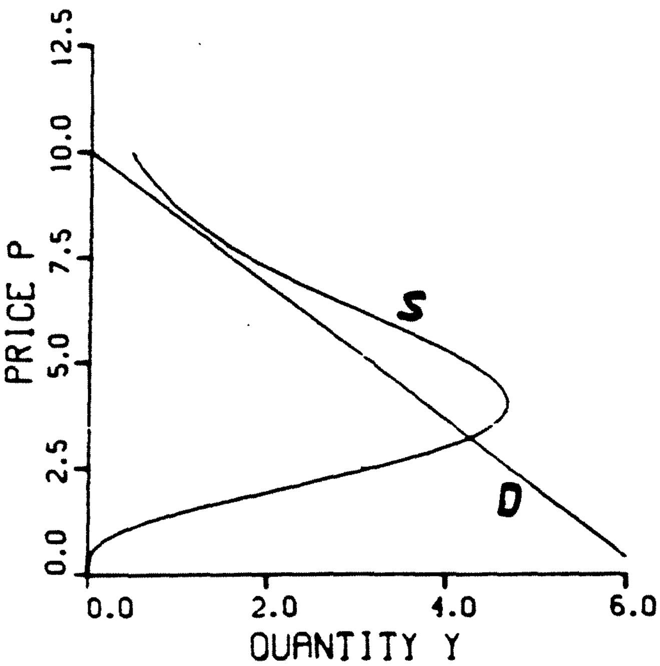
$$y_6 < y_0 < y_2$$

This means that the second iterated map

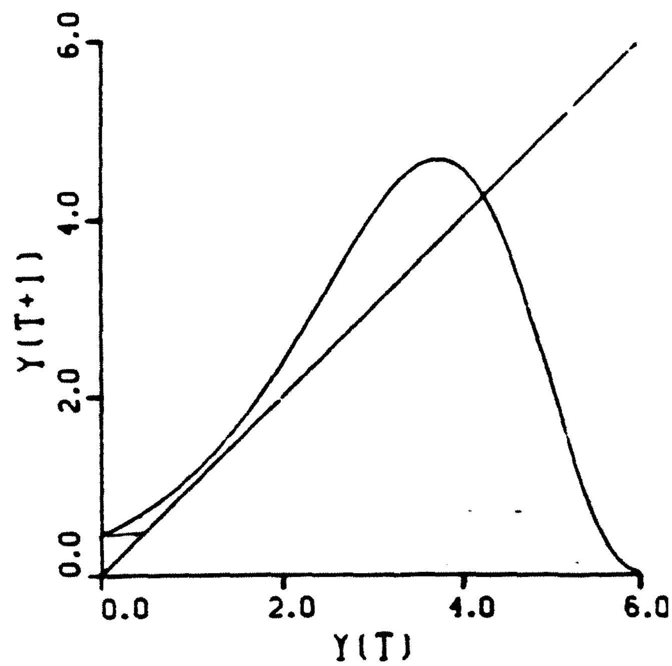
$$y_{t+2} = \Theta(\Theta(y_t))$$

derived from (12) satisfies the sufficient condition (9a) for chaos. This explains why the amplitudes are so highly irregular even though the turning points occur every period! We then have the appearance of a regular 2 period cycle with random shocks superimposed even though the model is deterministic.

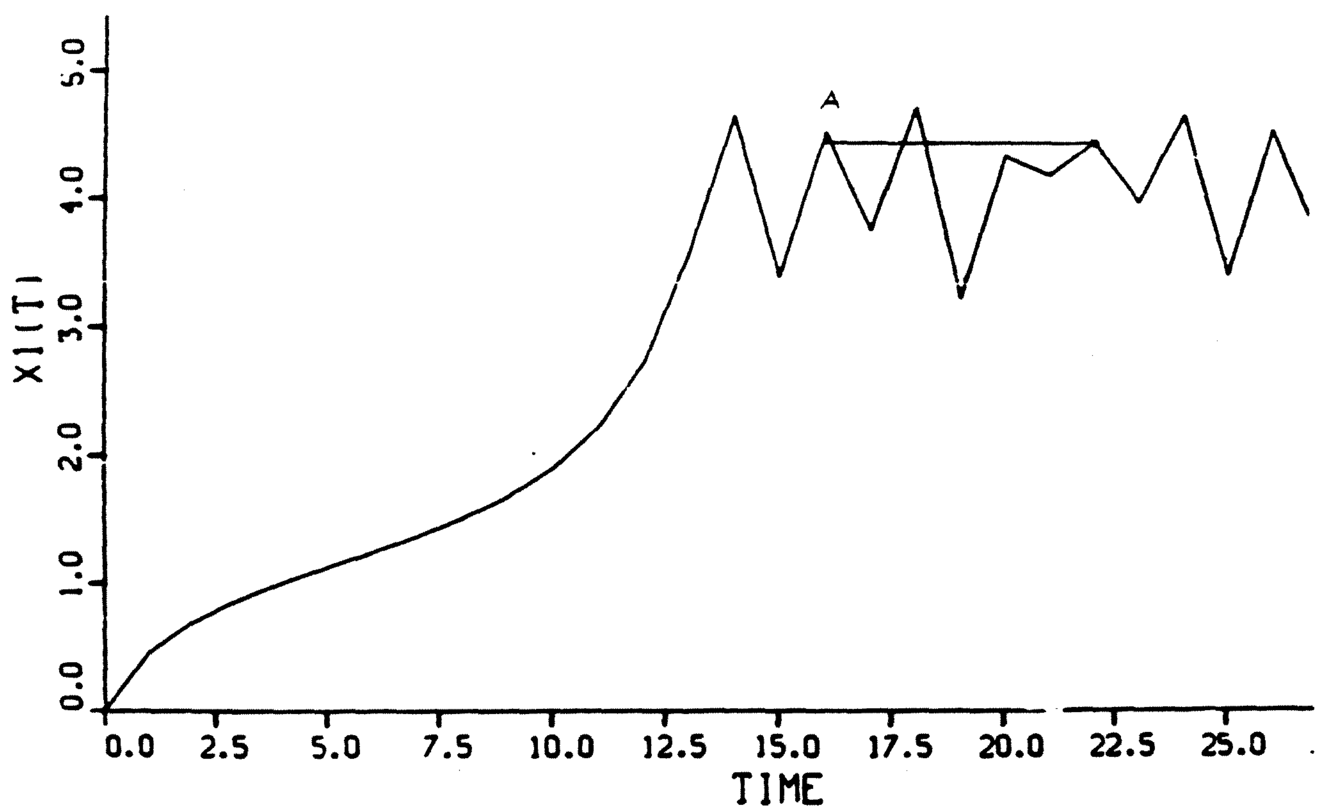
Figure 1 The endogenous switch in qualitative mode of behavior



(a) Supply and Demand Functions

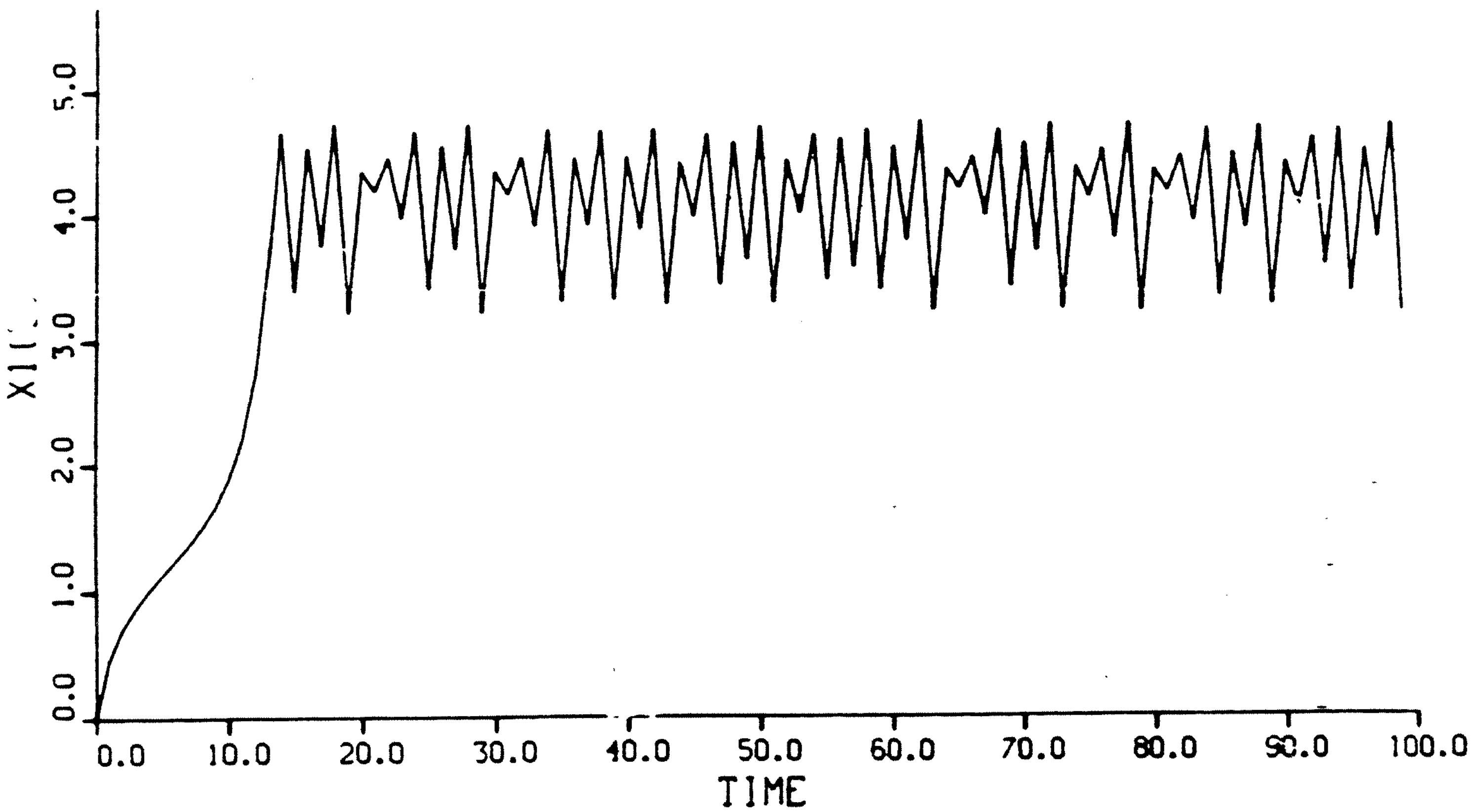


(b) The Phase Diagram

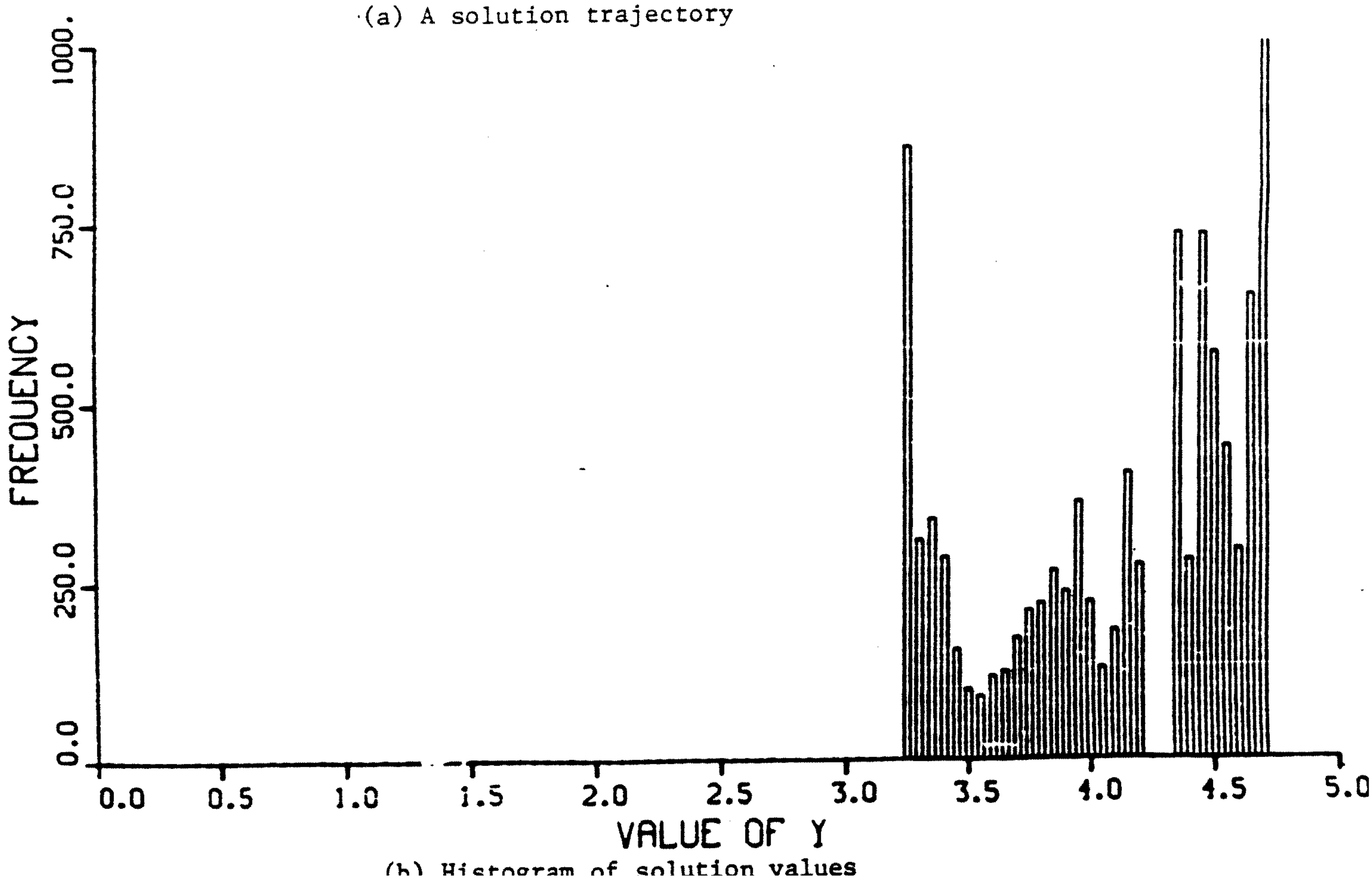


(c) A trajectory for output. Point A satisfies the LMPY condition for chaos if we use the second iterated map.

Figure 2 Irregular amplitudes



(a) A solution trajectory



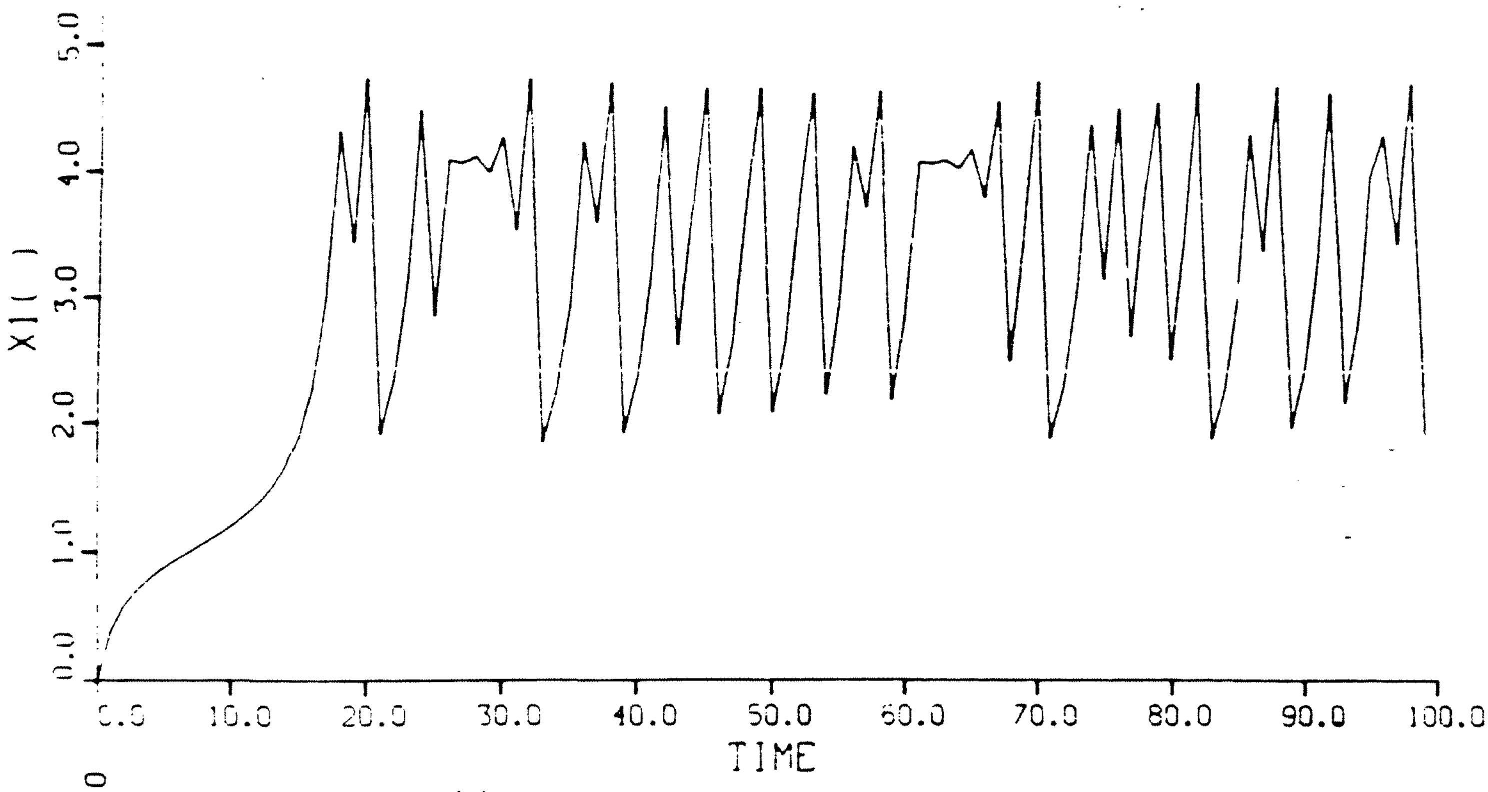
(b) Histogram of solution values

Figure 2b shows a histogram of 10,000 iterates in this solution. There are 28 nonzero cells indicating that if the solution converges it must very likely be to a cycle of periodicity of at least 28. Indeed, the periodicity would have to be much greater than that because the bars of the histogram would all be exactly the same height if the number of nonzero cells were exactly equal to the period of the cycle. This possibility deserves emphasis because high periodicity looks like short-run irregularity. Our simulations really can't distinguish between very high periodicity and chaos even when we know chaotic trajetories exist.

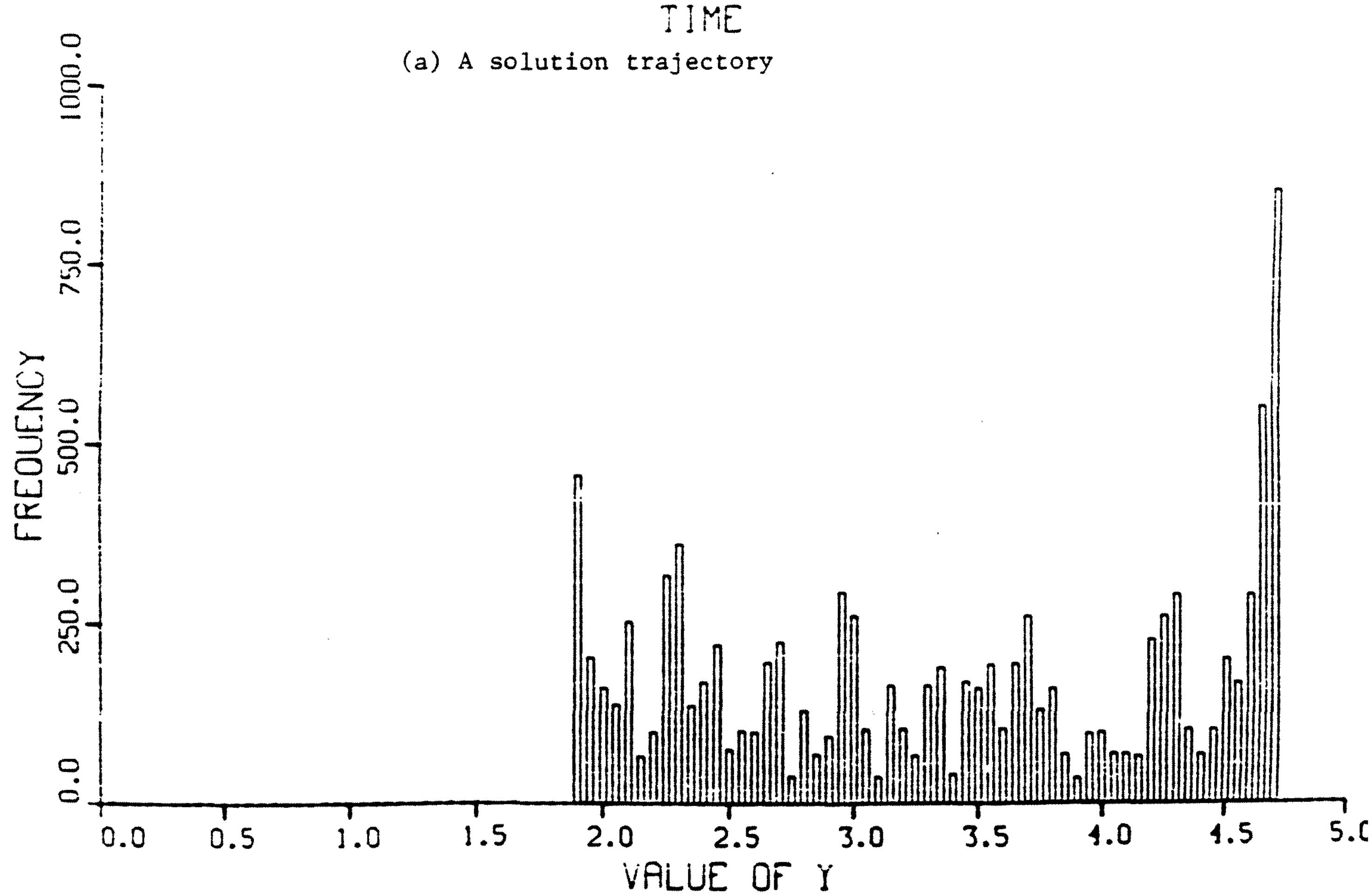
Figure 3a shows the time path for a second example in which the demand parameters "a" and "b" have been shifted slightly.[5] The amplitudes are highly irregular and points satisfying the sufficient conditions for chaos are readily found, an exercise that may be left to the reader. In addition turning points now occur at irregular intervals, sometimes every period, sometimes after three periods and occasionally, with somewhat diminished frequency, after two periods. The histogram of 10,000 iterates in this solution is shown in Figure 3b. There are 57 nonzero cells with highly varying frequencies indicating that the fluctuation, if periodic, has a period much greater than that number. Again, as the sufficient conditions for chaos are satisfied we may have here an example of chaos, though strictly speaking, we can't be sure.

The backward bending supply curve used here to obtain unstable, yet bounded oscillations is not entirely a theoretical curiosity. It has been thought to have some empirical relevance for labor

Figure 3 An example with irregular turning points



(a) A solution trajectory



(b) Histogram of solution values

markets (Friedman, 1962) for example. the idea that farmers and others might behave in accordance with it does not seem wholly implausible either. For our purposes, however, it is merely a pedagogically convenient way to show how evolving modes of behavior and the emergence of irregular, unpredictable fluctuations can be explained endogenously, results not realized by the early developers of the cobweb theory (Hanau, 1927; Tinbergen, 1931; Kaldor, 1934; Leontief, 1934; Ezekiel, 1938) or even by its later expositors (Baumol, 1959; Henderson and Quandt, 1958; Samuelson, 1983). Obviously, given appropriate configurations of supply and demand, all of the other types of cobweb paths, including converging and periodic cycles can be found.

3 THE ROBERTSON-WILLIAMS COBWEB MODEL

In order to dispell the notion that our results derive solely from the backward bending supply function and could not occur otherwise, we consider an alternative version of market dynamics based on the "Robertsonian Lag", that is the idea that current expenditures must come from previous incomes. As in the standard cobweb model prices are determined by market clearing, while anticipated prices are assumed (again for convenience of analysis) to be based on naive, adaptive expectations. Given that firms attempt to maximize anticipated profits (when the latter are positive) or to maximize sales (Baumol, 1959a) when anticipated profits are negative, output is determined by the equation: Current Production Costs = Reinvestment Income, where Reinvestment Income = Lagged Total Revenue - Overhead Costs - (Dividends or Consumption Expenditures). See Williams (1967).

Suppose output is initially very small but sufficient to cover overhead and other deductions so that initial reinvestment income is positive. A period of growth may occur so long as demand is elastic and total revenue increasing. Eventually, however, when supply reaches the inelastic portion of the demand curve revenues fall; reinvestment income declines so that output must subsequently be reduced. Later, because market supplies are reduced, prices increase and revenues recover. That firms may over-expand and "spoil their market" is not news to any student of Marshall. That the resulting fluctuation could appear to be highly irregular may come as something of a surprise.

Let $C^i(y^i)$ be the total cost function and let k_t^i be the reinvestment income or working capital of the ith firm. By consumption output in time t+1 is determined as the solution y_{t+1}^i to the problem

$$\max_{y^i}\{\hat{p}_{t+1}\ y^i \mid C^i(y^i) \leqslant k_{t+1}^i\} \tag{15}$$

or by the solution of the implicit function in y^i

$$C^i(y_{t+1}^i) = k_{t+1}^i \tag{16}$$

Suppose the reinvestment function is

$$k_{t+1}^i = H^i(p_t,\ y_t) \tag{17}$$

Solving for y_{t+1}^i in (16) and using (17) we get the supply function corresponding to (3)

$$y_{t+1}^i = S^i(p_t,\ y_t) = C^{i-}\big(H(p_t,\ y_t)\big) \tag{18}$$

Substituting the inverse demand equation (4) we get

$$y_{t+1}^i = S^i\big(D^-(y_t),\ y_t\big) \tag{19}$$

which is analogous to (5).

Our aggregation conditions (6) will hold if each firm has identical cost and distribution functions C^i and H^i and identical initial conditions. In this way we can arrive at a difference equation representing the dependence of aggregate supply (or that of the representative firm) on output for the preceding period.

$$Y_{t+1} = \Theta(y_t) := \Sigma_i S^i\left(D^{-1}(y_t),\ y_t\right) = S\left(D^{-1}(y_t),\ y_t\right) \quad (20)$$

Indeed, if for simplicity we assume identical, constant unit costs, c, and a degenerate distribution function $\left(H^i(y^i,\ p_i) \equiv 0\right)$ then the model boils down to the equation

$$y_{t+1} = \Theta(y_y) := y_t D^{-}(y_y)/c \quad (21)$$

Given the simplifications we have imposed, the dynamics of industry output must depend on unit cost and on the parameters of demand.

This is most easily seen by considering the linear inverse demand curve (11) so that (12) becomes

$$y_{t+1} = ay_t(1-by_t)/c \quad (22)$$

Setting x = by we obtain the difference equation

$$x_{t+1} = mx_t(1-x_t) \quad (23)$$

which shows that behavior is completely determined by the ratio m = a/c.

The parameter "m" has a simple intuitive meaning: it is the extent of the market "a" divided by an "efficiency index", "c"; or, we could say that "m" is the "extent of the market" measured in "cost efficiency" units.

It is easy to show that when $m \leqslant 1$, $\bar{x}^0 = 0$ is the single nonnegative and stable stationary state. For $1 < m \leqslant 2$ we have monotonic growth convering to the positive stationary state $\bar{x}^1 = (m-1)/m = (a-c)/c$; and for $2 \leqslant m < 3$ damped cycles must eventually occur (after a period of growth if x_0 is close to zero), converging to the positive stationary state x^1. Thus, the parameters m = 1 and m = 2 are <u>bifurcation points</u> at which the qualitative properties of trajectories satisfying (3) suddenly change.

When $m > 3$ two things occur simultaneously: the positive stationary state becomes unstable and a stable, two period cycle emerges. For $0 < (m+1) \cdot (m-3) \leq 1$ the two period cycle is approached monotonically, i.e., from "inside" or from the "outside". For $1 < (m+1)(m-3) \leq 2$, or $m = 1 + \sqrt{6}$, the two period cycle becomes unstable and a four period cycle emerges. We have thus found new bifurcation points at $m = 3$, $M = 1 + \sqrt{5}$ and $m = 1 + \sqrt{6}$.

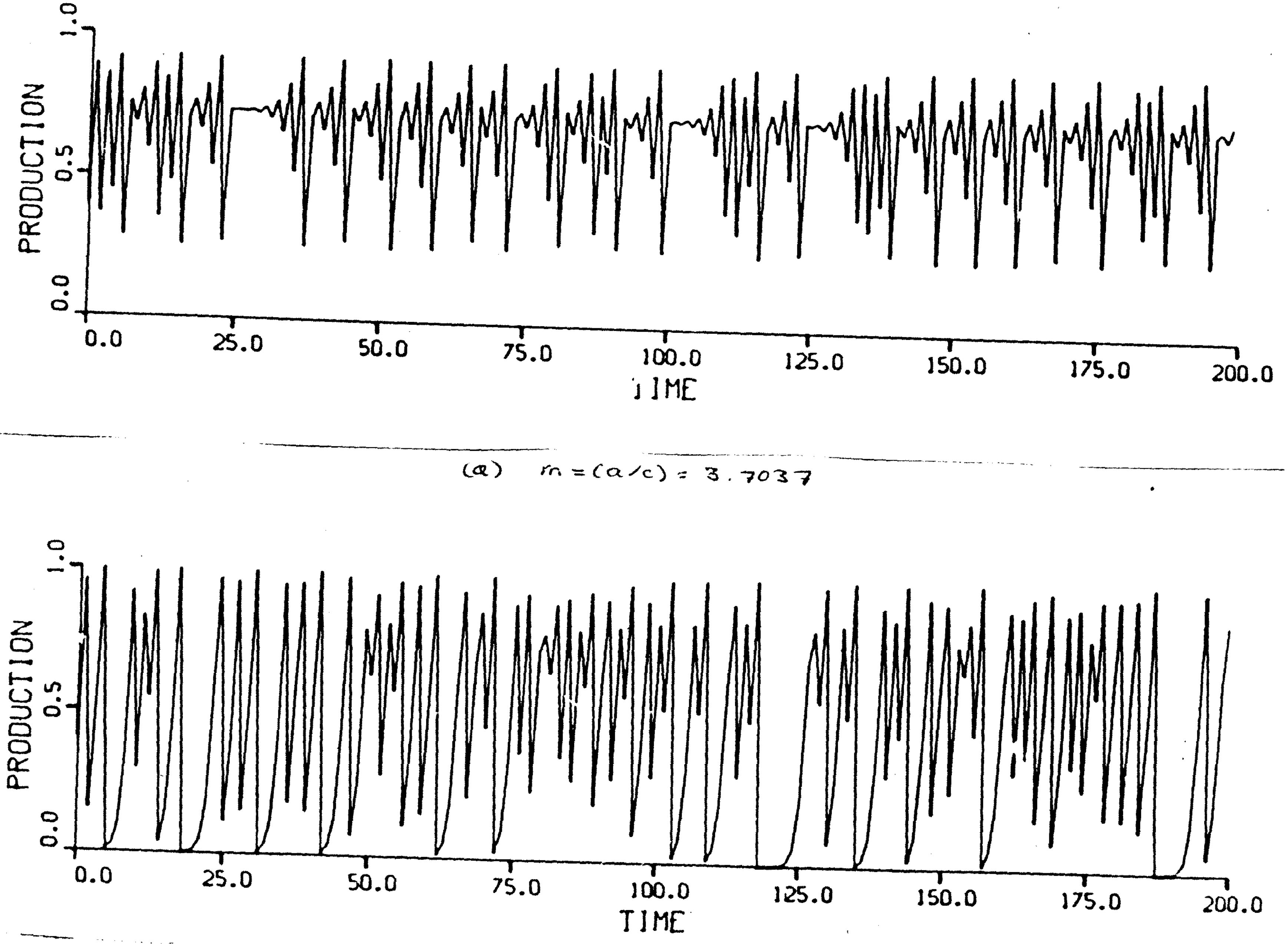
In this manner bifurcation points are generated at each of which the qualitative dynamics of the model change. Thus, as the extent of the market increases ("a" gets larger) or as unit cost decreases ("c" gets smaller) contraction is replaced by growth, then cycles of higher and higher even order emerge. Day (1967), who worked directly with a version equivalent to the equation (22), left the bifurcation analysis here, having shown how the qualitative dynamics of the market can change abruptly as the parameters of demand and/or cost are changed.

In the meantime Hoppensteadt and Hyman (1977), who were motivated by Lorenz (1963), (1964), carried out a definitive bifurcation analysis of equation (23). Let m_k be the value of m at which the stable cycle of order k emerges and the cycle of order k - 1 becomes unstable. Hoppensteadt and Hyman showed that $\lim_{\ell \to \infty} m_{2^\ell} \sim 3.57$. Moreover by direct calculation they found the value $m_3 = 3.83$ and a sequence of values $k = 3 \cdot 2^{\ell}$, $\ell = 0,1,2,3...$ such that $... < m_{3\cdot 2^{\ell+1}} < m_{3\cdot 2^{\ell}} < ... m_3$ and such that $\lim_{\ell\to\infty} m_{3\cdot 2^{\ell}} \sim 3.57$. Thus as m increases from 3 toward the value of 3.57, higher and higher order even cycles emerge of order 2^n, $n = 1,2,....$ Beyond 3.57, very high odd cycles of order $3\cdot 2^n$ appear. These diminish in order until at 3.83 a cycle of order 3 exists.

Clearly, as m increases beyond 3 the behavior of (23) becomes very complicated. Just how complex was already evident in computations of this model conducted by Richard Benson under the first author's direction in the mid 60s. He simulated equation (22) for various parameter values. Some of the computations displayed unusual irregularities. Figure 4 presents two examples of this kind which we have reproduced for this paper using the canonical form (23). Diagram (a) shows a two-period oscillation that appears to jump to a seven-period cycle. The latter approximately reproduces itself several times; the pattern is broken and then apparently reestablished only to wander away in a quite different, irregular pattern. Diagram (b) develops extreme fluctuations with periods of growth followed by cycles and occasional plunges at erratic intervals that nearly eliminate the industry altogether. It is easy to find examples of points satisfying the sufficient conditions (9a) in either of the two series so we "know" chaotic trajectories exist, though, of course, as in the examples of Section 2, we can't be entirely sure these are not just trajectories with extremely high periodicity.

Using an argument developed by one of us elsewhere (Day, 1982, 1983) a sufficient condition that is

Figure 4 **Simulations of the Robertson-Williams cobweb model**



(a) $m = (a/c) = 3.7037$

(b) $m = (a/c) = 4.0$

very easy to construct for a large class of demand functions can be developed. Fix all the parameters of the industry demand curve except for one representing the extent of the market "a" so that we can write (21) as

$$y_{t+1} = \Theta(y_t;\ m) := my_tD^-(y_t) \qquad (24)$$

where $m = a/c$, the extent of the market in cost efficiency units. Suppose also that inverse demand is continuous, downward sloping and such that $\lim_{y\to 0} + yD^-(y) = \lim_{y\to\infty} yD^-(y) = 0$. Then $\Theta(y;\ m)$ has a single-humped shape with maximizer y^* independent of m and maximum $y^m(m) = my^m$ where $y^m = y^*D^-(y^*)$. Finally let y^c be the smaller of the two roots (if m is big enough for them to exist) of the equation $myD(y) = y^*$.

Obviously when m is big enough $y^m(m) = my^m$ becomes bigger than y^* which is a sufficient condition for $y^c(m)$ to exist and for $y^c(m) < y^*$. Clearly $\Theta(y;\ m) \in J = [0,\ y^m(m)]$. Hence, from the Li-Yorke (1975) condition chaos exists for all Robertson-Williams cobweb models such that

$$A = (a/c) \in \{m \mid \Theta(y^m(m)) \leq y^c(m)\} \qquad (25)$$

Consider the semi-log linear demand form

$$aD^-(y;\ \beta) := ae^{-\beta y},\ y \geq 0 \qquad (26)$$

Substituting in (24) we get

$$y_{t+1} = my_te^{-\beta y_t} \qquad (27)$$

or multiplying both sides by β and setting $x_t = \beta y_t$,

$$x_{t+1} = mx_t e^{-x_t} \tag{28}$$

The maximizer is $x^* = 1$ with maximum $x^m = me$. According to our definition above $x^c = \min\{x \mid mxe^{-x} = 1\}$ so chaotic trajetories exist for all models with $(a/c) \in A = \{m>0 \mid m^2e^{me-1} \leq x^c\}$, or using a numerical result of May (1975) for all $m > 2.692$. Mueller and Day (1978, p. 240) estimate "a" for West German demand for pork circa 1970 to $e^{2.875}$. Accordingly, chaotic cobweb cycles could exist for average unit production costs below $e^{2.875}/2.692$.

Of course when the extent of the market is small enough, all trajectories fall to zero: the industry dies out. When m is big enough growth converging to the stationary state $\bar{x} = \log m$ emerges. When m is increased still more cycles of period doubling orders occur until eventually the chaos condition is satisfied where cycles of all orders and a chaotic set of initial conditions exists. Thus, the rising complexity of industry behavior depends on the extent of the market (now measured by a) and the cost coefficient, c, in exactly the same way as for the linear demand case.

The effect of (a/c) on market dynamics can be dramatic as shown by May in the reference cited. His results may be interpreted for the Robertson-Williams Industry as follows. As the extent of the market increases or as unit costs decrease, stable growth gives way to successively higher cycles until the chaos regime is reached. As those parameters change still more, the cycles eventually display irregular sharp peaks separated by periods of extremely slow growth, somewhat in the manner of clothing fashions or faddish sporting goods such as hula hoops or roller skates. In the present model these cycles are not due to changing tastes but to a Marshallian "spoiling the market" phenomenon. Each competitor, in an effort to produce all he can, ploughs back working capital in an effort to maximize sales. Total revenues increase explosively leading to an eventual collapse in the market. The result is an industry characterized by a very small output much of the time but which enjoys great booms at sporadic intervals.[7]

4 CONCLUSIONS

Two forms of the basic cobweb model have been investigated, one in which a backward bending supply function is used, and one in which supply depends on reinvestment income or "working capital". Although both versions have limitations, particularly in their use as a simplifying device, of naive, adaptive expectations, they still serve very well the pedagogical purpose of showing how complicated dynamics can be generated within a completely deterministic framework, one in which shocks play no role and in which all the changes, structural ones and even those of an essentially random nature, are explained entirely by the endogenous forces of supply and demand.

Paraphrasing a passage by the physicist P.W. Bridgeman, Schultz (1982) remarked that "... economic behavior is much more complex than our thoughts about it." That, unfortunately, is likely always to remain a problem. Still, progress along the lines illustrated here may enable our thoughts to become complex enough eventually to provide a better foundation for policy formulation and evaluation.

NOTES

* The first author's work on this paper was conducted at The Netherlands Institute for Advanced Study while that of the second was carried out at the Industrial Institute of Industrial Research in Stockholm. The model in Section 2 was presented at the Institute Henri Poincaré in Paris in the spring of 1982. Related works by Jensen and Urban (undated), Cigno and Montrucchio (1984) and Gabisch (undated) have recently come to our attention. The model of Section 3 was originally studied by Day (1967). The new results obtained here were first presented to the NBER Conference on General Equilibrium Theory in the spring of 1981. See Dury (1981).

[1] For discussions of mathematical chaos in an economic context see Benhabib and Day (1982), Day (1982, 1983), Dana and Malgrange (1984), Grandmont (1983), Stutzer (1981), and Pojohla (1981). References to the related technical literature will be found in these papers.

[2] Obviously, alternative, more "rational" expectations might be introduced, and such alternative assumptions would, of course, affect the analysis. But they would either force us to assume a stationary equilibrium in the case of perfect foresight or introduce considerable analytical complications in the case of adaptive or other kinds of more sophisticated expectation models. Such considerations have to be taken up eventually but we shall not do so here.

[3] We should note that this example is quite a tractable one from the econometric point of view. Indeed, (12) may be expressed in the semi-log linear form $\log y_{t+1} = \log A + \alpha \log p_t - \beta B + \beta\gamma y_t$ which with (11) gives an identifiable system.

[4] The parameters for this example are $\alpha = 4$, $\beta = 1$, $\gamma = 1.6$, $A = 1$, $B = 10$, $c = 0$, $y(0) = 0$.

[5] The parameters are the same as the previous example (note 4) except $\gamma = 1.8$ and $B = 10.3$.

[6] Benson, now an investment banker on Wall Street, was then an undergraduate student at the University of Wisconsin.

[7] Returning to the linear demand function (11) which gives the quadratic difference equation or (23) we have $x^* = 1/2$ and $x^m(m) = m/4$. Consequently, chaotic trajectories exist when $(a/c) = m \in A = \{m(m/4)^2\ (4-m) \leqslant m-\sqrt{m^2-2m},\ m < 4\}$. From the results of Hoppensteadt and Hyman drawn above we know that A contains the interval [3.83, 4]. Moreover, if we use iterates of $\Theta(\cdot)$ in (25) instead of $\Theta(\cdot)$ itself then we know that an $m \to 3.54$ and cycles of order 3.2^n emerge. Thus if we consider the map $\Theta^{2n}(y; m)$ then the set A is approximately [3.57; 4] and in this sense $m = 3.57$ is the threshold for chaos for the Robertson-Williams cobweb model when demand is linear and has the form (11).

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