Exam Le Mans

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Answers

Question 1

Consider the following game with two players and where the payoffs of players 1 and 2 are given by:

		Player 2		
		C	D	E
Player 1	A	$(1,1)$	$(2,-2)$	$(5,6)$
	B	$(3,1)$	$(1,4)$	$(0,0)$

The strategies of player 1 are A, B and those of player 2 are: C, D, E.

(1a) Assume that the beliefs of all players are such that player 1 plays strategy A with probability p and strategy B with probability $1-p$, and that player 2 plays strategy C with probability q, strategy D with probability r and strategy E with probability $1-q-r$.

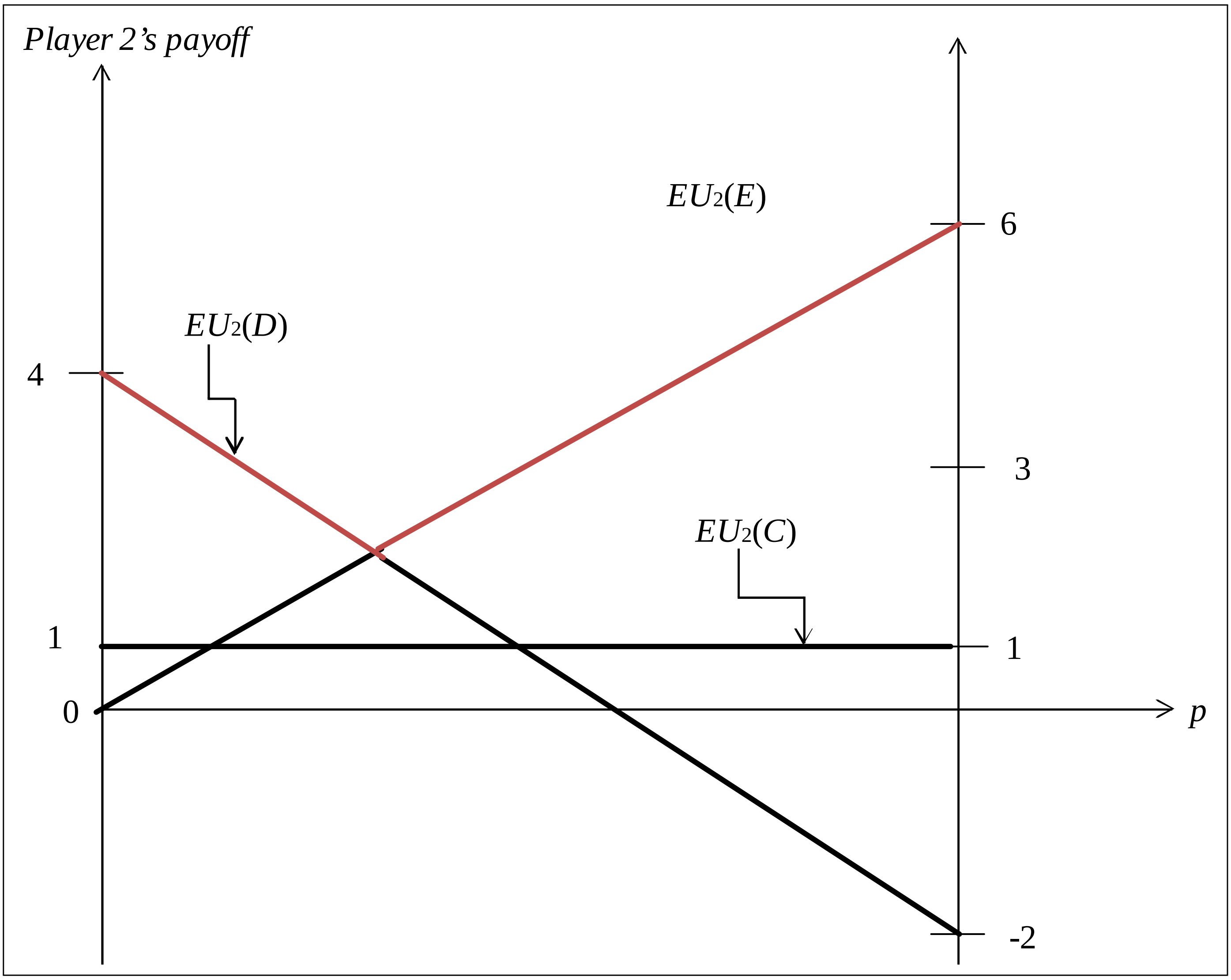
Among the three strategies of player 2, is there any of them which is *strictly dominated* by any other pure or mixed strategies?

For the domination of a pure strategy by a mixed strategy, you need to answer this question in two steps. First, draw a figure where on the vertical axis you have the (expected) payoff of player 2 and, on the horizontal axis, you have the probability p that player 1 plays strategy A. This will give you which strategy is dominated by a mixed strategy.

Second, you need to demonstrate formally which pure strategy is strictly dominated by a mixed one. For that, you need to use the beliefs q, r and $1-r-q$ and solve a system of equations. You then need to give the values of q and r for which a pure strategy is strictly dominated by a mixed strategy for player 2.

It should be clear that no pure strategy of player 2 is strictly dominated by any pure strategy of player 2. Let us look for mixed strategies.

For that, we draw the following figure:



It should be clear from this figure that the expected payoff of playing strategy C is always lower than the expected payoff of playing either strategy D or strategy E. As a result, strategy C is dominated by a mixed strategy. Let us show this formally and determine the value of q, r and $1-q-r$.

To show that strategy C is strictly dominated by a mixed strategy of C, D and E, we need to solve the two following equations simultaneously:

$$\begin{cases} q - 2r + 6\left(1-q-r\right) > 1 \\ q + 4r + 0\left(1-q-r\right) > 1 \end{cases}$$

This is equivalent to:

$$q + 4r > 1$$

$$\begin{cases} q < 1 - \frac{8}{5}r \\ q > 1 - 4r \end{cases}$$

Since, for $1 - 4r < 1 - \frac{8}{5}r$, this is equivalent to

$$1 - 4r < q < 1 - \frac{8}{5}r$$

Can we find values of q and r such that $1 - 4r < q < 1 - \frac{8}{5}r$, with $0 < q < 1$, $0 < r < 1$ and $0 < q + r < 1$? The answer is clearly yes. For example, set $r = 1/9$, then

$$1 - 4r = 1 - \frac{4}{9} = \frac{5}{9} = 0.555 \quad \text{and } 1 - \frac{8}{5}r = 1 - \frac{8}{5 \times 9} = \frac{37}{45} = 0.822$$

As a result, if we set any value of q for which $0.555 < q < 0.822$, this will work. Assume for example $q = 0.6$. Then we can say that the mixed strategy for which player 2 plays strategy C with probability $q = 0.6$, strategy D with probability $r = 1/9 = 0.111$ and strategy E with probability $1 - q - r = 0.289$ strictly dominates the pure strategy C (i.e. $q = 1$).

(1b) Using the result of the previous question, is there a solution to this game using the concept of Iterated Deletion of Strictly Dominated Strategies?

Because we have seen that player 2 will never play strategy C, the payoff matrix can now be written as follows:

		Player 2	
		D	E
Player 1	A	$(2, -2)$	$(5, 6)$
	B	$(1, 4)$	$(0, 0)$

We can see now that player 1 will never plays strategy B since it is strictly dominated by strategy A and thus the payoff matrix can now be written as follows:

		Player 2	
		D	E
Player 1	A	$(2, -2)$	$(5, 6)$

We can now see that strategy D will never be played by player 2 because it is strictly dominated by strategy E and thus the unique solution of this game using the concept of Iterated Deletion of Strictly Dominated Strategies is (A, E) with payoff $(5, 6)$.

(1c) If there is a solution using the concept of Iterated Deletion of Strictly Dominated Strategies, is it equivalent to the unique Nash equilibrium in pure strategies of this game? Show it if this is true.

Since we have seen that there is a unique solution using the concept of Iterated Deletion of Strictly Dominated Strategies, then it has to be equal to the unique Nash equilibrium in pure strategies of this game. Indeed, the best-reply functions of this game are (a number which is underlined is a BR):

		Player 2		
		C	D	E
Player 1	A	$(1, 1)$	$(\underline{2}, -2)$	$(\underline{5}, \underline{6})$
	B	$(\underline{3}, 1)$	$(1, \underline{4})$	$(0, 0)$

It is clear that there is a unique Nash equilibrium in pure strategies, which is (A, E) with payoff $(5, 6)$ and it is exactly equal to the unique solution using the concept of Iterated Deletion of Strictly Dominated Strategies.

Question 2

Firm 1 and Firm 2 currently both have a constant production cost of 3 per unit. Firm 1 has to decide whether to install a new technology – that would guarantee a cost of 0 per unit – or to keep the current one. Installing the new technology would cost F. Once Firm 1 investment decision is observed by Firm 2, the two firms will simultaneously choose output levels q_1 and q_2 as in Cournot competition. Suppose that the market demand is given by

$$p(Q) = 21 - Q$$

where $Q = q_1 + q_2$.

(2a) Solve the second stage of this game, that is the optimal choice of q_1 and q_2 by firms 1 and 2.

Firm 1's payoff is

$$\pi_1^{NI} = [21 - 3 - (q_1 + q_2)]\, q_1$$

if it does not invest, and

$$\pi_1^I = [21 - (q_1 + q_2)]\, q_1 - F$$

if it does. Firm 2's payoff is

$$\pi_2 = [18 - (q_1 + q_2)]\, q_2$$

To find the subgame perfect equilibria, we work backward. The game admits two final (proper) sub-games, where the two firms compete simultaneously on quantity given the investment decision made by firm 1 in the initial stage.

• If firm 1 *does not* invest, both firms have unit cost 3, and hence their reaction functions are

$$BR_i^{NI}(q_j) = 9 - \frac{q_j}{2}$$

These reaction functions intersect at the point ($q_1^{NI*} = q_2^* = 6$), with payoff of 36 each, i.e.

$$\pi_1^{NI*} = \pi_2^* = 36$$

• If firm 1 *does* invest, her reaction function becomes

$$q_1 \equiv BR_1^I(q_2) = \frac{21}{2} - \frac{q_2}{2}$$

and the second stage equilibrium is

$$q_1^{I*} = 8 \text{ and } q_2^* = 5$$

and firms' total payoffs are

$$\pi_1^I = 64 - F \text{ and } \pi_2^* = 25$$

(2b) Find the values of the parameter F such that the game admits a Subgame Perfect Nash Equilibrium in which Firm 1 installs the new technology.

Let us consider the improper sub-game starting with firm 1's investment decision. If firm 1 invests, she would get a payoff of $64 - F$; if she does not, her payoff would be 36. Therefore, firm 1 should make the investment if

$$64 - F \geq 36$$

or

$$F \leq 28$$

Thus when $F \leq 28$, there is a Subgame Perfect Nash Equilibrium in which Firm 1 installs the new technology.

Remark: Making the investment increases firm 1's second-stage profit in two ways. First, firm 1's profit is higher at any fixed pair of outputs, because its cost of production has gone down. Second, firm 1 gains because firm 2's second-stage output is decreased. The reason firm 2's output is lower is because by lowering its cost, firm 1 altered its own second-period incentives, and in particular made itself "more aggressive" in the sense that $BR_1^I(q_2) > BR_1^{NI}(q_2)$ for all q_2.