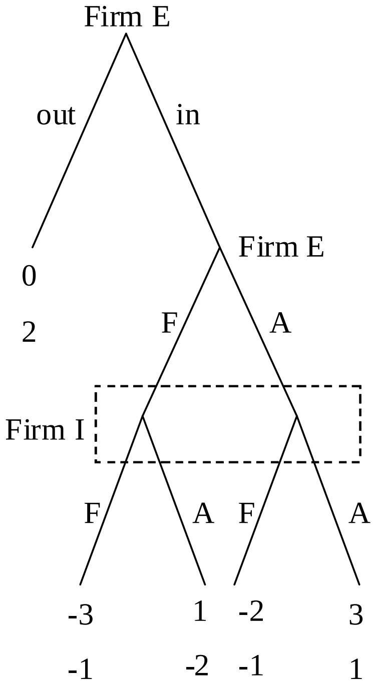
Microeconomic Theory EC104

Answers Problem Set 3

(* is easy, ** is difficult, *** is more difficult)

1. * Consider the 2 player game depicted in Figure 1.

Figure 1



1a. Find the unique subgame perfect equilibrium of this game.

The game has 1 proper subgame and an improper one (the game itself). Solve the game using subgame perfection:

Final subgame:

E/I	F	A
F	$-3, \underline{-1}$	$1, -2$
A	$\underline{-2}, -1$	$\underline{3}, \underline{1}$

The unique NE of this final subgame is (A,A).

Penultimate (initial) subgame:

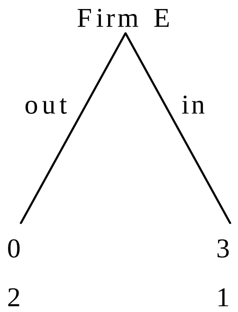


Figure 1:

The unique NE of this subgame is "in". It follows that the unique SPNE (and outcome) of the game is $(s_E; s_I) = (in, A; A)$.

1b. Identify all other pure strategy Nash equilibria of this game. Explain why none of these other equilibria are sub-game perfect.

The normal form representation of the game is as follows:

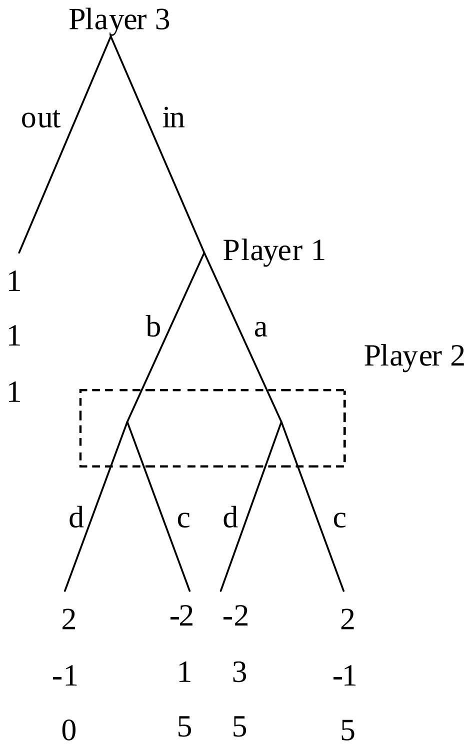
E/I	F	A
in,F	$-3, -1$	$1, -2$
in,A	$-2, -1$	$3, 1$
out,F	$0, -1$	$0, -2$
out,A	$0, -1$	$0, 1$

The game admits 2 pure strategies NE: (out,F;F) and (in,A;A), but only (in,A;A) is subgame perfect.

The equilibrium (out,F;F) is not subgame perfect because is based on firm E wrong belief that firm I will play F: firm I will never play F because F is dominated by A in the post-entry game ($-1 < 2$ and $-1 < 1$).

2. ** Consider the 3 player game depicted in Figure 2.

Figure 2



2a. Explain why there is no subgame perfect Nash equilibrium in pure strategies.

The game has 1 proper subgame plus the game itself. The final subgame can be represented by the following matrix:

1/2	c	d
a	**2**,-1	-2,**3**
b	-2,**1**	**2**,-1

The game does not have pure strategies SPNE because the final subgame does not have pure strategies NE, but only a unique equilibrium in mixed strategies:

$$(r, 1-r; q, 1-q) = (1/3, 2/3; 1/2, 1/2)$$

where r is the probability with which player 1 plays "a", and q is the probability with which player 2 plays "c".

2b. Find the unique subgame perfect Nash equilibrium.

Consider the penultimate (initial) subgame: player 3 will decide to play "out" if the expected payoff from playing "in" is larger than 1. We have:

$$\begin{aligned} & EU_3(\text{“in” given } (1/3, 2/3; 1/2, 1/2)) \\ = \; & \frac{2}{3}\frac{1}{2} \times 0 + \frac{2}{3}\frac{1}{2} \times 5 + \frac{1}{3}\frac{1}{2} \times 5 + \frac{1}{3}\frac{1}{2} \times 5 = \frac{20}{6} > 1 \end{aligned}$$

Therefore player 3 will play "in" and the unique SPNE is $(1/3, 2/3; 1/2, 1/2; \text{in})$.

3. * Suppose three firms compete in a market for a single product with industry inverse demand curve $p = A - Q$. All three firms have constant marginal cost m. Firm 1 is a leader and selects output level q_1. Firms 2 and 3 are followers and select q_2 and q_3 after q_1. Note, q_2 and q_3 are chosen simultaneously. Total output is $Q = q_1 + q_2 + q_3$. Find a subgame perfect Nash equilibrium solution.

This game has 1 proper (final) subgame in which firm 2 and 3 simultaneously choose their quantities, and an improper (initial) subgame - the game itself- in which firm 1chooses its quantity given its beliefs about the outcome of the competition between firm 2 and 3 in the final subgame.

Final subgame: Firm i's payoff is given by: $(A - Q - m)\, q_i$, where $Q = q_i + q_j + q_1$, $i, j = 2, 3$. Firm i's best response function is therefore:

$$BR_i(q_1, q_j) = (A - m - q_1 - q_j)/2$$

Firm 2 and firm 3 best response functions cross at the point

$$\left(\frac{A - m - q_1}{3} \,; \, \frac{A - m - q_1}{3} \right)$$

Penultimate (initial subgame): Firm 1 chooses q_1 that maximizes its payoff given by $(A - Q - m)\, q_1$, where $Q = 2(A - m - q_1)/3 + q_1$. Therefore the unique SPNE of the game is

$$\left(\frac{A - m}{2} \,; \, \frac{A - m - q_1}{3} ; \, \frac{A - m - q_1}{3} \right)$$

The unique subgame perfect outcome is thus:

$$\left(\frac{A - m}{2} \,; \, \frac{A - m}{6} \,; \, \frac{A - m)}{6} \right)$$

4. ** Consider three firms in an industry such that each firm $i = 1, 2, 3$ produces quantity q_i. In fact, firms produce their output sequentially, that is firm 1 produces first q_1, then, in the second stage, firm 2 produces q_2, and then, in the third stage, firm 3 produces q_3. The market price is given by:

$$p = 120 - Q$$

where $Q = q_1 + q_2 + q_3$. The marginal cost of production is assumed to be the same for all firms and constant and equals to c.

4a. Determine the subgame perfect Nash equilibrium of this game by giving the equilibrium quantities of each firm q_1^*, q_2^* and q_3^*, the equilibrium price p^* as well as the equilibrium profits π_1^*, π_2^* and π_3^*. All these equilibrium values should be expressed in terms of only c.

At the last stage , firm 3 takes as given q_1 and q_2 and chooses q_3 that maximizes its profit, which is:

$$\pi_3 = pq_3 - cq_3 = \left[120 - (q_1 + q_2 + q_3)\right] q_3 - cq_3$$

Thus, it solves the following program:

$$\max_{q_3} \left\{\left[120 - (q_1 + q_2 + q_3)\right] q_3 - cq_3\right\}$$

The first-order condition is:

$$-q_3 + \left[120 - (q_1 + q_2 + q_3)\right] - c = 0$$

and the BR function for firm 3 is thus:

$$q_3(q_1, q_2) = \frac{120 - c}{2} - \frac{(q_1 + q_2)}{2} \tag{1}$$

Let us solve the second stage. Firm 2 takes q_1 and $q_3(q_1, q_2)$ as given and chooses q_2 that maximizes its profit. Thus, it solves the following program:

$$\max_{q_2} \left\{\left[120 - (q_1 + q_2 + q_3(q_1, q_2))\right] q_2 - cq_2\right\}$$

which, by using (1), is equivalent to:

$$\max_{q_2} \left\{\left[120 - \left(q_1 + q_2 + \frac{120 - c}{2} - \frac{(q_1 + q_2)}{2}\right)\right] q_2 - cq_2\right\}$$

$$\Leftrightarrow \quad \max_{q_2} \left\{\left[120 - \frac{1}{2}(q_1 + q_2 + 120 - c)\right] q_2 - cq_2\right\}$$

The first-order condition is:

$$-\frac{1}{2}q_2 + 120 - \frac{1}{2}(q_1 + q_2 + 120 - c) - c = 0$$

and the BR function for firm 2 is thus:

$$q_2(q_1) = \frac{120 - q_1 - c}{2} \tag{2}$$

Let us now solve the first stage. Firm 1 takes $q_2(q_1)$ and $q_3(q_1, q_2(q_1))$ (respectively defined by (2) et (1)) as given and chooses q_1 that maximizes its profit. Thus, it solves the following program:

$$\max_{q_1} \left\{\left[120 - \left[q_1 + q_2(q_1) + q_3(q_1, q_2(q_1))\right]\right] q_1 - cq_1\right\}$$

By using (2) and (1), this program can be written as:

$$\max_{q_1} \left\{\left[120 - \left(q_1 + \frac{120 - q_1 - c}{2} + \frac{120 - c}{2} - \frac{\left(q_1 + \frac{120 - q_1 - c}{2}\right)}{2}\right)\right] q_1 - cq_1\right\}$$

$$\Longleftrightarrow \quad \max_{q_1} \left\{\left[120 - q_1 - \frac{120 - q_1 - c}{2} - \frac{120 - c}{2} + \frac{q_1}{2} + \frac{120 - q_1 - c}{4}\right] q_1 - cq_1\right\}$$

$$\Longleftrightarrow \quad \max_{q_1} \left\{\left[30 + \frac{3}{4}c - \frac{q_1}{4}\right] q_1 - cq_1\right\}$$

The first-order condition is:

$$-\frac{q_1}{4} + 30 + \frac{3}{4}c - \frac{q_1}{4} - c = 0$$

and we obtain:

$$q_1^* = 60 - \frac{c}{2} \tag{3}$$

By replacing the value of q_1^* given in (3) into (2), we obtain:

$$q_2^* = 30 - \frac{c}{4} \tag{4}$$

Finally, replacing the value of q_1^* given in (3) and the value of q_2^* given in (4) into (1), we get:

$$q_3^* = 15 - \frac{c}{8} \tag{5}$$

We are now able to calculate the equilibrium market price:

$$\begin{aligned} p^* &= 120 - Q = 120 - 60 + \frac{c}{2} - 30 + \frac{c}{4} - 15 + \frac{c}{8} \\ &= 15 + \frac{7c}{8} \end{aligned} \tag{6}$$

and firm i's profit:

$$\begin{aligned} \pi_i^* &= (p^* - c)\, q_i^* \\ &= \left(15 - \frac{c}{8}\right) q_i^* \end{aligned}$$

By using (3), (4) et (5), we obtain:

$$\begin{aligned} \pi_1^* &= \left(15 - \frac{c}{8}\right)\left(60 - \frac{c}{2}\right) \\ &= 900 - 15c + \frac{c^2}{16} = \frac{(120 - c)^2}{16} \end{aligned} \tag{7}$$

$$\begin{aligned} \pi_2^* &= \left(15 - \frac{c}{8}\right)\left(30 - \frac{c}{4}\right) \\ &= 450 - 7.5c + \frac{c^2}{32} = \frac{(120 - c)^2}{32} \end{aligned} \tag{8}$$

$$\begin{aligned} \pi_3^* &= \left(15 - \frac{c}{8}\right)\left(15 - \frac{c}{8}\right) \\ &= 225 - 3.75c + \frac{c^2}{64} = \frac{(120 - c)^2}{64} \end{aligned} \tag{9}$$

4b. Which firm produces the most and which firm produces the least in equilibrium? Which firm has the highest profit and which firm has the lowest profit in equilibrium? Explain why.

For firms to make profit, it has to be that $p^* > c$, which using (6) can be written as:

$$15 + \frac{7c}{8} > c$$

$$\Leftrightarrow c < 120 \tag{10}$$

We also want that: $q_1^* > q_2^* > q_3^*$, which is equivalent to:

$$60 - \frac{c}{2} > 30 - \frac{c}{4} > 15 - \frac{c}{8}$$

If (10) is true, then these inequalities are also true and therefore

$$q_1^* > q_2^* > q_3^*$$

Since the profit of each firm is: $\pi_i^* = (p^* - c)\, q_i^*$ and $p^* - c$ is the same for all firms, this implies that

$$\pi_1^* > \pi_2^* > \pi_3^*$$

There is thus a first-mover advantage.

4c. How each firm i's $(i = 1, 2, 3)$ profit varies with c? Explain.

By differentiating (7), we obtain:

$$\frac{\partial \pi_1^*}{\partial c} = -15 + \frac{c}{8}$$

We have thus:

$$\frac{\partial \pi_1^*}{\partial c} = -15 + \frac{c}{8} < 0 \Leftrightarrow c < 120$$

which is always true (see (10)).

By differentiating (8), we obtain:

$$\frac{\partial \pi_2^*}{\partial c} = -7.5 + \frac{2c}{32}$$

We have thus:

$$\frac{\partial \pi_2^*}{\partial c} = -7.5 + \frac{2c}{32} < 0 \Leftrightarrow c < 120$$

which is always true (see (10)).

Finally, by differentiating (9), we obtain:

$$\frac{\partial \pi_3^*}{\partial c} = -3.75 + \frac{2c}{64}$$

We have therefore:

$$\frac{\partial \pi_3^*}{\partial c} = -3.75 + \frac{2c}{64} < 0 \Leftrightarrow c < 120$$

which is always true (see (10)).

This is very intuitive since when c increases, the quantity produced by each firm is reduced and thus each firm's profit is also reduced.

4d. Assume now that the firm 1 is a potential entrant whereas firms 2 and 3 are incumbents. If firm 1 enters the market, the sequence of actions is as before, that is firm 1 first chooses q_1, then firm 2 chooses q_2 and finally firm 3 chooses q_3. The entry cost for firm 1 is $F > 0$. If firm 1 chooses to not enter, then its profit is zero. We assume that if firm 1 has profit equals to zero whether it enters or not enters in the market, it always prefers enter in the market. For which value of F the firm will decide to enter in this market?

We have calculated firm 1's profit if it enters. By using (7), this profit is equal to:

$$\pi_1^E = 900 - 15c + \frac{c^2}{16} - F$$

If the firm does not enter, its profit is zero. As a result, the firm will enter if and only if:

$$900 - 15c + \frac{c^2}{16} - F \geq 0$$

which is equivalent to:

$$F \leq 900 - 15c + \frac{c^2}{16} = \frac{(120-c)^2}{16} \tag{11}$$

4e. Assume now that $F = c^2/16$. Does firm 1 enter in the market? Discuss the results in terms of entry versus not entry for different values of c. Calculate firm 1's profit in all possible cases.

Condition (11), which determines when firm 1 will enter in the market can now be written as:

$$\frac{c^2}{16} \leq 900 - 15c + \frac{c^2}{16}$$

which is equivalent to:

$$0 \leq 900 - 15c \Leftrightarrow c \leq \frac{900}{15} = 60$$

Since $c < 120$ (for the market to be viable), we have the following conclusion:

If $0 < c < 60$, firm 1 enter in the market and its profit is equal to:

$$\pi_1^{E*} = 900 - 15c$$

If $c = 60$, firm 1 enter in the market and its profit is equal to $\pi_1^{E*} = 0$.
If $60 < c < 120$, firm 1 does not enter in the market and its profit is equal to zero.

5. *** Exercise 177.2 (Osborne, only questions 5a and 5b) (The "rotten kid theorem")

A child's action a affects both her own private income $c(a)$ and her parent's income $p(a)$; for all values of a we have

$$c(a) < p(a)$$

The child is selfish: she cares only about the amount of money she has. Her loving parent cares both about how much money she has and how much her child has.

Specifically, the preferences of the child are represented by a payoff function U_c equals to the smaller of the amount of money she has and the amount of money her child has.

The parent may transfer money to the child. This transfer is denoted by $t \geq 0$. In that case, the utility U_c of the child is given by:

$$U_c(a, t) = c(a) + t$$

whereas the utility of the parent is:

$$U_p(a,t) = \min\{p(a) - t \ , \ c(a) + t\}$$

The timing is as follows. First the child takes an action, then the parent decides how much money to transfer.

5a. Model this situation as an extensive game.

Players: The parents and the child.

Terminal histories: The set of sequences (a, t), where a is an action of the child and t is a transfer from the parents to the child; a and t are positive numbers.

Players' function: $P(\varnothing)$ is the child, $P(a)$ is the parents for each value of a.

Preferences: For the child, they are given by:

$$U_c(a,t) = c(a) + t$$

while for the parents, we have:

$$U_p(a,t) = \min\{p(a) - t \ , \ c(a) + t\}$$

5b. Show that in a subgame perfect equilibrium the child takes an action that maximizes the sum of her private income and the parent's income. (In particular, the child's action does not maximize her own private income.)

To find the subgame perfect equilibria of this game, let us first solve the *second stage* of this game, that is the optimal choice of the parents in terms of transfer t.

By assumption, we have:

$$c(a) < p(a)$$

Thus, if the parents do not give any transfer, i.e. $t = 0$, then their utility is:

$$\begin{aligned} U_p(a,0) &= \min\{p(a) - t \ , \ c(a) + t\} \\ &= \min\{p(a) \ , \ c(a)\} \\ &= c(a) \end{aligned}$$

If the parents transfer 1 dollar to the child, then their payment increases by 1 dollar since it is equal to:

$$\begin{aligned} U_p(a,1) &= \min\{p(a) - 1 \ , \ c(a) + 1\} \\ &= c(a) + 1 \end{aligned}$$

In fact, it is easily seen that by augmenting t, the utility of the parents increases up to:

$$p(a) - t^* = c(a) + t^*$$

If the parents pay $t > t^*$, then their utility will be lower since they utility will be lower than that of their child, i.e.

$$U_p(a, t > t^*) = \min\{p(a) - t \ , \ c(a) + 1\} = c(a) + t$$

In order to see this in a clear way, we can represent in a figure the functions $p(a) - t$ and $c(a) + t$. If we denote by $f(t) = p(a) - t$ and $g(t) = c(a) + t$, then it is easily checked that:

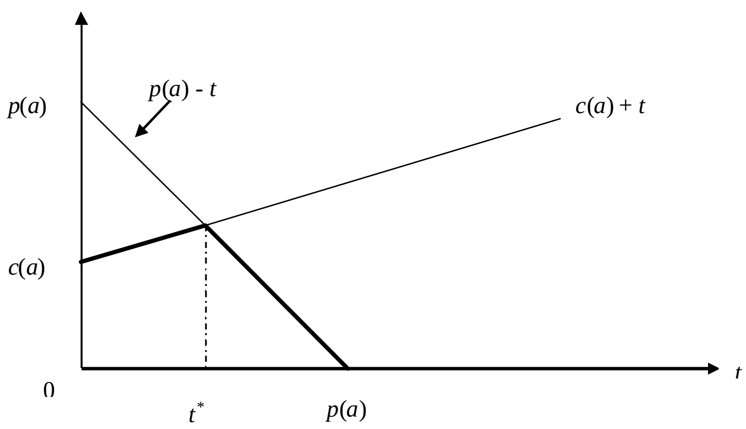
$$f(0) = p(a) > g(0) = c(a)$$
$$f'(t) = -1,\ g'(t) = 1$$

and

$$f(t) = g(t) \text{ when } t = \frac{p(a) - c(a)}{2}$$

In the following figure, the utility of parents corresponds to the bold part of the two lines.

Figure 1



By solving equation (1), we obtain:

$$t^* = \frac{p(a) - c(a)}{2} \tag{12}$$

We have solved the second stage.

Now consider the whole game. Given the parents' optimal action in each subgame, a child who chooses a receives the payoff

$$U_c(a, t^*) = c(a) + t^* = \frac{p(a) + c(a)}{2}$$

Thus in a subgame perfect equilibrium, the child chooses the action that maximizes $p(a)+c(a)$, the sum of her own private income and her parents' income.

5c. We now assume that $c(a) = a$ and $p(a) = 2a$. We also assume that a can only take values between 0 and 1, i.e. $a \in [0, 1]$. Determine the unique subgame perfect Nash equilibrium of this game. Give the equilibrium utilities of the parent and the child.

We have solved the second stage and the optimal transfer t^* is given by (12).

Now consider the whole game and let us solve the first stage where the child decides the optimal action a, anticipating the parents' optimal action in final subgame given by (12).

The child solves the following program:

$$\max_a \left\{ U_c(a,t^*) = c(a) + t^* = \frac{p(a)+c(a)}{2} \right\}$$

The first order condition is:

$$\frac{\partial U_c}{\partial a} = \frac{p'(a^*)+c'(a^*)}{2} = 0$$

We assume that $c(a) = a$ and $p(a) = 2a$. This means that (using (12)):

$$t^* = \frac{p(a)-c(a)}{2} = \frac{a^*}{2} \tag{13}$$

and

$$\frac{\partial U_c}{\partial a} = p'(a^*) + c'(a^*) = 3 > 0$$

which implies that

$$a^* = 1$$

since a can only take values between 0 and 1, i.e. $a \in [0,1]$. Plugging this value into (13), we obtain:

$$t^* = \frac{1}{2}$$

So the unique SPNE is $(a^*, t^*) = (1, 1/2)$ and the equilibrium utilities are given by:

$$U_c(a^*, t^*) = a^* + t^* = \frac{3}{2}$$

$$\begin{aligned} U_p(a^*, t^*) &= \min\{p(a^*) - t \ , \ c(a^*) + t\} \\ &= p(a^*) - t \\ &= c(a^*) + t \\ &= \frac{3}{2} \end{aligned}$$

5d. Assume now assume that $c(a) = -a^2$ and $p(a) = a$. Determine the unique subgame perfect Nash equilibrium of this game. Interpret the results.

The optimal transfer is still given by:

$$t^* = \frac{p(a)-c(a)}{2}$$

and the optimal child's action by:

$$\frac{\partial U_c}{\partial a} = \frac{p'(a^*)+c'(a^*)}{2} = 0$$

which is now equivalent to:

$$\frac{\partial U_c}{\partial a} = \frac{1-2a^*}{2} = 0$$

Solving these two equation leads to:

$$a^* = \frac{1}{2} \text{ and } t^* = \frac{3}{8} = 0.375$$

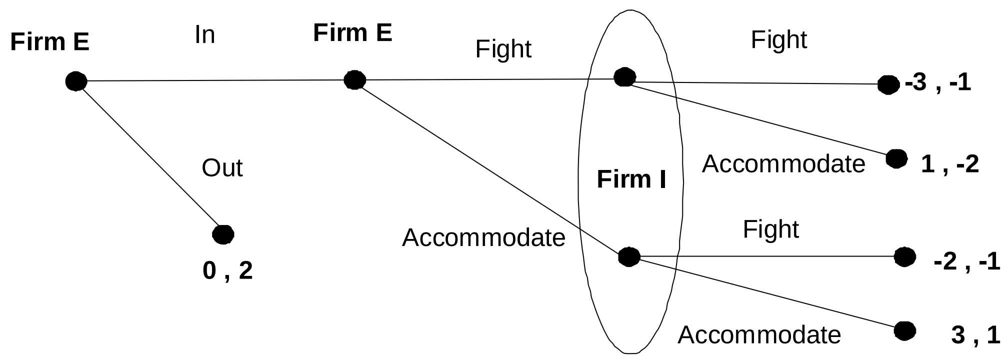
So the unique SPNE is $(a^*, t^*) = (1/2, 3/8)$ and the equilibrium utilities are given by:

$$U_c(a^*, t^*) = c(a^*) + t^* = \frac{1}{8} = 0.125$$

$$\begin{aligned} U_p(a^*, t^*) &= \min\{p(a^*) - t \ , \ c(a^*) + t\} \\ &= p(a^*) - t \\ &= c(a^*) + t \\ &= \frac{1}{8} = 0.125 \end{aligned}$$

It is easy to interpret this model. The action a is how much effort the child works at school (for example, how many hours she spends doing her homework). Effort a is now costly, $c(a) = -a^2$, which means the higher a the lower the utility of the child. For the parent, it is exactly the contrary. The more the child works hard in school, the higher are her grades, and the happier are the parents. We thus have $p(a) = a$, with $p'(a) > 0$. When choosing the optimal effort , the child faces the following trade off. On the one hand, she does not like to exert effort since $c'(a) < 0$. On the other, she knows that the higher a, the higher is the transfer t^* from the parent since $p'(a) > 0$. This is why the optimal effort is not anymore equal to 1 but to a lower value $(a^* = 1/2)$, which is such that the marginal benefit of effort $(p'(a))$ is equal to the marginal cost of effort $(c'(a))$.

6. ** (Mas-Colell, Whinston and Green) Consider the following entry game:



There are two stages. In the first one, firm E has to decide to enter or not. In the second stage, firms E and I play a simultaneous game which is given by:

E/I	Accommodate	Fight
Accommodate	$3, 1$	$-2, -1$
Fight	$1, -2$	$-3, -1$

6a. Calculate the pure-strategy subgame perfect Nash equilibria of this game.

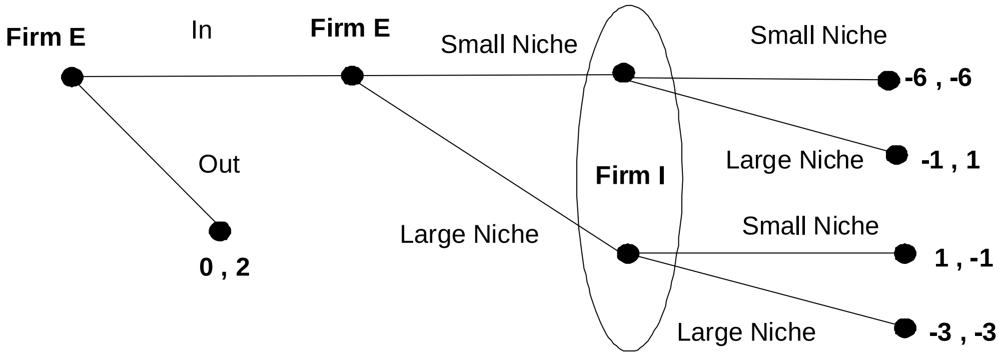
Let us solve the second stage. We have:

E/I	Accommodate	Fight
Accommodate	$\underline{3}, \underline{1}$	$\underline{-2}, -1$
Fight	$1, -2$	$-3, \underline{-1}$

There is a unique NE in pure strategies, which is (A,A) and gives payoffs (3,1).

Consider now the first stage. If firm E does not enter, it gets 0 while if it enters, it obtains 3. As a result, the unique pure-strategy subgame perfect Nash equilibrium of this game is (In, A, A) and payoffs are (3,1).

Let us modify this game in the following way. Instead of having the two firms to choose whether to fight or accommodate each other, we suppose that there are actually two niches in the market, one large and one small. After entry, the two firms E and I decide simultaneously which niche they will be in. For example, the niches might correspond to two types of customers, and the firms may be to which type they are targeting their product design. Both firms lose money if they choose the same niche, with more lost if it is the small niche. If they choose different niches, the firm that targets the large niche earns a profit, and the firm with the small niche incurs a loss, but a smaller loss than if the two firms targeted the same niche. The extensive form of the game is depicted in the following figure:



As before, this means that there are two stages. In the first one, firm E has to decide to enter or not. In the second stage, firms E and I play a simultaneous game which is given by:

E/I	Small Niche	Large Niche
Small Niche	$-6, -6$	$-1, 1$
Large Niche	$1, -1$	$-3, -3$

6b. Calculate the pure-strategy subgame perfect Nash equilibria of this new game.

Let us solve the second stage:

E/I	Small Niche	Large Niche
Small Niche	$-6, -6$	$\underline{-1}, \underline{1}$
Large Niche	$\underline{1}, \underline{-1}$	$-3, -3$

There are two pure strategy Nash equilibria in this second stage: (LN, SN) and (SN, LN), with corresponding payoffs (1, −1) and (−1, 1).

In any pure strategy SPNE, the firms' strategies must induce one of these two NE in the post-entry subgame.

Suppose, first, that the firms will play (LN, SN). In this case, the payoffs from reaching the post-entry game are (1, −1) and thus firm E will enter since $0 < 1$, i.e. the utility from not entering is lower than that of entering. Hence, one SPNE is

$$s_E = (\text{in} \ , \ LN \text{ if in})$$

$$s_I = (SN \text{ if firm } E \text{ plays in})$$

Now suppose that post-entry play is (SN, LN). Then the payoffs from reaching the post-entry game are (−1, 1) and thus firm E will enter since $0 > -1$, i.e. the utility from not entering is higher than that of entering. Hence, the second SPNE is

$$s_E = (\text{out} \ , \ SN \text{ if in})$$

$$s_I = (LN \text{ if firm } E \text{ plays in})$$

6c. Solve the mixed strategy equilibrium involving actual randomization in the post-entry subgame. Is there an SPNE that induces that behavior in the post-entry subgame? What are the SPNE strategies?

To find the mixed strategy equilibrium in the post-entry subgame, assume that firm E plays SN with probability p and firm I plays SN with probability q. Then, the q that makes firm E indifferent between strategies SN and LN is given by:

$$EU_E(SN) = EU_E(LN)$$

$$\Leftrightarrow -6q - (1-q) = q - 3(1-q)$$

$$\Leftrightarrow q = \frac{2}{9}$$

Similarly, the p that makes firm I indifferent between strategies SN and LN is given by:

$$EU_I(SN) = EU_I(LN)$$

$$\Leftrightarrow -6p - (1-p) = p - 3(1-p)$$

$$\Leftrightarrow p = \frac{2}{9}$$

Thus, in a mixed-strategy equilibrium of the post-entry game, firms E and I play "Small Niche" (SN) with probability 2/9. This gives both firms a payoff of

$$EU_I(SN) = EU_I(LN) = EU_E(SN) = EU_E(LN) = -\frac{19}{9}$$

which causes firm E to choose not to enter since $0 > -19/9$.

Therefore the following strategies constitute a SPNE:

$$s_E = (\text{out}\ ,\ SN \text{ with proba } 2/9 \text{ and } LN \text{ with proba } 7/9 \text{ if in})$$

$$s_I = (SN \text{ with proba } 2/9 \text{ and } LN \text{ with proba } 7/9 \text{ if firm } E \text{ plays in})$$

7. ** (Exercise 192.1 Osborne) (Sequential variant of Bertrand's duopoly game)

Consider the variant of Bertrand's duopoly game in which first firm 1 chooses a price (first stage), then firm 2 chooses a price (second stage). Assume that each firm is restricted to choose a price that is an integral number of cents, that each firm's unit cost is constant, equal to c (an integral number of cents), and that the monopoly profit is positive. The payoff function of firm $i = 1, 2$ $(i \neq j)$ is given by:

$$\pi_i(p_i, x_j) = \begin{cases} (p_i - c)\, D(p) & \text{if} \quad p_i < p_j \\ \frac{1}{2}(p_i - c)\, D(p) & \text{if} \quad p_i = p_j \\ 0 & \text{if} \quad p_i > p_j \end{cases}$$

where $D(p)$ is the market demand.

7a. Specify an extensive game with perfect information that models this situation.

Players: The two firms.

Terminal histories: The set of all sequences (p_1, p_2) of prices (where each p_i is a nonnegative number).

Player function: $P(\varnothing) = 1$ and $P(p_1) = 2$ for all p_1.

Preferences: The payoff of each firm $i = 1, 2$ $(i \neq j)$ to the terminal history (p_1, p_2) is its profit:

$$\pi_i(p_i, x_j) = \begin{cases} (p_i - c)\, D(p) & \text{if} \quad p_i < p_j \\ \frac{1}{2}(p_i - c)\, D(p) & \text{if} \quad p_i = p_j \\ 0 & \text{if} \quad p_i > p_j \end{cases}$$

7b. Give an example of a strategy of firm 1 and an example of a strategy of firm 2.

A strategy of firm 1 is a price (e.g. the price c). A strategy of firm 2 is a function that associates a price with every price chosen by firm 1 (e.g. $s_2(p_1) = p_1 - 1$, the strategy in which firm 2 always charges 1 cent less than firm 1).

7c. Find the subgame perfect equilibria of the game.

First consider firm 2's best responses to each price p_1 chosen by firm 1.

• *(i)* If $p_1 < c$, any price greater than p_1 is a best response for firm 2.

• *(ii)* If $p_1 = c$, any price at least equal to c is a best response for firm 2.

• *(iii)* If $p_1 = c + 1$, firm 2's unique best response is to set the same price.

• *(iv)* If $p_1 > c + 1$, firm 2's unique best response is to set the price $\min\{p^m, p_1 - 1\}$ (where p^m is the monopoly price).

In other words, firm 2's best responses are:

$$s_2(p_1) = \begin{cases} k(p_1) & \text{if} \quad p_1 < c \\ k' & \text{if} \quad p_1 = c \\ c+1 & \text{if} \quad p_1 = c+1 \\ \min\{p^m, p_1 - 1\} & \quad\; p_1 > c+1 \end{cases} \tag{14}$$

where $k(p_1) > p_1$ for all p_1 and $k' \geq c$.

Now consider the optimal action of firm 1. Given firm 2's best responses,
• (i) if $p_1 < c$, firm 1's profit is negative.
• (ii) if $p_1 = c$, firm 1's profit is zero.
• (iii) if $p_1 = c + 1$, firm 1's profit is positive.
• (iv) if $p_1 > c + 1$, firm 1's profit is zero.

Thus the only price p_1 for which there is a best response of firm 2 that leads to a positive profit for firm 1 is $p_1 = c + 1$.

We conclude that in every subgame perfect equilibrium firm 1's strategy is $p_1 = c + 1$, and, given firm 2's best responses (see (14)), firm 2 also chooses $p_2 = c + 1$.

The outcome of every subgame perfect equilibrium is that both firms choose the price $c + 1$, i.e. $p_1^* = p_2^* = c+1$. There no first-mover advantage and we end up with the same outcome than in the case of simultaneous moves.

8. *** (Exercise 211.1 Osborne) (Timing claims on an investment)

An amount of money accumulates; in period t ($t = 1, 2, ..., T$), its size is \$$2t$. In each period two people simultaneously decide whether to claim the money. If only one person does so, she gets all the money; if both people do so, they split the money equally; and if neither person does so, both people have the opportunity to do so in the next period. If neither person claims the money in period T, each person obtains \$$T$. Each person cares only about the amount of money she obtains.

8a. Formulate this situation as an extensive game with perfect information and simultaneous moves.

The following extensive game models the situation.

Players: The two people.

Terminal histories: The sequences of the form$((N, N), (N, N), ..., (N, N), x_t)$, where $1 \leq t \leq T$, x_t is (C, C), (C, N), or (N, C) if $t \leq T - 1$ and $(C, C), (C, N), (N, C)$, or (N, N) if $t = T$, where C means "claim", and N means "do not claim".

Player function: The set of players assigned to every nonterminal history is $\{1, 2\}$ (the two people).

Actions: The set of actions of each player after every nonterminal history is $\{C, N\}$.

Preferences: Each player's preferences are represented by a payoff equal to the amount of money she obtains.

8b. Find the subgame perfect equilibrium (equilibria?) of this game. (Start by considering the cases $T = 1$ and $T = 2$.)

Let us consider period T. The consequences of the players' actions in period T are given by the following matrix:

1/2	C	N
C	T, T	$2T, 0$
N	$0, 2T$	T, T

We see that the subgame starting in period T has a unique Nash equilibrium, (C, C), in which each player's payoff is T. Indeed,

1/2	C	N
C	$\underline{T}, \underline{T}$	$\underline{2T}, 0$
N	$0, \underline{2T}$	T, T

Thus if $T = 1$ the game has a unique subgame perfect equilibrium, in which both players claim (C, C).

Now suppose that $T \geq 2$, and consider period $T - 1$. The consequences of the players' actions in this period, given the equilibrium in the subgame starting in period T, are shown in the following matrix:

1/2	C	N
C	$T - 1, T - 1$	$2\,(T - 1)\,, 0$
N	$0, 2\,(T - 1)$	T, T

Observe that the entry in the bottom right box for strategies (N, N), whose payoffs are (T, T), is *the pair of equilibrium payoffs in the subgame in period T*.

If $T > 2$, then $2(T - 1) > T$, so that the subgame starting in period $T - 1$ has a unique subgame perfect equilibrium, (C, C), in which each player's payoff is $T - 1$. Indeed,

1/2	C	N
C	$\underline{T-1}, \underline{T-1}$	$\underline{2\,(T-1)}, 0$
N	$0, \underline{2\,(T-1)}$	T, T

If $T = 2$, then $2(T - 1) = T$, the whole game has two subgame perfect equilibria, in one of which both players claim in both periods (C, C) and (C, C), and another in which neither claims in period 1 and both claim in period 2.

To see that, consider $t = 1, 2$ where $T = 2$. In the last period, $t = 2 = T$, we have

1/2	C	N
C	$\underline{2}, \underline{2}$	$\underline{4}, 0$
N	$0, \underline{4}$	$2, 2$

So the NE of this last subgame is unique and is (C, C). Consider now period $t = 1 = T - 1$, the payoffs are given by:

1/2	C	N
C	$\underline{1}, \underline{1}$	$\underline{2}, 0$
N	$0, \underline{2}$	$\underline{2}, \underline{2}$

So in period $1 (= T - 1)$, there are two NE of this subgame. Either (C, C) or (N, N). As a result, the whole game has two subgame perfect equilibria: (i) one in which both players claim in both periods (C, C) and (C, C), (ii) another in which neither claims in period 1, i.e. (N, N) and both claim in period 2, i.e. (C, C).

For $T > 2$, working back to period 1, we see that the game has two subgame perfect equilibria: one in which each player claims in every period, i.e. (C, C) for all periods, and one in which neither player claims in period 1 but both players claim in every subsequent period.

To understand this last result, take the case of $T = 3$, i.e. $t = 1, 2, 3$. In the last period $T = 3$, we have:

1/2	C	N
C	$\underline{3}, \underline{3}$	$\underline{6}, 0$
N	$0, \underline{6}$	$3, 3$

There is a unique NE which is (C, C).

Consider now period $t = 2$. We have:

1/2	C	N
C	$\underline{2}, \underline{2}$	$\underline{4}, 0$
N	$0, \underline{4}$	$3, 3$

There is a unique NE which is (C, C).

Consider now period $t = 1$. We have:

1/2	C	N
C	$\underline{1}, \underline{1}$	$\underline{2}, 0$
N	$0, \underline{2}$	$\underline{2}, \underline{2}$

Observe again that the entry in the bottom right box for strategies (N, N), whose payoffs are $(2, 2)$, is *the pair of equilibrium payoffs in the subgame in period* $T - 1$ (next period). In this first period, there are two NE, which are (C, C) and (N, N).

To summarize,

(i) If $T = 1$, there is a unique subgame perfect equilibrium, in which both players claim (C, C).

(ii) If $T = 2$, there are two subgame perfect equilibria, in one of which both players claim in both periods (C, C) and (C, C), and another in which neither claims in period 1 and both claim in period 2.

(iii) If $T > 2$, there are two subgame perfect equilibria, in one of which each player claims in every period, i.e. (C, C) for all periods, and another in which neither player claims in period 1 but both players claim in every subsequent period.

9. *** There are two players, a seller and a buyer, and two dates. At date 1, the seller chooses her investment level $I \geq 0$ at cost I. At date 2, the seller may sell *one unit* of a good and the seller has cost $c(I)$ of supplying it, where $\lim_{I \to 0} c'(I) = -\infty$, $c'(I) < 0$, $c''(I) > 0$, and $c(0)$ is less than the buyer's valuation of one unit of the good, which is $V > 0$, so that $c(0) < V$. Suppose that at date 2, the buyer observes the investment I and makes a take-it-or-leave-it offer in terms of price of the good (denoted by P) to the seller.

9a. What is the unique subgame-perfect equilibrium of this game.

In a subgame-perfect equilibrium, after the buyer offers a price P, the seller will sell the good if

$$c(I) \leq P$$

Therefore the buyer offers the lowest price which will be accepted, i.e. $P = c(I)$.

In the first stage of subgame-perfect equilibrium, the seller knows that $P = c(I)$ will result after her choice of I. Her profit is then:

$$U_S = -I + P - c(I) = -I + c(I) - c(I) = -I$$

This profit is clearly maximized when the seller chooses $I = 0$.

As a result, the unique subgame-perfect equilibrium of this game is $(I^*, P^*) = (0, c(0))$ and the corresponding payoffs are: $U_S = 0$ and $U_B = V - c(0)$.

9b. Determine the socially efficient outcome of this game, which you denote by I^*. Show that this is a unique maximum and that it is strictly positive, i.e. $I^* > 0$. Explain why the subgame-perfect equilibrium and the socially efficient outcome are different.

The planner chooses I that maximizes the social welfare W, which is the sum of the utilities of the two agents, $U_B + U_S$. We have:

$$\begin{aligned} W &= U_B + U_S = (V - P) + [-I + P - c(I)] \\ &= V - I - c(I) \end{aligned}$$

Observe that the price P is just a transfer and thus does not enter in the social welfare function W. As a result, the planner solves:

$$\max_I \{V - I - c(I)\}$$

First-order condition gives:

$$-1 - c'(I) = 0$$

and we obtain that the socially optimal level of investment I^* is implicitly defined by the following equation.

$$1 + c'(I^*) = 0 \tag{15}$$

where $c'(I^*) < 0$. Let us verify that I^* is a unique maximum and that it is "interior", i.e. different that zero. First, for a unique maximum, we need to check that the second-order condition holds, i.e. $\partial^2 W / \partial I^2 < 0$, that is

$$\frac{\partial^2 W}{\partial I^2} = -c''(I) < 0$$

since $c''(I) > 0$.

Second, let us show that I^* is interior. Equation (15) can be written as:

$$-c'(I^*) = 1 \tag{16}$$

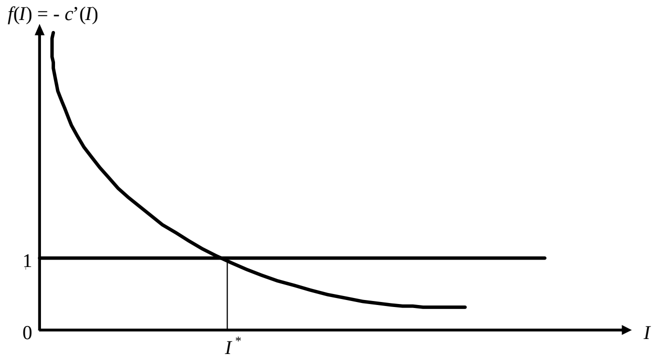
We know that $c'(0) = -\infty$, i.e. $\lim_{I \to 0} c'(I) = -\infty$, $c'(I) < 0$, and $c''(I) > 0$. If we denote

$$f(I) = -c'(I)$$

then we have:

$$f'(I) < 0 \quad \text{and} \quad \lim_{I \to 0} f(I) = +\infty$$

We can therefore easily plot equation (16) and obtain:



As a result, the planner's solution I^* is always interior.

The planner's solution and the SPNE differ because P is just a transfer and I only affects the seller (and not the buyer). As a result, the planner just maximizes the utility of the seller.

9c. Assume now that

$$C(I) = \frac{1}{\left(I + \frac{1}{V} + \frac{1}{2}\right)}$$

What are the conditions on $C(I)$ that are now satisfied? Determine the socially efficient outcome of this game. What is the condition on V that is needed to be assumed for I^* to be strictly positive?

When

$$C(I) = \frac{1}{\left(I + \frac{1}{V} + \frac{1}{2}\right)} \tag{17}$$

then

$$C(0) = \frac{1}{\frac{1}{V} + \frac{1}{2}} < V$$

$$C'(I) = -\frac{1}{\left(I + \frac{1}{V} + \frac{1}{2}\right)^2} < 0$$

$$C''(I) = \frac{2}{\left(I + \frac{1}{V} + \frac{1}{2}\right)^3} > 0$$

$$\lim_{I \to 0} C'(I) = -\frac{1}{\left(\frac{1}{V} + \frac{1}{2}\right)^2} < 0$$

$$\lim_{I \to +\infty} C(I) = 0$$

Solving the planner's problem leads to:

$$-c'(I^*) = 1$$

which is equivalent here to:

$$\frac{1}{\left(I^* + \frac{1}{V} + \frac{1}{2}\right)^2} = 1$$

Solving this equation leads to:

$$I^* = \frac{1}{2} - \frac{1}{V}$$

So for $I^* > 0$, it has to be that: $V > 2$.

9d. Can you think of a contractual way of avoiding the inefficient outcome of the subgame perfect equilibrium derived in question 9a? In particular, can you find a contract that leads to the socially optimal outcome?

One solution is to have a contract that specify a price P before any investment takes place. Then the buyer will choose I to

$$\max_I \{P - [c(I) + I]\}$$

First-order condition leads to:

$$1 + c'(I) = 0$$

which is the socially optimal level.