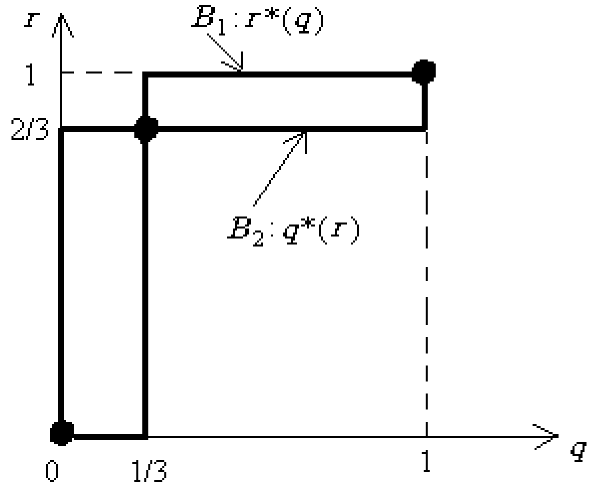
Correction of Lecture Notes 8bis

Some examples of Bayes-Nash Equilibria

Correction of Example 1: Relation between mixed-strategy equilibrium and Bayesian equilibrium (Gibbons)

1a) We have seen in Lecture Notes 8bis that this game has three mixed strategy Nash equilibria:



(i) $(q, 1-q) = (0, 1)$ for Patrick and $(r, 1-r) = (0, 1)$ for Lindsey, i.e. the pure strategy equilibrium (Fight, Fight)

(ii) $(q, 1-q) = (1, 0)$ for Patrick and $(r, 1-r) = (1, 0)$ for Lindsey, i.e. the pure strategy equilibrium (Opera, Opera),

(iii) the mixed strategies $(q, 1-q) = (1/3, 2/3)$ for Patrick and $(r, 1-r) = (2/3, 1/3)$ for Lindsey are a Nash equilibrium.

1b) Formal definition of this static Bayesian game in normal form (action spaces, type spaces, beliefs). In terms of the abstract static Bayesian game in normal form we have:

$$G = \{A_L, A_P; T_L, T_P; p_L, p_P; u_L, u_P\}$$

Players: Lindsey and Patrick;

Strategies: $A_L = A_P = \{\text{Opera , Fight}\}$;

Type spaces: $T_L = T_P = [0, x]$;

Beliefs: $p_L(t_P) = p_P(t_L) = 1/x$ for all t_L and t_P;

Payoffs: there are given the matrix above.

1c) When the distribution is uniform, the probability that Lindsey plays "Opera" is $(x - c)/x$ and the probability that Patrick plays "Fight" is $(x - p)/x$.

If the distribution was not uniform, then the probability that Lindsey plays "Opera" would be $1 - F(c)$ since $F(c) = \Pr(t_L \leq c)$ and the probability that Patrick plays "Fight" would be $1 - F(p)$ since $F(p) = \Pr(t_P \leq p)$.

1d) For a given value of x, let us determine values of c and p such that these strategies are a Bayesian Nash equilibrium.

Given Patrick's strategy, Lindsey's expected payoff from playing "Opera" is

$$\frac{p}{x}(2 + t_L) + \left[1 - \frac{p}{x}\right].0 = \frac{p}{x}(2 + t_L)$$

and Lindsey's expected payoff from playing "Fight" is

$$\frac{p}{x}.0 + \left[1 - \frac{p}{x}\right].1 = 1 - \frac{p}{x}$$

Thus, for Lindsey, playing "Opera" is optimal if and only if:

$$t_L \geq \frac{x}{p} - 3 \equiv c \tag{1}$$

Similarly, given Lindsey's strategy, Patrick's expected payoff from playing "Fight" is

$$\left[1 - \frac{c}{x}\right].0 + \frac{c}{x}(2 + t_P) = \frac{c}{x}(2 + t_P)$$

and Patrick's expected payoff from playing "Opera" is

$$\left[1 - \frac{c}{x}\right].1 + \frac{c}{x}.0 = 1 - \frac{c}{x}$$

Thus, for Patrick, playing "Fight" is optimal if and only if:

$$t_P \geq \frac{x}{c} - 3 \equiv p \tag{2}$$

Solving (1) and (2) simultaneously yields

$$\begin{cases} x - 3p = cp \\ x - 3c = cp \end{cases}$$

or equivalently

$$\begin{cases} p = c \\ p^2 + 3p - x = 0 \end{cases}$$

The discriminant of the quadratic equation $p^2 + 3p - x = 0$ is:

$$\Delta = 9 + 4x$$

and the two roots are given by

$$p_1 = \frac{-3 - \sqrt{9 + 4x}}{2} < 0$$

and

$$p_2 = \frac{-3 + \sqrt{9 + 4x}}{2} > 0 \tag{3}$$

Since p has to be positive, then only p_2 is possible. Thus the probability that Lindsey plays "Opera", namely $(x-c)/x$, and the probability that Patrick plays "Fight", namely $(x-p)/x$, are both equal to

$$\frac{x-c}{x} = \frac{x-p}{x} = 1 - \frac{-3+\sqrt{9+4x}}{2x}$$

To summarize, the values of c and p such the strategies decribe above (i.e. Lindsey plays "Opera" if $t_L \geq c$ and plays "Fight" otherwise; Patrick plays "Fight" if $t_P \geq p$ and plays "Opera" otherwise) are a Bayesian Nash equilibrium are

$$c = p = \frac{-3+\sqrt{9+4x}}{2}$$

1e) We have to show that both $(x-c)/x$ and $(x-p)/x$ approach $2/3$ as x approaches 0. It is easy to see that

$$\lim_{x\to 0} \frac{x-c}{x} = \lim_{x\to 0} \frac{x-p}{x} = \lim_{x\to 0} \left[1 - \frac{-3+\sqrt{9+4x}}{2x}\right]$$

is not determined. We have to use the l'Hopital rule. The latter says:

If two functions $f()$ and $g()$ are continuous and differentiable, then

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{f'(x)}{g'(x)}$$

Denote by $f(x) = -3 + \sqrt{9+4x}$ and $g(x) = 2x$. Then, since $f'(x) = 2/\sqrt{9+4x}$ and $g'(x) = 2$

$$\lim_{x\to 0} \left[\frac{-3+\sqrt{9+4x}}{2x}\right] = \lim_{x\to 0} \left[\frac{1}{\sqrt{9+4x}}\right] = \frac{1}{3}$$

As a result,

$$\lim_{x\to 0} \frac{x-c}{x} = \lim_{x\to 0} \frac{x-p}{x} = \lim_{x\to 0} \left[1 - \frac{-3+\sqrt{9+4x}}{2x}\right] = 1 - \frac{1}{3} = \frac{2}{3}$$

Thus, as the incomplete information disappears, the players' behavior in the pure-strategy Bayesian Nash equilibrium of the incomplete-information game approaches their behavior in the mixed-strategy Nash equilibrium in the original game of complete information.

This shows that a mixed-strategy Nash equilibrium in a game of complete information can (almost always) be intrepreted as a pure-strategy Bayesian Nash equilibrium in a closely related game with a little bit of incomplete information.