Université du Maine
Théorie des Jeux

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(1 heure 30)

Problem (1) (8 points)

Consider the following lobbying game between two firms. Each firm may lobby the government in hopes of persuading the government to make a decision that is favorable to the firm. The two firms, X and Y, *independently* and *simultaneously* decide whether to lobby (L) or not (N). Lobbying entails a cost of 15. Not lobbying costs nothing. If both firms lobby or neither firm lobbies then the government takes a neutral decision, which yields 10 to both firms. If firm Y lobbies and X does not lobby, then the government makes a decision that favors firm Y, yielding zero to firm X and 30 to firm Y. Finally, if firm X lobbies and Y does not, the government's decision yields x to firm X and zero to firm Y. Assume that $x > 25$. The normal form of this game is (player 1 is firm X and player 2 is firm Y):

$X \setminus Y$	L	N
L	-5 , -5	$x-15$, 0
N	0 , 15	10 , 10

(1a) Determine the pure-strategy Nash equilibrium of this game (if it has any). **(1 point)**

The BR functions are given by (since $x > 25$, then $x - 15 > 10$):

$X \setminus Y$	L	N
L	-5 , -5	$\underline{x-15}$, $\underline{0}$
N	$\underline{0}$, $\underline{15}$	10 , 10

As a result, there are two pure-strategy Nash equilibria: (L, N) and (N, L).

(1b) Compute the mixed-strategy Nash equilibrium of this game (if it has any). **(3 points)**

Assume that firm X believes that firm Y plays strategy L with probability q and strategy N with probability $1-q$. Assume also that firm Y believes that firm X plays strategy L with probability p and strategy N with probability $1-p$.

In that case, the value of q that makes firm X indifferent between L and N is given by:

$$\underbrace{-5q+(x-15)(1-q)}_{\text{Firm } X\text{'s expected payoff of playing } L} \quad = \quad \underbrace{0\times q+10(1-q)}_{\text{Firm } X\text{'s expected payoff of playing } N}$$

Rearranging yields

$$q=\frac{x-25}{x-20}$$

i.e., this is the probability that firm Y chooses to play L that makes firm X indifferent between playing L and N. It is easily verified that $0<q<1$.

Similarly, the value of p that makes firm Y indifferent between L and N is given by:

$$\underbrace{-5p+15(1-p)}_{\text{Firm } Y\text{'s expected payoff of playing } L} \quad = \quad \underbrace{0\times p+10(1-p)}_{\text{Firm } Y\text{'s expected payoff of playing } N}$$

Rearranging yields

$$p=\frac{1}{2}$$

i.e., this is the probability that firm X chooses to play L that makes firm Y indifferent between playing L and N.

The mixed strategy NE is thus:

$$\left\{\left(\frac{5}{x-20},\frac{25-x}{20-x}\right),\left(\frac{1}{2},\frac{1}{2}\right)\right\}$$

(1c) Given the mixed-strategy Nash equilibrium computed in part (1b), what is the probability that the government makes a decision that favors firm X? **(2 points)**

We would like to calculate the probability that (L, N) occurs, i.e. firm X plays L and firm Y plays N. This probability is:

$$p(1-q)=\frac{1}{2}\left(\frac{5}{x-20}\right)$$

(1d) As x rises, does the probability that the government makes a decision favoring firm X rise or fall? Is this good from an economic standpoint? **(2 points)**

We need to calculate:

$$\frac{\partial\left[p(1-q)\right]}{\partial x} = -\frac{1}{2}\left[\frac{5}{(x-20)^2}\right] < 0$$

Thus, as x increases, the probability of (L, N) decreases. However, as x becomes larger, (L, N) is a "better" outcome.

Problem (2) (12 points)

Consider a Cournot game with two firms where the (inverse) demand function is given by:

$$p(Q) = \begin{cases} a - b\,f(Q) & \text{if } Q < \widehat{Q} \\ 0 & \text{otherwise} \end{cases}$$

where $Q = q_1 + q_2$ is the market demand and $\widehat{Q} = f^{-1}(\frac{a}{b})$. It is assumed that $f'(Q) > 0$. Each firm $i = 1, 2$ has a *total* cost function equals to

$$C(q_i) = c_i\,q_i^2$$

The two firms simultaneously choose their quantities and we want to determine the Nash equilibrium of this game in quantities, which we denote by (q_1^*, q_2^*).

(2a) Show under which conditions there is a *unique* and *interior* Nash equilibrium. **(2 points)**

We need to verify that the second-order conditions and the boundary conditions are always satisfied. The profit of firm i is given by:

$$\pi_i = p(Q)q_i - C(q_i) = \left[a - b\,f(Q)\right]q_i - c_i q_i^2 = \left[a - b\,f(q_i + q_j)\right]q_i - c_i q_i^2$$

We have for $i \neq j$:

$$\frac{\partial \pi_i(q_i, q_j)}{\partial q_i} = a - b\,f'(Q)q_i - b\,f(Q) - 2c_i q_i = 0$$

The **second-order condition** for $i = 1, 2$ is given by:

$$\frac{\partial^2 \pi_i(q_i, q_j)}{\partial q_i^2} = -b\,f''(Q)q_i - 2b\,f'(Q) - 2c_i < 0$$

A sufficient condition for the second-order condition to be satisfied is: $f''(Q) \geq 0$.

Boundary conditions: We want to show that $\left(q_i^*, q_j^*\right) = (0,0)$ is not a Nash equilibrium and that $\left(q_i^*, q_j^*\right) = \left(\frac{\widehat{Q}}{2}, \frac{\widehat{Q}}{2}\right)$ is not a Nash equilibrium.

(i) Let us show that $\left(q_i^*, q_j^*\right) = (0,0)$ is not a Nash equilibrium. The first boundary condition (fix at $q_j^* = 0$ and show that firm i wants to deviate from $q_i^* = 0$) is given by:

$$\frac{\partial \pi_i(q_i, q_j)}{\partial q_i} \mid_{q_i=0, q_j^*=0} = a - b\, f(0) > 0$$

As a result, if $f(0) < a/b$, then firm i always wants to deviate from $q_i^* = 0$ and thus $\left(q_i^*, q_j^*\right) = (0,0)$ cannot be a Nash equilibrium.

(ii) Let us now show that $\left(q_i^*, q_j^*\right) = \left(\frac{\widehat{Q}}{2}, \frac{\widehat{Q}}{2}\right)$ is not a Nash equilibrium. Fix q_j^* at $q_j^* = \widehat{Q}/2$ and let us show that firm i wants to deviate from $q_i^* = \widehat{Q}/2$. The second boundary condition is thus given by:

$$\frac{\partial \pi_i(q_i, q_j)}{\partial q_i} \mid_{q_i^*=\widehat{Q}/2, q_j^*=\widehat{Q}/2} = a - b\, f'(\widehat{Q})\frac{\widehat{Q}}{2} - b\, f(\widehat{Q}) - c_i \widehat{Q} < 0$$

This is clearly always negative since $a - c_i \widehat{Q} = 0$ and $f'(\widehat{Q}) > 0$.

To summarize, if both $f''(Q) \geq 0$ and $f(0) < a/b$, then there is a *unique* and *interior* Nash equilibrium.

(2b) Assume that $b = 1$ and that $f(Q) = Q$. Calculate the unique Nash equilibrium of this game. Give the Nash equilibrium values of the quantities, q_1^{NE}, q_2^{NE}. **(4 points)**

First-order conditions:

$$\frac{\partial \pi_1(q_1, q_2)}{\partial q_1} = -2q_1\left(1 + c_1\right) + a - q_2 = 0$$

$$\frac{\partial \pi_2(q_1, q_2)}{\partial q_2} = -2q_2\left(1 + c_2\right) + a - q_1 = 0$$

which lead to the following best-reply functions:

$$BR_1(q_2) \equiv q_1(q_2) = \frac{(a - q_2)}{2\left(1 + c_1\right)}$$

$$BR_2(q_1) \equiv q_2(q_1) = \frac{(a - q_1)}{2\left(1 + c_2\right)}$$

Combining these two equations, we obtain:

$$q_1 = \frac{a}{2(1+c_1)} - \frac{(a-q_1)}{4(1+c_1)(1+c_2)}$$

that is

$$q_1^{NE} = \frac{a(1+2c_2)}{3+4(c_1+c_2+c_1c_2)}$$

Plugging back this value in $BR_2(q_1)$, we obtain:

$$q_2^{NE} = \frac{a(1+2c_1)}{3+4(c_1+c_2+c_1c_2)}$$

(2c) Assume that $b = 1$ and that $f(Q) = Q$. Determine the social optimum solution in quantities, q_1^{SO}, q_2^{SO}. **(2 points)**

The social optimum is defined such that:

$$\max_{q_1,q_2} \{\Pi(q_1,q_2) = \pi_1(q_1,q_2) + \pi_2(q_1,q_2)\}$$

Here

$$\Pi(q_1,q_2) = (a-Q)\,Q - c_1q_1^2 - c_2q_2^2$$

First-order conditions:

$$\frac{\partial \Pi}{\partial q_1} = a - 2Q - 2c_1q_1 = 0 \quad (1)$$

$$\frac{\partial \Pi}{\partial q_2} = a - 2Q - 2c_2q_2 = 0 \quad (2)$$

If we substract these two equations, we first obtain that:

$$q_2 = \frac{c_1}{c_2}q_1 \quad (3)$$

Plugging the value of q_2 from (3) into (1), we obtain:

$$q_1^{SO} = \frac{ac_2}{2(c_1+c_2+c_1c_2)}$$

Plugging this value into (3), we finally get:

$$q_2^{SO} = \frac{ac_1}{2(c_1+c_2+c_1c_2)}$$

(2d) Assume that $b = 1$ and that $f(Q) = Q$. Assume also that firm 1 decides *first* its quantity $q_1$and *then* firm 2 decides its quantity q_2. Determine the subgame-perfect equilibrium of this game where you denote the equilibrium quantities by q_1^{SPNE} and q_2^{SPNE}. Under which condition there is a first-mover advantage in terms of quantities, i.e. $q_1^{SPNE} > q_2^{SPNE}$? When is there no first-mover advantage? **(4 points)**

We solve the model backward. The best reply function of firm 2 is (as above) given by:

$$BR_2(q_1) \equiv q_2(q_1) = \frac{a - q_1}{2(1 + c_2)}$$

Plugging this value into the profit function of firm 1, we obtain:

$$\begin{aligned}
\pi_1 &= (a - q_1 - q_2)\, q_1 - c_1 q_1^2 \\
&= \left[a - q_1 - \frac{(a - q_1)}{2(1 + c_2)}\right] q_1 - c_1 q_1^2 \\
&= \left[\frac{a + 2ac_2 + q_1}{2(1 + c_2)}\right] q_1 - (1 + c_1)\, q_1^2
\end{aligned}$$

First-order condition yields:

$$\frac{\partial \pi_1}{\partial q_1} = \frac{a + 2ac_2}{2(1 + c_2)} + \frac{q_1}{2(1 + c_2)} - 2(1 + c_1)\, q_1 = 0$$

This is equivalent to:

$$q_1^{SPNE} = \frac{a(1 + 2c_2)}{2(1 + 2c_1 + 2c_2 + 2c_1c_2)}$$

By plugging this value into the best-reply function of firm 2, we obtain:

$$\begin{aligned}
q_2^{SPNE} &= \frac{a - q_1^{SPNE}}{2(1 + c_2)} \\
&= \frac{a(1 + 4c_1 + 2c_2 + 4c_1c_2)}{4(1 + 2c_1 + 2c_2 + 2c_1c_2)(1 + c_2)}
\end{aligned}$$

Thus $q_1^{SPNE} > q_2^{SPNE}$ if and only if:

$$c_1 < c_2 + \frac{1}{4(1 + c_2)}$$

which is equivalent to:

$$c_1 < \frac{1 + 4(1 + c_2)\, c_2}{4(1 + c_2)}$$

Observe that $\frac{1+4(1+c_2)c_2}{4(1+c_2)}$ is increasing in c_2. As a result, there is a first-mover advantage if c_1 is low enough or c_2 is large enough.