Urban Labor Economics

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Appendix 2: Derivation of Bellman equations

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1. An intuitive way of deriving the Bellman equations

Let us first derive the expected lifetime-utility of an unemployed worker I_U. During a small interval of time dt, the unemployed worker obtains W_U and during this time he/she may find a job and enjoys a expected lifetime-utility level of I_E at time $t+dt$. The probability that he/she finds a job is: $adt+o(dt)$ with $\lim_{dt\to 0} o(dt)/dt = 0$. If he/she does not find a job, then he/she enjoys utility I_U at time $t+dt$. Observe that, during this small interval of time dt, the probability that the unemployed worker leave unemployment and loses straightaway his/her new job is negligible with respect to dt (since it is a term in $(dt)^2$). We have:[1]

$$I_U(t) = W_U(t)dt + \frac{1}{1+rdt}\left[adt\, I_L(t+dt) + (1-adt)I_U(t+dt)\right]$$

This is equivalent to

$$(1+rdt)\, I_U(t) = (1+rdt)\, W_U(t)dt + adt\, I_L(t+dt) + (1-adt)I_U(t+dt)$$

$$\Leftrightarrow rI_U(t)dt = W_U(t)dt + rW_U(t)(dt)^2 + adt\left[I_L(t+dt) - I_U(t+dt)\right] + I_U(t+dt) - I_U(t)$$

Now, by dividing everything by dt we obtain (observe that $(dt)^2$ is negligible compared to dt)

$$rI_U(t) = W_U(t) + a\left[I_L(t+dt) - I_U(t+dt)\right] + \frac{I_U(t+dt) - I_U(t)}{dt}$$

By taking the limit when $dt \to 0$, we get

$$rI_U(t) = W_U(t) + a\left[I_L(t) - I_U(t)\right] + \dot{I}_U(t)$$

since

$$\dot{I}_U(t) \equiv \frac{dI_U(t)}{dt} = \lim_{dt\to 0}\frac{I_U(t+dt) - I_U(t)}{dt}$$

In steady-state, $\dot{I}_U(t) = 0$ and $I_U(t) = I_U$ and $I_L(t) = I_L$. The steady-state lifetime expected utility of an unemployed worker is given by:

$$rI_U = W_U + a\left(I_L - I_U\right)$$

[1] We could have discounted in a different way,

$$I_U(t) = \frac{1}{1+rdt}\left[W_U(t)dt + adtI_E(t+dt) + (1-adt)I_U(t+dt)\right]$$

i.e. before starting the period. In this case, we would haveand it is easy to check that the Bellman equation would have been exactly the same.

Let us now determine the expected lifetime-utility of a non-shirker employed worker I_L. We have:

$$I_L(t) = W_L^{NS}(t)dt + \frac{1}{1+rdt}\left[\delta dt\, I_U(t+dt) + (1-\delta dt)I_L^{NS}(t+dt)\right]$$

By replicating the same analysis than for the unemployed, we easily obtain:

$$rI_L = W_L^{NS} - \delta\left(I_E - I_U\right)$$

Determining the case of shirking is more complicated since shirker workers can lose their jobs either by an exogenous shock δ or by being caught shirking at rate m.

We have

$$I_L^S(t) = W_L^S(t)dt + \frac{1}{1+rdt}\left[\delta dt\, I_U(t+dt) + mdt\, I_U(t+dt) + (1-(\delta+m)\,dt)\, I_L^S(t+dt)\right]$$

By replicating the same analysis, we obtain:

$$rI_L^S = W_L^S - (\delta+m)\left(I_L^S - I_U\right) \tag{1.1}$$

2. A formal way of deriving the Bellman equations

As stated above, changes in employment status are assumed to be governed by a Poisson (or Markov) process with two states: employed or unemployed. The key feature of these stochastic processes is that the duration time spent in each state is a random variable with exponential distribution (Poisson Process). More precisely, if we denote by τ_a and τ_δ the (random) unemployment and employment duration times, then

$$F(\tau_a) = \mathbb{P}\left[\tau_a < t\right] = 1 - e^{-a\,\tau_a}$$

$$F(\tau_\delta) = \mathbb{P}\left[\tau_\delta < t\right] = 1 - e^{-\delta\,\tau_\delta}$$

This implies that the probability densities are given by:

$$f(\tau_a) = a\, e^{-a\,\tau_a} \tag{2.1}$$

$$f(\tau_\delta) = \delta\, e^{-\delta\,\tau_p} \tag{2.2}$$

As a result, the average time spent in each state is equal to:

$$\begin{aligned}
\mathbb{E}\left[\tau_a\right] &= \int_0^{+\infty} \tau_a \, f(\tau_a) d\tau_a \\
&= \int_0^{+\infty} a\, \tau_a \, e^{-a\,\tau_a} d\tau_a \\
&= \int_0^{+\infty} e^{-a\,\tau_a} d\tau_a - \left[\tau_a \, e^{-a\,\tau_a}\right]_0^{+\infty} \\
&= -\frac{1}{a}\left[e^{-a\,\tau_a}\right]_0^{+\infty} - \left[\tau_a \, e^{-a\,\tau_a}\right]_0^{+\infty} \\
&= \frac{1}{a}
\end{aligned}$$

since

$$\lim_{\tau_a \to +\infty} \tau_a \, e^{-a\,\tau_a} = \lim_{\tau_a \to 0} \tau_a \, e^{-a\,\tau_a} = 0$$

Similarly,

$$\mathbb{E}\left[\tau_\delta\right] = \int_0^{+\infty} \tau_\delta \, f(\tau_\delta) d\tau_\delta = \frac{1}{\delta}$$

As above, let us first determine the expected lifetime-utility of an unemployed worker I_U. It is given by:

$$I_U = \mathbb{E}_{\tau_a}\left[\int_0^{\tau_a} W_U e^{-rt} dt + e^{-r\tau_a} I_L\right] \tag{2.3}$$

I_U is thus the discounted value at time $t = 0$. The unemployed worker stays unemployed during a random period of time τ_a. During this period he/she earns w_U discounted at rate r. Then, after this period τ_a, he/she becomes employed and obtains an expected utility of I_L discounted at rate r starting at time $t = \tau_a$.

By developing (2.3), we obtain:

$$I_U = \int\limits_0^{+\infty} \left[\int_0^{\tau_a} W_U e^{-rt} dt\right] f(\tau_a) d\tau_p + \int\limits_0^{+\infty} e^{-r\tau_a} I_E \, f(\tau_a) d\tau_a$$

Since

$$\int_0^{\tau_a} W_U e^{-rt} dt = W_U \frac{1 - e^{-r\tau_a}}{r}$$

and using (2.1), it can be rewritten as:

$$\begin{aligned} I_U &= a\frac{W_U}{r}\int_0^{+\infty}\left(1-e^{-r\tau_a}\right)e^{-a\tau_a}d\tau_a + aI_L\int_0^{+\infty}e^{-r\tau_a}e^{-a\tau_a}d\tau_a \\ &= a\frac{W_U}{r}\left[\int_0^{+\infty}\left(e^{-a\tau_a}-e^{-(r+a)\tau_a}\right)d\tau_a\right] + a\,I_L\int_0^{+\infty}e^{-(r+a)\tau_a}\,d\tau_a \\ &= \frac{W_U}{r+a}+\frac{a}{r+a}I_L \end{aligned}$$

We finally obtain:

$$rI_U = W_U + a\left(I_L - I_U\right) \tag{2.4}$$

Similarly, the expected lifetime-utility of a non shirker employed worker I_L is:

$$I_L^{NS} = \mathbb{E}_{\tau_\delta}\left[\int_0^{\tau_\delta} W_L^{NS}e^{-rt}dt + e^{-r\tau_\delta}I_U\right] \tag{2.5}$$

By doing exactly the same analysis, one easily obtains :

$$rI_L^{NS} = W_L^{NS} - \delta\left(I_L - I_U\right) \tag{2.6}$$

The case of a shirker is again more complicated since, when employed, a shirker can lose his/her job because either he/has been caught shirking or the job has been destroyed. Denote by τ_m the (random) length of time until the next control of shirking occurs. This implies that τ_a is still the (random) unemployment duration time whereas $\min(\tau_\delta,\tau_m)$ is now the employment duration time for a shirker. Since we know (see for example Kulkarni, 1995, ch. 5) that $\min(\tau_\delta,\tau_m)$ is a random variable characterized by an exponential distribution of parameter $\delta+m$, i.e.

$$F(\min(\tau_\delta,\tau_m)) = \mathbb{P}\left[\min(\tau_\delta,\tau_m) < t\right] = 1 - e^{-(\delta+m)\,\min(\tau_\delta,\tau_m)}$$

then the expected lifetime-utility of a shirker I_L^S is equal to:

$$I_L^S = \mathbb{E}\left[\int_0^{\min(\tau_\delta,\tau_m)} W_L^S\,e^{-rt}dt + e^{-r\min(\tau_\delta,\tau_m)}I_U\right]$$

By doing exactly the same kind of manipulations as above, we obtain (1.1).

References

[1] Kulkarni, V.G. (1995), *Modeling and Analysis of Stochastic Systems*, London: Chapman & Hall.