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Supply Function Equilibria in Networks with Transport Constraints

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Abstract

Transport constraints limit competition and arbitrageurs' possibilities of exploiting price differences between commodities in neighbouring markets. We analyze a transport-constrained network with local demand shocks, where spatially distributed oligopoly producers compete with supply functions, as in wholesale electricity markets. Uniqueness and existence results are proven, and we are able to explicitly solve for symmetric supply-function equilibria in some special cases.

Key words: Spatial competition, Multi-unit auction, Supply-function equilibrium, Transmission network, Wholesale electricity markets

JEL Classification C72, D43, D44, L91

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1 Introduction

The transport of commodities can be conducted via air, road, sea or rail routes (in case of freight) or through pipelines (in case of gas or oil), or transmission lines (in case of electricity). In this paper, we theoretically analyse how transport constraints influence competition in commodity markets with spatially distributed oligopoly producers.

Transport constraints limit trade, which makes production and consumption less efficient. Moreover, transport constraints reduce competition between agents situated in separated markets, which worsens market efficiency even further. Congestion is of particular importance for markets with negligible storage possibilities, such as wholesale electricity markets. Then demand and supply must be instantly balanced and temporary congestion in the network can result in large local price spikes. Hence, the same market can at times exhibit very little market power and, at other times, suffer from the exercise of a great deal of market power. Borenstein et al. (1999) show that standard concentration measures such as the Herfindahl-Hirschman index (HHI) work poorly to assess the degree of competition in such markets. Thus competition authorities who seek to predict the use of market power under various counterfactuals – what might happen if a merger or acquisition is accepted or transport capacity is expanded – need more detailed analytical tools.

We consider a homogeneous commodity that is produced and consumed in local oligopoly markets connected by a network of transport links. Our analysis has other applications, but it is mainly motivated by the design and operation of wholesale electricity markets. Transportation through a network link is costless up to the link's transport capacity. Demand is inelastic up to a reservation price and production costs are common knowledge. We consider a simultaneous-move game, where each strategic producer first commits to a supply function, as in wholesale electricity markets, and then a local exogenous additive demand shock is realized in each local market. After demand shocks have been realized, the price-taking transport sector buys the commodity at the cheap end of a transport link and sells it at the more expensive end, until the transport capacity is exhausted or until market prices are equal at both ends of the link. We solve for a Nash equilibrium of supply functions, also called a supply-function equilibrium (SFE).

The SFE for a single market with marginal (uniform) pricing was originally developed by Klemperer and Meyer (1989) and first applied to electricity markets by Green and Newbery (1992). They show that the optimal output of a producer is proportional to its mark-up and the slope of its residual demand at every price. In our setting, transport constraints and multiple local demand shocks mean that a firm does not know the slope of its residual demand with certainty at a given price. We prove that Klemperer and Meyer's (1989) condition can be generalized to such cases; the optimal output of a producer is proportional to its mark-up and the expected slope of its residual demand at every price. This relationship can be used to numerically solve for equilibria in general networks. Related results have been derived by Wilson (2008), but he considers a system with shocks in the transmission capacities, whereas we consider local demand shocks. In addition to

Wilson (2008) we also contribute by deriving global second-order conditions for general networks and by establishing uniqueness and existence results for networks with two nodes. Moreover, we contribute by deriving explicit expressions for symmetric SFE when producers have identical costs, producers are symmetrically distributed in a radial (tree) network and multi-dimensional demand shocks are uniform.

Normally a local market would represent the geographical location of a market place, and with transport we normally mean that the commodity is moved from one geographical location to another location. But local markets and transports could be interpreted in a more general sense. For example, a local market could represent a geographical location at a particular point in time. Thus storage can be represented by transport links that allow for transports of the commodity to the same place but at a later point in time. The transport capacity of such a link would then correspond to the local storage capacity.

Section 2 compares our results with the previous literature. In Section 3 we analyse radial networks. Meshed networks are studied in Section 4. The paper concludes in Section 5. All proofs are in the Appendices.

2 Comparisons with the previous literature

The setting of the SFE is particularly well suited for markets where producers submit supply functions to a uniform-price auction before demand has been realized, as in wholesale electricity markets (Green and Newbery, 1992; Bolle 1992; Holmberg and Newbery, 2010). This has also been confirmed qualitatively and quantitatively in several empirical studies of bidding behaviour in electricity markets.[1]

Our paper has many parallels with Wilson (2008).[2] He uses calculus of variations to derive Euler conditions that extend Klemperer and Meyer's (1989) model to consider the network's influence on bidding strategies. However, Wilson does not establish uniqueness, derive any second-order conditions, nor does he study examples where he (numerically or analytically) solves for SFE, so his analysis is missing some fundamental components that our analysis provides.

Verifying global optimality conditions is important for oligopoly markets with transport constraints, because previous research has shown that such conditions

[1] Empirical studies of the electricity market in Texas (ERCOT) show that supply functions of the two to three largest firms in this market roughly match Klemperer and Meyer's first-order condition (Hortaçsu and Puller, 2008; Sioshansi and Oren, 2007). The fit is worse for small producers. According to Wolak (2007) the reason is that these studies did not consider that supply functions are stepped. He shows that both large and small electricity producers in Australia choose stepped supply functions in order to maximize profits; at least observed data does not reject this hypothesis.

[2] Related is also Anderson et al. (2007), who investigate a two-node transmission network with both independent and correlated demand at the nodes. They derive formulae to represent the market distribution function for a producer when its network becomes interconnected to a previously separate grid under the assumption that the interconnection does not change competitors' supply functions.

are often violated in such settings. The reason is that transport constraints can result in a producer's residual demand curve having discontinuous changes in its slope. These kinks are such that the slope becomes discontinuously less price sensitive when net imports to the producer's local market are congested. In the neighbourhood of such a kink the residual demand curve is sufficiently convex to yield a profitable deviation from a first-order solution in which imports to a local market are nearly congested (namely by withholding production in order to push the price above the next breakpoint in its residual demand curve). This type of deviation will often rule out pure-strategy Nash equilibria in networks with transport constraints, especially if there are no market uncertainties, so that each bidder can perfectly predict the location of the kinks/breakpoints of its residual demand curve in equilibrium.[3] Borenstein et al. (2000) for example rule out Cournot NE when the transport capacity between two symmetric markets is sufficiently small and demand is certain. Downward et al. (2010) analyse similar problems in general networks with transport constraints.[4]

We use Anderson and Philpott's (2002a) market-distribution-function approach to verify that monotonic solutions to our first-order conditions are supply-function equilibria (SFE) when the probability density of the demand shocks (shock density) is sufficiently evenly distributed, i.e. sufficiently close to a uniform multi-dimensional distribution. In this case the producers react to the expectation of the residual demand slope over different congestion conditions. The uncertainty has a smoothing effect, which reduces problems with local convexities in the residual demand curve, so that profitable deviations from first-order solutions can be precluded. But existence of SFE cannot be taken for granted. Profitable deviations from the first-order solution will for example exist for perfectly correlated demand shocks or for steep slopes and discontinuities in the probability density of the demand shocks.[5]

Our paper also differs from Wilson (2008) in the source of randomness. Wilson focuses on shocks in the transmission capacities, e.g. due to tripped electric power lines. His general model has a multi-dimensional shock in the transmission capacities, so that each transmission line could have a capacity that is imperfectly correlated with the capacity of other transmission lines. Wilson also allows for a shock in total demand in the network. Local demand could be a function of this

[3]But Escobar and Jofre (2006,2008) show that there is normally a mixed-strategy NE in those cases. Adler et al. (2008) and Hu and Ralph (2007) show that existence of pure-strategy Cournot NE depends on the assumptions made about the rationality of the players. Hobbs et al. (2004) bypasses the existence issue by using conjectural variations instead of a Nash equilibrium. Existence of equilibria is more straightforward in competitive networks with infinitesimally small producers (Cho, 2003; Escobar and Jofre, 2006, 2008; Holmberg and Lazarczyk, 2015).

[4]Willems (2002) analyses how a network operator's rule to allocate transmission capacity influences the Cournot NE. Wei and Smeers (1999) calculate Cournot NE in transmission networks with regulated transmission prices. Oren (1997) calculates Cournot NE in a network with transmission rights. Neuhoff et al (2005) use Cournot NE to analyse competition in the northwestern European wholesale electricity market.

[5]Note that a discontinuity in the demand shock density of a local market is acceptable as long as it occurs when transport capacities in all transport links to the local market are binding.

total demand shock, but demand shocks are still one-dimensional.[6] Our model is different in that we consider multi-dimensional demand shocks, so that demand in a local market could be imperfectly correlated with demand in other local markets. On the other hand, we do not consider shocks in transmission capacities. Thus our model is more relevant for commodity markets where the uncertainty is dominated by local net-demand shocks, which for example is the case for power systems with significant amounts of spatially distributed, intermittent wind power plants.

It follows from Klemperer and Meyer (1989) and Genc and Reynolds (2011) that there will be multiple SFE in a single market when the demand shock is sufficiently bounded such that a producer is certain to sell a strictly positive output that is strictly lower than its production capacity. The reason is that a producer has a lot of freedom when choosing the shape of sections of a supply function that is never going to be price-setting. As illustrated by Klemperer and Meyer (1989), producers can use this freedom to support a wide range of equilibria. However, as shown by Holmberg (2008) and Anderson (2013), a unique equilibrium will normally exist if demand shocks are such that any point of a producer's supply function would be marginal for some possible demand shock outcome. Our local demand shocks have this property, so our uniqueness results are consistent with the previous SFE literature for single markets.

The SFE model has mainly been used in studies of producers' strategic bidding in wholesale electricity markets. But there are some exceptions. Laussel (1992) and Pehlivan and Vuong (2013) have for example used the SFE to study competition between exporters in a global economy. In this context our model with transport constraints would for example be useful when analysing the effect of trading quotas on the strategic interaction between exporters. Krishna (1989) has previously analysed such problems for the case with Bertrand competition between exporters. Related is also Malamud and Rostek's (2013) study of traders that compete with linear supply functions in a network of decentralized exchanges without transport constraints.

3 Radial networks

Radial networks have a tree structure. Such graphs have no loops, so there is a unique chain of transport links between any two local markets. Radial networks for example include hub-and-spoke and line networks. Although most electric power networks contain loops, radial networks are often used as a first approximation. Some market operators in Europe use radial approximations when clearing the wholesale electricity market. Thus producers in such markets would consider this radial approximation when preparing their bids. Cho (2003) uses an extensive radial network to approximate the electric power grid in California. In the Nordic countries, Britain, New Zealand and Germany, the dominating transmission capacity constraints approximately separate the electricity market into a northern

[6] This is the case for Wilson's (2008) general set-up. An exception is his introductory analysis, where he discusses SFE in two-node networks with two local demand shocks.

and southern market, giving a simple two-node radial representation.

3.1 Model set-up

A homogenous commodity is traded across a network of M local markets (nodes). They are connected by K directed[7] transport links (arcs) in the following way: at most one arc connects any pair of nodes, but there is a path (series of arcs) that can be followed from any one node to any other (i.e. the network is connected). Radial networks have $K = M - 1$, so that there is a unique path between any two nodes. As is standard in graph theory, the topology of the network can be described by a *node-arc incidence* matrix $\mathbf{A}$ (Bazaraa et al., 2009). This matrix $\mathbf{A}$ has a row for every node and a column for every arc, and $mk-$th element a_{mk} defined as follows:[8]

$$a_{mk} = \begin{cases} -1, & \text{if arc } k \text{ is oriented away from node } m, \\ 1, & \text{if arc } k \text{ is oriented towards node } m, \\ 0, & \text{otherwise.} \end{cases}$$

Every arc starts in one node and ends in another node, so by definition we have that the rows of $\mathbf{A}$ add up to a row vector with zeros. Thus the rows are linearly dependent. It can be shown that the incidence matrix $\mathbf{A}$ of a connected network has rank $M - 1$ (Bazaraa et al., 2009).

The transported quantity in arc k is represented by the variable t_k which can be positive or negative, the latter indicating a flow in the opposite direction from the orientation of the arc. Thus the mth row of $\mathbf{At}$ represents the flow of the commodity into node m (imports) from the rest of the network. Transportation is assumed to be lossless and costless, but each arc k has a capacity $\bar{t}_k$, so the vector $\mathbf{t}$ of arc flows satisfies

$$-\bar{\mathbf{t}} \leq \mathbf{t} \leq \bar{\mathbf{t}}. \tag{1}$$

Each arc is in one of three states depending on whether the flow is uncongested, at capacity in the positive direction, or at capacity in the reverse direction. Since our network has K arcs there are 3^K different combinations of states for the arcs. We denote each of these combinations by an integer $\omega \in \Omega = \{1, 2, \ldots, 3^K\}$ called a *congestion state*.

At each node m there are N_m producers who play a simultaneous move, one shot game. Each producer offers a nondecreasing differentiable supply function

$$Q_{mn}(p), \; n = 1, 2, \ldots, N_m,$$

that defines how much each firm is prepared to supply at price p.[9] We denote the total nodal supply in each node by $S_m(p_m) = \sum_{n=1}^{N_m} Q_{mn}(p_m)$ and the vector with

[7] It matters for the mass/material balance in a node whether a flow is imported or exported from the node. The direction of an arc defines the positive direction of the flow through the arc. A negative flow indicates that the transported quantity is flowing in the opposite direction.

[8] Some authors adopt a different convention in which $a_{ik} = 1$ if arc k is oriented away from node i.

[9] Similar to Wilson (2008), forward contracts and similar contracts could be considered by assuming that the output, offered supply and production capacity are net of contracts.

such components by $\mathbf{s}(\mathbf{p})$. We also introduce $S_{m,-n}(p_m) = \sum_{j=1,j\neq n}^{N_m} Q_{mj}(p_m)$, which excludes the supply of firm n from the nodal supply in node m. For simplicity we assume that each firm is only active in one node.

Demand in each node m is given by a random local shock ε_m having a known probability distribution with joint density $f(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_M)$.[10] The demand shocks are realized after firms have committed to their supply functions. We let $D_{mn}^{\boldsymbol{\varepsilon}}(p_m)$ be the realized residual demand function of firm n in node m. This is the market demand that is not met by other firms in the industry at price p_m in node m for a realized vector of local demand shocks $\boldsymbol{\varepsilon} = \{\varepsilon_m\}_{m=1}^{M}$.

We assume that the commodity is traded at the local market price of each node. In electric power networks this is called nodal pricing or locational marginal pricing (LMP) (Chao and Peck, 1996; Hogan, 1992; Bohn et al., 1984). There is no storage in the nodes. Hence, for each realization $\boldsymbol{\varepsilon}$, the market clearing must result in network flows $\mathbf{t}$ and local market prices $\mathbf{p}$, such that net-imports are equal to net-consumption in each node (consumption net of production).

$$\mathbf{A}\mathbf{t} = \boldsymbol{\varepsilon} - \mathbf{s}(\mathbf{p}). \tag{2}$$

A price-taking transport sector, such as a regulated network operator, clears the market after the demand shocks have been realized. In a radial network, the transport sector buys the commodity at the cheap end of a transport link and sells it at the more expensive end, until the link's transport capacity is exhausted or until market prices are equal at both ends of the link.

We let $C_{mn}(q)$ be the production cost of firm n in node m. It is differentiable, convex and increasing up to its capacity constraint $\bar{q}_{mn}$. The reservation price $\bar{p}$ is such that $\bar{p} > C'_{mn}(\bar{q}_{mn})$ for all $m \in \{1, \ldots, M\}$ and all $n \in \{1, \ldots, N_m\}$.[11] When solving for an SFE, i.e. a Nash equilibrium of supply-function bids, we assume that each producer is risk-neutral and chooses its supply function in order to maximize its expected profit.

3.2 Market-clearing conditions

Given supply functions and a vector of local demand realizations, the price-taking transport sector buys the commodity at the cheap end of a transport link and sells it at the more expensive end, until the link's transport capacity is exhausted or until market prices are equal at both ends of the link. This corresponds to the market-clearing conditions below, where ρ is the vector of non-negative shadow prices (one for each arc) for flows in the positive direction, and $\boldsymbol{\sigma}$ is the vector of

[10] Our model does not consider price-response on the demand side. However, similar to Wilson (2008), the model could be generalized by representing a strategic consumer with demand $D_{mn}(p)$ by a supply function $S_{mn}(p) = -D_{mn}(p)$.

[11] The reservation price is the highest price that the auctioneer is willing to pay for the commodity. Most auctions and wholesale electricity markets have reservation prices.

non-negative shadow prices (one for each arc) for flows in the negative direction.

$$\begin{array}{l}\mathbf{A}^{\top}\mathbf{p} = \boldsymbol{\rho} - \boldsymbol{\sigma} \\ \rho_k \geq 0,\ t_k \leq \bar{t}_k,\ \rho_k\left(\bar{t}_k - t_k\right) = 0,\ k = 1 \ldots K. \\ \sigma_k \geq 0,\ t_k \geq -\bar{t}_k,\ \sigma_k\left(\bar{t}_k + t_k\right) = 0,\ k = 1 \ldots K. \\ \mathbf{A}\mathbf{t} + \mathbf{s}(\mathbf{p}) = \boldsymbol{\varepsilon}.\end{array} \tag{3}$$

The first condition states that the shadow price for an arc equals the difference in nodal prices between its endpoints. The second and third set of conditions are called complementary slackness. They ensure that there are no feasible profitable arbitrage trades in the radial network. If two nodes are connected by a congested arc then the price at the importing end will be at least as large as the price in the exporting end. The fourth condition ensures that net-demand equals net-imports in every node. These market-clearing conditions would for example be satisfied for a capacity constrained, perfectly competitive transport sector. As shown by Chao and Peck (1996), the same conditions would also arise for a regulated network operator that treats an inverse supply function, $Q_{mn}^{-1}(x)$, as if it is a statement of the actual marginal cost of firm n in node m, and that accepts production in order to minimize the total stated production cost subject to that local demand in each node is satisfied without violating any transport constraints.

Recall that $\mathbf{A}$ has rank $M-1$ for radial networks, so it is not invertible. However, we can remove any row from the matrix $\mathbf{A}$ to make it invertible (Bapat, 2010). The removed row corresponds to a node, which we denote by m and refer to as the *slack node*.[12] We can now write the market-clearing condition that net-imports equal net-demand in the remaining nodes as follows

$$\mathbf{A}_{-m}\mathbf{t} = \boldsymbol{\varepsilon}_{-m} - \mathbf{s}_{-m}(\mathbf{p}),$$

where we use the subscript $-m$ to indicate that row m has been removed. Flows in the network can now be determined from net-exports in the remaining nodes as follows:

$$\mathbf{t} = \underbrace{-\left(\mathbf{A}_{-m}\right)^{-1}}_{\mathbf{H}}\left(\mathbf{s}_{-m}(\mathbf{p}) - \boldsymbol{\varepsilon}_{-m}\right). \tag{4}$$

The components of the matrix $\mathbf{H}$ define power transfer distribution factors (PTDFs) $H_{k\ell}$ that give the flow on arc k that would result from a net-injection of one unit at node $\ell \neq m$ and a withdrawal of one unit at the slack node m.

Similar to Wilson (2008) it is convenient to choose the slack node m to be a trading hub with nodal price $p = p_m$. As shown by Xu and Baldick (2007), Hogan (2000) and Chao et al. (2000), one can express the vector of other nodal prices $\mathbf{p}_{-m}$ in terms of the price of the trading hub and the shadow prices of the arcs.

$$\mathbf{p}_{-m} = p\mathbf{1}_{M-1} - \mathbf{H}^{T}\left(\boldsymbol{\rho} - \boldsymbol{\sigma}\right), \tag{5}$$

where $\mathbf{1}_{M-1}$ is a column vector of $M-1$ ones. Thus the injection of one unit at node $\ell \neq m$ that is withdrawn at the trading hub m is paid p (the local price at the trading hub) minus resulting shadow price payments for resulting flows on congested lines.

[12] In power system studies the slack node is sometimes also referred to as the swing bus.

3.3 Optimality conditions

In this subsection, we consider firm n in node m of a radial network. We derive optimality conditions for its supply-function offer $Q_{mn}(\cdot)$, given supply functions of its competitors. In these calculations, it is convenient to let node m be the trading hub and slack node of the network. Thus the trading hub and slack node is moved, when deriving optimality conditions for firms in other nodes.

To derive the optimality conditions for a supply function $Q_{mn}(\cdot)$, we consider a candidate point $(\hat{p}, \hat{q}) \in (0, \bar{p}) \times (0, \bar{q}_{mn})$ with the nodal market price $\hat{p}$ at node m (the trading hub) and the output $\hat{q}$ of firm n, and investigate whether such a point could be part of an optimal supply function $Q_{mn}(\cdot)$. When testing this for $(\hat{p}, \hat{q})$, we restrict attention to the set of demand shocks such that the residual demand curve passes through the candidate point $(\hat{p}, \hat{q})$. Thus we introduce

$$E(\hat{p}, \hat{q}) = \{\boldsymbol{\varepsilon} \mid D^{\boldsymbol{\varepsilon}}_{mn}(\hat{p}) = \hat{q}\}.$$

We partition $E(\hat{p}, \hat{q})$ into 3^K disjoint subsets $E(\hat{p}, \hat{q}, \omega)$ each of which consists of the demand shocks in $E(\hat{p}, \hat{q})$ that produce a congestion state $\omega \in \Omega$. To simplify our presentation, we only consider demand outcomes $\boldsymbol{\varepsilon}$ that are strictly inside a congestion state. The probability of the outcome where an arc is just binding, i.e. on the boundary between two congestion states, is negligible. Given differentiable supply functions for each competitor and a vector of demand shocks $\boldsymbol{\varepsilon}$, the realized residual demand curve of producer n in node m is piecewise differentiable. Kinks in the realized residual demand curve occur at points where a transmission-line switches between being uncongested and congested as producer n slightly varies its output. However, the probability of being cleared at a kink is negligible. For any considered demand outcome $\boldsymbol{\varepsilon}$, all arcs will have an unchanged congestion status for marginal changes in the bid price or output of producer n.

Let $\Xi(\omega)$ be a set with node m and all nodes that are connected to node m through an uncongested chain of arcs for congestion state ω. We say that nodes in this set are completely integrated with node m.

Lemma 1 *In a radial network all nodes in $\Xi(\omega)$ have the same market price p for a congestion state ω. The slope of residual demand facing producer n in node m is given by*

$$D'_{mn}(p, \omega) = \frac{d}{dp} D^{\varepsilon}_{mn}(p) = -S'_{m,-n}(p) - \sum_{\ell \in \Xi(\omega) \setminus \{m\}} S'_{\ell}(p) \tag{6}$$

for $\boldsymbol{\varepsilon} \in E(p, q, \omega)$.

Thus $D'_{mn}(p, \omega)$ equals the sum over competitors' supply function slopes in the integrated area consisting of nodes that are linked to node m by uncongested lines in congestion state ω.

For considered (non-negligible) shock outcomes, $E(\hat{p}, \hat{q}, \omega_1)$ and $E(\hat{p}, \hat{q}, \omega_2)$ are disjoint when $\omega_1 \neq \omega_2$. Thus the probability that a residual demand curve for producer n at node m passes through the interval $[\hat{q}, \hat{q} + dq]$ at price $\hat{p}$ equals $P_{mn}(\hat{p}, \hat{q})dq$, where

$$P_{mn}(\hat{p}, \hat{q}) = \sum_{\omega} P_{mn}(\hat{p}, \hat{q}, \omega) \tag{7}$$

and

$$P_{mn}(\hat{p}, \hat{q}, \omega) = \lim_{dq \to 0} \frac{\Pr(\cup_{0 \leq \eta \leq dq} E(\hat{p}, \hat{q} + \eta, \omega))}{dq}. \tag{8}$$

The conditional probability that the congestion state is $\hat{\omega}$ given that $D^{\varepsilon}_{mn}(\hat{p}) = \hat{q}$ can now be defined by:

$$\hat{P}_{mn}(\hat{\omega} \mid \hat{p}, \hat{q}) = \frac{P_{mn}(\hat{p}, \hat{q}, \hat{\omega})}{\sum_{\omega} P_{mn}(\hat{p}, \hat{q}, \omega)}. \tag{9}$$

Given the residual demand slopes $D'_{mn}(p, \omega)$ for each state ω, and $\hat{P}_{mn}(\hat{\omega} \mid p, q)$, we find it useful to introduce the function $Z_{mn}(p, q)$:[13]

Definition 1 *In a radial network,*

$$Z_{mn}(p, q) = (p - C'_{mn}(q)) \sum_{\omega} -D'_{mn}(p, \omega) \hat{P}_{mn}(\omega \mid p, q) - q. \tag{10}$$

The combination of (6) and Definition 1 yields:

Corollary 1 *In a radial network*

$$Z_{mn}(p, q) = (p - C'_{mn}(q)) \sum_{\omega} \left(S'_{m,-n}(p) + \sum_{\ell \in \Xi(\omega) \setminus \{m\}} S'_{\ell}(p) \right) \hat{P}_{mn}(\omega | p, q) - q. \tag{11}$$

The sum $\sum_{\omega} -D'_{mn}(p, \omega) \hat{P}_{mn}(\omega \mid p, q)$ in (10) is the expected slope in the residual demand that firm n is facing at a point (p, q), in other words

$$\mathbb{E}_{\omega}\left[-D'_{mn}(p, \omega) \mid D^{\varepsilon}_{mn}(p) = q\right] = \sum_{\omega} -D'_{mn}(p, \omega) \hat{P}_{mn}(\omega \mid p, q). \tag{12}$$

This sum can be interpreted as a quantity effect, i.e. how many units are lost in expectation when producer n increases the local price in node m by one unit, provided that residual demand passes through the point (p, q). Multiplying the lost quantity by the mark-up $(p - C'_{mn}(q))$ gives the lost value due to the quantity effect. The second term, q, is the price effect. This is what producer n would gain in expectation from increasing the price in node m by one unit if its output is fixed at q. There is an extremum when the price effect equals the lost value due to the quantity effect, so that $Z_{mn}(p, q) = 0$. The extremum is a profit maximum if the quantity effect dominates the price effect at prices above the extremum price and if the price effect dominates at prices below the extremum price. For a monotonic increasing supply function $Q_{mn}(p)$ this is equivalent to the following statement, which is formally proven in Appendix A by means of Anderson and Philpott's (2002a) market-distribution-approach.

[13]Note that our definition of $Z_{mn}(p, q)$ should be multiplied by $\sum_{\omega} P_{mn}(\hat{p}, \hat{q}, \omega)$ to make it consistent with the Z function that was originally introduced by Anderson and Philpott (2002a).

Proposition 1 *In radial networks, a monotonic increasing supply function* $Q_{mn}(p)$ *is globally optimal if it satisfies*

$$\begin{cases} Z_{mn}(p,q) \geq 0, & \text{if } q < Q_{mn}(p) \\ Z_{mn}(p,q) = 0, & \text{if } q = Q_{mn}(p) \\ Z_{mn}(p,q) \leq 0, & \text{if } q > Q_{mn}(p). \end{cases} \tag{13}$$

The following necessary first-order condition follows from Definition 1, (12) and $Z_{mn}(p,q) = 0$.

Corollary 2 *For radial networks, a monotonic increasing optimal supply function* $Q_{mn}(p_m)$ *satisfies*

$$Q_{mn}(p_m) = (p_m - C'_{mn}(Q_{mn}(p_m)))\, \mathbb{E}_\omega \left[-D'_{mn}(p_m,\omega) \mid D^{\varepsilon}_{mn}(p_m) = Q_{mn}(p_m) \right]. \tag{14}$$

This generalizes the first-order condition of Klemperer and Meyer (1989) to multi-dimensional shocks, by saying that the optimal output of a producer at its local price p_m is proportional to its mark-up and the expected slope of the residual demand that it is facing at p_m. Wilson (2008) derives a similar first-order condition for his setting. The necessary first-order condition could be used to empirically test the optimal bidding behaviour of producers in the presence of transmission congestion. $\mathbb{E}_\omega \left[-D'_{mn}(p_m,\omega) \mid D^{\varepsilon}_{mn}(p_m) = Q_{mn}(p_m) \right]$ would then be estimated from the historical average slope of residual demand at price p_m when the output of firm n in node m is $Q_{mn}(p_m)$. Previous empirical studies of bidding behaviour in wholesale electricity markets (e.g. Sioshansi and Oren, 2007; Hortaçsu and Puller, 2008; Wolak, 2007) have neglected transmission constraints.

In this paper we will use optimality conditions to solve for Nash equilibria in networks. Lemma 8 in Appendix A presents an explicit expression for how conditional probabilities $\hat{P}_{mn}(\omega|p,q)$ can be calculated in a general radial network. Thus we can use (11) and the optimality condition in Proposition 1 to solve for the best response of firm n in node m to competitors' supply functions. We could in principle do this for each firm in the network and then solve for an asymmetric Nash equilibrium from a system of such equations. However, we will focus on solving for symmetric equilibria. In this case, we find it useful to introduce a market integration function as below.

Definition 2 *For firm* n *in node* m *we define the* market integration function *by*

$$\mu_{mn}(p,q) = \sum_{\omega} M_{\Xi(\omega)} \hat{P}_{mn}(\omega|p,q),$$

where $M_{\Xi(\omega)}$ *is the number of nodes in the set* $\Xi(\omega)$.

Thus, the market integration function is equal to the expected number of nodes (including node m itself) that are completely integrated with node m given that firm n has output q and node m has the market price p.

Lemma 2 *Suppose each node has* $N_m = N \geq 1$ *identical producers with production costs* $C(Q)$*, and each producer chooses a supply function* $Q(p)$ *in order to maximize its expected profit. A necessary condition for* $Q(p)$ *to be a SFE is*

$$Q = (p - C'(Q))\left(\mu\left(p, Q\left(p\right)\right) N - 1\right) Q', \tag{15}$$

where $\mu\left(p, Q\left(p\right)\right) = \mu_{mn}\left(p, Q\left(p\right)\right)$ *for each node* m *and firm* n*.*

The next lemma shows that $\mu\left(p, Q\left(p\right)\right)$ simplifies in networks with symmetric producers.

Lemma 3 *Consider a network that is symmetric with respect to producers and where each node has* $N_m = N \geq 1$ *producers, which are identical. If each producer chooses a monotonic supply function* $Q(p)$*, then* $P_{mn}(p, Q(p), \omega)$ *is determined by the nodal output; there is a function* $\tilde{P}(NQ(p), \omega)$*, such that*

$$P_{mn}(p, Q(p), \omega) = \tilde{P}(NQ(p), \omega) \tag{16}$$

for each node m*, each firm* n*, any positive integer* N *and any* $Q(p)$*. Similarly,*

$$\mu_{mn}\left(p, Q\left(p\right)\right) = \widetilde{\mu}\left(NQ\left(p\right)\right) = \frac{\sum_{\omega} M_{\Xi(\omega)} \tilde{P}(NQ(p), \omega)}{\sum_{\omega} \tilde{P}(NQ(p), \omega)} \tag{17}$$

for each node m*, each firm* n*, any positive integer* N *and any* $Q(p)$*.*

Symmetric offers $Q(p)$ and realized nodal prices depend on the number of firms per node. Still, it follows from Lemma 3 that market integration of a node is determined by its nodal output irrespective of the number of symmetric firms per node. The reason is that by assumption consumers are insensitive to the price, so the number of production units that are needed to meet a given demand shock outcome does not depend on market competition. Moreover, the order in which production units of symmetric firms are accepted is the same irrespective of their symmetric mark-ups. Thus $\widetilde{\mu}(\cdot)$ can be determined from a competitive market with price-taking firms. Once this market integration function is known, one can use (15) and (17) to determine symmetric SFE of a finite number of spatially distributed producers. It follows from (15) that symmetric oligopoly producers will increase their mark-ups at output levels where the (exogenous) market integration function $\widetilde{\mu}(NQ)$ is small, i.e. when arcs to node m are congested with a high conditional probability. Similarly, oligopoly producers will decrease their mark-ups at output levels where the market integration function $\widetilde{\mu}(NQ)$ is large.

We have derived the optimality conditions above for our setting with multi-dimensional demand shocks. However, with minor edits in the proofs in Appendix A, the optimality conditions can be proven to hold for other settings where the residual demand curve of a firm is shifted horizontally by shocks $\boldsymbol{\varepsilon} \in E(p, q, \omega)$ for any given ω. Thus the slope of a firm's residual demand curve is determined by the congestion state and the firm's local price. It can for example be shown that the optimality conditions above would hold also for strategic producers competing with supply functions in networks with shocks in the transmission capacities. The computation of conditional probabilities $\hat{P}_{mn}(\omega|p, q)$ in Appendix A is less general; it only applies to our setting with multi-dimensional demand shocks.

3.4 Examples

By means of Corollary 1 and the first-order condition $Z_{mn}(p,q) = 0$, we are able to construct a first-order condition for each firm in a radial network. The SFE can be solved from a system of such first-order conditions for general radial networks. The global second-order condition of an available first-order solution can be verified by (13). In this section we use these optimality conditions to derive SFE for two-node and star networks with symmetric firms.

3.4.1 Two-node network

Consider a simple network with two nodes connected by one arc from node 1 to node 2 with flow $t \in [-\bar{t}, \bar{t}]$. There are three congestion states: ω_1 corresponds to $t \in (-\bar{t}, \bar{t})$, ω_2 to $t = \bar{t}$, and ω_3 to $t = -\bar{t}$. We derive the optimality condition for a firm in node 1, and thus we pick node 1 as being the slack node and trading hub with price $p = p_1$. It can be shown that:

Lemma 4 *In a two-node network, the optimal supply function of firm n in node 1 can be determined from:*

$$\begin{aligned} Z_{1n}(p,q) = (p - C'_{1n}(q))(S'_{1,-n}(p) + S'_2(p))\hat{P}(\omega_1 \mid p,q) \\ +(p - C'_{1n}(q))S'_{1,-n}(p)\left(\hat{P}(\omega_2 \mid p,q) + \hat{P}(\omega_3 \mid p,q)\right) - q = 0, \end{aligned} \tag{18}$$

where $\hat{P}(\omega \mid p,q)$ is defined by (9) from

$$\begin{aligned} P(p,q,\omega_1) &= \textstyle\int_{-\bar{t}}^{\bar{t}} f(q + S_{1,-n}(p) - t, S_2(p) + t)\,dt \\ P(p,q,\omega_2) &= \textstyle\int_{S_2(p)+\bar{t}}^{\infty} f(q + S_{1,-n}(p) - \bar{t}, \varepsilon_2)\,d\varepsilon_2 \\ P(p,q,\omega_3) &= \textstyle\int_{-\infty}^{S_2(p)-\bar{t}} f(q + S_{1,-n}(p) + \bar{t}, \varepsilon_2)\,d\varepsilon_2. \end{aligned} \tag{19}$$

Below we consider symmetric NE for symmetric firms and symmetric shock densities. The existence of an equilibrium depends on the partial derivatives $f_m(\varepsilon_1, \varepsilon_2) = \frac{\partial f(\varepsilon_1,\varepsilon_2)}{\partial \varepsilon_m}$, $m = 1, 2$, of the shock density which must be sufficiently small. It can be shown that symmetric solutions to (15) are equilibria under the following circumstances.

Proposition 2 *Consider a two-node network with N symmetric firms in each node, each firm having identical production capacities $\overline{q}$ and identical marginal costs that are either constant or strictly increasing. If demand has a bounded shock density that satisfies $f(\varepsilon_1, \varepsilon_2) = f(\varepsilon_2, \varepsilon_1) > 0$ and $2N\overline{q}\,|f_m(\varepsilon_1, \varepsilon_2)| \leq (3N-2) f(\varepsilon_1, \varepsilon_2)$ when $(\varepsilon_1, \varepsilon_2) \in [-\bar{t}, N\overline{q} + \bar{t}] \times [-\bar{t}, N\overline{q} + \bar{t}] : \{0 \leq \varepsilon_1 + \varepsilon_2 \leq 2N\overline{q}\}$, then there exists a unique symmetric SFE in the network. Each firm's monotonic equilibrium offer, $Q(p)$, can be calculated for $p \in (C'(0), \overline{p}]$ from the initial condition $Q(\overline{p}) = \overline{q}$ and*

$$Q'(p) = \frac{Q(p)}{(p - C'(Q(p)))\left(N\widetilde{\mu}(NQ(p)) - 1\right)} \tag{20}$$

$$\widetilde{\mu}(NQ) = 1 + \frac{\widetilde{P}(NQ, \omega_1)}{\sum_\omega \widetilde{P}(NQ, \omega)}, \tag{21}$$

The functions $\widetilde{P}(NQ,\omega)$ *are given by*

$$\begin{array}{c} \widetilde{P}(NQ,\omega_1) = \int_{-\bar{t}}^{\bar{t}} f(NQ-t, NQ+t)\,dt \\ \widetilde{P}(NQ,\omega_2) = \int_{NQ+\bar{t}}^{N\overline{q}+\bar{t}} f(NQ-\bar{t},\varepsilon_2)\,d\varepsilon_2 \\ \widetilde{P}(NQ,\omega_3) = \int_{-\bar{t}}^{NQ-\bar{t}} f(NQ+\bar{t},\varepsilon_2)\,d\varepsilon_2. \end{array} \tag{22}$$

In the next step we will explicitly solve for the unique symmetric SFE in the two-node network. To simplify the optimality conditions we consider the case where demand shocks follow a bivariate uniform distribution.

Assumption 1: *Consider a network with two nodes connected by an arc with capacity* $\bar{t}$ and *with* N *symmetric firms in each node. Inelastic demand in each node is given by the shock* ε_m*. We assume that shocks are uniformly distributed with a constant density,* $\frac{1}{V_1}$*, on the surface* $(\varepsilon_1,\varepsilon_2) \in [-\bar{t}, N\overline{q}+\bar{t}] \times [-\bar{t}, N\overline{q}+\bar{t}] :$ $\{0 \le \varepsilon_1+\varepsilon_2 \le 2N\overline{q}\}$ *and zero elsewhere.*

Proposition 3 *Under Assumption 1, the symmetric market integration function for the two-node network is given by*

$$\mu = \frac{4\bar{t}+N\overline{q}}{2\bar{t}+N\overline{q}}. \tag{23}$$

There is a unique symmetric SFE with inverse symmetric supply functions that can be calculated from:

$$p(Q) = Q^{-1}(Q) = \frac{\overline{p}Q^{\mu N-1}}{\overline{q}^{\mu N-1}} + (\mu N-1)\,Q^{\mu N-1}\int_Q^{\overline{q}} \frac{C'(u)\,du}{u^{\mu N}}. \tag{24}$$

It follows from Proposition 3 that the market integration function μ simplifies to a constant for uniformly distributed demand shocks. In this case, the equilibrium offer of a firm in the two-node network with N symmetric firms per node is identical to the equilibrium offer of a firm in an isolated node with μN symmetric firms. We note that the market integration function μ is close to 2 when the transmission capacity $\bar{t}$ is significantly larger than the nodal production capacity $N\overline{q}$, so that the two nodes are almost completely integrated. In the other extreme when the transmission capacity is much smaller than the nodal production capacity, then the market integration factor is close to 1, i.e. the two markets are almost isolated from each other, so that a node is approximately only integrated with itself. Fig. 1 plots (24) for the special case with constant marginal costs. The figure illustrates how the aggregated supply function in a node depends on μN if the total production capacity in each node is kept fixed. As the equations are identical, the symmetric SFE of the network also inherits the following properties from the single node case (Holmberg, 2008).

Corollary 3 *Solutions to (24) have the following properties:*

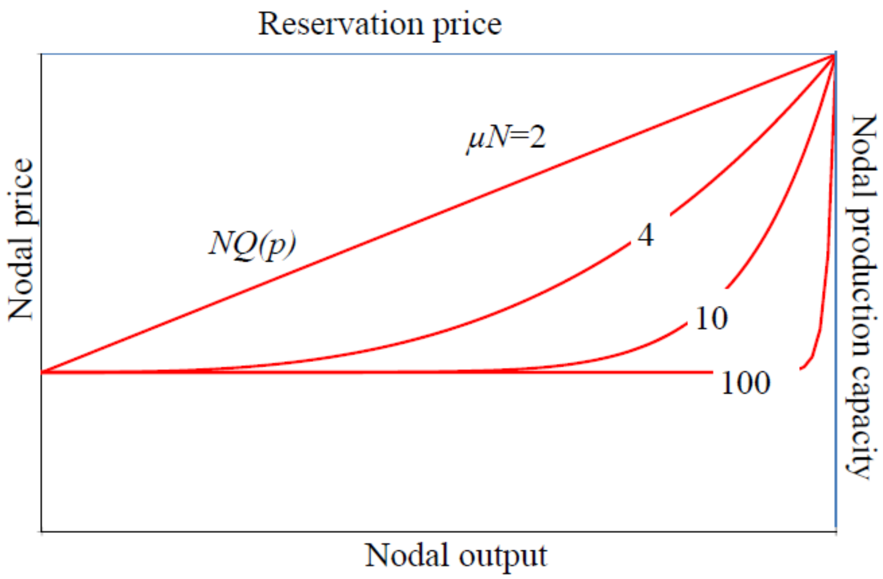

Figure 1: Aggregated nodal supply function for SFE in symmetric network with N firms per node. In this example, marginal costs are constant up to a fixed nodal production capacity. Demand shocks are uniformly distributed so that the market integration function μ is constant.

1. Mark-ups are positive for a positive output.
2. For a given nodal production cost function, mark-ups decrease at every positive nodal output level with more symmetric firms in the market.

Proposition 2 ensures existence of equilibria when slopes in the probability density of the demand shocks are sufficiently small, which is a new contribution. In previous studies, which assume certain demand, existence of pure-strategy Nash equilibria in transport-constrained networks has only been established for cases where transport constraints are either far from binding or firmly binding (Borenstein et al., 2000). With certain demand, producers would typically have profitable deviations from first-order solutions when transport constraints are close to binding. In our case, where demand is uncertain we find a pure-strategy NE although the transmission-line can be binding, non-binding or close to binding with positive probabilities. However, existence is problematic for steep slopes in the shock density and especially so when it has discontinuities. This is illustrated by the non-existence example below.

Example 1 ***Shock densities with discontinuities:*** *Assume that the support of the shock ε_m, $m \in \{1,2\}$ is given by $[0,\overline{\varepsilon}]$. The probability density is differentiable inside the support set, but decreases discontinuously to zero when $\varepsilon_1 = \overline{\varepsilon}$ and $\varepsilon_2 \in [0,\overline{\varepsilon}]$, where*

$$\overline{t} < \overline{\varepsilon} < \overline{q} + \overline{t}, \tag{25}$$

Thus the maximum demand shock is sufficiently large to congest the line, provided the output at the importing node is sufficiently small. However, the maximum *demand shock is not large enough to exhaust both the import capacity and local production capacity. Thus assumptions in Proposition 2 are violated. Consider a potential symmetric NE of a duopoly market with one firm in each node with identical costs $C(q)$ and identical supply functions $Q(p)$. Assume that the symmetric supply functions $Q(p)$ are monotonic. In the following we will show that the producer in node 1 will have a profitable deviation from the potential symmetric pure-strategy NE. In particular we will consider the point (q_0, p_0), where*

$$q_0 = Q(p_0) = \overline{\varepsilon} - \overline{t} \in (0, \overline{q}), \tag{26}$$

because of the inequality in (25). There is only one firm per node in our example, so $S_2(p) = Q(p)$ and $S_{1,-n}(p) = 0$. It now follows from (19) that outcomes where the price in node 1 is p, the firm in this node has output q, and imports to node 1 are congested occur with the following probability:

$$P(p, q, \omega_3) = \int_{-\infty}^{Q(p)-\overline{t}} f(q + \overline{t}, \varepsilon) d\varepsilon$$

and accordingly

$$\lim_{q \uparrow \overline{\varepsilon} - \overline{t}} P(p, q, \omega_3) > \lim_{q \downarrow \overline{\varepsilon} - \overline{t}} P(p, q, \omega_3) = \lim_{q \downarrow \overline{\varepsilon} - \overline{t}} \int_{-\infty}^{Q(p)-\overline{t}} f(q + \overline{t}, \varepsilon) d\varepsilon = 0. \tag{27}$$

Thus at the point $(p_0, Q(p_0))$ of its supply function $Q(p)$, the producer in node 1 can discontinuously increase the probability that imports are congested by slightly withholding output. However, $P(p, q, \omega_1)$ (the line is uncongested) and $P(p, q, \omega_2)$ (exports are congested) are still continuous at the point (q_0, p_0). From (19) we have:

$$\begin{aligned} P(p_0, q_0, \omega_1) &= \int_{-\overline{t}}^{\overline{t}} f(q_0 - t, q_0 + t) dt = \int_{-\overline{t}}^{\overline{t}} f(\overline{\varepsilon} - \overline{t} - t, \overline{\varepsilon} - \overline{t} + t) dt > 0 \\ P(p_0, q_0, \omega_2) &= \int_{\overline{t}+q_0}^{\infty} f(q_0 - \overline{t}, \varepsilon) d\varepsilon = \int_{\overline{\varepsilon}}^{\infty} f(\overline{\varepsilon} - 2\overline{t}, \varepsilon) d\varepsilon = 0, \end{aligned}$$

so (27) implies that

$$\lim_{q \downarrow \overline{\varepsilon} - \overline{t}} \sum_{\omega} P(p_0, q, \omega) < \lim_{q \uparrow \overline{\varepsilon} - \overline{t}} \sum_{\omega} P(p_0, q, \omega)$$

and accordingly

$$\begin{aligned} \lim_{q \downarrow \overline{\varepsilon} - \overline{t}} \hat{P}(\omega_1 \mid p_0, q) &= \lim_{q \downarrow \overline{\varepsilon} - \overline{t}} \frac{P(p_0, q, \omega_1)}{\sum_{\omega} P(p_0, q, \omega)} \\ &> \lim_{q \uparrow \overline{\varepsilon} - \overline{t}} \frac{P(p_0, q, \omega_1)}{\sum_{\omega} P(p_0, q, \omega)} = \lim_{q \uparrow \overline{\varepsilon} - \overline{t}} \hat{P}(\omega_1 \mid p_0, q). \end{aligned} \tag{28}$$

Thus the producer in node 1 discontinuously decreases the conditional probability that the line will be uncongested by slightly withholding output. A necessary condition for the symmetric supply functions being an equilibrium is that the optimality *condition in (13) is locally satisfied at the point* $(p_0, \overline{\varepsilon} - \bar{t})$*. Thus we must have* $\lim_{q \uparrow \overline{\varepsilon} - \bar{t}} Z_{mn}(p_0, q) \geq 0$*, but together with (18) and (28) this would imply that*

$$\begin{aligned}
\lim_{q \downarrow \overline{\varepsilon} - \bar{t}} Z_{mn}(p_0, q) &= (p - C'(q_0))Q'(p_0) \lim_{q \downarrow \overline{\varepsilon} - \bar{t}} \hat{P}(\omega_1 \mid p_0, q) - q_0 \\
&> (p - C'(q_0))Q'(p_0) \lim_{q \uparrow \overline{\varepsilon} - \bar{t}} \hat{P}(\omega_1 \mid p_0, q) - q_0 \\
&= \lim_{q \uparrow \overline{\varepsilon} - \bar{t}} Z_{mn}(p_0, q) \geq 0,
\end{aligned}$$

which would violate the local second-order condition implied by (13), and accordingly there is a profitable deviation from the equilibrium candidate.

Assumptions in Proposition 2 are also violated if demand shocks in the two nodes are sufficiently correlated. In particular, existence of symmetric pure-strategy NE is ruled out in the example below, where demand shocks are perfectly correlated. In such an extreme case, a producer would be able to infer the slope of its residual demand curve from the market price and thereby locate at what price convex kinks would occur. Similar to the incentives to congest for certain demand analysed by Borenstein et al. (2000), perfectly correlated shocks give a producer in a node where imports are nearly congested the incentive to unilaterally deviate from the first-order solution by withholding power in order to congest imports so as to increase the price of the importing node.

Example 2 ***Perfectly correlated shocks:*** *Consider two nodes connected by one arc. Demand shocks in the two nodes are perfectly correlated. This means that market prices are driven by a one-dimensional uncertainty. We assume that the demand shocks in both nodes are strictly increasing with respect to this underlying one-dimensional shock.*[14] *We also assume that* $D'_n < 0$*,* $n \in \{1, 2\}$*, so that* $S'_1(p)$ *and* $S'_2(p)$ *are always strictly positive. Thus both nodal prices are strictly increasing in the underlying shock, and there is a one-to-one mapping between the underlying shock and each nodal price. In the candidate equilibrium, firms maximize their profits by choosing a supply function that optimizes the output for each price and shock, so that the equilibrium becomes ex-post optimal as in Klemperer and Meyer's model of single markets (Klemperer and Meyer, 1989). Without loss of generality, assume that the arc from node 1 to 2 is congested at the price* p^* *and uncongested in some range* $(\widehat{p}, p^*)$*. Assume that the first-order condition results in a well-behaved monotonic solution for each firm where mark-ups are strictly positive in the range*

[14] In his analysis (Wilson, 2008) of perfectly correlated shocks, Wilson focuses on the special case when the shock at node 1 is fixed to zero. This means that regardless of deviations in node 2, exports from node 1 can never congest the arc below the price p^*. Thus the profitable deviation that is outlined in our example does not exist in this special case. We have found that ex-post optimal SFE can be constructed for such special cases. For similar reasons we have found that SFE can be constructed when demand shocks in the two nodes are negatively correlated. However, these equilibria are more complicated as one of the nodal shocks will decrease with respect to the underlying shock. The price in this node will first increase with respect to the one-dimensional underlying shock until the arc is congested and then decrease with respect to the underlying shock. Thus such SFE are not ex-post optimal.

$(\widehat{p}, p^*]$. *Consider a firm* i *in node 2 (the importing node), with the first-order solution* $Q_{2i}(p)$. *We choose* $\widehat{p}$ *sufficiently close to* p^* *and assume that the shock density is well-behaved so that* $P(p,q,\omega_1)+P(p,q,\omega_2)+P(p,q,\omega_3)$ *is well-defined and bounded away from zero for* $(p,q) \in (\widehat{p}, p^*) \times (Q_{2i}(\widehat{p}), Q_{2i}(p^*))$. *To simplify the analysis we consider the case when firms have constant marginal costs. We use (18) and consider*

$$\begin{aligned} Z_{2i}(p,q) = & (p - C'_{2i})(S'_1(p) + S'_{2,-i}(p))P(\omega_1|p,q) \\ & +(p - C'_{2i})S'_{2,-i}(p)\,P(\omega_2 \cup \omega_3|p,q) - q, \end{aligned}$$

where $P(\omega_1|p,q) = \frac{P(p,q,\omega_1)}{P(p,q,\omega_1)+P(p,q,\omega_2)+P(p,q,\omega_3)}$ *is the conditional probability that the arc is uncongested and* $P(\omega_2 \cup \omega_3|p,q) = \frac{P(p,q,\omega_3)+P(p,q,\omega_2)}{P(p,q,\omega_1)+P(p,q,\omega_2)+P(p,q,\omega_3)}$ *is the conditional probability that the arc is congested. It follows from our assumptions that the first-order solution satisfies:*

$$\begin{aligned} P(\omega_1|p, Q_{2i}(p)) &= \begin{cases} 1 & \text{if } p < p^* \\ 0 & \text{if } p \geq p^* \end{cases} \\ P(\omega_2 \cup \omega_3|p, Q_{2i}(p)) &= \begin{cases} 0 & \text{if } p < p^* \\ 1 & \text{if } p \geq p^* \end{cases} \end{aligned} \qquad (29)$$

and that

$$Z_{2i}(p, Q_{2i}(p)) = 0. \qquad (30)$$

Consider a price $p_0 \in (\widehat{p}, p^*)$. *Since* $S'_1(p) > 0$ *and mark-ups are strictly positive for* $p \in (\widehat{p}, p^*)$,

$$p_0 - C'_{2i}S'_1(p_0) \geq \inf_{p \in (\widehat{p}, p^*)} \{(p - C'_{2i})S'_1(p)\} = \Delta > 0.$$

We proceed to construct a deviation for the function $Q_{2i}(p)$ *that improves the payoff of firm* i. *The shock at node 1 is increasing in the underlying one-dimensional shock, so for prices* p_0 *sufficiently close to* p^* *it is possible for firm* i *to withhold an amount of production* $\delta_0 \in (0, \Delta)$ *so that* $P(\omega_1|p_0, Q_{2i}(p_0) - \delta_0) = 0$ *and* $P(\omega_2 \cup \omega_3|p_0, Q_{2i}(p_0) - \delta_0) = 1$. *Let* δ_1 *be the infimum of such* δ_0. *This implies that for every* $\delta \in \left(\delta_1, \frac{\Delta+\delta_1}{2}\right)$

$$\begin{aligned} Z_{2i}(p_0, Q_{2i}(p_0)) - Z_{2i}(p_0, Q_{2i}(p_0) - \delta) &= (p_0 - C'_{2i})(S'_1(p_0) + S'_{2,-i}(p_0)) - Q_{2i}(p_0) \\ &\quad -((p_0 - C'_{2i})S'_{2,-i}(p_0) - (Q_{2i}(p_0) - \delta)) \\ &= (p_0 - C'_{2i})S'_1(p_0) - \delta \\ &> \frac{\Delta - \delta_1}{2}. \end{aligned}$$

It follows from (30) that for every $\delta \in \left(\delta_1, \frac{\Delta+\delta_1}{2}\right)$, $Z_{2i}(p_0, Q_{2i}(p_0) - \delta) < -(\frac{\Delta-\delta_1}{2}) < 0$, *and so*

$$Z_{2i}(p_0, Q_{2i}(p_0) - \delta) < -h(\frac{\Delta - \delta_1}{2}) \qquad (31)$$

for some constant $h > 0$, *where* h *is less than or equal to the infimum of* $P(p_0, Q_{2i}(p_0) - \delta, \omega_1) + P(p_0, Q_{2i}(p_0) - \delta, \omega_2) + P(p_0, Q_{2i}(p_0) - \delta, \omega_3)$ *over* $\delta \in \left(\delta_1, \frac{\Delta+\delta_1}{2}\right)$. *Withholding* *less than* δ_1 *units at* p_0 *only has a second-order effect on* $\widehat{Z}_{2i}$ *and* Z_{2i}*. The deviation in* $Q_{2i}(p_0)$ *starts at* $p_\delta < p_0$*, which we define by*

$$Q_{2i}(p_\delta) = Q_{2i}(p_0) - \delta.$$

We assume that p_0 *is sufficiently close to* p^**, so that we can find a sufficiently small* $\delta \in \left(\delta_1, \frac{\Delta+\delta_1}{2}\right)$ *to ensure that* $p_\delta > \widehat{p}$*. For some* $\eta_1 > 0$*, when* $p \in (p_\delta + \eta_1, p_0)$*, the line is congested when the offer is* $Q_{2i}(p_\delta)$ *at price* p*. It follows from (30) and (29) that*

$$\begin{aligned} Z_{2i}(p, Q_{2i}(p_\delta)) &= Z_{2i}(p, Q_{2i}(p_\delta)) - Z_{2i}(p, Q_{2i}(p)) \\ &= (p - C'_{2i})S'_{2,-i}(p) - Q_{2i}(p_\delta) \\ &\quad -((p - C'_{2i})(S'_1(p) + S'_{2,-i}(p)) - Q_{2i}(p)) \\ &= Q_{2i}(p) - Q_{2i}(p_0) + \delta - (p - C'_{2i})S'_1(p) \\ &< -(\Delta - \delta) < -(\frac{\Delta - \delta_1}{2}) \end{aligned}$$

for $p \in (p_\delta + \eta_1, p_0)$*. Thus*

$$Z_{2i}(p, Q_{2i}(p_\delta)) < -k(\frac{\Delta - \delta_1}{2})$$

for $p \in (p_\delta + \eta_1, p_0)$ *and some positive*

$$k \leq \inf_{p \in (p_\delta + \eta_1, p_0)} \{P(p, Q_{2i}(p_\delta), \omega_1) + P(p, Q_{2i}(p_\delta), \omega_2) + P(p, Q_{2i}(p_\delta), \omega_3)\}.$$

Together with (31) this implies that if we integrate Z *along the deviation defined by* δ*, then*

$$\int_{p_\delta}^{p_0} Z_{2i}(p, Q_{2i}(p_\delta))dp + \int_{Q_{2i}(p_\delta)}^{Q_{2i}(p_0)} Z_{2i}(p_0, q)dq < 0, \tag{32}$$

if we choose p_0 *sufficiently close to* p^* *and* $\delta \in \left(\delta_1, \frac{\Delta+\delta_1}{2}\right)$ *sufficiently small, so that first-order effects dominate second-order effects. (32) violates a necessary local optimality condition (Anderson and Philpott, 2002a). The intuition is that a producer in an importing node always has an incentive to unilaterally deviate from the first-order solution by withholding power in order to congest the arc at lower prices than* p^**, which increases the price of the importing node.*

3.4.2 Star network

Next, we consider a star network with four nodes and three radial lines with capacity $\bar{t}$ as shown in Figure 2. Firms are located in nodes $1-3$ and each arc has the same number as the starting node, i.e. $1, 2$ or 3.

Demand shocks are defined on the following region Θ_2:

$$\Theta_2 = \left\{ \begin{array}{c} (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) \in \mathbb{R}^4 | -\bar{t} \leq \varepsilon_m \leq N\overline{q} + \bar{t}, \ -3\bar{t} \leq \varepsilon_4 \leq 3\bar{t}, \\ -2\bar{t} \leq \varepsilon_m + \varepsilon_4 \leq N\overline{q} + 2\bar{t}, \ -\bar{t} \leq \varepsilon_m + \varepsilon_\ell + \varepsilon_4 \leq 2N\overline{q} + \bar{t}, \\ 0 \leq \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 \leq 3N\overline{q}, \\ \forall m \in \{1,2,3\} \text{ and } \forall \ell \in \{1,2,3\}, \text{ where } \ell \neq m \end{array} \right\}$$

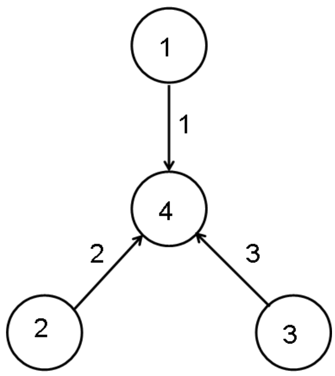

Figure 2: Star network example.

and we let V_2 be the volume of this region.

Assumption 2. *There are N firms with identical costs $C(q)$ in each node $1-3$. There are no producers in node 4 (the center node). Inelastic demand in nodes $m \in \{1,2,3,4\}$ is given by ε_m. Demand shocks are uniformly distributed such that:*

$$f(\boldsymbol{\varepsilon}) = \begin{cases} \frac{1}{V_2} & \text{if } \boldsymbol{\varepsilon} \in \Theta \\ 0 & \text{otherwise.} \end{cases}$$

Thus the shock density and network are symmetric with respect to nodes $1, 2, 3$. We can show the following under these circumstances:

Proposition 4 *Under Assumption 2, the symmetric market-integration function is a constant given by:*

$$\mu = \frac{3(N\overline{q})^2 + 12\overline{t}N\overline{q} + 12\overline{t}^2}{3(N\overline{q})^2 + 8\overline{t}N\overline{q} + 4\overline{t}^2}. \tag{33}$$

There is a unique symmetric SFE with inverse symmetric supply functions that can be calculated from (24).

It follows from (33) that the market integration function is close to 3 when each transmission capacity $\overline{t}$ is significantly larger than the nodal production capacity $N\overline{q}$, so that the three nodes with producers are almost completely integrated. In the other extreme when the transmission capacity is much smaller than the nodal production capacity, then the market integration factor is close to 1, i.e. the local markets are almost isolated from each other. Figure 1 and Corollary 3 apply to the star network as well.

4 Meshed network

So far we have studied radial networks, where there is a unique path between every pair of nodes. However, most electric power grids are meshed to some extent, i.e. they have loops. To consider such cases we generalize our results to include more complicated networks consisting of M nodes and K arcs, where $K \geq M$. This means that there will be at least one loop in the network, and there will be at least two paths between any two nodes in the loop (Bazaraa et al., 2009). Thus we need to make assumptions of how transport flows are divided between transport routes for cases when there are multiple possible paths. Similar to Wilson (2008) we assume that flows are determined by physical laws (Kirchhoff's laws) that are valid for electricity and incompressible mediums with laminar (non-turbulent) flows. Such flows are sometimes called potential flows, because one can model them as being driven by the potentials $\boldsymbol{\phi}$ in the nodes. In case the commodity is a gas or liquid (e.g. oil), the potential is the pressure at the node. In a DC network it is the voltage that is the potential.[15] For DC networks and laminar flows it can be shown that the electricity and flow choose paths that minimize total energy losses.

In a potential flow model, the flow in the arc k is driven by the potential difference between its endpoints. Given a vector of potentials $\boldsymbol{\phi}$, we have

$$t_k = \frac{-\left(\mathbf{A}^\top \boldsymbol{\phi}\right)_k}{X_k}, \tag{34}$$

where $-\left(\mathbf{A}^\top \boldsymbol{\phi}\right)_k$ is the potential difference and X_k is the impedance resisting the the flow through the arc. The impedance parameter is determined by the geometrical and material properties of the line/pipe that transports the commodity, and is independent of the flow in the arc k. In a DC network, the impedance is given by the resistance of the line.[16] To simplify the analysis we rule out some unrealistic or degenerate cases: we assume that the impedance is positive and that the capacities of the arcs and impedance factors are such that for any feasible flow, the set of arcs with flows at a lower or upper bound contains no loops.[17]

In principle, the meshed network could be cleared by a decentralized competitive transport sector. However, in practice this would be complicated as it would require significant coordination within the transport sector and between flows in the arcs to ensure that all loop constraints are satisfied in the network. Thus the relevant meshed case is to assume that the market is cleared by a regulated price-taking network operator. The standard assumption for wholesale electricity networks is that the network operator accepts production and determines feasible network flows in order to minimize the total stated production cost subject to realized local demand (Chao and Peck, 1996; Downward et al., 2010; Escobar and Jofre, 2006, 2008; Holmberg and Lazarczyk, 2015; Wilson, 2008).

[15]For AC networks it is standard to calculate electric power flows by means of a *DC-load flow approximation*, where $\boldsymbol{\phi}$ is the vector of voltage phase angles at the nodes (Chao and Peck, 1996).

[16]In a DC-load flow approximation of an AC network, X_k represents the reactance of the transmission line.

[17]This precludes certain degenerate solutions which can only arise if the values of the bounds and impedances for arcs forming a loop $\mathcal{L}$, coincidentally satisfy equations of the form

$$\sum_{k \in \mathcal{L}} \delta_k X_k \bar{t}_k = 0$$

where $\delta_k = 1$ if arc k is oriented in the direction that $\mathcal{L}$ is traversed and $\delta_k = -1$ otherwise. We can preclude instances having such solutions by perturbing the line capacities if necessary.

When analysing the optimal supply function of a producer in node m, it is convenient to choose node m to be a slack node and trading hub with nodal price p, as in the radial case. As before the vector of other nodal prices $\mathbf{p}_{-m}$ can be expressed in terms of the price of the trading hub and the shadow prices of the arcs (Xu and Baldick, 2007; Hogan, 2000; Chao et al., 2000):

$$\mathbf{p}_{-m} = p\mathbf{1}_{M-1} - \mathbf{H}^T (\boldsymbol{\rho} - \boldsymbol{\sigma}) . \tag{35}$$

In the meshed case, the matrix $\mathbf{H}$ with power transfer distribution factors $H_{k\ell}$ is determined from (Xu and Baldick, 2007; Hogan, 2000; Chao et al., 2000):

$$\mathbf{H} = -\mathbf{X}^{-1} (\mathbf{A}_{-m})^T \left(\mathbf{A}_{-m}\mathbf{X}^{-1} (\mathbf{A}_{-m})^T\right)^{-1}, \tag{36}$$

where $\mathbf{X}$ is a diagonal matrix with $\{X_k\}_{k=1}^{K}$ as diagonal entries. $\mathbf{A}_{-m}$ is nonsingular in the radial case, which gives $\mathbf{H} = -((\mathbf{A}_{-m})^T)^{-1}$ as in (4).

For each congestion state, we denote by $L(\omega)$, $B(\omega)$, and $U(\omega)$ the sets of arcs where flows are at their lower bound (i.e. congested in the negative direction), between their bounds or at their upper bound, respectively. For each congestion state, we also partition the shadow prices $\boldsymbol{\sigma}$ and $\boldsymbol{\rho}$ into $(\boldsymbol{\sigma}_{L(\omega)},\ \mathbf{0}_{B(\omega)},\ \mathbf{0}_{U(\omega)})$ and $(\mathbf{0}_{L(\omega)},\ \mathbf{0}_{B(\omega)},\ \boldsymbol{\rho}_{U(\omega)})$ corresponding to flows at their lower bounds, strictly between their bounds, and at their upper bounds. Similarly we find it convenient to partition the rows of $\mathbf{H}$ into $(\mathbf{H}_{L(\omega)},\ \mathbf{H}_{B(\omega)},\ \mathbf{H}_{U(\omega)})$.

As in the radial case, we let $\Xi(\omega)$ be a set with node m and all nodes that are connected to node m through uncongested paths only for congestion state ω. We refer to such nodes as completely integrated nodes. We also introduce a set $\Upsilon(\omega)$ with nodes that are semi-integrated with node m for congestion state ω, i.e. connected to node m through both congested and uncongested paths. We let $\boldsymbol{\Psi}(\omega)$ be a diagonal matrix with nodal supply slopes $S_\ell'(p_\ell))$ for $\ell \in \Upsilon(\omega)$. We also introduce

$$(\mathbf{W}(\omega))^T = \left(\left(\begin{bmatrix} \mathbf{H}_{U(\omega)} \\ -\mathbf{H}_{L(\omega)} \end{bmatrix}\right)^T\right)_{\Upsilon(\omega)}, \tag{37}$$

i.e. all columns in $\begin{bmatrix} \mathbf{H}_{U(\omega)} \\ -\mathbf{H}_{L(\omega)} \end{bmatrix}$ are removed that do not correspond to nodes in the set $\Upsilon(\omega)$. The following can now be shown by means of a related result in Xu and Baldick (2007).

Lemma 5 *All nodes in $\Xi(\omega)$ have the same market price p for a congestion state ω. The slope of residual demand facing producer n in node m is given by*

$$\begin{aligned} D'_{mn}(p,\omega,\boldsymbol{\rho},\boldsymbol{\sigma}) = \tfrac{d}{dp} D^{\varepsilon}_{mn}(p) = -S'_{m,-n}(p) - \textstyle\sum_{\ell\in\Xi(\omega)\setminus\{m\}} S'_\ell(p) \\ - \textstyle\sum_{\ell\in\Upsilon(\omega)} S'_\ell(p_\ell) + \mathbf{1}_{M-1}^\top \boldsymbol{\Psi}\mathbf{W}^\top(\mathbf{W}\boldsymbol{\Psi}\mathbf{W}^\top)^{-1}\mathbf{W}\boldsymbol{\Psi}\mathbf{1}_{M-1} \end{aligned} \tag{38}$$

for $\boldsymbol{\varepsilon} \in E(p,q,\omega)$.

Thus the contribution from completely integrated nodes to the residual demand slope of firm n in node m is similar to the radial case. However, the contribution from semi-integrated nodes is more complicated. The reason is that when producer n makes a change in quantity or price at node m then contributions from semi-integrated nodes have to be balanced by the network operator in order not to violate line capacities or Kirchhoff's laws. As flows between semi-integrated nodes and node m are semi-constrained, this also means that semi-integrated nodes will typically have prices different from p. Thus semi-integration implies that the slope of the residual demand at price p of firm n in node m depends on the slope of the nodal supply (in the matrix $\mathbf{\Psi}(\omega)$) at prices different from p for nodes $\ell \in \Upsilon(\omega)$.

We find it convenient to define the following Z function for meshed networks. Note that it simplifies to Definition 1 for radial networks, where $D'_{mn}(p, \omega, \boldsymbol{\rho}, \boldsymbol{\sigma})$ and $P_{mn}(p, q, \omega, \boldsymbol{\rho}, \boldsymbol{\sigma})$ are not functions of the shadow prices $\boldsymbol{\rho}$ and $\boldsymbol{\sigma}$.

Proposition 5 *The optimality conditions in a meshed network are the same as in Proposition 1, but with*

$$Z_{mn}(p,q) = (p - C'_{mn}(q)) \times \sum_{\omega} \int_{\boldsymbol{\rho}_{U(\omega)}} \int_{\boldsymbol{\sigma}_{L(\omega)}} -D'_{mn}(p, \omega, \boldsymbol{\rho}, \boldsymbol{\sigma}) \hat{P}_{mn}(\omega, \boldsymbol{\rho}, \boldsymbol{\sigma}, \mid p, q)\, d\boldsymbol{\sigma}_{L(\omega)} d\boldsymbol{\rho}_{U(\omega)} - q \tag{39}$$

where

$$\hat{P}_{mn}(\omega, \boldsymbol{\rho}, \boldsymbol{\sigma} \mid p, q) = \frac{P_{mn}(p, q, \omega, \boldsymbol{\rho}, \boldsymbol{\sigma})}{\sum_{\omega} \int_{\boldsymbol{\rho}_{U(\omega)}} \int_{\boldsymbol{\sigma}_{L(\omega)}} P_{mn}(p, q, \omega, \boldsymbol{\rho}, \boldsymbol{\sigma})\, d\boldsymbol{\sigma}_{L(\omega)} d\boldsymbol{\rho}_{U(\omega)}}.$$

Lemma 10 in Appendix D presents an explicit expression for how $P(p, q, \omega, \boldsymbol{\rho}, \boldsymbol{\sigma})$ can be calculated in a meshed network. As implied by the optimality condition in Proposition 1, the first-order condition of producer n in node m is given by $Z_{mn} = 0$. Thus for given supply functions of its competitors, the optimal supply function $Q_{mn}(p)$ of producer n can be determined from the following implicit equation:

$$Q_{mn}(p) = (p - C'_{mn}(Q_{mn}(p)))\, \mathbb{E}\left[-D'_{mn}(p, \omega, \boldsymbol{\rho}, \boldsymbol{\sigma}) \middle| D^{\varepsilon}_{mn}(p) = Q_{mn}(p)\right]. \tag{40}$$

This is similar to the radial case, with the difference that in a meshed network the slope of the residual demand of producer n at price p will often depend on the slope of competitors' supply functions at prices other than p, i.e. D'_{mn} depends on shadow prices of the arcs.

4.1 Delta network

As an example of a meshed network, we consider a delta network as shown in Figure 3.

Assumption 3. *Each line has capacity $\bar{t}$ and equal impedance. There are N firms with identical costs $C(q)$ in each node. Inelastic demand in nodes $m \in \{1,2,3\}$ is given by ε_m. Demand shocks are uniformly distributed such that:*

$$f(\boldsymbol{\varepsilon}) = \begin{cases} \frac{1}{V_3} & \text{if } \boldsymbol{\varepsilon} \in \Theta_3 \\ 0 & \text{otherwise,} \end{cases}$$

where

$$\Theta_3 = \left\{ \begin{array}{c} (\varepsilon_1, \varepsilon_2, \varepsilon_3) \in \mathbb{R}^3 | , 0 \leq \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \leq 3N\overline{q}, \\ -2\overline{t} \leq \varepsilon_m + \varepsilon_\ell \leq 2N\overline{q} + 2\overline{t}, \\ -2\overline{t} \leq \varepsilon_m \leq N\overline{q} + 2\overline{t}, \\ \forall m \in \{1,2,3\} \text{ and } \forall \ell \in \{1,2,3\}, \text{ where } \ell \neq m \end{array} \right\}$$

and V_3 is the volume of this region.

In Table 3 and Table 4 in Appendix E we present expressions for

$$\int_{\boldsymbol{\rho}_{U(\omega)}} \int_{\boldsymbol{\sigma}_{L(\omega)}} -D'_{mn}(p, \omega, \boldsymbol{\rho}, \boldsymbol{\sigma}) P\left(p, q, \boldsymbol{\rho}, \boldsymbol{\sigma}, \omega\right) d\boldsymbol{\sigma}_{L(\omega)} d\boldsymbol{\rho}_{U(\omega)}$$

and

$$\sum_\omega \int_{\boldsymbol{\rho}_{U(\omega)}} \int_{\boldsymbol{\sigma}_{L(\omega)}} P\left(p, q, \boldsymbol{\rho}, \boldsymbol{\sigma}, \omega\right) d\boldsymbol{\sigma}_{L(\omega)} d\boldsymbol{\rho}_{U(\omega)}$$

for each of the feasible congestion states of the delta network. Summing over the congestion states yields:

$$\begin{array}{c} \sum_\omega \int_{\boldsymbol{\rho}_{U(\omega)}} \int_{\boldsymbol{\sigma}_{L(\omega)}} -D'_{mn}(p, \omega, \boldsymbol{\rho}, \boldsymbol{\sigma}) P\left(p, q, \boldsymbol{\rho}, \boldsymbol{\sigma}, \omega\right) d\boldsymbol{\sigma}_{L(\omega)} d\boldsymbol{\rho}_{U(\omega)} \\ = \frac{2\overline{t}(N-1)Q'(p)}{V_3} \left[-NQ\left(p - \frac{\rho_2}{3}\right) + NQ\left(p + \frac{\rho_2}{3}\right)\right]_0^{\min[3p, 3(\overline{p}-p)]} \\ + \frac{8\overline{t}N^2}{3V_3} \int_0^{\min[3p, 3(\overline{p}-p)]} Q'\left(p - \frac{\rho_2}{3}\right) Q'\left(p + \frac{\rho_2}{3}\right) d\rho_2 \\ + \frac{9(3N-1)\overline{t}^2}{V_3} Q'(p) \\ + \frac{2\overline{t}(N-1)Q'(p)}{V_3} \left[2NQ\left(\overline{p}\right) + NQ\left(\frac{p+\overline{p}}{2}\right) - NQ\left(\frac{p}{2}\right)\right] \\ + \frac{2\overline{t}N^2}{3V_3} \int_{-\frac{3p}{2}}^{\frac{3(\overline{p}-p)}{2}} Q'\left(p + \frac{2\rho_1}{3}\right) Q'\left(p + \frac{\rho_1}{3}\right) d\rho_1 \\ + \frac{(N-1)N^2 Q'(p)}{V_3} \left(3Q^2\left(\overline{p}\right) + 2Q^2\left(p\right) - 2Q\left(p\right) Q\left(\overline{p}\right)\right). \end{array} \tag{41}$$

and

$$\begin{array}{c} \sum_\omega \int_{\boldsymbol{\rho}_{U(\omega)}} \int_{\boldsymbol{\sigma}_{L(\omega)}} P\left(p, q, \boldsymbol{\rho}, \boldsymbol{\sigma}, \omega\right) d\boldsymbol{\sigma}_{L(\omega)} d\boldsymbol{\rho}_{U(\omega)} \\ = \frac{2\overline{t}}{V_3} \left[-NQ\left(p - \frac{\rho_2}{3}\right) + NQ\left(p + \frac{\rho_2}{3}\right)\right]_0^{\min[3p, 3(\overline{p}-p)]} \\ + \frac{9\overline{t}^2}{V_3} + \frac{2\overline{t}}{V_3} \left[2NQ\left(\overline{p}\right) + NQ\left(\frac{p+\overline{p}}{2}\right) - NQ\left(p/2\right)\right] \\ + \frac{N^2}{V_3} \left(3Q^2\left(\overline{p}\right) + 2Q^2\left(p\right) - 2Q\left(p\right) Q\left(\overline{p}\right)\right), \end{array} \tag{42}$$

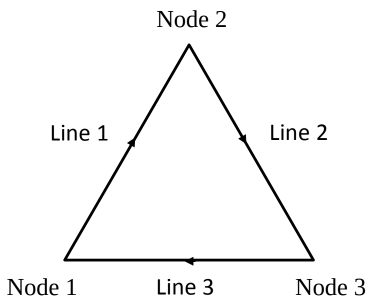

Figure 3: Delta network example

which can be used to compute

$$
\begin{aligned}
&\mathbb{E}\left[-D'_{mn}(p,\omega,\boldsymbol{\rho},\boldsymbol{\sigma})\right| D^{\varepsilon}_{mn}(p) = Q(p)] \\
= \; & \frac{\sum_{\omega}\int_{\boldsymbol{\rho}_{U(\omega)}}\int_{\boldsymbol{\sigma}_{L(\omega)}} -D'_{mn}(p,\omega,\boldsymbol{\rho},\boldsymbol{\sigma})P\left(p,Q(p),\boldsymbol{\rho},\boldsymbol{\sigma},\omega\right)d\boldsymbol{\sigma}_{L(\omega)}d\boldsymbol{\rho}_{U(\omega)}}{\sum_{\omega}\int_{\boldsymbol{\rho}_{U(\omega)}}\int_{\boldsymbol{\sigma}_{L(\omega)}} P\left(p,Q(p),\boldsymbol{\rho},\boldsymbol{\sigma},\omega\right)d\boldsymbol{\sigma}_{L(\omega)}d\boldsymbol{\rho}_{U(\omega)}}.
\end{aligned}
$$

As we would expect, we note from (41) and (42) that the expected slope of the residual demand has the following properties:

$$
\mathbb{E}\left[-D'_{mn}(p,\omega,\boldsymbol{\rho},\boldsymbol{\sigma})\right| D^{\varepsilon}_{mn}(p) = Q(p)] \to (N-1)\,Q'
$$

when $\bar{t} \to 0$, so that nodes are nearly isolated, and that

$$
\mathbb{E}\left[-D'_{mn}(p,\omega,\boldsymbol{\rho},\boldsymbol{\sigma})\right| D^{\varepsilon}_{mn}(p) = Q(p)] \to (3N-1)\,Q'
$$

when $\bar{t}$ is large, so that all nodes are nearly completely integrated.

It follows from (40) that symmetric supply function equilibria in the delta network can be determined from the following implicit relation:

$$
Q(p) = (p - C'(Q(p)))\,\mathbb{E}\left[-D'_{mn}(p,\omega,\boldsymbol{\rho},\boldsymbol{\sigma})\right| D^{\varepsilon}_{mn}(p) = Q(p)].
$$

However, solving for supply function equilibria in a delta network becomes complicated as supply functions for semi-integrated nodes are evaluated at different prices, as shown in (41) and (42). Hence, solutions are given by a system of delay differential equations rather than a system of ordinary differential equations, which is the case for radial and single node networks. In addition delay is endogenous and both negative and positive in the differential equations of the delta network. Thus non-standard numerical methods would generally be required to solve for SFE in a delta network. The development of such tools is an interesting research topic in itself.

5 Conclusions

We derive optimality conditions for supply functions of producers competing in a network with transport constraints and local demand shocks. We show that the optimal output of a producer is proportional to its mark-up and the expected slope of its residual demand curve at every local price of the producer. In principle, a system of such optimality conditions can be used to numerically calculate asymmetric supply-function equilibria (SFE) in a general network, including meshed networks with loop flows.

In the paper, we focus on characterizing symmetric SFE in radial networks with inelastic demand. We verify that there is a unique symmetric monotonic solution to the first-order condition and that this solution is an SFE when the joint probability density of the local demand shocks is sufficiently evenly distributed, i.e. sufficiently close to a uniform multi-dimensional distribution. But existence of SFE cannot be taken for granted. Profitable deviations from the first-order solution will for example exist for perfectly correlated demand shocks or steep slopes and discontinuities in the demand shock density.

For symmetric equilibria in radial networks, it is useful to define a market integration function, which equals the expected number of nodes that are completely integrated with the node of the producer under study. Firms' mark-ups depend on the number of firms in the market. Still it can be shown that in a symmetric equilibrium, market integration is a function of the total production in a node. This function can be determined from exogenous parameters: the network topology, the demand shock distribution and production and transport capacities. The implication is that oligopoly producers will have high mark-ups at output levels for which the (exogenous) market integration function returns small values, and lower mark-ups at output levels where market integration is expected to be high.

The market integration function simplifies to a constant for symmetric equilibria in radial networks with multi-dimensional uniformly distributed shocks. In this case, we use our optimality conditions to explicitly solve for symmetric equilibria in two-node and star networks. We also show that these symmetric equilibria are well-behaved: (i) mark-ups are positive for a positive output, and (ii) for a given total production cost, mark-ups decrease with more firms in the market.

In a connected radial network, two nodes are either connected by one uncongested path or by a congested path. Nodes that are connected via an uncongested path have the same market price and are completely integrated. In a radial network, the slope of the residual demand of a firm is only influenced by supply function of competitors located in nodes that are completely integrated with the firm's node. In this case, supply-function equilibria can be determined from a system of ordinary differential equations as for single node networks. Meshed networks are more complicated, because two nodes can be connected by both congested and uncongested paths. Prices in such nodes are generally different, so we refer to this situation as semi-integration. In meshed networks, the slope of the residual demand of a firm is partly influenced by supply functions of competitors located in nodes that are semi-integrated with the firm's node. This means that supply-function equilibria are given by a system of delay differential equations, where delay is endogenous and both negative and positive, and such equations are harder to solve.

In a previous version of the working paper (Holmberg and Philpott, 2012), we also show how our optimality conditions can be modified to consider Cournot NE in networks and SFE in networks with discriminatory pricing.

6 References

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Appendix A: Radial networks

6.1 Market clearing

Proof. (Lemma 1) For each congestion state ω, it is convenient to partition the rows of $\mathbf{H}$ into $(\mathbf{H}_{L(\omega)}, \mathbf{H}_{B(\omega)}, \mathbf{H}_{U(\omega)})$ corresponding arcs where flows are at their lower bounds, strictly between their bounds, and at their upper bounds. Consider a node ℓ that is connected to node m through an uncongested chain of arcs for congestion state ω. In a radial network this is the unique path between node ℓ and node m. Thus, if a commodity is injected at node ℓ and consumed at node m, then no flow is going to pass through a congested arc. Thus $\left(\left(\mathbf{H}_{L(\omega)}\right)^T\right)_\ell$ and $\left(\left(\mathbf{H}_{U(\omega)}\right)^T\right)_\ell$ are either empty or zero row vectors, and consequently it follows from (5) that $p_\ell = p$. Flows from the rest of the network to the completely integrated area with price p are fixed on margin by congested arcs. Let $\widetilde{t}_{\Xi(\omega)}$ be the net flow (through congested lines) into the integrated area with nodes in $\Xi(\omega)$. Thus as a consequence of the argument above, we have

$$q = D^{\boldsymbol{\varepsilon}}_{mn}(p) = \sum_{\ell \in \Xi(\omega)} \varepsilon_\ell + \widetilde{t}_{\Xi(\omega)} - \sum_{\ell \in \Xi(\omega) \setminus \{m\}} S_\ell(p) - S_{m,-n}(p), \tag{43}$$

for $\boldsymbol{\varepsilon} \in E(p, q, \omega)$. (6) now follows from differentiation of the above. ■

Supply functions are assumed to be continuously differentiable, so we note that $D'_{mn}(p, \omega)$ is continuous for $\boldsymbol{\varepsilon}$ such that the congestion state is ω.

Optimality conditions

We make use of Anderson and Philpott's (2002a) market distribution function when deriving the optimality condition in Proposition 1. The market distribution function of a firm for a point (p, q) is given by the probability that the realized residual demand of the firm is lower than q at price p.

Definition 3 *The* market distribution function *for firm n at location m is*

$$\psi_{mn}(p, q) = \Pr(D^{\boldsymbol{\varepsilon}}_{mn}(p) < q) = \Pr(\cup_{\eta<0} E(p, q + \eta)).$$

$1 - \psi_{mn}(p, q)$ is equivalent to Wilson's "probability distribution of the sale price" in auctions of shares (Wilson, 1979). We find it useful to define the market distribution function conditional on a congestion state ω:

$$\psi_{mn}(p, q, \omega) = \Pr(\cup_{\eta<0} E(p, q + \eta, \omega)),$$

so

$$\left[\frac{\partial \psi_{mn}(p, q, \omega)}{\partial q}\right]_{(\hat{p}, \hat{q})} = \lim_{dq \to 0} \frac{\Pr(\cup_{0 \le \eta \le dq} E(\hat{p}, \hat{q} + \eta, \omega))}{dq}. \tag{44}$$

We show below how this derivative can be constructed for radial networks. Observe that (8) and (44) gives

$$P_{mn}(\hat{p}, \hat{q}, \omega) = \left[\frac{\partial \psi_{mn}(p, q, \omega)}{\partial q}\right]_{(\hat{p}, \hat{q})}. \tag{45}$$

Lemma 6 $\dfrac{\left[\frac{\partial \psi_{mn}(p,q,\omega)}{\partial p}\right]_{(\hat{p},\hat{q})}}{\left[\frac{\partial \psi_{mn}(p,q,\omega)}{\partial q}\right]_{(\hat{p},\hat{q})}} = -D'_{mn}(\hat{p}, \omega).$

Proof. Consider a particular state ω. For this state, we have from (43) that $\sum_{\ell \in \Xi(\omega)} \varepsilon_\ell$ is simply an additive demand shock that shifts the residual demand curve horizontally. Thus for a given additive shock $\boldsymbol{\varepsilon}$ corresponding to congestion state ω, $\psi_{mn}(p, D^{\boldsymbol{\varepsilon}}_{mn}(p), \omega) = g(\boldsymbol{\varepsilon})$ along such a curve for some function g. We know from Lemma 1 that the slope $D'_{mn}(\hat{p}, \omega)$ of the residual demand curve at $\hat{p}$, is the same for all demand shocks that result in the state ω. Implicit differentiation of $\psi_{mn}(p, D^{\boldsymbol{\varepsilon}}_{mn}(p), \omega) = g(\boldsymbol{\varepsilon})$ with respect to p gives

$$\left[\frac{\partial \psi_{mn}(p,q,\omega)}{\partial p}\right]_{(\hat{p},\hat{q})} + \left[\frac{\partial \psi_{mn}(p,q,\omega)}{\partial q}\right]_{(\hat{p},\hat{q})} D'_{mn}(\hat{p},\omega) = 0$$

from which the result follows. ■

Lemma 7 $\sum_\omega D'_{mn}(p,\omega)\hat{P}_{mn}(\omega \mid \hat{p},\hat{q}) = -\frac{\left[\frac{\partial \psi_{mn}(p,q)}{\partial p}\right]_{(\hat{p},\hat{q})}}{\left[\frac{\partial \psi_{mn}(p,q)}{\partial q}\right]_{(\hat{p},\hat{q})}}$.

Proof. First using (9) we have

$$\sum_\omega D'_{mn}(p,\omega)\hat{P}_{mn}(\omega \mid \hat{p},\hat{q}) = \frac{\sum_\omega P_{mn}(\hat{p},\hat{q},\omega) D'_{mn}(p,\omega)}{\sum_\omega P_{mn}(\hat{p},\hat{q},\omega)}.$$

Thus (45) and Lemma 6 gives

$$\begin{aligned}\sum_\omega D'_{mn}(p,\omega)\hat{P}_{mn}(\omega \mid \hat{p},\hat{q}) & \\ &= \frac{\sum_\omega \left[\frac{\partial \psi_{mn}(p,q,\omega)}{\partial q}\right]_{(\hat{p},\hat{q})} D'_{mn}(p,\omega)}{\sum_\omega \left[\frac{\partial \psi_{mn}(p,q,\omega)}{\partial q}\right]_{(\hat{p},\hat{q})}} \\ &= -\frac{\sum_\omega \left[\frac{\partial \psi_{mn}(p,q,\omega)}{\partial p}\right]_{(\hat{p},\hat{q})}}{\sum_\omega \left[\frac{\partial \psi_{mn}(p,q,\omega)}{\partial q}\right]_{(\hat{p},\hat{q})}}. \qquad (46)\end{aligned}$$

Now since $E(p,q,\omega_1)$ and $E(p,q,\omega_2)$ are disjoint when $\omega_1 \neq \omega_2$ for considered (non-negligible) demand shock outcomes, we have

$$\begin{aligned}\sum_\omega \left[\frac{\partial \psi_{mn}(p,q,\omega)}{\partial q}\right]_{(\hat{p},\hat{q})} &= \sum_\omega \lim_{dq \to 0} \frac{\Pr(\cup_{0 \le \eta \le dq} E(\hat{p},\hat{q}+\eta,\omega))}{dq} \\ &= \lim_{dq \to 0} \frac{\Pr(\cup_{0 \le \eta \le dq} E(\hat{p},\hat{q}+\eta))}{dq} \\ &= \left[\frac{\partial \psi_{mn}(p,q)}{\partial q}\right]_{(\hat{p},\hat{q})}\end{aligned}$$

and similarly

$$\sum_\omega \left[\frac{\partial \psi_{mn}(p,q,\omega)}{\partial p}\right]_{(\hat{p},\hat{q})} = \left[\frac{\partial \psi_{mn}(p,q)}{\partial p}\right]_{(\hat{p},\hat{q})},$$

which yields the result when substituted in (46). ■

Proof. (Proposition 1) As shown by Anderson and Philpott (2002a) a monotonic increasing supply function $Q_{mn}(p)$ is globally optimal if it satisfies:

$$\begin{cases} \tilde{Z}\,(p,q) \geq 0 & \text{if } q < Q_{mn}(p) \\ \tilde{Z}\,(p,q) = 0 & \text{if } q = Q_{mn}(p) \\ \tilde{Z}\,(p,q) \leq 0 & \text{if } q > Q_{mn}(p). \end{cases}$$

where

$$\tilde{Z}(p,q) = (p - C'_{mn}(q))\,\frac{\partial \psi_{mn}}{\partial p} - q\frac{\partial \psi_{mn}}{\partial q}.$$

If $\frac{\partial \psi_{mn}}{\partial q} > 0$ then we have from Lemma 7 and Definition 1 that

$$\begin{aligned} Z(p,q) &= \tilde{Z}(p,q)/\frac{\partial \psi_{mn}}{\partial q} \\ &= (p - C'_{mn}(q))\sum_{\omega} -D'_{mn}(p,\omega))\hat{P}(\omega \mid p,q) - q \\ &= (p - C'_{mn}(q))\left(\frac{\partial \psi_{mn}}{\partial p}/\frac{\partial \psi_{mn}}{\partial q}\right) - q. \end{aligned}$$

Since

$$\tilde{Z}(p,q\left\{\begin{array}{c} > \\ = \\ < \end{array}\right\}0 \iff Z(p,q)\left\{\begin{array}{c} > \\ = \\ < \end{array}\right\}0$$

we have that a supply function $Q_{mn}(p)$ is globally optimal if it satisfies

$$\begin{cases} Z_{mn}\,(p,q) \geq 0 & \text{if } q < Q_{mn}(p) \\ Z_{mn}\,(p,q) = 0 & \text{if } q = Q_{mn}(p) \\ Z_{mn}\,(p,q) \leq 0 & \text{if } q > Q_{mn}(p). \end{cases}$$

as required. ■

The first-order condition implied by $Z_{mn}\,(p,q) = 0$ and (11) can be simplified for symmetric equilibria.

Proof. (Lemma 2) We first substitute $q = Q\,(p)$, $S_{\ell}\,(p) = NQ\,(p)$ and $S_{m,-n}\,(p) = (N-1)\,Q\,(p)$ into (11), and then use the first-order condition $Z_{mn}\,(p,q) = 0$.

$$Q = (p - C'(Q))\sum_{\omega}\left((N-1)\,Q' + \sum_{\ell\in\Xi(\omega)\setminus\{m\}} NQ'\right)\hat{P}_{mn}\,(\omega|p,Q)\,.$$

We have $\sum_{\omega}\hat{P}_{mn}\,(\omega|p,q) = 1$, $M_{\Xi(\omega)} = \sum_{\ell\in\Xi(\omega)} 1$, and by definition $m \in \Xi\,(\omega)$, so it follows from Definition 2 that $\sum_{\omega}\sum_{\ell\in\Xi(\omega)\setminus\{m\}}\hat{P}_{mn}\,(\omega|p,q) = \mu_{mn}\,(p,Q) - 1$. Thus

$$Q = (p - C'(Q))\,((N-1)\,Q' + (\mu_{mn}\,(p,Q) - 1)\,NQ'),$$

which gives

$$Q = (p - C'(Q))\,(\mu_{mn}\,(p,Q\,(p))\,N - 1)\,Q'.$$

$Q(p)$ is the same for all firms, which is only possible if $\mu_{mn}(p, Q(p))$ is the same for all firms. ■

Proof. (Lemma 3) Consider a situation where the number of firms per node changes from N to some arbitrary number $\widehat{N}$ and where all firms change their offers from $Q(p)$ to some arbitrary monotonic supply function $\widehat{Q}(p)$. Such a change would typically change nodal prices. But for a given realized vector of demand shocks $\boldsymbol{\varepsilon}$, we conjecture that changing N in each node and $Q(p)$ for each firm will not change cleared nodal production and network flows. We prove that the conjecture is correct by verifying that it satisfies the market-clearing conditions in (3) for any vector of shock outcomes. First we note that conjectured nodal output and flows will satisfy (2). Next, the new (conjectured) nodal price $\hat{p}_\ell$ for a node ℓ can be calculated for each shock outcome from the old nodal price p_ℓ and the conjecture that $\widehat{N}\widehat{Q}(\hat{p}_\ell) = NQ(p_\ell)$. New supply functions are monotonic and identical in each node and it has been conjectured that nodal output is unchanged. Thus for a given shock outcome, new prices $\widehat{\mathbf{p}}$ are such that

$$\widehat{p}_m \geq \widehat{p}_\ell \text{ if and only if } p_m \geq p_\ell. \tag{47}$$

New shadow prices can be calculated for each arc from the first market-clearing condition in (3) by calculating price differences between nodes that are connected by the arc. It follows from (47) that the new shadow prices will have the same sign as the old shadow prices. Thus the new shadow prices will satisfy the complementary slackness conditions in (3), because flows are the same and the old shadow prices satisfied those conditions, which verify our conjecture. The argument is true for any vector of local demand shocks. Thus $P_{\ell n}(p_\ell, Q(p_\ell), \omega)$ for old offers is identical to $P^*_{\ell n}(\hat{p}_\ell, \widehat{Q}(\hat{p}_\ell), \omega)$ for new offers, where $\widehat{N}\widehat{Q}(\hat{p}_\ell) = NQ(p_\ell)$. It follows for symmetric supply functions that $P_{\ell n}$ for a congestion state ω can be determined from the nodal output $NQ(p_\ell)$. Symmetry of supply functions and the network implies that $P_{\ell n}(p_\ell, Q(p_\ell), \omega)$ is the same for each firm and node. Hence, there is a function $\tilde{P}(NQ(p_\ell), \omega)$, such that $P_{\ell n}(p_\ell, Q(p_\ell), \omega) = \tilde{P}(NQ(p_\ell), \omega)$ for any N and $Q(p)$. (17) now follows from (9) and Definition 2. ■

Computing $P(p, q, \omega)$

In the calculation of $P(p, q, \omega)$ for each congestion state ω, we find it useful to denote by $L(\omega)$, $B(\omega)$, and $U(\omega)$ the sets of arcs where flows are at their lower bound (i.e. congested in the negative direction), between their bounds or at their upper bound, respectively. Hence, the complementary slackness conditions, i.e. the second and third set of conditions in (3), can be equivalently written as follows:

$$\begin{array}{llll} t_k = \bar{t}_k, & \sigma_k = 0, & \rho_k > 0, & k \in U(\omega), \\ t_k \in (-\bar{t}_k, \bar{t}_k) & \sigma_k = 0, & \rho_k = 0, & k \in B(\omega), \\ t_k = -\bar{t}_k, & \sigma_k > 0, & \rho_k = 0, & k \in L(\omega). \end{array}$$

Observe that given a congestion state ω and arc k, there is exactly one variable t_k, ρ_k or σ_k that is not at a bound.

We partition $\mathbf{t}$ and the shadow prices $\boldsymbol{\sigma}$ and $\boldsymbol{\rho}$ into ($\mathbf{t}_L$, $\mathbf{t}_B$, $\mathbf{t}_U$), ($\boldsymbol{\sigma}_L$, $\mathbf{0}_B$, $\mathbf{0}_U$) and ($\mathbf{0}_L$, $\mathbf{0}_B$, $\boldsymbol{\rho}_U$) corresponding to flows at their lower bounds, strictly between their bounds, and at their upper bounds. Below we define the volume $\mathcal{T}(B(\omega))$ in $\mathbf{t}$ space that the flows in the set of uncongested arcs $B(\omega)$ can span. $\mathcal{U}(U(\omega))$ and $\mathcal{L}(L(\omega))$ are the volumes in $\boldsymbol{\sigma}$ and $\boldsymbol{\rho}$ space spanned by the shadow prices of congested arcs in the sets $U(\omega)$ and $L(\omega)$, respectively. Note that these volumes are all open sets, as we neglect outcomes at the boundary between two congestion states.

$$\begin{aligned} \mathcal{T}(B(\omega)) &= \{\mathbf{t}_{B(\omega)}\colon -\bar{\mathbf{t}}_{B(\omega)} < \mathbf{t}_{B(\omega)} < \bar{\mathbf{t}}_{B(\omega)}\}, \\ \mathcal{U}(U(\omega)) &= \{\boldsymbol{\rho}_{U(\omega)} : \mathbf{0} < \boldsymbol{\rho}_{U(\omega)}\}, \\ \mathcal{L}(L(\omega)) &= \{\boldsymbol{\sigma}_{L(\omega)} : \mathbf{0} < \boldsymbol{\sigma}_{L(\omega)}\}. \end{aligned} \tag{48}$$

In particular we are interested in $\mathcal{S}(\omega) \subseteq \mathbb{R}^K$, which we define by

$$\mathcal{S}(\omega) = \mathcal{L}(L(\omega)) \times \mathcal{U}(U(\omega)) \times \mathcal{T}(B(\omega)). \tag{49}$$

Hence, $\mathcal{S}(\omega)$ is the total volume in $\mathbf{t}$, $\boldsymbol{\sigma}$ and $\boldsymbol{\rho}$ space that is spanned for a congestion state ω.

We introduce additional notation in order to analyse the radial network in detail. For each state ω, we partition the nodes into the sets $\Xi(\omega)$ and $\digamma(\omega)$. $\Xi(\omega)$ includes all nodes that are connected to node m (the trading hub) through some uncongested chain of arcs. $M_{\Xi(\omega)}$ is the number of nodes in $\Xi(\omega)$ and we note that they must be connected by $M_{\Xi(\omega)} - 1$ uncongested arcs. The set $\digamma(\omega)$ contains all other nodes in the network. Similarly we partition the shock vector into $\boldsymbol{\varepsilon}_{\Xi(\omega)}$ and $\boldsymbol{\varepsilon}_{\digamma(\omega)}$. Let $\kappa(\omega)$ be the set of uncongested arcs that connect nodes in $\Xi(\omega)$. Other arcs are in the set $\vartheta(\omega)$. We let $\mathbf{t}_{\kappa(\omega)}$ be the flows in the uncongested arcs between nodes in the set $\Xi(\omega)$ and we let $\mathbf{t}_{\vartheta(\omega)}$ be the vector of flows in the other arcs. The node-arc incidence matrix $\mathbf{A}_{\Lambda(\omega)}$ describes the subtree with nodes in $\Xi(\omega)$ that are connected by arcs in $\kappa(\omega)$. We let $\mathbf{A}_{\backslash\Lambda(\omega)}$ be a node-arc incidence matrix with $M - M_{\Xi(\omega)}$ rows and $M - M_{\Xi(\omega)}$ columns, describing the rest of the network.[18]

Lemma 8

$$P_{mn}(p, q, \omega) = \int_{\mathcal{S}(\omega)} f(\mathbf{At} + \mathbf{s}(\mathbf{p}(p, \boldsymbol{\rho}, \boldsymbol{\sigma}), q))\, J(\omega) \mathbf{dt}_{B(\omega)} \mathbf{d}\boldsymbol{\rho}_{U(\omega)} \mathbf{d}\boldsymbol{\sigma}_{L(\omega)}, \tag{50}$$

where

$$\begin{aligned} \mathbf{s}_{-m}(\mathbf{p}(p, \boldsymbol{\rho}, \boldsymbol{\sigma}), q) &= \mathbf{s}_{-m}(\mathbf{p}(p, \boldsymbol{\rho}, \boldsymbol{\sigma})) \\ s_m(\mathbf{p}(p, \boldsymbol{\rho}, \boldsymbol{\sigma}), q) &= q + S_{m,-n}(p) \end{aligned}$$

and

$$J(\omega) = \left| \frac{\partial \boldsymbol{\varepsilon}_{\digamma(\omega)}}{\partial\left(\left(\mathbf{t}_{\vartheta(\omega)}\right)_{B(\omega)}, \boldsymbol{\rho}_{U(\omega)}, \boldsymbol{\sigma}_{L(\omega)}\right)} \right|. \tag{51}$$

[18]Note that the remainder of the network has at least one arc that is lacking its start or end node. Also the remainder of the network is not necessarily connected.

Proof. As defined by (8), $P_{mn}(p,q,\omega)\,dq$ is the probability that firm n in node m sells between q and $q+dq$ units at price p and that the system is in congestion state ω. The calculation of $P_{mn}(p,q,\omega)$ involves determining a market outcome for every realization of the vector $\boldsymbol{\varepsilon}$, and then integrating the multivariate density function f over the volume in $\boldsymbol{\varepsilon}$-space that corresponds to events where firm n in node m sells q units at price p and that the system is in congestion state ω. In the general case this volume is complicated. Similar to Wilson (2008), we avoid this by transforming the problem into one where we instead integrate over the flows and shadow prices that arise in each congestion state. Thus we want to transform the volume in $\boldsymbol{\varepsilon}-$space into a corresponding volume in $\mathbf{t}$, $\boldsymbol{\sigma}$ and $\boldsymbol{\rho}$ space for variables that are not at a bound. To make this substitution of variables when computing the multi-dimensional integral, we need the following factor to represent the change in measure Apostol (1974).

$$J_V(\omega)=\left|\frac{\partial\boldsymbol{\varepsilon}}{\partial\left(\mathbf{t}_{B(\omega)},\boldsymbol{\rho}_{U(\omega)},\boldsymbol{\sigma}_{L(\omega)},\varepsilon_{\ell}\right)}\right|, \tag{52}$$

the absolute value of the determinant of the Jacobian matrix representing the change of variables. The node ℓ can be chosen freely, but in the following we will choose it such that $\ell=m$. It can be shown from the identity $\boldsymbol{\varepsilon}=\mathbf{A}\mathbf{t}+\mathbf{s}(\mathbf{p})$ that:

$$\begin{aligned}\frac{\partial\boldsymbol{\varepsilon}}{\partial\left(\mathbf{t}_{B(\omega)},\boldsymbol{\rho}_{U(\omega)},\boldsymbol{\sigma}_{L(\omega)},\varepsilon_m\right)} &= \begin{bmatrix} \frac{\partial\boldsymbol{\varepsilon}_{\Xi(\omega)}}{\partial\left(\mathbf{t}_{\kappa(\omega)}\right)} & \frac{\partial\boldsymbol{\varepsilon}_{\Xi(\omega)}}{\partial\left(\left(\mathbf{t}_{\vartheta(\omega)}\right)_{B(\omega)},\boldsymbol{\rho}_{U(\omega)},\boldsymbol{\sigma}_{L(\omega)}\right)} & \frac{\partial\boldsymbol{\varepsilon}_{\Xi(\omega)}}{\partial\varepsilon_m} \\ \frac{\partial\boldsymbol{\varepsilon}_{F(\omega)}}{\partial\left(\mathbf{t}_{\kappa(\omega)}\right)} & \frac{\partial\boldsymbol{\varepsilon}_{F}}{\partial\left(\left(\mathbf{t}_{\vartheta(\omega)}\right)_{B(\omega)},\boldsymbol{\rho}_{U(\omega)},\boldsymbol{\sigma}_{L(\omega)}\right)} & \frac{\partial\boldsymbol{\varepsilon}_{F(\omega)}}{\partial\varepsilon_m}\end{bmatrix} \\ &= \begin{bmatrix} \mathbf{A}_{\Lambda(\omega)} & \mathbf{0} & \frac{\partial\boldsymbol{\varepsilon}_{\Xi(\omega)}}{\partial\varepsilon_m} \\ \mathbf{0} & \frac{\partial\boldsymbol{\varepsilon}_{F}}{\partial\left(\left(\mathbf{t}_{\vartheta(\omega)}\right)_{B(\omega)},\boldsymbol{\rho}_{U(\omega)},\boldsymbol{\sigma}_{L(\omega)}\right)} & \frac{\partial\boldsymbol{\varepsilon}_{F(\omega)}}{\partial\varepsilon_m}\end{bmatrix}. \end{aligned} \tag{53}$$

Let

$$\mathbf{B}=\begin{bmatrix} \mathbf{A}_{\Lambda(\omega)} & \mathbf{0} \\ \mathbf{0} & \frac{\partial\boldsymbol{\varepsilon}_{F}}{\partial\left(\left(\mathbf{t}_{\vartheta(\omega)}\right)_{B(\omega)},\boldsymbol{\rho}_{U(\omega)},\boldsymbol{\sigma}_{L(\omega)}\right)}\end{bmatrix}. \tag{54}$$

We can expand the determinant $\left|\frac{\partial\boldsymbol{\varepsilon}}{\partial\left(\mathbf{t}_{B(\omega)},\boldsymbol{\rho}_{U(\omega)},\boldsymbol{\sigma}_{L(\omega)},\varepsilon_{\ell}\right)}\right|$ in (53) along its Mth column $\begin{bmatrix}\frac{\partial\boldsymbol{\varepsilon}_{\Xi(\omega)}}{\partial\varepsilon_m} \\ \frac{\partial\boldsymbol{\varepsilon}_{F(\omega)}}{\partial\varepsilon_m}\end{bmatrix}$, which has a one in row m and zeros in the other rows, so it follows from the definition of the determinant that:

$$\begin{aligned} J_V(\omega) &= \left|(-1)^{m+M}\det\left(\mathbf{B}_{-m}\right)\right| = \left|\det\left(\mathbf{B}_{-m}\right)\right| \\ &= \left|\left(\mathbf{A}_{\Lambda(\omega)}\right)_{-m}\right| J(\omega), \end{aligned}$$

because $\mathbf{B}_{-m}$ is a block matrix with determinant $\left|\left(\mathbf{A}_{\Lambda(\omega)}\right)_{-m}\right| J(\omega)$. $\mathbf{A}_{\Lambda(\omega)}$ is a node-arc incidence matrix of a connected radial network. Thus it follows from Bapat (2010, p. 13) that $\left|\det\left(\mathbf{A}_{\Lambda(\omega)}\right)_{-m}\right|$ is 1, which gives the stated result. ■

$J(\omega)$ can be calculated as follows.

Lemma 9 *The Jacobian matrix* $\frac{\partial \boldsymbol{\varepsilon}_{F(\omega)}}{\partial\left(\left(\mathbf{t}_{\vartheta(\omega)}\right)_{B(\omega)}, \boldsymbol{\rho}_{U(\omega)}, \boldsymbol{\sigma}_{L(\omega)}\right)}$ *can be constructed for the state* ω *from the following results for nodes* $\ell \in F(\omega)$*:*

$$\begin{aligned}
\frac{\partial \varepsilon_\ell}{\partial \rho_k} &= -S'_\ell(p_\ell)\,\mathbf{H}_{k\ell} \text{ for } k \in U(\omega) \\
\frac{\partial \varepsilon_\ell}{\partial \sigma_k} &= S'_\ell(p_\ell)\,\mathbf{H}_{k\ell} \text{ for } k \in L(\omega) \\
\frac{\partial \varepsilon_\ell}{\partial t_k} &= A_{\ell k} \text{ for } k \in B(\omega).
\end{aligned}$$

Proof. We partition the columns of $\mathbf{A}_{\backslash\Lambda(\omega)}$ into $\left(\mathbf{A}_{\backslash\Lambda(\omega)}\right)_{L(\omega)}$, $\left(\mathbf{A}_{\backslash\Lambda(\omega)}\right)_{B(\omega)}$ and $\left(\mathbf{A}_{\backslash\Lambda(\omega)}\right)_{U(\omega)}$, corresponding to flows $\mathbf{t}_\vartheta$ being at their lower bounds, strictly between their bounds, and at their upper bounds. Thus the flow balance in (2) can be written as follows

$$\left(\mathbf{A}_{\backslash\Lambda}\right)_B (\mathbf{t}_\vartheta)_B + \left(\mathbf{A}_{\backslash\Lambda}\right)_U (\mathbf{t}_\vartheta)_U + \left(\mathbf{A}_{\backslash\Lambda}\right)_L (\mathbf{t}_\vartheta)_L + \mathbf{s}_F(\mathbf{p}) = \boldsymbol{\varepsilon}_F. \tag{55}$$

Observe that (5) implies that

$$\frac{\partial \varepsilon_\ell}{\partial \rho_k} = \frac{\partial \varepsilon_\ell}{\partial p_\ell}\frac{\partial p_\ell}{\partial \rho_k} = -S'_\ell(p_\ell)\,\mathbf{H}_{k\ell} \text{ for } \ell \in F(\omega) \text{ and } k \in U(\omega)$$

and

$$\frac{\partial \varepsilon_\ell}{\partial \sigma_k} = \frac{\partial \varepsilon_\ell}{\partial p_\ell}\frac{\partial p_\ell}{\partial \sigma_k} = S'_\ell(p_\ell)\,\mathbf{H}_{k\ell} \text{ for } \ell \in F(\omega) \text{ and } k \in L(\omega).$$

Moreover,

$$\frac{\partial \varepsilon_\ell}{\partial t_k} = A_{\ell k} \text{ for } \ell \in F(\omega) \text{ and } k \in B(\omega),$$

which gives the result. ■

Appendix B: Proofs for the two-node network

Proof. (Lemma 4). Below we list the congestion states of the network and how we partition the nodes for each state:

State	t	ρ	σ	Ξ	$F(\omega)$
ω_1	$\in(-\bar{t},\bar{t})$	0	0	$\{1,2\}$	$\emptyset$
ω_2	$\bar{t}$	$\in(0,\infty)$	0	$\{1\}$	$\{2\}$
ω_3	$-\bar{t}$	0	$\in(0,\infty)$	$\{1\}$	$\{2\}$

We have from (2) that

$$\begin{bmatrix}\varepsilon_1\\ \varepsilon_2\end{bmatrix} = \underbrace{\begin{bmatrix}S_1(p_1)\\ S_2(p_2)\end{bmatrix}}_{\mathbf{s}(\mathbf{p})} + \underbrace{\begin{bmatrix}-1\\ 1\end{bmatrix}}_{\mathbf{A}} t \tag{56}$$

and from (4) that

$$\mathbf{H} = -\left(\mathbf{A_{-1}}\right)^{-\mathbf{1}} = -\,1. \tag{57}$$

Thus it follows from (5) that

$$p_2 = p + \rho - \sigma. \tag{58}$$

The network is completely integrated in state ω_1, so $\boldsymbol{\varepsilon}_{F(\omega_1)}$ is empty. We only need the substitution factor $J(\omega)$ for states ω_2 and ω_3. It follows from Lemma 9, (51) and (57) that

$$J(\omega_2) = \left|\frac{\partial \varepsilon_2}{\partial \rho}\right| = S_2'(p_2) = S_2'(p+\rho)$$
$$J(\omega_3) = \left|\frac{\partial \varepsilon_2}{\partial \sigma}\right| = \left|-S_2'(p_2)\right| = S_2'(p-\sigma).$$

(50) and (56) now yields:

$$P(p,q,\omega_1) = \int_{-\bar{t}}^{\bar{t}} f(\mathbf{A}t+\mathbf{s}(\mathbf{p},q))\,dt = \int_{-\bar{t}}^{\bar{t}} f(q+S_{1,-n}(p)-t, S_2(p)+t)\,dt,$$

$$P(p,q,\omega_2) = \int_0^\infty f(\mathbf{A}t+\mathbf{s}(\mathbf{p},q))\,J(\omega_2)d\rho$$
$$= \int_0^\infty f\left(q+S_{1,-n}(p)-\bar{t}, S_2(p+\rho)+\bar{t}\right)S_2'(p+\rho)\,d\rho$$

and

$$P(p,q,\omega_3) = \int_0^\infty f(\mathbf{A}t+\mathbf{s}(\mathbf{p},q))\,J(\omega_3)d\sigma$$
$$= \int_0^\infty f\left(q+S_{1,-n}(p)+\bar{t}, S_2(p-\sigma)-\bar{t}\right)S_2'(p-\sigma)\,d\sigma.$$

This gives us (19) after the substitutions $\varepsilon_2 = S_2(p+\rho)+\bar{t}$ and $\varepsilon_2 = S_2(p-\sigma)-\bar{t}$, respectively, have been applied to the integrals of the states ω_2 and ω_3. The equation (18) follows from (11) and that the two nodes are only completely integrated in state ω_1. ■

Proof. (Proposition 2). Symmetry of the network, costs and shock densities ensure that the optimal supply functions of all producers are given by identical optimality conditions. We have $S_2(p) = q + S_{1,-n}(p) = NQ(p)$ in a symmetric equilibrium with inelastic demand, so (22) follows from (19). The differential equation in the statement follows from Lemma 2 and Lemma 3. In case that production capacity would bind at some price $p_b < \overline{p}$ then $Q(p)$ is inelastic in the range $(p_b, \overline{p})$, and it follows from (18) that $Z(p,q) < 0$ when $q \in (0, \overline{q})$ and $p \in (p_b, \overline{p})$. This would violate the second-order condition in (13), and it is necessary that this condition is locally satisfied (Anderson and Philpott, 2002a). Thus the production capacity must bind at the reservation price, which gives our initial condition.

Next we show that the symmetric solution is unique. It follows from the assumptions for $f(\varepsilon_1, \varepsilon_2)$, our definition of $\tilde{P}(NQ,\omega)$ and from (21) that

$$\frac{1}{(N\tilde{\mu}(NQ)-1)} > 0,$$

and that $\frac{1}{(N\tilde{\mu}(NQ)-1)}$ is Lipschitz continuous in Q. Consider a price $\tilde{p} \in (C'(0), \overline{p})$. We now want to show that $p - C'(Q(p))$ is bounded away from zero in the range $[\tilde{p}, \overline{p}]$. This is obvious for constant marginal costs, as we then have that $\tilde{p} -$

$C'(Q(\tilde{p})) = \tilde{p} - C'(0) > 0$. For strictly increasing marginal costs we can use the following argument. It follows from Picard-Lindelöf's theorem and $\overline{p} > C'(\overline{q})$ that a unique solution to (20) must exist for some range $[p_0, \overline{p}]$. In this price range the mark-up, $p - C'(Q(p))$, is smallest at some price p^* where the inverse supply function is at least as steep as the marginal cost curve, i.e. $Q'(p^*) \leq \frac{1}{C''(Q(p^*))}$. Thus it follows from (20) that

$$p^* - C'(Q(p^*)) \geq \frac{Q(p^*)C''(Q(p^*))}{(N\tilde{\mu}(NQ(p^*))-1)}.$$

This is bounded away from zero whenever $Q(p^*)$ is bounded away from zero if marginal costs are strictly increasing. In case $Q(p^*) = 0$ for some price $p^* > C'(0)$, it follows from (20) that $Q'(p) = 0$ for $p \in (\tilde{p}, p^*)$. Thus it follows from Picard-Lindelöf's theorem and the properties of (20) that a unique monotonic symmetric solution will exist for the price interval $[\tilde{p}, \overline{p}]$. We can repeat the argument for any $\tilde{p} \in (C'(0), \overline{p})$ to show that a unique monotonic symmetric solution will exist for the price interval $(C'(0), \overline{p}]$.

We now verify the global second-order conditions. To simplify notation let

$$\beta(p,q) = (2N-1) P(p,q,\omega_1) + (N-1) P(p,q,\omega_2) + (N-1) P(p,q,\omega_3). \quad (59)$$

We also use (7), so that:

$$P(p,q) = P(p,q,\omega_1) + P(p,q,\omega_2) + P(p,q,\omega_3). \quad (60)$$

We have from (9) and (18) that

$$Z(p,q) = \frac{(p - C'(q))\beta(p,q)Q'(p)}{P(p,q)} - q.$$

We also have $C'' \geq 0$ and $Q'(p) \geq 0$, so

$$Z_q \leq \frac{(p - C'(q))\beta_q Q'P - (p - C'(q))\beta Q' P_q}{(P(p,q))^2} - 1.$$

In particular, whenever $Z(p,q) = 0$, we have

$$Z_q \leq \frac{qP\beta_q - \beta P - q\beta P_q}{\beta P}.$$

We know from (13) that the solution is an equilibrium if $Z(p,q) \geq 0$ when $q \leq Q(p)$ and $Z(p,q) \leq 0$ when $q \geq Q(p)$. This follows if $Z_q(p,q) \leq 0$ whenever $Z(p,q) = 0$. To verify this sufficiency condition, it suffices to show that

$$\beta(p,q)P(p,q) + q\beta(p,q)P_q(p,q) - qP(p,q)\beta_q(p,q) \geq 0. \quad (61)$$

To show this observe that the assumption

$$2N\overline{q}\,|f_m(\varepsilon_1,\varepsilon_2)| \leq (3N-2) f(\varepsilon_1,\varepsilon_2)$$

implies from (19) that

$$\begin{aligned}
2Nq\left|P_q(p,q,\omega_1)\right| &= 2Nq\left|\int_{-\bar{t}}^{\bar{t}}\frac{\partial}{\partial q}f\left(q+S_{1,-n}(p)-t,S_2(p)+t\right)dt\right| \\
&\leq 2N\overline{q}\left|\int_{-\bar{t}}^{\bar{t}}\frac{\partial}{\partial q}f\left(q+S_{1,-n}(p)-t,S_2(p)+t\right)dt\right| \\
&\leq \int_{-\bar{t}}^{\bar{t}}2N\overline{q}\left|f_1\left(q+S_{1,-n}(p)-t,S_2(p)+t\right)\right|dt \\
&\leq (3N-2)\int_{-\bar{t}}^{\bar{t}}f\left(q+S_{1,-n}(p)-t,S_2(p)+t\right)dt \\
&= (3N-2)\,P(p,q,\omega_1)\,.
\end{aligned}$$

Similarly $2Nq\left|P_q(p,q,\omega_3)\right| \leq (3N-2)\,P(p,q,\omega_3)$ and $2Nq\left|P_q(p,q,\omega_2)\right| \leq (3N-2)\,P(p,q,\omega_2)$. It follows from (60) and (59) that

$$\begin{aligned}
& q\beta(p,q)P_q(p,q) - qP(p,q)\beta_q(p,q) \\
= \;& qN\left(P(p,q,\omega_1)\left(P_q(p,q,\omega_2)+P_q(p,q,\omega_3)\right) - qNP_q(p,q,\omega_1)\left(P(p,q,\omega_2)+P(p,q,\omega_3)\right)\right) \\
\geq \;& -(3N-2)P(p,q,\omega_1)\left(P(p,q,\omega_2)+P(p,q,\omega_3)\right).
\end{aligned}$$

It can be deduced from (60) and (59) that

$$\beta(p,q)P(p,q) \geq (3N-2)P(p,q,\omega_1)\left(P(p,q,\omega_2)+P(p,q,\omega_3)\right).$$

Thus (61) is satisfied, which is sufficient for an equilibrium. ■

Proof. (Proposition 3) It follows from the definitions of $P(p,q,\omega_1)$, $P(p,q,\omega_2)$ and $P(p,q,\omega_3)$ in (19) that under Assumption 1 we get:

$$\begin{aligned}
P(p,q,\omega_1) &= \int_{-\bar{t}}^{\bar{t}} f\left(q+S_{1,-n}(p)-t,S_2(p)+t\right)dt = \int_{-\bar{t}}^{\bar{t}}\tfrac{dt}{V_1} = \tfrac{2\bar{t}}{V_1} \\
P(p,q,\omega_2) &= \int_{S_2(p)+\bar{t}}^{\infty} f\left(q+S_{1,-n}(p)-\bar{t},\varepsilon_2\right)d\varepsilon_2 = \int_{S_2(p)+\bar{t}}^{N\overline{q}+\bar{t}}\tfrac{d\varepsilon_2}{V_1} = \tfrac{N\overline{q}-S_2(p)}{V_1} \qquad (62)\\
P(p,q,\omega_3) &= \int_{-\infty}^{S_2(p)-\bar{t}} f\left(q+S_{1,-n}(p)+\bar{t},\varepsilon_2\right)d\varepsilon_2 = \int_{-\bar{t}}^{S_2(p)-\bar{t}}\tfrac{d\varepsilon_2}{V_1} = \tfrac{S_2(p)}{V_1}.
\end{aligned}$$

(23) now follows from (21). For constant μ, we note the similarities between (15) and the first-order condition for single-node networks with $\widetilde{N}$ symmetric firms (Klemperer and Meyer, 1989).

$$Q = (p - C'(Q))\left(\widetilde{N}-1\right)Q'. \qquad (63)$$

By comparing (15) and (63) we can conclude that the first-order solution of a firm in a symmetric two-node network with N firms per node is the same as for a firm in an isolated node with inelastic demand and μN symmetric firms. Thus analytical solutions to (63), derived by Anderson and Philpott (2002b) and Rudkevich et al. (1998), are also solutions to (15) when $\widetilde{N} = \mu N$, which gives us (24). We also know that such solutions are monotonic (Holmberg, 2008). It follows from our assumptions and Proposition 2 that this is a supply-function equilibrium. ■

Appendix C: Proof for star network

Proof. (Proposition 4) Local net-imports must equal net-demand in every node, so

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \end{bmatrix} = \underbrace{\begin{bmatrix} S_1(p_1) \\ S_2(p_2) \\ S_3(p_3) \\ S_4(p_4) \end{bmatrix}}_{\mathbf{s}(\mathbf{p})} + \underbrace{\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} t_1(\mathbf{p}) \\ t_2(\mathbf{p}) \\ t_3(\mathbf{p}) \end{bmatrix}}_{\mathbf{t}}. \tag{64}$$

We derive the optimal supply function for a producer in node 1, so we choose this node to be the slack node and trading hub. Thus

$$\mathbf{A}_{-1} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} \tag{65}$$

and we have from (4) that

$$\mathbf{H} = -\left(\mathbf{A}_{-1}\right)^{-1} = \begin{bmatrix} -1 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \tag{66}$$

so it follows from (5) that

$$\mathbf{p}_{-1} = p \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} (\boldsymbol{\rho} - \boldsymbol{\sigma}). \tag{67}$$

Each arc k has three states. In the uncongested state we have $\sigma_k = 0$, $\rho_k = 0$ and $t_k \in (-\bar{t}, \bar{t})$. When the arc is congested towards node 4 we have $t_k = \bar{t}$, $\sigma_k = 0$, and $\rho_k > 0$ and when the arc is congested away from node 4 we have $t_k = -\bar{t}$, $\sigma_k > 0$, and $\rho_k = 0$. Altogether there are $3 \times 3 \times 3 = 27$ congestion states. In Appendix C.1, we use (50) to calculate $P(p, q, \omega)$ for one state ω at a time. The results are summarized in Table 1. Each competitor is assumed to submit a symmetric offer $Q(p)$, so $S_2(p) \equiv S_3(p) \equiv S(p) := NQ(p)$. Adding the results in Table 1 yields:

$$\sum_{\omega} P(p, q, \omega) = \frac{6\bar{t}S^2(\overline{p})}{V_2} + \frac{16\bar{t}^2 S(\overline{p})}{V_2} + \frac{8\bar{t}^3}{V_2}. \tag{68}$$

Node 1 is completely integrated with either node 2 or 3 in states ω_{15}, ω_{17}, ω_{26}, ω_{27} and completely integrated with both nodes in state ω_{18}. In the other states node 1 is either isolated or only completely integrated with node 4, which does not have any producers and where demand is inelastic. We have

$$\begin{aligned} &P(p, q, \omega_{15}) + P(p, q, \omega_{17}) + P(p, q, \omega_{26}) + P(p, q, \omega_{27}) + 2P(p, q, \omega_{18}) \\ &= \tfrac{4\bar{t}^2 S(p)}{V_2} + \tfrac{4\bar{t}^2(S(\overline{p}) - S(p))}{V_2} + \tfrac{4\bar{t}^2 S(p)}{V_2} + \tfrac{4\bar{t}^2(S(\overline{p}) - S(p))}{V_2} + \tfrac{16\bar{t}^3}{V_2} = \tfrac{8\bar{t}^2 S(\overline{p}) + 16\bar{t}^3}{V_2}, \end{aligned} \tag{69}$$

Table 1: The 27 congestion states of the star network.

State	$t_1(\omega)$	$t_2(\omega)$	$t_3(\omega)$	$P(p,q,\omega)$
ω_1	$\bar{t}$	$\bar{t}$	$\bar{t}$	0
ω_2	$\bar{t}$	$\bar{t}$	$-\bar{t}$	0
ω_3	$\bar{t}$	$\bar{t}$	$\in\left(-\bar{t},\bar{t}\right)$	$\frac{\bar{t}\left(S^2(\overline{p})-S^2(p)\right)}{V_2}$
ω_4	$\bar{t}$	$-\bar{t}$	$-\bar{t}$	0
ω_5	$\bar{t}$	$-\bar{t}$	$\in\left(-\bar{t},\bar{t}\right)$	$\frac{\bar{t}(S(\overline{p})-S(p))^2}{V_2}$
ω_6	$\bar{t}$	$\in\left(-\bar{t},\bar{t}\right)$	$\in\left(-\bar{t},\bar{t}\right)$	$\frac{8\bar{t}^2(S(\overline{p})-S(p))}{V_2}$
ω_7	$-\bar{t}$	$\bar{t}$	$\bar{t}$	0
ω_8	$-\bar{t}$	$\bar{t}$	$-\bar{t}$	0
ω_9	$-\bar{t}$	$\bar{t}$	$\in\left(-\bar{t},\bar{t}\right)$	$\frac{\bar{t}S^2(p)}{V_2}$
ω_{10}	$-\bar{t}$	$-\bar{t}$	$-\bar{t}$	0
ω_{11}	$-\bar{t}$	$-\bar{t}$	$\in\left(-\bar{t},\bar{t}\right)$	$\frac{\bar{t}S(p)(2S(\overline{p})-S(p))}{V_2}$
ω_{12}	$-\bar{t}$	$\in\left(-\bar{t},\bar{t}\right)$	$\in\left(-\bar{t},\bar{t}\right)$	$\frac{8\bar{t}^2S(p)}{V_2}$
ω_{13}	$\in\left(-\bar{t},\bar{t}\right)$	$\bar{t}$	$\bar{t}$	$\frac{2\bar{t}S^2(p)}{V_2}$
ω_{14}	$\in\left(-\bar{t},\bar{t}\right)$	$\bar{t}$	$-\bar{t}$	$\frac{2\bar{t}S(p)(S(\overline{p})-S(p))}{V_2}$
ω_{15}	$\in\left(-\bar{t},\bar{t}\right)$	$\bar{t}$	$\in\left(-\bar{t},\bar{t}\right)$	$\frac{4\bar{t}^2S(p)}{V_2}$
ω_{16}	$\in\left(-\bar{t},\bar{t}\right)$	$-\bar{t}$	$-\bar{t}$	$\frac{2\bar{t}(S(\overline{p})-S(p))^2}{V_2}$
ω_{17}	$\in\left(-\bar{t},\bar{t}\right)$	$-\bar{t}$	$\in\left(-\bar{t},\bar{t}\right)$	$\frac{4\bar{t}^2(S(\overline{p})-S(p))}{V_2}$
ω_{18}	$\in\left(-\bar{t},\bar{t}\right)$	$\in\left(-\bar{t},\bar{t}\right)$	$\in\left(-\bar{t},\bar{t}\right)$	$\frac{8\bar{t}^3}{V_2}$
ω_{19}	$\bar{t}$	$-\bar{t}$	$\bar{t}$	0
ω_{20}	$\bar{t}$	$\in\left(-\bar{t},\bar{t}\right)$	$\bar{t}$	$\frac{\bar{t}\left(S^2(\overline{p})-S^2(p)\right)}{V_2}$
ω_{21}	$\bar{t}$	$\in\left(-\bar{t},\bar{t}\right)$	$-\bar{t}$	$\frac{\bar{t}(S(\overline{p})-S(p))^2}{V_2}$
ω_{22}	$-\bar{t}$	$-\bar{t}$	$\bar{t}$	0
ω_{23}	$-\bar{t}$	$\in\left(-\bar{t},\bar{t}\right)$	$\bar{t}$	$\frac{\bar{t}S^2(p)}{V_2}$
ω_{24}	$-\bar{t}$	$\in\left(-\bar{t},\bar{t}\right)$	$-\bar{t}$	$\frac{\bar{t}S(p)(2S(\overline{p})-S(p))}{V_2}$
ω_{25}	$\in\left(-\bar{t},\bar{t}\right)$	$-\bar{t}$	$\bar{t}$	$\frac{2\bar{t}S(p)(S(\overline{p})-S(p))}{V_2}$
ω_{26}	$\in\left(-\bar{t},\bar{t}\right)$	$\in\left(-\bar{t},\bar{t}\right)$	$\bar{t}$	$\frac{4\bar{t}^2S(p)}{V_2}$
ω_{27}	$\in\left(-\bar{t},\bar{t}\right)$	$\in\left(-\bar{t},\bar{t}\right)$	$-\bar{t}$	$\frac{4\bar{t}^2(S(\overline{p})-S(p))}{V_2}$

and

$$
\begin{array}{l}
\hat{P}\left(\omega_{15}|p,q\right)+\hat{P}\left(\omega_{17}|p,q\right)+\hat{P}\left(\omega_{26}|p,q\right)+\hat{P}\left(\omega_{27}|p,q\right)+2\hat{P}\left(\omega_{18}|p,q\right)\\
=\frac{P(p,q,\omega_{15})+P(p,q,\omega_{17})+P(p,q,\omega_{26})+P(p,q,\omega_{27})+2P(p,q,\omega_{18})}{\sum_{\omega}P(p,q,\omega)}=\frac{4\bar{t}S(\overline{p})+8\bar{t}^2}{3S^2(\overline{p})+8\bar{t}S(\overline{p})+4\bar{t}^2}.
\end{array}
\tag{70}
$$

This gives (33), because $S\left(\overline{p}\right):=N\overline{q}$, and

$$
\mu=1+\hat{P}\left(\omega_{15}|p,q\right)+\hat{P}\left(\omega_{17}|p,q\right)+\hat{P}\left(\omega_{26}|p,q\right)+\hat{P}\left(\omega_{27}|p,q\right)+2\hat{P}\left(\omega_{18}|p,q\right).
$$

It follows from (11), (68) and (69) that

$$
Z\left(p,q\right)=\frac{\tilde{Z}\left(p,q\right)}{\frac{2\bar{t}}{V_2}\left[3S^2\left(\overline{p}\right)+8\bar{t}S\left(\overline{p}\right)+4\bar{t}^2\right]}
$$

where

$$
\begin{array}{l}
\tilde{Z}\left(p,q\right)=\left(p-C'(q)\right)\left(S'_{1,-i}\left(p\right)\left(\frac{6\bar{t}S^2(\overline{p})}{V_2}+\frac{16\bar{t}^2S(\overline{p})}{V_2}\right)+\frac{8\bar{t}^2S(\overline{p})+16\bar{t}^3}{V_2}S'\left(p\right)\right)\\
\qquad -q\frac{2\bar{t}}{V_2}\left[3S^2\left(\overline{p}\right)+8\bar{t}S\left(\overline{p}\right)+4\bar{t}^2\right].
\end{array}
$$

We note that $\frac{\partial Z(p,q)}{\partial q}=\frac{\partial\tilde{Z}(p,q)}{\partial q}\leq 0$, so if we find a monotonic stationary solution, then it is an equilibrium. The explicit equilibrium expression and monotonicity of this solution can be established as in the proof of Proposition 3. ■

Appendix C.1: Calculations for congestion states in star network

Viewing the network from node 1 we realize that it is symmetric with respect to nodes 2 and 3, and it is sufficient to make these calculations for the first eighteen states. Results for the last nine states follow from symmetry of the problem.

State ω_1

State	$t_1(\omega)$	$t_2(\omega)$	$t_3(\omega)$	$L(\omega)$	$B(\omega)$	$U(\omega)$	$\Xi(\omega)$	$F(\omega)$
ω_1	$\bar{t}$	$\bar{t}$	$\bar{t}$	$\varnothing$	$\varnothing$	$\{1,2,3\}$	$\{1\}$	$\{2,3,4\}$

In this state we have from (66) and Lemma 9 that

$$
J(\omega_1)=\left|\frac{\partial\boldsymbol{\varepsilon}_{F(\omega)}}{\partial(\left(\mathbf{t}_{\vartheta(\omega)}\right)_{B(\omega)},\boldsymbol{\rho}_{U(\omega)},\boldsymbol{\sigma}_{L(\omega)})}\right|=\left|\det\begin{bmatrix}S'_2\left(p_2\right)&-S'_2\left(p_2\right)&0\\S'_3\left(p_3\right)&0&-S'_3\left(p_3\right)\\S'_4\left(p_4\right)&0&0\end{bmatrix}\right|=0,
$$

because $S'_4\left(p_4\right)=0$. Now, we have from (50) that

$$
P\left(p,q,\omega_1\right)=0.
$$

State ω_2

State	$t_1(\omega)$	$t_2(\omega)$	$t_3(\omega)$	$L(\omega)$	$B(\omega)$	$U(\omega)$	$\Xi(\omega)$	$F(\omega)$
ω_2	$\bar{t}$	$\bar{t}$	$-\bar{t}$	$\{3\}$	$\varnothing$	$\{1,2\}$	$\{1\}$	$\{2,3,4\}$

With similar calculations as for state ω_1, one gets

$$P(p,q,\omega_2) = 0.$$

State ω_3

State	$t_1(\omega)$	$t_2(\omega)$	$t_3(\omega)$	$L(\omega)$	$B(\omega)$	$U(\omega)$	$\Xi(\omega)$	$F(\omega)$
ω_3	$\bar{t}$	$\bar{t}$	$\in\left(-\bar{t},\bar{t}\right)$	$\varnothing$	$\{3\}$	$\{1,2\}$	$\{1\}$	$\{2,3,4\}$

In this state we have from (66), (65) and Lemma 9 that:

$$\begin{aligned} J(\omega_3) &= \left|\frac{\partial \boldsymbol{\varepsilon}_{F(\omega)}}{\partial\left(\left(\mathbf{t}_{\vartheta(\omega)}\right)_{B(\omega)}, \boldsymbol{\rho}_{U(\omega)}, \boldsymbol{\sigma}_{L(\omega)}\right)}\right| = \left|\det\begin{bmatrix} 0 & S_2'(p_2) & -S_2'(p_2) \\ -1 & S_3'(p_3) & 0 \\ 1 & S_4'(p_4) & 0 \end{bmatrix}\right| \\ &= S_3'(p_3)\, S_2'(p_2), \end{aligned}$$

because $S_4'(p_4) = 0$. Now, we have from (50) that

$$\begin{aligned} P(p,q,\omega_3) &= \int_{\rho_1=0}^{\overline{p}-p}\int_{\rho_2=0}^{p+\rho_1}\int_{t_3=-\bar{t}}^{\bar{t}} f\left(\mathbf{At}+\mathbf{s}\left(\mathbf{p}(p,\boldsymbol{\rho},\boldsymbol{\sigma})\right)\right) S_3'(p_3)\, S_2'(p_2)\, dt_3 d\rho_2 d\rho_1 \\ &= \frac{2\bar{t}}{V_2}\int_{\rho_1=0}^{\overline{p}-p} S_3'(p+\rho_1)\int_{\rho_2=0}^{p+\rho_1} S_2'(p+\rho_1-\rho_2)\, d\rho_2 d\rho_1 \\ &= \frac{2\bar{t}}{V_2}\int_{\rho_1=0}^{\overline{p}-p} S_3'(p+\rho_1)\, S_2(p+\rho_1)\, d\rho_1 \\ &= \frac{\bar{t}\left(S^2(\overline{p}) - S^2(p)\right)}{V_2}, \end{aligned}$$

where we have used that $S_2(p) = S_3(p) = S(p)$.

State ω_4

State	$t_1(\omega)$	$t_2(\omega)$	$t_3(\omega)$	$L(\omega)$	$B(\omega)$	$U(\omega)$	$\Xi(\omega)$	$F(\omega)$
ω_4	$\bar{t}$	$-\bar{t}$	$-\bar{t}$	$\{2,3\}$	$\varnothing$	$\{1\}$	$\{1\}$	$\{2,3,4\}$

With similar calculations as for state ω_1, one gets

$$P(p,q,\omega_4) = 0.$$

State ω_5

State	$t_1(\omega)$	$t_2(\omega)$	$t_3(\omega)$	$L(\omega)$	$B(\omega)$	$U(\omega)$	$\Xi(\omega)$	$F(\omega)$
ω_5	$\bar{t}$	$-\bar{t}$	$\in\left(-\bar{t},\bar{t}\right)$	$\{2\}$	$\{3\}$	$\{1\}$	$\{1\}$	$\{2,3,4\}$

In this state we have from (66), (65) and Lemma 9 that:

$$\begin{aligned} J(\omega_5) &= \left|\frac{\partial \boldsymbol{\varepsilon}_{F(\omega)}}{\partial(\left(\mathbf{t}_{\vartheta(\omega)}\right)_{B(\omega)}, \boldsymbol{\rho}_{U(\omega)}, \boldsymbol{\sigma}_{L(\omega)})}\right| = \left|\det\begin{bmatrix} 0 & S_2'(p_2) & S_2'(p_2) \\ -1 & S_3'(p_3) & 0 \\ 1 & S_4'(p_4) & 0 \end{bmatrix}\right| \\ &= S_2'(p_2)\, S_3'(p_3), \end{aligned}$$

because $S_4'(p_4)=0$. Now, we have from (50) that

$$\begin{aligned} P(p,q,\omega_5) &= \int_{\rho_1=0}^{\overline{p}-p}\int_{\sigma_2=0}^{\overline{p}-p-\rho_1}\int_{t_3=-\bar{t}}^{\bar{t}} f\left(\mathbf{A}\mathbf{t}+\mathbf{s}\left(\mathbf{p}(p,\boldsymbol{\rho},\boldsymbol{\sigma})\right)\right) S_2'(p_2)\, S_3'(p_3)\, dt_3 d\sigma_2 d\rho_1 \\ &= \frac{2\bar{t}}{V_2}\int_{\rho_1=0}^{\overline{p}-p} S_3'(p+\rho_1)\int_{\sigma_2=0}^{\overline{p}-p-\rho_1} S_2'(p+\rho_1+\sigma_2)\, d\sigma_2 d\rho_1 \\ &= \frac{2\bar{t}}{V_2}\int_{\rho_1=0}^{\overline{p}-p} S'(p+\rho_1)\left[S_2(\overline{p})-S_2(p+\rho_1)\right] d\rho_1 \\ &= \frac{\bar{t}}{V_2}\left[2S(\overline{p})\,S(p+\rho_1)-S^2(p+\rho_1)\right]_0^{\overline{p}-p} \\ &= \frac{\bar{t}\left(S(\overline{p})-S(p)\right)^2}{V_2}, \end{aligned}$$

where we have used that $S_2(p)=S_3(p)=S(p)$.

State ω_6

State	$t_1(\omega)$	$t_2(\omega)$	$t_3(\omega)$	$L(\omega)$	$B(\omega)$	$U(\omega)$	$\Xi(\omega)$	$F(\omega)$
ω_6	$\bar{t}$	$\in\left(-\bar{t},\bar{t}\right)$	$\in\left(-\bar{t},\bar{t}\right)$	$\varnothing$	$\{2,3\}$	$\{1\}$	$\{1\}$	$\{2,3,4\}$

In this state we have from (66), (65) and Lemma 9 that:

$$\begin{aligned} J(\omega_6) &= \left|\frac{\partial \boldsymbol{\varepsilon}_{F(\omega)}}{\partial(\left(\mathbf{t}_{\vartheta(\omega)}\right)_{B(\omega)}, \boldsymbol{\rho}_{U(\omega)}, \boldsymbol{\sigma}_{L(\omega)})}\right| = \left|\det\begin{bmatrix} -1 & 0 & S_2'(p_2) \\ 0 & -1 & S_3'(p_3) \\ 1 & 1 & S_4'(p_4) \end{bmatrix}\right| \\ &= S_2'(p_2)+S_3'(p_3), \end{aligned}$$

because $S_4'(p_4)=0$. Now, we have from (50) that

$$\begin{aligned} P(p,q,\omega_6) &= \int_{\rho_1=0}^{\overline{p}-p}\int_{t_3=-\bar{t}}^{\bar{t}}\int_{t_2=-\bar{t}}^{\bar{t}} f\left(\mathbf{A}\mathbf{t}+\mathbf{s}\left(\mathbf{p}(p,\boldsymbol{\rho},\boldsymbol{\sigma})\right)\right)\left(S_2'(p_2)+S_3'(p_3)\right) dt_2 dt_3 d\rho_1 \\ &= \frac{4\bar{t}^2}{V_2}\int_{\rho_1=0}^{\overline{p}-p} 2S'(p+\rho_1)\, d\rho_1 \\ &= \frac{8\bar{t}^2\left(S(\overline{p})-S(p)\right)}{V_2}, \end{aligned}$$

where we have used that $S_2(p) = S_3(p) = S(p)$.

State ω_7

State	$t_1(\omega)$	$t_2(\omega)$	$t_3(\omega)$	$L(\omega)$	$B(\omega)$	$U(\omega)$	$\Xi(\omega)$	$F(\omega)$
ω_7	$-\bar{t}$	$\bar{t}$	$\bar{t}$	$\{1\}$	$\varnothing$	$\{2,3\}$	$\{1\}$	$\{2,3,4\}$

With similar calculations as for state ω_1, one gets

$$P(p,q,\omega_7) = 0.$$

State ω_8

State	$t_1(\omega)$	$t_2(\omega)$	$t_3(\omega)$	$L(\omega)$	$B(\omega)$	$U(\omega)$	$\Xi(\omega)$	$F(\omega)$
ω_8	$-\bar{t}$	$\bar{t}$	$-\bar{t}$	$\{1,3\}$	$\varnothing$	$\{2\}$	$\{1\}$	$\{2,3,4\}$

With similar calculations as for state ω_1, one gets

$$P(p,q,\omega_8) = 0.$$

State ω_9

State	$t_1(\omega)$	$t_2(\omega)$	$t_3(\omega)$	$L(\omega)$	$B(\omega)$	$U(\omega)$	$\Xi(\omega)$	$F(\omega)$
ω_9	$-\bar{t}$	$\bar{t}$	$\in(-\bar{t},\bar{t})$	$\{1\}$	$\{3\}$	$\{2\}$	$\{1\}$	$\{2,3,4\}$

In this state we have from (66), (65) and Lemma 9 that:

$$\begin{aligned} J(\omega_9) &= \left|\frac{\partial \boldsymbol{\varepsilon}_{F(\omega)}}{\partial\left(\left(\mathbf{t}_{\vartheta(\omega)}\right)_{B(\omega)}, \boldsymbol{\rho}_{U(\omega)}, \boldsymbol{\sigma}_{L(\omega)}\right)}\right| = \left|\det\begin{bmatrix} 0 & -S_2'(p_2) & -S_2'(p_2) \\ -1 & 0 & -S_3'(p_3) \\ 1 & 0 & -S_4'(p_4)\end{bmatrix}\right| \\ &= S_3'(p_3)\, S_2'(p_2), \end{aligned}$$

because $S_4'(p_4) = 0$. Now, we have from (50) that

$$\begin{aligned} P(p,q,\omega_9) &= \int_{\sigma_1=0}^{p}\int_{\rho_2=0}^{p-\sigma_1}\int_{t_3=-\bar{t}}^{\bar{t}} f\left(\mathbf{At} + \mathbf{s}\left(\mathbf{p}(p,\boldsymbol{\rho},\boldsymbol{\sigma})\right)\right) S_3'(p_3)\, S_2'(p_2)\, dt_3 d\rho_2 d\sigma_1 \\ &= \frac{2\bar{t}}{V_2}\int_{\sigma_1=0}^{p} S_3'(p-\sigma_1)\int_{\rho_2=0}^{p-\sigma_1} S_2'(p-\sigma_1-\rho_2)\, d\rho_2 d\sigma_1 \\ &= \frac{\bar{t}}{V_2}\int_{\sigma_1=0}^{p} 2S'(p-\sigma_1)\, S(p-\sigma_1)\, d\sigma_1 \\ &= \frac{\bar{t}S^2(p)}{V_2}, \end{aligned}$$

where we have used that $S_2(p) = S_3(p) = S(p)$.

State ω_{10}

State	$t_1(\omega)$	$t_2(\omega)$	$t_3(\omega)$	$L(\omega)$	$B(\omega)$	$U(\omega)$	$\Xi(\omega)$	$F(\omega)$
ω_{10}	$-\bar{t}$	$-\bar{t}$	$-\bar{t}$	$\{1,2,3\}$	$\varnothing$	$\varnothing$	$\{1\}$	$\{2,3,4\}$

With similar calculations as for state ω_1, one gets

$$P(p,q,\omega_{10}) = 0.$$

State ω_{11}

State	$t_1(\omega)$	$t_2(\omega)$	$t_3(\omega)$	$L(\omega)$	$B(\omega)$	$U(\omega)$	$\Xi(\omega)$	$F(\omega)$
ω_{11}	$-\bar{t}$	$-\bar{t}$	$\in(-\bar{t},\bar{t})$	$\{1,2\}$	$\{3\}$	$\varnothing$	$\{1\}$	$\{2,3,4\}$

In this state we have from (66), (65) and Lemma 9 that:

$$\begin{aligned} J(\omega_{11}) &= \left|\frac{\partial \boldsymbol{\varepsilon}_{F(\omega)}}{\partial(\left(\mathbf{t}_{\vartheta(\omega)}\right)_{B(\omega)}, \boldsymbol{\rho}_{U(\omega)}, \boldsymbol{\sigma}_{L(\omega)})}\right| = \left|\det\begin{bmatrix} 0 & -S_2'(p_2) & S_2'(p_2) \\ -1 & -S_3'(p_3) & 0 \\ 1 & -S_4'(p_4) & 0 \end{bmatrix}\right| \\ &= S_2'(p_2)\,S_3'(p_3), \end{aligned}$$

because $S_4'(p_4) = 0$. Now, we have from (50) that

$$\begin{aligned} P(p,q,\omega_{11}) &= \int_{\sigma_1=0}^{p}\int_{\sigma_2=0}^{\overline{p}-p+\sigma_1}\int_{t_3=-\bar{t}}^{\bar{t}} f\left(\mathbf{At}+\mathbf{s}\left(\mathbf{p}(p,\boldsymbol{\rho},\boldsymbol{\sigma})\right)\right) S_2'(p_2)\,S_3'(p_3)\,dt_3 d\sigma_2 d\sigma_1 \\ &= \frac{2\bar{t}}{V_2}\int_{\sigma_1=0}^{p} S_3'(p-\sigma_1)\int_{\sigma_2=0}^{\overline{p}-p+\sigma_1} S_2'(p-\sigma_1+\sigma_2)\,d\sigma_2 d\sigma_1 \\ &= \frac{2\bar{t}}{V_2}\int_{\sigma_1=0}^{p} S'(p-\sigma_1)\left(S(\overline{p})-S(p-\sigma_1)\right)d\sigma_1 \\ &= \frac{\bar{t}\left(2S(\overline{p})\,S(p)-S^2(p)\right)}{V_2}, \end{aligned}$$

where we have used that $S_2(p) = S_3(p) = S(p)$.

State ω_{12}

State	$t_1(\omega)$	$t_2(\omega)$	$t_3(\omega)$	$L(\omega)$	$B(\omega)$	$U(\omega)$	$\Xi(\omega)$	$F(\omega)$
ω_{12}	$-\bar{t}$	$\in(-\bar{t},\bar{t})$	$\in(-\bar{t},\bar{t})$	$\{1\}$	$\{2,3\}$	$\varnothing$	$\{1\}$	$\{2,3,4\}$

In this state we have from (66), (65) and Lemma 9 that:

$$\begin{aligned} J(\omega_{12}) &= \left|\frac{\partial \boldsymbol{\varepsilon}_{F(\omega)}}{\partial(\left(\mathbf{t}_{\vartheta(\omega)}\right)_{B(\omega)}, \boldsymbol{\rho}_{U(\omega)}, \boldsymbol{\sigma}_{L(\omega)})}\right| = \left|\det\begin{bmatrix} -1 & 0 & -S_2'(p_2) \\ 0 & -1 & -S_3'(p_3) \\ 1 & 1 & -S_4'(p_4) \end{bmatrix}\right| \\ &= S_3'(p_3)+S_2'(p_2), \end{aligned}$$

because $S_4'(p_4) = 0$. Now, we have from (50) that

$$
\begin{aligned}
P(p, q, \omega_{12}) &= \int_{\sigma_1=0}^{p} \int_{t_2=-\bar{t}}^{\bar{t}} \int_{t_3=-\bar{t}}^{\bar{t}} f(\mathbf{At} + \mathbf{s}(\mathbf{p}(p, \boldsymbol{\rho}, \boldsymbol{\sigma}))) (S_3'(p_3) + S_2'(p_2))\, dt_3 dt_2 d\sigma_1 \\
&= \frac{4\bar{t}^2}{V_2} \int_{\sigma_1=0}^{p} 2S'(p - \sigma_1)\, d\sigma_1 \\
&= \frac{8\bar{t}^2 S(p)}{V_2},
\end{aligned}
$$

where we have used that $S_2(p) = S_3(p) = S(p)$.

State ω_{13}

State	$t_1(\omega)$	$t_2(\omega)$	$t_3(\omega)$	$L(\omega)$	$B(\omega)$	$U(\omega)$	$\Xi(\omega)$	$F(\omega)$
ω_{13}	$\in(-\bar{t},\bar{t})$	$\bar{t}$	$\bar{t}$	$\varnothing$	$\{1\}$	$\{2,3\}$	$\{1,4\}$	$\{2,3\}$

In this state we have from (66), (65) and Lemma 9 that:

$$
\begin{aligned}
J(\omega_{13}) &= \left| \frac{\partial \boldsymbol{\varepsilon}_{F(\omega)}}{\partial(\left(\mathbf{t}_{\vartheta(\omega)}\right)_{B(\omega)}, \boldsymbol{\rho}_{U(\omega)}, \boldsymbol{\sigma}_{L(\omega)})} \right| = \left| \det \begin{bmatrix} -S_2'(p_2) & 0 \\ 0 & -S_3'(p_3) \end{bmatrix} \right| \\
&= S_2'(p_2)\, S_3'(p_3).
\end{aligned}
$$

Now, we have from (50) that

$$
\begin{aligned}
P(p, q, \omega_{13}) &= \int_{\rho_2=0}^{p} \int_{\rho_3=0}^{p} \int_{t_1=-\bar{t}}^{\bar{t}} f(\mathbf{At} + \mathbf{s}(\mathbf{p}(p, \boldsymbol{\rho}, \boldsymbol{\sigma}))) S_2'(p_2)\, S_3'(p_3)\, dt_1 d\rho_3 d\rho_2 \\
&= \frac{2\bar{t}}{V_2} \int_{\rho_2=0}^{p} S_2'(p - \rho_2)\, d\rho_2 \int_{\rho_3=0}^{p} S_3'(p - \rho_3)\, d\rho_3 \\
&= \frac{2\bar{t} S^2(p)}{V_2} = \frac{2\bar{t} S^2(p)}{V_2},
\end{aligned}
$$

where we have used that $S_2(p) = S_3(p) = S(p)$.

State ω_{14}

State	$t_1(\omega)$	$t_2(\omega)$	$t_3(\omega)$	$L(\omega)$	$B(\omega)$	$U(\omega)$	$\Xi(\omega)$	$F(\omega)$
ω_{14}	$\in(-\bar{t},\bar{t})$	$\bar{t}$	$-\bar{t}$	$\{3\}$	$\{1\}$	$\{2\}$	$\{1,4\}$	$\{2,3\}$

In this state we have from (66), (65) and Lemma 9 that:

$$
\begin{aligned}
J(\omega_{14}) &= \left| \frac{\partial \boldsymbol{\varepsilon}_{F(\omega)}}{\partial(\left(\mathbf{t}_{\vartheta(\omega)}\right)_{B(\omega)}, \boldsymbol{\rho}_{U(\omega)}, \boldsymbol{\sigma}_{L(\omega)})} \right| = \left| \det \begin{bmatrix} -S_2'(p_2) & 0 \\ 0 & S_3'(p_3) \end{bmatrix} \right| \\
&= S_2'(p_2)\, S_3'(p_3).
\end{aligned}
$$

Now, we have from (50) that

$$\begin{aligned} P\left(p,q,\omega_{14}\right) &= \int_{\rho_2=0}^{p}\int_{\sigma_3=0}^{\overline{p}-p}\int_{t_1=-\bar{t}}^{\bar{t}} f\left(\mathbf{At}+\mathbf{s}\left(\mathbf{p}(p,\boldsymbol{\rho},\boldsymbol{\sigma})\right)\right) S_2'\left(p_2\right) S_3'\left(p_3\right) dt_1 d\sigma_3 d\rho_2 \\ &= \frac{2\bar{t}}{V_2}\int_{\rho_2=0}^{p} S_2'\left(p-\rho_2\right) d\rho_2 \int_{\sigma_3=0}^{\overline{p}-p} S_3'\left(p+\sigma_3\right) d\sigma_3 \\ &= \frac{2\bar{t}S\left(p\right)\left(S\left(\overline{p}\right)-S\left(p\right)\right)}{V_2}, \end{aligned}$$

where we have used that $S_2\left(p\right)=S_3\left(p\right)=S(p)$.

State ω_{15}

State	$t_1(\omega)$	$t_2(\omega)$	$t_3(\omega)$	$L(\omega)$	$B(\omega)$	$U(\omega)$	$\Xi(\omega)$	$\digamma(\omega)$
ω_{15}	$\in\left(-\bar{t},\bar{t}\right)$	$\bar{t}$	$\in\left(-\bar{t},\bar{t}\right)$	$\emptyset$	$\{1,3\}$	$\{2\}$	$\{1,3,4\}$	$\{2\}$

In this state we have from (66), (65) and Lemma 9 that:

$$J(\omega_{15})=\left|\frac{\partial\boldsymbol{\varepsilon}_{\digamma(\omega)}}{\partial(\left(\mathbf{t}_{\vartheta(\omega)}\right)_{B(\omega)},\boldsymbol{\rho}_{U(\omega)},\boldsymbol{\sigma}_{L(\omega)})}\right|=\left|\left[-S_2'\left(p_2\right)\right]\right|=S_2'\left(p_2\right).$$

Now, we have from (50) that

$$\begin{aligned} P\left(p,q,\omega_{15}\right) &= \int_{t_3=-\bar{t}}^{\bar{t}}\int_{\rho_2=0}^{p}\int_{t_1=-\bar{t}}^{\bar{t}} f\left(\mathbf{At}+\mathbf{s}\left(\mathbf{p}(p,\boldsymbol{\rho},\boldsymbol{\sigma})\right)\right) S_2'\left(p_2\right) dt_1 d\rho_2 dt_3 \\ &= \frac{4\bar{t}^2}{V_2}\int_{\rho_2=0}^{p} S'\left(p-\rho_2\right) d\rho_2 \\ &= \frac{4\bar{t}^2 S\left(p\right)}{V_2}, \end{aligned}$$

where we have used that $S_2\left(p\right)=S_3\left(p\right)=S(p)$.

State ω_{16}

State	$t_1(\omega)$	$t_2(\omega)$	$t_3(\omega)$	$L(\omega)$	$B(\omega)$	$U(\omega)$	$\Xi(\omega)$	$\digamma(\omega)$
ω_{16}	$\in\left(-\bar{t},\bar{t}\right)$	$-\bar{t}$	$-\bar{t}$	$\{2,3\}$	$\{1\}$	$\emptyset$	$\{1,4\}$	$\{2,3\}$

In this state we have from (66), (65) and Lemma 9 that:

$$J(\omega_{16})=\left|\frac{\partial\boldsymbol{\varepsilon}_{\digamma(\omega)}}{\partial(\left(\mathbf{t}_{\vartheta(\omega)}\right)_{B(\omega)},\boldsymbol{\rho}_{U(\omega)},\boldsymbol{\sigma}_{L(\omega)})}\right|=\left|\det\begin{bmatrix} S_2'\left(p_2\right) & 0 \\ 0 & S_3'\left(p_3\right)\end{bmatrix}\right|=S_2'\left(p_2\right)S_3'\left(p_3\right).$$

Now, we have from (50) that

$$
\begin{aligned}
P(p,q,\omega_{16}) &= \int_{\sigma_2=0}^{\overline{p}-p}\int_{\sigma_3=0}^{\overline{p}-p}\int_{t_1=-\bar{t}}^{\bar{t}} f\left(\mathbf{At}+\mathbf{s}\left(\mathbf{p}(p,\boldsymbol{\rho},\boldsymbol{\sigma})\right)\right) S_2'\left(p_2\right) S_3'\left(p_3\right) dt_1 d\sigma_3 d\sigma_2 \\
&= \frac{2\bar{t}}{V_2}\int_{\sigma_2=0}^{\overline{p}-p} S'\left(p+\sigma_2\right) d\sigma_2 \int_{\sigma_3=0}^{\overline{p}-p} S'\left(p+\sigma_3\right) d\sigma_3 \\
&= \frac{2\bar{t}\left(S\left(\overline{p}\right)-S\left(p\right)\right)^2}{V_2},
\end{aligned}
$$

where we have used that $S_2(p) = S_3(p) = S(p)$.

State ω_{17}

State	$t_1(\omega)$	$t_2(\omega)$	$t_3(\omega)$	$L(\omega)$	$B(\omega)$	$U(\omega)$	$\Xi(\omega)$	$F(\omega)$
ω_{17}	$\in\left(-\bar{t},\bar{t}\right)$	$-\bar{t}$	$\in\left(-\bar{t},\bar{t}\right)$	$\{2\}$	$\{1,3\}$	$\emptyset$	$\{1,3,4\}$	$\{2\}$

In this state we have from (66), (65) and Lemma 9 that:

$$
J(\omega_{17}) = \left|\frac{\partial \boldsymbol{\varepsilon}_{F(\omega)}}{\partial\left(\left(\mathbf{t}_{\vartheta(\omega)}\right)_{B(\omega)}, \boldsymbol{\rho}_{U(\omega)}, \boldsymbol{\sigma}_{L(\omega)}\right)}\right| = S_2'\left(p_2\right)
$$

and we have from (50) that

$$
\begin{aligned}
P(p,q,\omega_{17}) &= \int_{\sigma_2=0}^{\overline{p}-p}\int_{t_3=-\bar{t}}^{\bar{t}}\int_{t_1=-\bar{t}}^{\bar{t}} f\left(\mathbf{At}+\mathbf{s}\left(\mathbf{p}(p,\boldsymbol{\rho},\boldsymbol{\sigma})\right)\right) S_2'\left(p_2\right) dt_1 dt_3 d\sigma_2 \\
&= \frac{4\bar{t}^2}{V_2}\int_{\sigma_2=0}^{\overline{p}-p} S'\left(p+\sigma_2\right) d\sigma_2 \\
&= \frac{4\bar{t}^2\left(S\left(\overline{p}\right)-S\left(p\right)\right)}{V_2}.
\end{aligned}
$$

State ω_{18}

State	$t_1(\omega)$	$t_2(\omega)$	$t_3(\omega)$	$L(\omega)$	$B(\omega)$	$U(\omega)$	$\Xi(\omega)$	$F(\omega)$
ω_{18}	$\in\left(-\bar{t},\bar{t}\right)$	$\in\left(-\bar{t},\bar{t}\right)$	$\in\left(-\bar{t},\bar{t}\right)$	$\emptyset$	$\{1,2,3\}$	$\emptyset$	$\{1,2,3,4\}$	$\emptyset$

In this state we have from (50) that

$$
P(p,q,\omega_{18}) = \int_{t_3=-\bar{t}}^{\bar{t}}\int_{t_2=-\bar{t}}^{\bar{t}}\int_{t_1=-\bar{t}}^{\bar{t}} f\left(\mathbf{At}+\mathbf{s}\left(\mathbf{p}(p,\boldsymbol{\rho},\boldsymbol{\sigma})\right)\right) dt_1 dt_2 dt_3 = \frac{8\bar{t}^3}{V_2}.
$$

Appendix D: Meshed networks

Proof. (Lemma 5) The proof is similar to the radial version in Lemma 1. Consider a node ℓ that is connected to node m through one or more uncongested path(s), so that there is no fully or partly congested chain of arcs between node ℓ and node m for congestion state ω. Thus, if a commodity is injected at node ℓ and consumed at node m, then no flow is going to pass through a congested arc. Thus $\left((\mathbf{H}_{L(\omega)})^T\right)_\ell$ and $\left((\mathbf{H}_{U(\omega)})^T\right)_\ell$ are both empty or row vectors of zeros, and consequently it follows from (35) that $p_\ell = p$. The expression in (38) is a simplified version of a related expression derived by Xu and Baldick (2007). In Xu and Baldick's expression (equation 29) all nodes $\ell \neq m$ are treated as if they are semi-integrated with node m. We observe that the contribution to the residual demand from nodes $\ell \in \Xi(\omega)$ can be simplified; here $\left((\mathbf{H}_L)^T\right)_\ell$ and $\left((\mathbf{H}_U)^T\right)_\ell$ are both empty or row vectors of zeros. We also note that nodes ℓ that do not have any uncongested path to node m will not have any contribution to the slope of the residual demand in node m. ■

Proof. (Proposition 5) This follows from a proof analogous to the proofs of Lemma 6, Lemma 7 and Proposition 1 in Appendix A. ■

Probabilities for different market outcomes are calculated similar to the radial case. Instead of integrating over demand shocks for a given congestion state, we transform the problem by integrating over the flows and shadow prices that arise in the congestion state. However, the volume in t, σ and ρ space is more complicated in the meshed case, as flows around a loop are coordinated in order to satisfy Kirchhoff's law. Generally, $\mathbf{A}_{-m}$ will have $M-1$ rows and $K > M-1$ columns, and so it will not have an inverse. We let $\mathbf{G}$ be a matrix with $K-(M-1)$ rows forming a basis for the null space of $\mathbf{A}_{-m}$. $\mathbf{G}$ could for example be the rows of the orientation vectors of a set of $K-(M-1)$ cycles in the network (Strang, 1986). We have from (34) that

$$\mathbf{Xt} = -\mathbf{A}^\top \boldsymbol{\phi}$$

so

$$\mathbf{GXt} = -\mathbf{GA}^\top \boldsymbol{\phi} = \mathbf{0}.$$

Thus feasible flows $\mathbf{t}$ must lie in the nullspace of $\mathbf{Y} = \mathbf{GX}$. We let $T(B(\omega))$ be the volume in $\mathbf{t}$ space that the flows in a set of uncongested arcs $B(\omega)$ can span:

$$\mathcal{T}(B(\omega)) = \{\mathbf{t}_B : \mathbf{Y}_B\mathbf{t}_B = -\mathbf{Y}_L\mathbf{t}_L - \mathbf{Y}_U\mathbf{t}_U, \quad -\bar{\mathbf{t}}_B \leq \mathbf{t}_B \leq \bar{\mathbf{t}}_B\}. \tag{71}$$

We can first choose $M-1$ flows $\mathbf{t}'_B$ and then let them determine the remaining $K-(M-1)$ flows $\mathbf{t}''_B$ in order to satisfy (71).

Lemma 10

$$P_{mn}(p, q, \boldsymbol{\rho}, \boldsymbol{\sigma}, \omega) = \int\limits_{\mathcal{T}(B(\omega))} f\left(\mathbf{At} + \mathbf{s}\left(\mathbf{p}(p, \boldsymbol{\rho}, \boldsymbol{\sigma}), q\right)\right) J_V(\omega)\mathbf{dt}'_{B(\omega)}, \tag{72}$$

where

$$\begin{aligned} s_{-m}\left(\mathbf{p}(p,\boldsymbol{\rho},\boldsymbol{\sigma}),q\right) &= s_{-m}\left(\mathbf{p}(p,\boldsymbol{\rho},\boldsymbol{\sigma})\right) \\ s_{m}\left(\mathbf{p}(p,\boldsymbol{\rho},\boldsymbol{\sigma}),q\right) &= q+S_{m,-n}\left(p\right) \end{aligned}$$

and

$$J_V(\omega)=\left|\frac{\partial\boldsymbol{\varepsilon}}{\partial(\mathbf{t}'_{B(\omega)},\boldsymbol{\rho}_{U(\omega)},\boldsymbol{\sigma}_{L(\omega)},\varepsilon_m)}\right|.$$

Proof. This follows from a similar argument as in the first part of Lemma 8. Unfortunately $J_V(\omega)$ does not simplify as in the radial case. ■

Appendix E: Example: a delta network

The delta network can be described by the following node-arc incidence matrix.

$$\mathbf{A}=\begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}.$$

Thus (2) can be written as follows:

$$\begin{bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\end{bmatrix}=\underbrace{\begin{bmatrix}S_1\left(p_1\right)\\ S_2\left(p_2\right)\\ S_3\left(p_3\right)\end{bmatrix}}_{\mathbf{s}(\mathbf{p})}+\underbrace{\begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}}_{\mathbf{A}}\underbrace{\begin{bmatrix}t_1\\ t_2\\ t_3\end{bmatrix}}_{\mathbf{t}}. \tag{73}$$

We let node 1 be the trading hub and slack node, so we calculate

$$\mathbf{A}_{-1}=\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}.$$

Impedances are assumed to be the same, x, in all arcs. Thus

$$\mathbf{X}=\begin{bmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{bmatrix}.$$

It now follows from (36) that

$$\begin{aligned} \mathbf{H} &= -\mathbf{X}^{-1}\mathbf{A}_{-1}^{\top}\left(\mathbf{A}_{-1}\mathbf{X}^{-1}\mathbf{A}_{-1}^{\top}\right)^{-1}=-\mathbf{A}_{-1}^{\top}\left(\mathbf{A}_{-1}\mathbf{A}_{-1}^{\top}\right)^{-1} \\ &= \begin{bmatrix} -\frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} \end{aligned} \tag{74}$$

Table 2: The 13 feasible congestion states of the delta network.

State	$t_1(\omega)$	$t_2(\omega)$	$t_3(\omega)$
ω_1	$\in (-\bar{t}, 0)$	$\bar{t}$	$-t_1 - \bar{t}$
ω_2	$\in (0, \bar{t})$	$-\bar{t}$	$-t_1 + \bar{t}$
ω_3	$\in [-\bar{t}, \bar{t}]$	$\in [\max(-\bar{t} - t_1, -\bar{t}), \min(\bar{t} - t_1, \bar{t})]$	$-t_1 - t_2$
ω_4	$\bar{t}$	$\in (-\bar{t}, 0)$	$-t_2 - \bar{t}$
ω_5	$-\bar{t}$	$\in (0, \bar{t})$	$-t_2 + \bar{t}$
ω_6	$\in (-\bar{t}, 0)$	$-t_1 - \bar{t}$	$\bar{t}$
ω_7	$\in (0, \bar{t})$	$-t_1 + \bar{t}$	$-\bar{t}$
ω_8	$\bar{t}$	$-\bar{t}$	0
ω_9	$\bar{t}$	0	$-\bar{t}$
ω_{10}	$-\bar{t}$	$\bar{t}$	0
ω_{11}	$-\bar{t}$	0	$\bar{t}$
ω_{12}	0	$\bar{t}$	$-\bar{t}$
ω_{13}	0	$-\bar{t}$	$\bar{t}$

Thus we have from (35) and (74) that:

$$\begin{aligned}\begin{bmatrix} p_2 \\ p_3 \end{bmatrix} &= \begin{bmatrix} p \\ p \end{bmatrix} - \begin{bmatrix} -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} \rho_1 - \sigma_1 \\ \rho_2 - \sigma_2 \\ \rho_3 - \sigma_3 \end{bmatrix} \\ &= \begin{bmatrix} p \\ p \end{bmatrix} - \begin{bmatrix} \frac{2}{3}\sigma_1 - \frac{1}{3}\sigma_2 - \frac{1}{3}\sigma_3 - \frac{2}{3}\rho_1 + \frac{1}{3}\rho_2 + \frac{1}{3}\rho_3 \\ \frac{1}{3}\sigma_1 + \frac{1}{3}\sigma_2 - \frac{2}{3}\sigma_3 - \frac{1}{3}\rho_1 - \frac{1}{3}\rho_2 + \frac{2}{3}\rho_3 \end{bmatrix}. \end{aligned} \tag{75}$$

Each arc of the delta network can be in one of three states: uncongested, congested in the positive direction or congested in the negative direction. As in the star network this would in principle give $3 \times 3 \times 3 = 27$ congestion states in the delta network. However, many of those congestion states are not feasible in the studied delta network. Kirchhoff's law cannot be satisfied for three congested lines in our example. Two lines could be congested, but then they have to be congested in opposite directions. Thus only the 13 states listed in Table 2 are feasible in the studied delta network. Note that we have considered that too large or too small flows in arc 1 and 2 results in a flow in arc 3 that is outside its allowed range $[-\bar{t}, \bar{t}]$.

In Appendix E.1, we use Lemma 5 and Lemma 10 to calculate

$$\int_{\boldsymbol{\rho}_U} \int_{\boldsymbol{\sigma}_L} P\left(p, q, \boldsymbol{\rho}, \boldsymbol{\sigma}, \omega\right) d\sigma_L d\rho_U$$

and

$$\int_{\boldsymbol{\rho}_U} \int_{\boldsymbol{\sigma}_L} D'_{mn}(p, \omega, \rho, \sigma) P\left(p, q, \boldsymbol{\rho}, \boldsymbol{\sigma}, \omega\right) d\sigma_L d\rho_U$$

for each feasible congestion state. The results are summarized in Table 3 and Table 4.

Table 3: $\int_{\boldsymbol{\rho}_U}\int_{\boldsymbol{\sigma}_L} P\left(p,q,\boldsymbol{\rho},\boldsymbol{\sigma},\omega\right) d\boldsymbol{\sigma}_L d\boldsymbol{\rho}_U$ for each feasible congestion state of the delta network.

State	$\int_{\boldsymbol{\rho}_U}\int_{\boldsymbol{\sigma}_L} P\left(p,q,\boldsymbol{\rho},\boldsymbol{\sigma},\omega\right) d\boldsymbol{\sigma}_L d\boldsymbol{\rho}_U$
ω_1	$\frac{\bar{t}}{V_3}\left[-NQ\left(p-\frac{\rho_2}{3}\right)+NQ\left(p+\frac{\rho_2}{3}\right)\right]_0^{\min[3p,3(\overline{p}-p)]}$
ω_2	$\frac{\bar{t}}{V_3}\left[-NQ\left(p-\frac{\rho_2}{3}\right)+NQ\left(p+\frac{\rho_2}{3}\right)\right]_0^{\min[3p,3(\overline{p}-p)]}$
ω_3	$\frac{9\bar{t}^2}{V_3}$
ω_4	$\frac{\bar{t}}{V_3}\left[2NQ\left(p+\frac{2\rho_1}{3}\right)+NQ\left(p+\frac{\rho_1}{3}\right)\right]_0^{\frac{3(\overline{p}-p)}{2}}$
ω_5	$\frac{\bar{t}}{V_3}\left[-2NQ\left(p-\frac{2\sigma_1}{3}\right)-NQ\left(p-\frac{\sigma_1}{3}\right)\right]_0^{\frac{3p}{2}}$
ω_6	$\frac{\bar{t}}{V_3}\left[-2NQ\left(p-\frac{2\sigma_1}{3}\right)-NQ\left(p-\frac{\sigma_1}{3}\right)\right]_0^{\frac{3p}{2}}$
ω_7	$\frac{\bar{t}}{V_3}\left[2NQ\left(p+\frac{2\sigma_3}{3}\right)+NQ\left(p+\frac{\sigma_3}{3}\right)\right]_0^{\frac{3(\overline{p}-p)}{2}}$
ω_8	$\frac{N^2}{V_3}\left(Q\left(\overline{p}\right)-Q\left(p\right)\right)Q\left(\overline{p}\right)$
ω_9	$\frac{N^2}{V_3}\left(Q\left(\overline{p}\right)-Q\left(p\right)\right)^2$
ω_{10}	$\frac{N^2}{V_3}Q\left(p\right)Q\left(\overline{p}\right)$
ω_{11}	$\frac{N^2}{V_3}Q^2\left(p\right)$
ω_{12}	$\frac{N^2}{V_3}\left(Q\left(\overline{p}\right)-Q\left(p\right)\right)Q\left(\overline{p}\right)$
ω_{13}	$\frac{N^2}{V_3}Q\left(p\right)Q\left(\overline{p}\right)$

Table 4: $\int_{\boldsymbol{\rho}_U}\int_{\boldsymbol{\sigma}_L} D'_{mn}(p,\omega,\boldsymbol{\rho},\boldsymbol{\sigma})P\left(p,q,\boldsymbol{\rho},\boldsymbol{\sigma},\omega\right) d\boldsymbol{\sigma}_L d\boldsymbol{\rho}_U$ for each feasible congestion state of the delta network. Note that $p^* = \min\left[3p, 3\left(\overline{p}-p\right)\right]$ and $\breve{p} = \frac{3(\overline{p}-p)}{2}$.

State	$-\int_{\boldsymbol{\rho}_U}\int_{\boldsymbol{\sigma}_L} D'_{mn}(p,\omega,\boldsymbol{\rho},\boldsymbol{\sigma})P\left(p,q,\boldsymbol{\rho},\boldsymbol{\sigma},\omega\right) d\boldsymbol{\sigma}_L d\boldsymbol{\rho}_U$
ω_1	$\frac{\bar{t}(N-1)Q'(p)}{V_3}\left[-NQ\left(p-\frac{\rho}{3}\right)+NQ\left(p+\frac{\rho}{3}\right)\right]_0^{p^*}+\frac{4\bar{t}N^2}{3V_3}\int_0^{p^*}Q'\left(p-\frac{\rho}{3}\right)Q'\left(p+\frac{\rho}{3}\right)d\rho$
ω_2	$\frac{\bar{t}(N-1)Q'(p)}{V_3}\left[-NQ\left(p-\frac{\rho}{3}\right)+NQ\left(p+\frac{\rho}{3}\right)\right]_0^{p^*}+\frac{4\bar{t}N^2}{3V_3}\int_0^{p^*}Q'\left(p-\frac{\rho}{3}\right)Q'\left(p+\frac{\rho}{3}\right)d\rho$
ω_3	$\frac{9(3N-1)\bar{t}^2}{V_3}Q'(p)$
ω_4	$\frac{\bar{t}(N-1)Q'(p)}{3V_3}\left[6NQ\left(p+\frac{2\rho}{3}\right)+3NQ\left(p+\frac{\rho}{3}\right)\right]_0^{\breve{p}}+\frac{\bar{t}N^2}{3V_3}\int_0^{\breve{p}}Q'\left(p+\frac{2\rho}{3}\right)Q'\left(p+\frac{\rho}{3}\right)d\rho$
ω_5	$\frac{\bar{t}(N-1)Q'(p)}{3V_3}\left[-6NQ\left(p-\frac{2\sigma}{3}\right)-3NQ\left(p-\frac{\sigma}{3}\right)\right]_0^{\frac{3p}{2}}+\frac{\bar{t}N^2}{3V_3}\int_0^{\frac{3p}{2}}Q'\left(p-\frac{2\sigma}{3}\right)Q'\left(p-\frac{\sigma}{3}\right)d\sigma$
ω_6	$\frac{\bar{t}(N-1)Q'(p)}{3V_3}\left[-6NQ\left(p-\frac{2\sigma}{3}\right)-3NQ\left(p-\frac{\sigma}{3}\right)\right]_0^{\frac{3p}{2}}+\frac{\bar{t}N^2}{3V_3}\int_0^{\frac{3p}{2}}Q'\left(p-\frac{2\sigma}{3}\right)Q'\left(p-\frac{\sigma}{3}\right)d\sigma$
ω_7	$\frac{\bar{t}(N-1)Q'(p)}{3V_3}\left[6NQ\left(p+\frac{2\sigma}{3}\right)+3NQ\left(p+\frac{\sigma}{3}\right)\right]_0^{\breve{p}}+\frac{\bar{t}N^2}{3V_3}\int_0^{\breve{p}}Q'\left(p+\frac{2\sigma}{3}\right)Q'\left(p+\frac{\sigma}{3}\right)d\sigma$
ω_8	$\frac{(N-1)N^2Q'(p)}{V_3}\left(Q\left(\overline{p}\right)-Q\left(p\right)\right)Q\left(\overline{p}\right)$
ω_9	$\frac{(N-1)N^2Q'(p)}{V_3}\left(Q\left(\overline{p}\right)-Q\left(p\right)\right)^2$
ω_{10}	$\frac{(N-1)N^2Q'(p)}{V_3}Q\left(p\right)Q\left(\overline{p}\right)$
ω_{11}	$\frac{(N-1)N^2Q'(p)}{V_3}Q^2\left(p\right)$
ω_{12}	$\frac{(N-1)N^2Q'(p)}{V_3}\left(Q\left(\overline{p}\right)-Q\left(p\right)\right)Q\left(\overline{p}\right)$
ω_{13}	$\frac{(N-1)N^2Q'(p)}{V_3}Q\left(p\right)Q\left(\overline{p}\right)$

Appendix E.1: Congestion states in delta network

From the perspective of a producer in node 1, ω_1 is symmetric with ω_2, ω_4 is symmetric with ω_7, ω_5 is symmetric with ω_6, ω_8 is symmetric with ω_{12}, and ω_{10} is symmetric with ω_{13}. Thus it is sufficient to make the calculations for eight states.

State ω_1

State	$t_1(\omega)$	$t_2(\omega)$	$t_3(\omega)$	$L(\omega)$	$B(\omega)$	$U(\omega)$	$\Xi(\omega)$	$\digamma(\omega)$	$\Upsilon(\omega)$
ω_1	$\in(-\bar{t},0)$	$\bar{t}$	$-t_1-\bar{t}$	$\emptyset$	$\{1,3\}$	$\{2\}$	$\{1\}$	$\emptyset$	$\{2,3\}$

We have from (73) that

$$\begin{aligned}\begin{bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\end{bmatrix} &= \underbrace{\begin{bmatrix}S_1(p)\\ S_2(p_2)\\ S_3(p_3)\end{bmatrix}}_{\mathbf{s}(\mathbf{p})}+\underbrace{\begin{bmatrix}-1&0&1\\ 1&-1&0\\ 0&1&-1\end{bmatrix}}_{\mathbf{A}}\underbrace{\begin{bmatrix}t_1\\ \bar{t}\\ -t_1-\bar{t}\end{bmatrix}}_{\mathbf{t}} \qquad (76)\\ &= \begin{bmatrix}S_1(p)-2t_1-\bar{t}\\ S_2(p_2)+t_1-\bar{t}\\ S_3(p_3)+t_1+2\bar{t}\end{bmatrix},\end{aligned}$$

so

$$\frac{d}{dt_1}\begin{bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\end{bmatrix}=\begin{bmatrix}-2\\ 1\\ 1\end{bmatrix}. \qquad (77)$$

It follows from (75) that

$$\begin{bmatrix}p_2\\ p_3\end{bmatrix}=\begin{bmatrix}p-\frac{1}{3}\rho_2\\ p+\frac{1}{3}\rho_2\end{bmatrix}. \qquad (78)$$

We now have from (76) and (78) that

$$\frac{\partial}{\partial\rho_2}\begin{bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\end{bmatrix}=\begin{bmatrix}\frac{\partial\varepsilon_1}{\partial p_1}\frac{\partial p_1}{\partial\rho_2}\\ \frac{\partial\varepsilon_2}{\partial p_2}\frac{\partial p_2}{\partial\rho_2}\\ \frac{\partial\varepsilon_3}{\partial p_3}\frac{\partial p_3}{\partial\rho_2}\end{bmatrix}=\begin{bmatrix}0\\ -\frac{1}{3}S_2'(p_2)\\ \frac{1}{3}S_3'(p_3)\end{bmatrix}. \qquad (79)$$

We use (77) and (79) to calculate

$$\begin{aligned}J_V(\omega_1) &= \left|\frac{\partial\boldsymbol{\varepsilon}}{\partial(\mathbf{t}'_{B(\omega)},\boldsymbol{\rho}_{U(\omega)},\boldsymbol{\sigma}_{L(\omega)},\varepsilon_m)}\right|=\left|\det\begin{pmatrix}\frac{d\varepsilon_1}{dt_1}&\frac{\partial\varepsilon_1}{\partial\rho_2}&\frac{\partial\varepsilon_1}{\partial\varepsilon_1}\\ \frac{d\varepsilon_2}{dt_1}&\frac{\partial\varepsilon_2}{\partial\rho_2}&\frac{\partial\varepsilon_2}{\partial\varepsilon_1}\\ \frac{d\varepsilon_3}{dt_1}&\frac{\partial\varepsilon_3}{\partial\rho_2}&\frac{\partial\varepsilon_3}{\partial\varepsilon_1}\end{pmatrix}\right|\\ &= \left|\det\begin{pmatrix}-2&0&1\\ 1&-\frac{S_2'(p_2)}{3}&0\\ 1&\frac{S_3'(p_3)}{3}&0\end{pmatrix}\right|=\frac{S_2'(p_2)+S_3'(p_3)}{3}.\end{aligned}$$

Thus we have from Proposition 5 that

$$\begin{aligned} P\left(p,q,\boldsymbol{\rho},\boldsymbol{\sigma},\omega_1\right) &= \frac{1}{V_3}\int_{t_1=-\bar{t}}^{0} J_V(\omega_1)dt_1 = \frac{\bar{t}}{3V_3}\left(S'\left(p_2\right)+S'\left(p_3\right)\right) \\ &= \frac{\bar{t}}{3V_3}\left(S'\left(p-\frac{\rho_2}{3}\right)+S'\left(p+\frac{\rho_2}{3}\right)\right). \end{aligned} \tag{80}$$

It follows from (74) that

$$\mathbf{H}_{U(\omega)} = \left[\begin{array}{cc} \frac{1}{3} & -\frac{1}{3} \end{array}\right]$$

and

$$\mathbf{H}_{L(\omega)} = [\,],$$

so we have from (37) that

$$\mathbf{W} = \left[\begin{array}{cc} \frac{1}{3} & -\frac{1}{3} \end{array}\right]. \tag{81}$$

Moreover,

$$\boldsymbol{\Psi} = \left[\begin{array}{cc} S'(p_2) & 0 \\ 0 & S'(p_3) \end{array}\right]. \tag{82}$$

We can now use (81) and (82) to compute

$$\begin{aligned} & \mathbf{1}^T\boldsymbol{\Psi}\mathbf{W}^\top(\mathbf{W}\boldsymbol{\Psi}\mathbf{W}^\top)^{-1}\mathbf{W}\boldsymbol{\Psi}\mathbf{1} \\ = & \left[\begin{array}{cc} S'(p_2) & S'(p_3) \end{array}\right]\left[\begin{array}{c} \frac{1}{3} \\ -\frac{1}{3} \end{array}\right]\left(\left[\begin{array}{cc} \frac{1}{3} & -\frac{1}{3} \end{array}\right]\left[\begin{array}{cc} S'(p_2) & 0 \\ 0 & S'(p_3) \end{array}\right]\left[\begin{array}{c} \frac{1}{3} \\ -\frac{1}{3} \end{array}\right]\right)^{-1}\left[\begin{array}{cc} \frac{1}{3} & -\frac{1}{3} \end{array}\right]\left[\begin{array}{c} S'(p_2) \\ S'(p_3) \end{array}\right] \\ = & \frac{1}{S'(p_2)+S'(p_3)}\left(S'(p_2)-S'(p_3)\right)^2. \end{aligned}$$

We assume symmetric offers in each node, so $Q_n(p) = Q(p)$. The residual demand slope for this state can now be calculated from Lemma 5:

$$\begin{aligned} & D'_{mn}(p,\omega_1,\boldsymbol{\rho},\boldsymbol{\sigma}) \\ = & -(N-1)Q'(p) - NQ'(p_2) - NQ'(p_3) + \frac{N\left(Q'(p_2)-Q'(p_3)\right)^2}{Q'(p_2)+Q'(p_3)} \\ = & -(N-1)Q'(p) - NQ'(p-\frac{\rho_2}{3}) - NQ'(p+\frac{\rho_2}{3}) && (83) \\ & + \frac{N\left(Q'(p-\frac{\rho_2}{3})-Q'(p+\frac{\rho_2}{3})\right)^2}{Q'(p-\frac{\rho_2}{3})+Q'(p+\frac{\rho_2}{3})}. && (84) \end{aligned}$$

We now have from (80) and (83) that the contribution from this state to Z_{mn} in Proposition 5 is:

$$
\begin{aligned}
&\int_{\boldsymbol{\rho}_{U(\omega_1)}} \int_{\boldsymbol{\sigma}_{L(\omega_1)}} -D'_{mn}(p,\omega_1,\boldsymbol{\rho},\boldsymbol{\sigma}) P\left(p,q,\boldsymbol{\rho},\boldsymbol{\sigma},\omega_1\right) d\boldsymbol{\sigma}_{L(\omega_1)} d\boldsymbol{\rho}_{U(\omega_1)} \\
&= \int_0^{\min[3p,3(\overline{p}-p)]} \tfrac{\overline{t}}{3V_3} \left(NQ'\left(p-\tfrac{\rho_2}{3}\right) + NQ'\left(p+\tfrac{\rho_2}{3}\right)\right) \\
&\left((N-1)Q'(p) + NQ'(p-\tfrac{\rho_2}{3}) + NQ'(p+\tfrac{\rho_2}{3}) - \tfrac{N\left(Q'(p-\frac{\rho_2}{3})-Q'(p+\frac{\rho_2}{3})\right)^2}{Q'(p-\frac{\rho_2}{3})+Q'(p+\frac{\rho_2}{3})}\right) d\rho_2 \\
&= \tfrac{\overline{t}(N-1)Q'(p)}{3V_3} \int_0^{\min[3p,3(\overline{p}-p)]} \left(NQ'\left(p-\tfrac{\rho_2}{3}\right) + NQ'\left(p+\tfrac{\rho_2}{3}\right)\right) d\rho_2 \\
&+ \tfrac{\overline{t}N^2}{3V_3} \int_0^{\min[3p,3(\overline{p}-p)]} \left(Q'\left(p-\tfrac{\rho_2}{3}\right) + Q'\left(p+\tfrac{\rho_2}{3}\right)\right)^2 d\rho_2 \\
&- \tfrac{\overline{t}N^2}{3V_3} \int_0^{\min[3p,3(\overline{p}-p)]} \left(Q'\left(p-\tfrac{\rho_2}{3}\right) - Q'\left(p+\tfrac{\rho_2}{3}\right)\right)^2 d\rho_2 \\
&= \tfrac{\overline{t}(N-1)Q'(p)}{V_3} \left[-NQ\left(p-\tfrac{\rho_2}{3}\right) + NQ\left(p+\tfrac{\rho_2}{3}\right)\right]_0^{\min[3p,3(\overline{p}-p)]} \\
&+ \tfrac{4\overline{t}N^2}{3V_3} \int_0^{\min[3p,3(\overline{p}-p)]} Q'\left(p-\tfrac{\rho_2}{3}\right) Q'\left(p+\tfrac{\rho_2}{3}\right) d\rho_2.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\int_{\boldsymbol{\rho}_{U(\omega_1)}} \int_{\boldsymbol{\sigma}_{L(\omega_1)}} P\left(p,q,\boldsymbol{\rho},\boldsymbol{\sigma},\omega_1\right) d\boldsymbol{\sigma}_{L(\omega_1)} d\boldsymbol{\rho}_{U(\omega_1)} \\
&= \int_0^{\min[3p,3(\overline{p}-p)]} \tfrac{\overline{t}}{3V_3} \left(NQ'\left(p-\tfrac{\rho_2}{3}\right) + NQ'\left(p+\tfrac{\rho_2}{3}\right)\right) d\rho_2 \\
&= \tfrac{\overline{t}}{V_3} \left[-NQ\left(p-\tfrac{\rho_2}{3}\right) + NQ\left(p+\tfrac{\rho_2}{3}\right)\right]_0^{\min[3p,3(\overline{p}-p)]}.
\end{aligned}
$$

State ω_2

State	$t_1(\omega)$	$t_2(\omega)$	$t_3(\omega)$	$L(\omega)$	$B(\omega)$	$U(\omega)$	$\Xi(\omega)$	$\digamma(\omega)$	$\Upsilon(\omega)$
ω_2	$\in \left(0,\overline{t}\right)$	$-\overline{t}$	$-t_1+\overline{t}$	$\{2\}$	$\{1,3\}$	$\emptyset$	$\{1\}$	$\emptyset$	$\{2,3\}$

It follows from (75) that

$$
\begin{bmatrix} p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} p+\frac{1}{3}\sigma_2 \\ p-\frac{1}{3}\sigma_2 \end{bmatrix}.
$$

From the perspective of a producer in node 1, ω_2 is symmetric with ω_1, so

$$
\begin{aligned}
P\left(p,q,\boldsymbol{\rho},\boldsymbol{\sigma},\omega_2\right) &= \frac{\overline{t}}{3V_3}\left(S'\left(p_2\right)+S'\left(p_3\right)\right) \\
&= \frac{\overline{t}}{3V_3}\left(S'\left(p+\frac{\sigma_2}{3}\right)+S'\left(p-\frac{\sigma_2}{3}\right)\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&D'_{mn}(p,\omega_2,\boldsymbol{\rho},\boldsymbol{\sigma}) \\
&= -(N-1)Q'(p) - NQ'(p_2) - NQ'(p_3) + \frac{N\left(Q'(p_2)-Q'(p_3)\right)^2}{Q'(p_2)+Q'(p_3)} \\
&= -(N-1)Q'(p) - NQ'(p+\frac{\sigma_2}{3}) - NQ'(p-\frac{\sigma_2}{3}) \\
&+ \frac{N}{Q'(p+\frac{\sigma_2}{3})+Q'(p-\frac{\sigma_2}{3})}\left(Q'(p+\frac{\sigma_2}{3})-Q'(p-\frac{\sigma_2}{3})\right)^2. \qquad (85)
\end{aligned}
$$

Moreover, the contribution to Z_{mn} in Proposition 5 is the same as for state ω_1.

State ω_3

State	$t_1(\omega)$	$t_2(\omega)$	$t_3(\omega)$	$L(\omega)$	$B(\omega)$	$U(\omega)$	$\Xi(\omega)$	$\digamma(\omega)$	$\Upsilon(\omega)$
ω_3	$\in\left[-\bar{t},\bar{t}\right]$	$\in\left[\check{t},\hat{t}\right]$	$-t_1-t_2$	$\emptyset$	$\left\{\begin{array}{c}1,\\2,\\3\end{array}\right\}$	$\emptyset$	$\left\{\begin{array}{c}1,\\2,\\3\end{array}\right\}$	$\emptyset$	$\emptyset$,

where $\check{t}=\max(-\bar{t}-t_1,-\bar{t})$ and $\hat{t}=\min\left(\bar{t}-t_1,\bar{t}\right)$. We have from (73) that

$$\begin{aligned}\left[\begin{array}{c}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\end{array}\right] &= \underbrace{\left[\begin{array}{c}S_1(p)\\ S_2(p_2)\\ S_3(p_3)\end{array}\right]}_{\mathbf{s}(\mathbf{p})}+\underbrace{\left[\begin{array}{ccc}-1&0&1\\1&-1&0\\0&1&-1\end{array}\right]}_{\mathbf{A}}\underbrace{\left[\begin{array}{c}t_1\\ t_2\\ -t_1-t_2\end{array}\right]}_{\mathbf{t}}\\ &= \left[\begin{array}{c}S_1(p)-2t_1-t_2\\ S_2(p_2)+t_1-t_2\\ S_3(p_3)+t_1+2t_2\end{array}\right], \qquad (86)\end{aligned}$$

so

$$\frac{d}{dt_1}\left[\begin{array}{c}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\end{array}\right]=\left[\begin{array}{c}-2\\ 1\\ 1\end{array}\right] \qquad (87)$$

and

$$\frac{d}{dt_2}\left[\begin{array}{c}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\end{array}\right]=\left[\begin{array}{c}-1\\ -1\\ 2\end{array}\right]. \qquad (88)$$

It follows from (75) that

$$\left[\begin{array}{c}p_2\\ p_3\end{array}\right]=\left[\begin{array}{c}p\\ p\end{array}\right]. \qquad (89)$$

We use (87) and (88) to calculate

$$\begin{aligned}J_V(\omega_3) &= \left|\frac{\partial\boldsymbol{\varepsilon}}{\partial(\mathbf{t}'_{B(\omega)},\boldsymbol{\rho}_{U(\omega)},\boldsymbol{\sigma}_{L(\omega)},\varepsilon_m)}\right|=\left|\det\left(\begin{array}{ccc}\frac{d\varepsilon_1}{\partial t_1}&\frac{d\varepsilon_1}{\partial t_2}&\frac{\partial\varepsilon_1}{\partial\varepsilon_1}\\ \frac{d\varepsilon_2}{\partial t_1}&\frac{d\varepsilon_2}{\partial t_2}&\frac{\partial\varepsilon_2}{\partial\varepsilon_1}\\ \frac{d\varepsilon_3}{\partial t_1}&\frac{d\varepsilon_3}{dt_2}&\frac{\partial\varepsilon_3}{\partial\varepsilon_1}\end{array}\right)\right|\\ &= \left|\det\left(\begin{array}{ccc}-2&-1&1\\1&-1&0\\1&2&0\end{array}\right)\right|=3.\end{aligned}$$

Thus we have from Lemma 10 that

$$\begin{aligned}P(p,q,\boldsymbol{\rho},\boldsymbol{\sigma},\omega_3) &= \frac{1}{V_3}\int_{t_1=-\bar{t}}^{\bar{t}}\int_{t_2=\max(-\bar{t}-t_1,-\bar{t})}^{\min\left(\bar{t}-t_1,\bar{t}\right)}J_V(\omega_3)dt_2dt_1\\ =\frac{3}{V_3}\int_{-\bar{t}}^{0}\left(2\bar{t}+t_1\right)dt_1 &+\frac{3}{V_3}\int_{0}^{\bar{t}}\left(2\bar{t}-t_1\right)dt_1=\frac{9\bar{t}^2}{V_3}. \qquad (90)\end{aligned}$$

In the congestion state ω_3 all nodes belong to the set $\Xi(\omega)$. We assume symmetric offers in each node, so $Q_n(p)=Q(p)$. The residual demand slope for this state can now be calculated from Lemma 5:

$$D'_{mn}(p,\omega_3,\boldsymbol{\rho},\boldsymbol{\sigma})=-(N-1)Q'(p)-NQ'(p)-NQ'(p)=-\left(3N-1\right)Q'(p) \quad (91)$$

We now have from (90) and (91) that the contribution from this state to Z_{mn} in Proposition 5 is:

$$\int_{\boldsymbol{\rho}_{U(\omega_3)}} \int_{\boldsymbol{\sigma}_{L(\omega_3)}} -D'_{mn}(p,\omega_3,\boldsymbol{\rho},\boldsymbol{\sigma})P(p,q,\boldsymbol{\rho},\boldsymbol{\sigma},\omega_3)\,d\boldsymbol{\sigma}_{L(\omega_3)}d\boldsymbol{\rho}_{U(\omega_3)}$$
$$= \frac{9(3N-1)\bar{t}^2}{V_3}Q'(p).$$

Similarly,

$$\int_{\boldsymbol{\rho}_{U(\omega_3)}} \int_{\boldsymbol{\sigma}_{L(\omega_3)}} P(p,q,\boldsymbol{\rho},\boldsymbol{\sigma},\omega_3)\,d\boldsymbol{\sigma}_{L(\omega_3)}d\boldsymbol{\rho}_{U(\omega_3)} = \frac{9\bar{t}^2}{V_3}.$$

State ω_4

State	$t_1(\omega)$	$t_2(\omega)$	$t_3(\omega)$	$L(\omega)$	$B(\omega)$	$U(\omega)$	$\Xi(\omega)$	$\digamma(\omega)$	$\Upsilon(\omega)$
ω_4	$\bar{t}$	$\in(-\bar{t},0)$	$-t_2-\bar{t}$	$\emptyset$	$\{2,3\}$	$\{1\}$	$\{1,2,3\}$	$\emptyset$	$\{2,3\}$

We have from (73) that

$$\begin{bmatrix}\varepsilon_1\\\varepsilon_2\\\varepsilon_3\end{bmatrix} = \underbrace{\begin{bmatrix}S_1(p)\\S_2(p_2)\\S_3(p_3)\end{bmatrix}}_{\mathbf{s}(\mathbf{p})} + \underbrace{\begin{bmatrix}-1&0&1\\1&-1&0\\0&1&-1\end{bmatrix}}_{\mathbf{A}}\underbrace{\begin{bmatrix}\bar{t}\\t_2\\-t_2-\bar{t}\end{bmatrix}}_{\mathbf{t}} \tag{92}$$
$$= \begin{bmatrix}S_1(p)-2\bar{t}-t_2\\S_2(p_2)-t_2+\bar{t}\\S_3(p_3)+2t_2+\bar{t}\end{bmatrix}$$

so

$$\frac{d}{dt_2}\begin{bmatrix}\varepsilon_1\\\varepsilon_2\\\varepsilon_3\end{bmatrix} = \begin{bmatrix}-1\\-1\\2\end{bmatrix}. \tag{93}$$

It follows from (75) that

$$\begin{bmatrix}p_2\\p_3\end{bmatrix} = \begin{bmatrix}p+\frac{2}{3}\rho_1\\p+\frac{1}{3}\rho_1\end{bmatrix}. \tag{94}$$

We now have from (92) and (94) that

$$\frac{\partial}{\partial\rho_1}\begin{bmatrix}\varepsilon_1\\\varepsilon_2\\\varepsilon_3\end{bmatrix} = \begin{bmatrix}\frac{\partial\varepsilon_1}{\partial p_1}\frac{\partial p_1}{\partial\rho_1}\\\frac{\partial\varepsilon_2}{\partial p_2}\frac{\partial p_2}{\partial\rho_1}\\\frac{\partial\varepsilon_3}{\partial p_3}\frac{\partial p_3}{\partial\rho_1}\end{bmatrix} = \begin{bmatrix}0\\\frac{2}{3}S'_2(p_2)\\\frac{1}{3}S'_3(p_3)\end{bmatrix}. \tag{95}$$

We use (93) and (95) to calculate

$$J_V(\omega_4) = \left| \frac{\partial \boldsymbol{\varepsilon}}{\partial(\mathbf{t}'_{B(\omega)}, \boldsymbol{\rho}_{U(\omega)}, \boldsymbol{\sigma}_{L(\omega)}, \varepsilon_m)} \right| = \left| \det \begin{pmatrix} \frac{\partial \varepsilon_1}{\partial \rho_1} & \frac{d\varepsilon_1}{dt_2} & \frac{\partial \varepsilon_1}{\partial \varepsilon_1} \\ \frac{\partial \varepsilon_2}{\partial \rho_1} & \frac{d\varepsilon_2}{dt_2} & \frac{\partial \varepsilon_2}{\partial \varepsilon_1} \\ \frac{\partial \varepsilon_3}{\partial \rho_1} & \frac{d\varepsilon_3}{dt_2} & \frac{\partial \varepsilon_3}{\partial \varepsilon_1} \end{pmatrix} \right|$$

$$= \left| \det \begin{pmatrix} 0 & -1 & 1 \\ \frac{2S_2'(p_2)}{3} & -1 & 0 \\ \frac{S_3'(p_3)}{3} & 2 & 0 \end{pmatrix} \right| = \frac{4S_2'(p_2)}{3} + \frac{S_3'(p_3)}{3}.$$

Thus we have from Lemma 10 that

$$P(p, q, \boldsymbol{\rho}, \boldsymbol{\sigma}, \omega_4) = \frac{1}{V_3} \int_{t_2=-\bar{t}}^{0} J_V(\omega_4) dt_2 = \frac{\bar{t}\,(4S_2'(p_2) + S_3'(p_3))}{3V_3}$$
$$= \frac{\bar{t}\left(4S'\left(p + \frac{2\rho_1}{3}\right) + S'\left(p + \frac{\rho_1}{3}\right)\right)}{3V_3}. \quad (96)$$

It follows from (74) that

$$\mathbf{H}_{U(\omega)} = \begin{bmatrix} -\frac{2}{3} & -\frac{1}{3} \end{bmatrix}$$

and

$$\mathbf{H}_{L(\omega)} = [\,],$$

so we have from (37) that

$$\mathbf{W} = \begin{bmatrix} -\frac{2}{3} & -\frac{1}{3} \end{bmatrix} \quad (97)$$

Moreover,

$$\boldsymbol{\Psi} = \begin{bmatrix} S'(p_2) & 0 \\ 0 & S'(p_3) \end{bmatrix}. \quad (98)$$

We can now use (97) and (98) to compute

$$\mathbf{1}^T \boldsymbol{\Psi} \mathbf{W}^\top (\mathbf{W} \boldsymbol{\Psi} \mathbf{W}^\top)^{-1} \mathbf{W} \boldsymbol{\Psi} \mathbf{1}$$
$$= \begin{bmatrix} S'(p_2) & S'(p_3) \end{bmatrix} \begin{bmatrix} -\frac{2}{3} \\ -\frac{1}{3} \end{bmatrix} \left(\begin{bmatrix} -\frac{2}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} S'(p_2) & 0 \\ 0 & S'(p_3) \end{bmatrix} \begin{bmatrix} -\frac{2}{3} \\ -\frac{1}{3} \end{bmatrix} \right)^{-1}$$
$$\cdot \begin{bmatrix} -\frac{2}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} S'(p_2) \\ S'(p_3) \end{bmatrix}$$
$$= \frac{1}{4S'(p_2) + S'(p_3)} \left(2S'(p_2) + S'(p_3)\right)^2.$$

We assume symmetric offers in each node, so $Q_n(p) = Q(p)$. The residual demand slope for this state can now be calculated from Lemma 5:

$$D'_{mn}(p, \omega_4, \boldsymbol{\rho}, \boldsymbol{\sigma})$$
$$= -(N-1)Q'(p) - NQ'(p_2) - NQ'(p_3) + \frac{N\left(2Q'(p_2) + Q'(p_3)\right)^2}{4Q'(p_2) + Q'(p_3)}$$
$$= -(N-1)Q'(p) - NQ'(p + \frac{2\rho_1}{3}) - NQ'(p + \frac{1}{3}\rho_1)$$
$$+ \frac{N\left(2Q'(p + \frac{2\rho_1}{3}) + Q'(p + \frac{1}{3}\rho_1)\right)^2}{4Q'(p + \frac{2\rho_1}{3}) + Q'(p + \frac{1}{3}\rho_1)}. \quad (99)$$

We now have from (96) and (99) that the contribution from this state to Z_{mn} in Proposition 5 is:

$$\int_{\boldsymbol{\rho}_{U(\omega_4)}}\int_{\boldsymbol{\sigma}_{L(\omega_4)}} -D'_{mn}(p,\omega_4,\boldsymbol{\rho},\boldsymbol{\sigma})P\left(p,q,\boldsymbol{\rho},\boldsymbol{\sigma},\omega_4\right)d\boldsymbol{\sigma}_{L(\omega_4)}d\boldsymbol{\rho}_{U(\omega_4)}$$
$$= \int_0^{\frac{3(\overline{p}-p)}{2}} \frac{\bar{t}\left(4NQ'\left(p+\frac{2\rho_1}{3}\right)+NQ'\left(p+\frac{\rho_1}{3}\right)\right)}{3V_3}$$
$$\left((N-1)Q'(p)+NQ'(p+\tfrac{2\rho_1}{3})+NQ'(p+\tfrac{\rho_1}{3})-\frac{N\left(2Q'(p+\frac{2\rho_1}{3})+Q'(p+\frac{1}{3}\rho_1)\right)^2}{4Q'(p+\frac{2\rho_1}{3})+Q'(p+\frac{\rho_1}{3})}\right)d\rho_1$$
$$= \frac{\bar{t}(N-1)Q'(p)}{3V_3}\int_0^{\frac{3(\overline{p}-p)}{2}}\left(4NQ'\left(p+\tfrac{2\rho_1}{3}\right)+NQ'\left(p+\tfrac{\rho_1}{3}\right)\right)d\rho_1$$
$$+\frac{\bar{t}N^2}{3V_3}\int_0^{\frac{3(\overline{p}-p)}{2}}\left(Q'(p+\tfrac{2\rho_1}{3})+Q'(p+\tfrac{\rho_1}{3})\right)\left(4Q'\left(p+\tfrac{2\rho_1}{3}\right)+Q'\left(p+\tfrac{\rho_1}{3}\right)\right)d\rho_1$$
$$-\frac{\bar{t}N^2}{3V_3}\int_0^{\frac{3(\overline{p}-p)}{2}}\left(2Q'(p+\tfrac{2\rho_1}{3})+Q'(p+\tfrac{1}{3}\rho_1)\right)^2 d\rho_1$$
$$= \frac{\bar{t}(N-1)Q'(p)}{3V_3}\left[6NQ\left(p+\tfrac{2\rho_1}{3}\right)+3NQ\left(p+\tfrac{\rho_1}{3}\right)\right]_0^{\frac{3(\overline{p}-p)}{2}}$$
$$+\frac{\bar{t}N^2}{3V_3}\int_0^{\frac{3(\overline{p}-p)}{2}}Q'\left(p+\tfrac{2\rho_1}{3}\right)Q'\left(p+\tfrac{\rho_1}{3}\right)d\rho_1.$$

Similarly,

$$\int_{\boldsymbol{\rho}_{U(\omega_4)}}\int_{\boldsymbol{\sigma}_{L(\omega_4)}} P\left(p,q,\boldsymbol{\rho},\boldsymbol{\sigma},\omega_4\right)d\boldsymbol{\sigma}_{L(\omega_4)}d\boldsymbol{\rho}_{U(\omega_4)}$$
$$= \int_0^{\frac{3(\overline{p}-p)}{2}} \frac{\bar{t}\left(4NQ'\left(p+\frac{2\rho_1}{3}\right)+NQ'\left(p+\frac{\rho_1}{3}\right)\right)}{3V_3}d\rho_1$$
$$= \frac{\bar{t}}{V_3}\left[2NQ\left(p+\frac{2\rho_1}{3}\right)+NQ\left(p+\frac{\rho_1}{3}\right)\right]_0^{\frac{3(\overline{p}-p)}{2}}$$

State ω_5

State	$t_1(\omega)$	$t_2(\omega)$	$t_3(\omega)$	$L(\omega)$	$B(\omega)$	$U(\omega)$	$\Xi(\omega)$	$\digamma(\omega)$	$\Upsilon(\omega)$
ω_5	$-\bar{t}$	$\in\left(0,\bar{t}\right)$	$-t_2+\bar{t}$	$\{1\}$	$\{2,3\}$	$\emptyset$	$\{1\}$	$\{\}$	$\{2,3\}$

We have from (73) that

$$\begin{bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\end{bmatrix} = \underbrace{\begin{bmatrix}S_1(p)\\ S_2(p_2)\\ S_3(p_3)\end{bmatrix}}_{\mathbf{s(p)}} + \underbrace{\begin{bmatrix}-1 & 0 & 1\\ 1 & -1 & 0\\ 0 & 1 & -1\end{bmatrix}}_{\mathbf{A}}\underbrace{\begin{bmatrix}-\bar{t}\\ t_2\\ -t_2+\bar{t}\end{bmatrix}}_{\mathbf{t}} \quad (100)$$

$$= \begin{bmatrix}S_1(p)+2\bar{t}-t_2\\ S_2(p_2)-t_2-\bar{t}\\ S_3(p_3)+2t_2-\bar{t}\end{bmatrix} \quad (101)$$

so

$$\frac{d}{dt_2}\begin{bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\end{bmatrix} = \begin{bmatrix}-1\\ -1\\ 2\end{bmatrix}. \quad (102)$$

It follows from (75) that

$$\begin{bmatrix} p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} p - \frac{2}{3}\sigma_1 \\ p - \frac{1}{3}\sigma_1 \end{bmatrix}. \tag{103}$$

We now have from (101) and (103) that

$$\frac{\partial}{\partial \sigma_1} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix} = \begin{bmatrix} \frac{\partial \varepsilon_1}{\partial p_1}\frac{\partial p_1}{\partial \sigma_1} \\ \frac{\partial \varepsilon_2}{\partial p_2}\frac{\partial p_2}{\partial \sigma_1} \\ \frac{\partial \varepsilon_3}{\partial p_3}\frac{\partial p_3}{\partial \sigma_1} \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{2}{3}S_2'(p_2) \\ -\frac{1}{3}S_3'(p_3) \end{bmatrix}. \tag{104}$$

We use (102) and (104) to calculate

$$\begin{aligned} J_V(\omega_5) &= \left| \frac{\partial \boldsymbol{\varepsilon}}{\partial(\mathbf{t}'_{B(\omega)}, \boldsymbol{\rho}_{U(\omega)}, \boldsymbol{\sigma}_{L(\omega)}, \varepsilon_m)} \right| = \left| \det \begin{pmatrix} \frac{\partial \varepsilon_1}{\partial \sigma_1} & \frac{\partial \varepsilon_1}{\partial t_2} & \frac{\partial \varepsilon_1}{\partial \varepsilon_1} \\ \frac{\partial \varepsilon_2}{\partial \sigma_1} & \frac{\partial \varepsilon_2}{\partial t_2} & \frac{\partial \varepsilon_2}{\partial \varepsilon_1} \\ \frac{\partial \varepsilon_3}{\partial \sigma_1} & \frac{\partial \varepsilon_3}{\partial t_2} & \frac{\partial \varepsilon_3}{\partial \varepsilon_1} \end{pmatrix} \right| \\ &= \left| \det \begin{pmatrix} 0 & -1 & 1 \\ \frac{-2S_2'(p_2)}{3} & -1 & 0 \\ \frac{-S_3'(p_3)}{3} & 2 & 0 \end{pmatrix} \right| \\ &= \frac{4S_2'(p_2)}{3} + \frac{S_3'(p_3)}{3}. \end{aligned}$$

Thus we have from Lemma 10 that

$$\begin{aligned} P(p, q, \boldsymbol{\rho}, \boldsymbol{\sigma}, \omega_5) &= \frac{1}{V_3} \int_{t_2=0}^{\bar{t}} J_V(\omega_5) dt_2 = \frac{4\bar{t}S_2'(p_2)}{3V_3} + \frac{\bar{t}S_3'(p_3)}{3V_3} \\ &= \frac{\bar{t}\left(4S'\left(p - \frac{2\sigma_1}{3}\right) + S'\left(p - \frac{\sigma_1}{3}\right)\right)}{3V_3}. \end{aligned} \tag{105}$$

It follows from (74) that

$$\mathbf{H}_{U(\omega)} = []$$

and

$$\mathbf{H}_{L(\omega)} = \begin{bmatrix} -\frac{2}{3} & -\frac{1}{3} \end{bmatrix},$$

so we have from (37) that

$$\mathbf{W} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \end{bmatrix}. \tag{106}$$

Moreover,

$$\boldsymbol{\Psi} = \begin{bmatrix} S'(p_2) & 0 \\ 0 & S'(p_3) \end{bmatrix} \tag{107}$$

We can now use (106) and (107) to compute

$$\begin{aligned} &\mathbf{1}^T \boldsymbol{\Psi} \mathbf{W}^\top (\mathbf{W} \boldsymbol{\Psi} \mathbf{W}^\top)^{-1} \mathbf{W} \boldsymbol{\Psi} \mathbf{1} \\ &= \begin{bmatrix} S'(p_2) & S'(p_3) \end{bmatrix} \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix} \left(\begin{bmatrix} \frac{2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} S'(p_2) & 0 \\ 0 & S'(p_3) \end{bmatrix} \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix} \right)^{-1} \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} S'(p_2) \\ S'(p_3) \end{bmatrix} \\ &= \frac{1}{4S'(p_2) + S'(p_3)} \left(2S'(p_2) + S'(p_3)\right)^2. \end{aligned}$$

We assume symmetric offers in each node, so $Q_n(p) = Q(p)$. The residual demand slope for this state can now be calculated from Lemma 5:

$$
\begin{aligned}
& D'_{mn}(p, \omega_5, \boldsymbol{\rho}, \boldsymbol{\sigma}) \\
= & -(N-1)Q'(p) - NQ'(p_2) - NQ'(p_3) \\
& + \frac{N}{4Q'(p_2) + Q'(p_3)} \left(2Q'(p_2) + Q'(p_3)\right)^2 \\
= & -(N-1)Q'(p) - NQ'(p - \frac{2\sigma_1}{3}) - NQ'(p - \frac{1}{3}\sigma_1) \\
& + \frac{N\left(2Q'(p - \frac{2\sigma_1}{3}) + Q'(p - \frac{1}{3}\sigma_1)\right)^2}{4Q'(p - \frac{2\sigma_1}{3}) + Q'(p - \frac{1}{3}\sigma_1)}
\end{aligned} \tag{108}
$$

We now have from (105) and (108) that the contribution from this state to Z_{mn} in Proposition 5 is:

$$
\begin{aligned}
& \int_{\boldsymbol{\rho}_{U(\omega_5)}} \int_{\boldsymbol{\sigma}_{L(\omega_5)}} -D'_{mn}(p, \omega_5, \boldsymbol{\rho}, \boldsymbol{\sigma}) P\left(p, q, \boldsymbol{\rho}, \boldsymbol{\sigma}, \omega_5\right) d\boldsymbol{\sigma}_{L(\omega_5)} d\boldsymbol{\rho}_{U(\omega_5)} \\
& \int_0^{\frac{3p}{2}} \frac{\bar{t}\left(4NQ'\left(p - \frac{2\sigma_1}{3}\right) + NQ'\left(p - \frac{\sigma_1}{3}\right)\right)}{3V_3} \\
& \left((N-1)Q'(p) + NQ'(p - \frac{2\sigma_1}{3}) + NQ'(p - \frac{1}{3}\sigma_1) - \frac{N\left(2Q'(p - \frac{2\sigma_1}{3}) + Q'(p - \frac{1}{3}\sigma_1)\right)^2}{4Q'(p - \frac{2\sigma_1}{3}) + Q'(p - \frac{1}{3}\sigma_1)}\right) d\sigma_1 \\
& = \frac{\bar{t}(N-1)Q'(p)}{3V_3} \int_0^{\frac{3p}{2}} \left(4NQ'\left(p - \frac{2\sigma_1}{3}\right) + NQ'\left(p - \frac{\sigma_1}{3}\right)\right) d\sigma_1 \\
& + \frac{\bar{t}N^2}{3V_3} \int_0^{\frac{3p}{2}} \left(Q'(p - \frac{2\sigma_1}{3}) + Q'(p - \frac{1}{3}\sigma_1)\right) \left(4Q'\left(p - \frac{2\sigma_1}{3}\right) + Q'\left(p - \frac{\sigma_1}{3}\right)\right) d\sigma_1 \\
& - \frac{\bar{t}N^2}{3V_3} \int_0^{\frac{3p}{2}} \left(2Q'(p - \frac{2\sigma_1}{3}) + Q'(p - \frac{\sigma_1}{3})\right)^2 d\sigma_1 \\
& = \frac{\bar{t}(N-1)Q'(p)}{3V_3} \left[-6NQ\left(p - \frac{2\sigma_1}{3}\right) - 3NQ\left(p - \frac{\sigma_1}{3}\right)\right]_0^{\frac{3p}{2}} \\
& + \frac{\bar{t}N^2}{3V_3} \int_0^{\frac{3p}{2}} Q'\left(p - \frac{2\sigma_1}{3}\right) Q'\left(p - \frac{\sigma_1}{3}\right) d\sigma_1.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
& \int_{\boldsymbol{\rho}_{U(\omega_5)}} \int_{\boldsymbol{\sigma}_{L(\omega_5)}} P\left(p, q, \boldsymbol{\rho}, \boldsymbol{\sigma}, \omega_5\right) d\boldsymbol{\sigma}_{L(\omega_5)} d\boldsymbol{\rho}_{U(\omega_5)} \\
= & \int_0^{\frac{3p}{2}} \frac{\bar{t}\left(4NQ'\left(p - \frac{2\sigma_1}{3}\right) + NQ'\left(p - \frac{\sigma_1}{3}\right)\right)}{3V_3} d\sigma_1 \\
= & \frac{\bar{t}}{V_3} \left[-2NQ\left(p - \frac{2\sigma_1}{3}\right) - NQ\left(p - \frac{\sigma_1}{3}\right)\right]_0^{\frac{3p}{2}}
\end{aligned}
$$

State ω_6

State	$t_1(\omega)$	$t_2(\omega)$	$t_3(\omega)$	$L(\omega)$	$B(\omega)$	$U(\omega)$	$\Xi(\omega)$	$\digamma(\omega)$	$\Upsilon(\omega)$
ω_6	$\in \left(-\bar{t}, 0\right)$	$-t_1 - \bar{t}$	$\bar{t}$	$\emptyset$	$\{1,2\}$	$\{3\}$	$\{1\}$	$\{\}$	$\{2,3\}$

It follows from (75) that

$$
\begin{bmatrix} p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} p - \frac{1}{3}\rho_3 \\ p - \frac{2}{3}\rho_3 \end{bmatrix}
$$

From the perspective of a producer in node 1, state ω_6 is symmetric with ω_5, so

$$J_V(\omega_6) = \frac{S_2'(p_2)}{3} + \frac{4S_3'(p_3)}{3},$$

$$P(p, q, \boldsymbol{\rho}, \boldsymbol{\sigma}, \omega_6) = \frac{\bar{t}S_2'(p_2)}{3V_3} + \frac{4\bar{t}S_3'(p_3)}{3V_3} = \frac{\bar{t}\left(S'\left(p - \frac{\rho_3}{3}\right) + 4S'\left(p - \frac{2\rho_3}{3}\right)\right)}{3V_3},$$

and

$$\begin{aligned} & D'_{mn}(p, \omega_6, \boldsymbol{\rho}, \boldsymbol{\sigma}) \\ = \; & -(N-1)Q'(p) - NQ'(p_2) - NQ'(p_3) + \frac{N\left(Q'(p_2) + 2Q'(p_3)\right)^2}{Q'(p_2) + 4Q'(p_3)} \\ = \; & -(N-1)Q'(p) - NQ'(p - \frac{\rho_3}{3}) - NQ'(p - \frac{2\rho_3}{3}) \\ & + \frac{N}{Q'((p - \frac{\rho_3}{3}) + 4Q'(p - \frac{2\rho_3}{3})}\left(Q'((p - \frac{\rho_3}{3}) + 2Q'(p - \frac{2\rho_3}{3})\right)^2. \end{aligned}$$

Moreover, the contribution to Z_{mn} in Proposition 5 is the same as for state ω_5.

State ω_7

State	$t_1(\omega)$	$t_2(\omega)$	$t_3(\omega)$	$L(\omega)$	$B(\omega)$	$U(\omega)$	$\Xi(\omega)$	$F(\omega)$	$\Upsilon(\omega)$
ω_7	$\in (0, \bar{t})$	$-t_1 + \bar{t}$	$-\bar{t}$	$\emptyset$	$\{1,2\}$	$\{3\}$	$\{1,2,3\}$	$\{\}$	$\{2,3\}$

We have from (73) that

$$\begin{aligned} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix} &= \underbrace{\begin{bmatrix} S_1(p) \\ S_2(p_2) \\ S_3(p_3) \end{bmatrix}}_{\mathbf{s(p)}} + \underbrace{\begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} t_1 \\ -t_1 + \bar{t} \\ -\bar{t} \end{bmatrix}}_{\mathbf{t}} \\ &= \begin{bmatrix} S_1(p) - \bar{t} - t_1 \\ S_2(p_2) + 2t_1 - \bar{t} \\ S_3(p_3) - t_1 + 2\bar{t} \end{bmatrix}, \end{aligned} \tag{109}$$

so

$$\frac{d}{dt_1}\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}. \tag{110}$$

It follows from (75) that

$$\begin{bmatrix} p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} p + \frac{1}{3}\sigma_3 \\ p + \frac{2}{3}\sigma_3 \end{bmatrix}. \tag{111}$$

We now have from (109) and (111) that

$$\frac{\partial}{\partial \sigma_3}\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix} = \begin{bmatrix} \frac{\partial \varepsilon_1}{\partial p_1}\frac{\partial p_1}{\partial \sigma_3} \\ \frac{\partial \varepsilon_2}{\partial p_2}\frac{\partial p_2}{\partial \sigma_3} \\ \frac{\partial \varepsilon_3}{\partial p_3}\frac{\partial p_3}{\partial \sigma_3} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{3}S_2'(p_2) \\ \frac{2}{3}S_3'(p_3) \end{bmatrix}. \tag{112}$$

We use (110) and (112) to calculate

$$
\begin{aligned}
J_V(\omega_7) &= \left| \frac{\partial \boldsymbol{\varepsilon}}{\partial(\mathbf{t}'_{B(\omega)}, \boldsymbol{\rho}_{U(\omega)}, \boldsymbol{\sigma}_{L(\omega)}, \varepsilon_m)} \right| = \left| \det \begin{pmatrix} \frac{\partial \varepsilon_1}{\partial \sigma_3} & \frac{d\varepsilon_1}{dt_1} & \frac{\partial \varepsilon_1}{\partial \varepsilon_1} \\ \frac{\partial \varepsilon_2}{\partial \sigma_3} & \frac{d\varepsilon_2}{dt_1} & \frac{\partial \varepsilon_2}{\partial \varepsilon_1} \\ \frac{\partial \varepsilon_3}{\partial \sigma_3} & \frac{d\varepsilon_3}{dt_1} & \frac{\partial \varepsilon_3}{\partial \varepsilon_1} \end{pmatrix} \right| \\
&= \left| \det \begin{pmatrix} 0 & -1 & 1 \\ \frac{S_2'(p_2)}{3} & 2 & 0 \\ \frac{2S_3'(p_3)}{3} & -1 & 0 \end{pmatrix} \right| = \frac{S_2'(p_2)}{3} + \frac{4S_3'(p_3)}{3}.
\end{aligned}
$$

Thus we have from Lemma 10 that

$$
P\left(p, q, \boldsymbol{\rho}, \boldsymbol{\sigma}, \omega_7\right) = \frac{1}{V_3} \int_{t_1=0}^{\bar{t}} J_V(\omega_7) dt_1 = \frac{\bar{t}\left(S'\left(p + \frac{1}{3}\sigma_3\right) + 4S'\left(p + \frac{2}{3}\sigma_3\right)\right)}{3V_3}. \quad (113)
$$

It follows from (74) that

$$
\mathbf{H}_{U(\omega)} = []
$$

and

$$
\mathbf{H}_{L(\omega)} = \left[\begin{array}{cc} \frac{1}{3} & \frac{2}{3} \end{array} \right],
$$

so we have from (37) that

$$
\mathbf{W} = \left[\begin{array}{cc} -\frac{1}{3} & -\frac{2}{3} \end{array} \right]. \quad (114)
$$

Moreover,

$$
\boldsymbol{\Psi} = \left[\begin{array}{cc} S'(p_2) & 0 \\ 0 & S'(p_3) \end{array} \right]. \quad (115)
$$

We can now use (114) and (115) to compute

$$
\begin{aligned}
& \mathbf{1}^T \boldsymbol{\Psi} \mathbf{W}^\top (\mathbf{W} \boldsymbol{\Psi} \mathbf{W}^\top)^{-1} \mathbf{W} \boldsymbol{\Psi} \mathbf{1} \\
= & \left[\begin{array}{cc} S'(p_2) & S'(p_3) \end{array} \right] \left[\begin{array}{c} -\frac{1}{3} \\ -\frac{2}{3} \end{array} \right] \left(\left[\begin{array}{cc} -\frac{1}{3} & -\frac{2}{3} \end{array} \right] \left[\begin{array}{cc} S'(p_2) & 0 \\ 0 & S'(p_3) \end{array} \right] \left[\begin{array}{c} -\frac{1}{3} \\ -\frac{2}{3} \end{array} \right] \right)^{-1} \\
& \cdot \left[\begin{array}{cc} -\frac{1}{3} & -\frac{2}{3} \end{array} \right] \left[\begin{array}{c} S'(p_2) \\ S'(p_3) \end{array} \right] \\
= & \frac{\left(S'(p_2) + 2S'(p_3)\right)^2}{S'(p_2) + 4S'(p_3)}.
\end{aligned}
$$

We assume symmetric offers in each node, so $Q_n(p) = Q(p)$. The residual demand slope for this state can now be calculated from Lemma 5:

$$
\begin{aligned}
& D'_{mn}(p, \omega_7, \boldsymbol{\rho}, \boldsymbol{\sigma}) \\
= & -(N-1)Q'(p) - NQ'(p + \frac{1}{3}\sigma_3) - NQ'(p + \frac{2}{3}\sigma_3) \\
& + \frac{N\left(Q'(p + \frac{1}{3}\sigma_3) + 2Q'(p + \frac{2}{3}\sigma_3)\right)^2}{Q'(p + \frac{1}{3}\sigma_3) + 4Q'(p + \frac{2}{3}\sigma_3)}. \quad (116)
\end{aligned}
$$

We now have from (113) and (116) that the contribution from this state to Z_{mn} in Proposition 5 is:

$$\int_{\boldsymbol{\rho}_{U(\omega_7)}}\int_{\boldsymbol{\sigma}_{L(\omega_7)}} -D'_{mn}(p,\omega_7,\boldsymbol{\rho},\boldsymbol{\sigma})P\left(p,q,\boldsymbol{\rho},\boldsymbol{\sigma},\omega_7\right)d\boldsymbol{\sigma}_{L(\omega_7)}d\boldsymbol{\rho}_{U(\omega_7)}$$
$$= \int_0^{\frac{3(\bar{p}-p)}{2}} \frac{\bar{t}\left(4NQ'\left(p+\frac{2\sigma_3}{3}\right)+NQ'\left(p+\frac{\sigma_3}{3}\right)\right)}{3V_3}$$
$$\left((N-1)Q'(p)+NQ'(p+\tfrac{2\sigma_3}{3})+NQ'(p+\tfrac{\sigma_3}{3})-\frac{N\left(2Q'(p+\frac{2\sigma_3}{3})+Q'(p+\frac{\sigma_3}{3})\right)^2}{4Q'(p+\frac{2\sigma_3}{3})+Q'(p+\frac{\sigma_3}{3})}\right)d\sigma_3$$
$$= \frac{\bar{t}(N-1)Q'(p)}{3V_3}\int_0^{\frac{3(\bar{p}-p)}{2}}\left(4NQ'\left(p+\tfrac{2\sigma_3}{3}\right)+NQ'\left(p+\tfrac{\sigma_3}{3}\right)\right)d\sigma_3$$
$$+\frac{\bar{t}N^2}{3V_3}\int_0^{\frac{3(\bar{p}-p)}{2}}\left(Q'(p+\tfrac{2\sigma_3}{3})+Q'(p+\tfrac{\sigma_3}{3})\right)\left(4Q'\left(p+\tfrac{2\sigma_3}{3}\right)+Q'\left(p+\tfrac{\sigma_3}{3}\right)\right)d\sigma_3$$
$$-\frac{\bar{t}N^2}{3V_3}\int_0^{\frac{3(\bar{p}-p)}{2}}\left(2Q'(p+\tfrac{2\sigma_3}{3})+Q'(p+\tfrac{\sigma_3}{3})\right)^2 d\sigma_3$$
$$= \frac{\bar{t}(N-1)Q'(p)}{3V_3}\left[6NQ\left(p+\tfrac{2\sigma_3}{3}\right)+3NQ\left(p+\tfrac{\sigma_3}{3}\right)\right]_0^{\frac{3(\bar{p}-p)}{2}}$$
$$+\frac{\bar{t}N^2}{3V_3}\int_0^{\frac{3(\bar{p}-p)}{2}}Q'\left(p+\tfrac{2\sigma_3}{3}\right)Q'\left(p+\tfrac{\sigma_3}{3}\right)d\sigma_3.$$

Similarly,

$$\int_{\boldsymbol{\rho}_{U(\omega_7)}}\int_{\boldsymbol{\sigma}_{L(\omega_7)}} P\left(p,q,\boldsymbol{\rho},\boldsymbol{\sigma},\omega_7\right)d\boldsymbol{\sigma}_{L(\omega_7)}d\boldsymbol{\rho}_{U(\omega_7)}$$
$$= \int_0^{\frac{3(\bar{p}-p)}{2}} \frac{\bar{t}\left(4NQ'\left(p+\frac{2\sigma_3}{3}\right)+NQ'\left(p+\frac{\sigma_3}{3}\right)\right)}{3V_3}d\sigma_3$$
$$= \frac{\bar{t}}{V_3}\left[2NQ\left(p+\frac{2\sigma_3}{3}\right)+NQ\left(p+\frac{\sigma_3}{3}\right)\right]_0^{\frac{3(\bar{p}-p)}{2}}.$$

State ω_8

State	$t_1(\omega)$	$t_2(\omega)$	$t_3(\omega)$	$L(\omega)$	$B(\omega)$	$U(\omega)$	$\Xi(\omega)$	$F(\omega)$	$\Upsilon(\omega)$
ω_8	$\bar{t}$	$-\bar{t}$	0	$\{2\}$	$\emptyset$	$\{1\}$	$\{1\}$	$\{2,3\}$	$\emptyset$

We have from (73) that

$$\begin{bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\end{bmatrix} = \underbrace{\begin{bmatrix}S_1(p)\\ S_2(p_2)\\ S_3(p_3)\end{bmatrix}}_{\mathbf{s}(\mathbf{p})} + \underbrace{\begin{bmatrix}-1 & 0 & 1\\ 1 & -1 & 0\\ 0 & 1 & -1\end{bmatrix}}_{\mathbf{A}}\underbrace{\begin{bmatrix}\bar{t}\\ -\bar{t}\\ 0\end{bmatrix}}_{\mathbf{t}} \tag{117}$$
$$= \begin{bmatrix}S_1(p)-\bar{t}\\ S_2(p_2)+2\bar{t}\\ S_3(p_3)-\bar{t}\end{bmatrix}.$$

It follows from (75) that

$$\begin{bmatrix}p_2\\ p_3\end{bmatrix} = \begin{bmatrix}p+\frac{2}{3}\rho_1+\frac{1}{3}\sigma_2\\ p+\frac{1}{3}\rho_1-\frac{1}{3}\sigma_2\end{bmatrix}. \tag{118}$$

We now have from (92) and (94) that

$$\frac{\partial}{\partial \rho_1}\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix} = \begin{bmatrix} \frac{\partial \varepsilon_1}{\partial p_1}\frac{\partial p_1}{\partial \rho_1} \\ \frac{\partial \varepsilon_2}{\partial p_2}\frac{\partial p_2}{\partial \rho_1} \\ \frac{\partial \varepsilon_3}{\partial p_3}\frac{\partial p_3}{\partial \rho_1} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{2}{3}S_2'(p_2) \\ \frac{1}{3}S_3'(p_3) \end{bmatrix} \tag{119}$$

and

$$\frac{\partial}{\partial \sigma_2}\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix} = \begin{bmatrix} \frac{\partial \varepsilon_1}{\partial p_1}\frac{\partial p_1}{\partial \sigma_2} \\ \frac{\partial \varepsilon_2}{\partial p_2}\frac{\partial p_2}{\partial \sigma_2} \\ \frac{\partial \varepsilon_3}{\partial p_3}\frac{\partial p_3}{\partial \sigma_2} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{3}S_2'(p_2) \\ -\frac{1}{3}S_3'(p_3) \end{bmatrix}. \tag{120}$$

We use (119) and (120) to calculate

$$\begin{aligned} J_V(\omega_8) &= \left|\frac{\partial \boldsymbol{\varepsilon}}{\partial(\mathbf{t}'_{B(\omega)}, \boldsymbol{\rho}_{U(\omega)}, \boldsymbol{\sigma}_{L(\omega)}, \varepsilon_m)}\right| = \left|\det\begin{pmatrix} \frac{\partial \varepsilon_1}{\partial \rho_1} & \frac{\partial \varepsilon_1}{\partial \sigma_2} & \frac{\partial \varepsilon_1}{\partial \varepsilon_1} \\ \frac{\partial \varepsilon_2}{\partial \rho_1} & \frac{\partial \varepsilon_2}{\partial \sigma_2} & \frac{\partial \varepsilon_2}{\partial \varepsilon_1} \\ \frac{\partial \varepsilon_3}{\partial \rho_1} & \frac{\partial \varepsilon_3}{\partial \sigma_2} & \frac{\partial \varepsilon_3}{\partial \varepsilon_1} \end{pmatrix}\right| \\ &= \left|\det\begin{pmatrix} 0 & 0 & 1 \\ \frac{2S_2'(p_2)}{3} & \frac{S_2'(p_2)}{3} & 0 \\ \frac{S_3'(p_3)}{3} & -\frac{S_3'(p_3)}{3} & 0 \end{pmatrix}\right| = \frac{S_2'(p_2)\,S_3'(p_3)}{3}. \end{aligned}$$

Thus we have from Lemma 10 that

$$P(p, q, \boldsymbol{\rho}, \boldsymbol{\sigma}, \omega_8) = \frac{S_2'(p_2)\,S_3'(p_3)}{3V_3}. \tag{121}$$

The residual demand slope in this state is

$$-D'_{mn}(p, \omega_8, \boldsymbol{\rho}, \boldsymbol{\sigma}) = -(N-1)Q'(p), \tag{122}$$

because the flows in all lines into node 1 are fixed.

We make the substitution $u = p + \frac{2}{3}\rho_1 + \frac{1}{3}\sigma_2$ and $v = p + \frac{1}{3}\rho_1 - \frac{1}{3}\sigma_2$. We now have from (121) and (122) that the contribution from this state to Z_{mn} in Proposition 5 is:

$$\begin{aligned} &\int_{\boldsymbol{\rho}_{U(\omega_8)}}\int_{\boldsymbol{\sigma}_{L(\omega_8)}} -D'_{mn}(p, \omega_8, \boldsymbol{\rho}, \boldsymbol{\sigma}) P(p, q, \boldsymbol{\rho}, \boldsymbol{\sigma}, \omega_8)\, d\boldsymbol{\sigma}_{L(\omega_8)} d\boldsymbol{\rho}_{U(\omega_8)} \\ &= \frac{(N-1)Q'(p)}{3V_3}\int_p^{\overline{p}}\int_0^{\overline{p}} S_2'(u)\,S_3'(v)\,\frac{\partial(\rho_1, \sigma_2)}{\partial(u, v)} du dv \\ &= \frac{(N-1)N^2Q'(p)}{V_3}\int_p^{\overline{p}}\int_0^{\overline{p}} Q'(u)\,Q'(v)\, du dv \\ &= \frac{(N-1)N^2Q'(p)}{V_3}\left(Q(\overline{p}) - Q(p)\right)Q(\overline{p}) \end{aligned}$$

Similarly,

$$\begin{aligned}
&\int_{\boldsymbol{\rho}_{U(\omega_8)}}\int_{\boldsymbol{\sigma}_{L(\omega_8)}} P\left(p,q,\boldsymbol{\rho},\boldsymbol{\sigma},\omega_8\right) d\boldsymbol{\sigma}_{L(\omega_8)} d\boldsymbol{\rho}_{U(\omega_8)} \\
=\ & \frac{1}{3V_3}\int_p^{\overline{p}}\int_0^{\overline{p}} S_2'\left(u\right) S_3'\left(v\right)\frac{\partial\left(\rho_1,\sigma_2\right)}{\partial\left(u,v\right)} dudv \\
=\ & \frac{N^2}{V_3}\left(Q\left(\overline{p}\right)-Q\left(p\right)\right)Q\left(\overline{p}\right).
\end{aligned}$$

State ω_9

State	$t_1(\omega)$	$t_2(\omega)$	$t_3(\omega)$	$L(\omega)$	$B(\omega)$	$U(\omega)$	$\Xi(\omega)$	$\digamma(\omega)$	$\Upsilon(\omega)$
ω_9	$\bar{t}$	0	$-\bar{t}$	$\{2\}$	$\emptyset$	$\{1\}$	$\{1\}$	$\{2,3\}$	$\emptyset$

We have from (73) that

$$\begin{aligned}
\begin{bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\end{bmatrix} &= \underbrace{\begin{bmatrix}S_1\left(p\right)\\ S_2\left(p_2\right)\\ S_3\left(p_3\right)\end{bmatrix}}_{\mathbf{s}(\mathbf{p})}+\underbrace{\begin{bmatrix}-1&0&1\\ 1&-1&0\\ 0&1&-1\end{bmatrix}}_{\mathbf{A}}\underbrace{\begin{bmatrix}\bar{t}\\ 0\\ -\bar{t}\end{bmatrix}}_{\mathbf{t}} \\
&= \begin{bmatrix}S_1\left(p\right)-2\bar{t}\\ S_2\left(p_2\right)+\bar{t}\\ S_3\left(p_3\right)+\bar{t}\end{bmatrix}. \qquad (123)
\end{aligned}$$

It follows from (75) that

$$\begin{bmatrix}p_2\\ p_3\end{bmatrix}=\begin{bmatrix}p+\frac{2}{3}\rho_1+\frac{1}{3}\sigma_3\\ p+\frac{1}{3}\rho_1+\frac{2}{3}\sigma_3\end{bmatrix}. \qquad (124)$$

We now have from (123) and (124) that

$$\frac{\partial}{\partial\rho_1}\begin{bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\end{bmatrix}=\begin{bmatrix}\frac{\partial\varepsilon_1}{\partial p_1}\frac{\partial p_1}{\partial\rho_1}\\ \frac{\partial\varepsilon_2}{\partial p_2}\frac{\partial p_2}{\partial\rho_1}\\ \frac{\partial\varepsilon_3}{\partial p_3}\frac{\partial p_3}{\partial\rho_1}\end{bmatrix}=\begin{bmatrix}0\\ \frac{2}{3}S_2'\left(p_2\right)\\ \frac{1}{3}S_3'\left(p_3\right)\end{bmatrix} \qquad (125)$$

and

$$\frac{\partial}{\partial\sigma_3}\begin{bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\end{bmatrix}=\begin{bmatrix}\frac{\partial\varepsilon_1}{\partial p_1}\frac{\partial p_1}{\partial\sigma_3}\\ \frac{\partial\varepsilon_2}{\partial p_2}\frac{\partial p_2}{\partial\sigma_3}\\ \frac{\partial\varepsilon_3}{\partial p_3}\frac{\partial p_3}{\partial\sigma_3}\end{bmatrix}=\begin{bmatrix}0\\ \frac{1}{3}S_2'\left(p_2\right)\\ \frac{2}{3}S_3'\left(p_3\right)\end{bmatrix}. \qquad (126)$$

We use (125) and (126) to calculate

$$\begin{aligned}
J_V(\omega_9) &= \left|\frac{\partial\boldsymbol{\varepsilon}}{\partial(\mathbf{t}'_{B(\omega)},\boldsymbol{\rho}_{U(\omega)},\boldsymbol{\sigma}_{L(\omega)},\varepsilon_m)}\right| = \left|\det\begin{pmatrix}\frac{\partial\varepsilon_1}{\partial\rho_1} & \frac{\partial\varepsilon_1}{\partial\sigma_3} & \frac{\partial\varepsilon_1}{\partial\varepsilon_1}\\ \frac{\partial\varepsilon_2}{\partial\rho_1} & \frac{\partial\varepsilon_2}{\partial\sigma_3} & \frac{\partial\varepsilon_2}{\partial\varepsilon_1}\\ \frac{\partial\varepsilon_3}{\partial\rho_1} & \frac{\partial\varepsilon_3}{\partial\sigma_3} & \frac{\partial\varepsilon_3}{\partial\varepsilon_1}\end{pmatrix}\right| \\
&= \left|\det\begin{pmatrix}0 & 0 & 1\\ \frac{2S_2'(p_2)}{3} & \frac{S_2'(p_2)}{3} & 0\\ \frac{S_3'(p_3)}{3} & \frac{2S_3'(p_3)}{3} & 0\end{pmatrix}\right| = \frac{S_2'\left(p_2\right)S_3'\left(p_3\right)}{3}.
\end{aligned}$$

Thus we have from Lemma 10 that

$$P\left(p,q,\boldsymbol{\rho},\boldsymbol{\sigma},\omega_9\right)=\frac{S_2'\left(p_2\right)S_3'\left(p_3\right)}{3V_3}. \tag{127}$$

The residual demand slope in this state is

$$-D_{mn}'(p,\omega_9,\boldsymbol{\rho},\boldsymbol{\sigma})=-(N-1)Q'(p), \tag{128}$$

because the flows in all lines into node 1 are fixed.

We make the substitution $u=p+\frac{2}{3}\rho_1+\frac{1}{3}\sigma_3$ and $v=p+\frac{1}{3}\rho_1+\frac{2}{3}\sigma_3$. It can be shown that $\frac{\partial(\rho_1,\sigma_3)}{\partial(u,v)}=3$. We now have from (127) and (128) that the contribution from this state to Z_{mn} in Proposition 5 is:

$$\begin{aligned}
&\int_{\boldsymbol{\rho}_{U(\omega_9)}}\int_{\boldsymbol{\sigma}_{L(\omega_9)}}-D_{mn}'(p,\omega_9,\boldsymbol{\rho},\boldsymbol{\sigma})P\left(p,q,\boldsymbol{\rho},\boldsymbol{\sigma},\omega_9\right)d\boldsymbol{\sigma}_{L(\omega_9)}d\boldsymbol{\rho}_{U(\omega_9)}\\
=&\ \frac{(N-1)Q'(p)}{3V_3}\int_p^{\overline{p}}\int_p^{\overline{p}}S_2'\left(u\right)S_3'\left(v\right)\frac{\partial\left(\rho_1,\sigma_3\right)}{\partial\left(u,v\right)}dudv\\
=&\ \frac{(N-1)N^2Q'(p)}{V_3}\int_p^{\overline{p}}\int_p^{\overline{p}}Q'\left(u\right)Q'\left(v\right)dudv\\
=&\ \frac{(N-1)N^2Q'(p)}{V_3}\left(Q\left(\overline{p}\right)-Q\left(p\right)\right)^2.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int_{\boldsymbol{\rho}_{U(\omega_9)}}\int_{\boldsymbol{\sigma}_{L(\omega_9)}}P\left(p,q,\boldsymbol{\rho},\boldsymbol{\sigma},\omega_9\right)d\boldsymbol{\sigma}_{L(\omega_9)}d\boldsymbol{\rho}_{U(\omega_9)}\\
=&\ \frac{1}{3V_3}\int_p^{\overline{p}}\int_p^{\overline{p}}S_2'\left(u\right)S_3'\left(v\right)\frac{\partial\left(\rho_1,\sigma_3\right)}{\partial\left(u,v\right)}dudv\\
=&\ \frac{N^2}{V_3}\left(Q\left(\overline{p}\right)-Q\left(p\right)\right)^2.
\end{aligned}$$

State ω_{10}

State	$t_1(\omega)$	$t_2(\omega)$	$t_3(\omega)$	$L(\omega)$	$B(\omega)$	$U(\omega)$	$\Xi(\omega)$	$F(\omega)$	$\Upsilon(\omega)$
ω_{10}	$-\overline{t}$	$\overline{t}$	0	$\{1\}$	$\emptyset$	$\{2\}$	$\{1\}$	$\{2,3\}$	$\emptyset$

We have from (73) that

$$\begin{aligned}
\begin{bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\end{bmatrix}&=\underbrace{\begin{bmatrix}S_1\left(p\right)\\ S_2\left(p_2\right)\\ S_3\left(p_3\right)\end{bmatrix}}_{\mathbf{s(p)}}+\underbrace{\begin{bmatrix}-1&0&1\\ 1&-1&0\\ 0&1&-1\end{bmatrix}}_{\mathbf{A}}\underbrace{\begin{bmatrix}-\overline{t}\\ \overline{t}\\ 0\end{bmatrix}}_{\mathbf{t}}\\
&=\begin{bmatrix}S_1\left(p\right)+\overline{t}\\ S_2\left(p_2\right)-2\overline{t}\\ S_3\left(p_3\right)+\overline{t}\end{bmatrix}.
\end{aligned} \tag{129}$$

It follows from (75) that

$$\begin{bmatrix} p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} p - \frac{2}{3}\sigma_1 - \frac{1}{3}\rho_2 \\ p - \frac{1}{3}\sigma_1 + \frac{1}{3}\rho_2 \end{bmatrix}. \tag{130}$$

We now have from (129) and (130) that

$$\frac{\partial}{\partial \sigma_1}\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix} = \begin{bmatrix} \frac{\partial \varepsilon_1}{\partial p_1}\frac{\partial p_1}{\partial \sigma_1} \\ \frac{\partial \varepsilon_2}{\partial p_2}\frac{\partial p_2}{\partial \sigma_1} \\ \frac{\partial \varepsilon_3}{\partial p_3}\frac{\partial p_3}{\partial \sigma_1} \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{2}{3}S_2'(p_2) \\ -\frac{1}{3}S_3'(p_3) \end{bmatrix} \tag{131}$$

and

$$\frac{\partial}{\partial \rho_2}\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix} = \begin{bmatrix} \frac{\partial \varepsilon_1}{\partial p_1}\frac{\partial p_1}{\partial \rho_2} \\ \frac{\partial \varepsilon_2}{\partial p_2}\frac{\partial p_2}{\partial \rho_2} \\ \frac{\partial \varepsilon_3}{\partial p_3}\frac{\partial p_3}{\partial \rho_2} \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{3}S_2'(p_2) \\ \frac{1}{3}S_3'(p_3) \end{bmatrix}. \tag{132}$$

We use (131) and (132) to calculate

$$\begin{aligned} J_V(\omega_{10}) &= \left| \frac{\partial \boldsymbol{\varepsilon}}{\partial(\mathbf{t}'_{B(\omega)}, \boldsymbol{\rho}_{U(\omega)}, \boldsymbol{\sigma}_{L(\omega)}, \varepsilon_m)} \right| = \left| \det \begin{pmatrix} \frac{\partial \varepsilon_1}{\partial \sigma_1} & \frac{\partial \varepsilon_1}{\partial \rho_2} & \frac{\partial \varepsilon_1}{\partial \varepsilon_1} \\ \frac{\partial \varepsilon_2}{\partial \sigma_1} & \frac{\partial \varepsilon_2}{\partial \rho_2} & \frac{\partial \varepsilon_2}{\partial \varepsilon_1} \\ \frac{\partial \varepsilon_3}{\partial \sigma_1} & \frac{\partial \varepsilon_3}{\partial \rho_2} & \frac{\partial \varepsilon_3}{\partial \varepsilon_1} \end{pmatrix} \right| \\ &= \left| \det \begin{pmatrix} 0 & 0 & 1 \\ -\frac{2S_2'(p_2)}{3} & -\frac{S_2'(p_2)}{3} & 0 \\ -\frac{S_3'(p_3)}{3} & \frac{S_3'(p_3)}{3} & 0 \end{pmatrix} \right| = \frac{S_2'(p_2)\, S_3'(p_3)}{3}. \end{aligned}$$

Thus we have from Lemma 10 that

$$P(p, q, \boldsymbol{\rho}, \boldsymbol{\sigma}, \omega_{10}) = \frac{S_2'(p_2)\, S_3'(p_3)}{3V_3}. \tag{133}$$

The residual demand slope in this state is

$$-D'_{mn}(p, \omega_{10}, \boldsymbol{\rho}, \boldsymbol{\sigma}) = -(N-1)Q'(p), \tag{134}$$

because the flows in all lines into node 1 are fixed.

We make the substitution $u = p - \frac{2}{3}\sigma_1 - \frac{1}{3}\rho_2$ and $v = p - \frac{1}{3}\sigma_1 + \frac{1}{3}\rho_2$. It can be shown that $\frac{\partial(\rho_1,\sigma_3)}{\partial(u,v)} = 3$. We now have from (133) and (134) that the contribution from this state to Z_{mn} in Proposition 5 is:

$$\begin{aligned} &\int_{\boldsymbol{\rho}_{U(\omega_{10})}} \int_{\boldsymbol{\sigma}_{L(\omega_{10})}} -D'_{mn}(p, \omega_{10}, \boldsymbol{\rho}, \boldsymbol{\sigma}) P(p, q, \boldsymbol{\rho}, \boldsymbol{\sigma}, \omega_{10})\, d\boldsymbol{\sigma}_{L(\omega_{10})} d\boldsymbol{\rho}_{U(\omega_{10})} \\ &= \frac{(N-1)Q'(p)}{3V_3} \int_0^p \int_0^{\overline{p}} S_2'(u)\, S_3'(v)\, \frac{\partial(\rho_1, \sigma_3)}{\partial(u,v)} du dv \\ &= \frac{(N-1)N^2 Q'(p)}{V_3} \int_0^p \int_0^{\overline{p}} Q'(u)\, Q'(v)\, du dv \\ &= \frac{(N-1)N^2 Q'(p)}{V_3} Q(p)\, Q(\overline{p}). \end{aligned}$$

Similarly,

$$\begin{aligned}
&\int_{\boldsymbol{\rho}_{U(\omega_{10})}}\int_{\boldsymbol{\sigma}_{L(\omega_{10})}} P\left(p,q,\boldsymbol{\rho},\boldsymbol{\sigma},\omega_{10}\right) d\boldsymbol{\sigma}_{L(\omega_{10})} d\boldsymbol{\rho}_{U(\omega_{10})} \\
= \;& \frac{1}{3V_3}\int_0^p\int_0^{\overline{p}} S_2'\left(u\right) S_3'\left(v\right)\frac{\partial\left(\rho_1,\sigma_3\right)}{\partial\left(u,v\right)} dudv = \frac{N^2}{V_3} Q\left(p\right) Q\left(\overline{p}\right).
\end{aligned}$$

State ω_{11}

State	$t_1(\omega)$	$t_2(\omega)$	$t_3(\omega)$	$L(\omega)$	$B(\omega)$	$U(\omega)$	$\Xi(\omega)$	$F(\omega)$	$\Upsilon(\omega)$
ω_{11}	$-\overline{t}$	0	$\overline{t}$	$\{1\}$	$\emptyset$	$\{3\}$	$\{1\}$	$\{2,3\}$	$\emptyset$

We have from (73) that

$$\begin{aligned}
\begin{bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\end{bmatrix} &= \underbrace{\begin{bmatrix} S_1\left(p\right)\\ S_2\left(p_2\right)\\ S_3\left(p_3\right)\end{bmatrix}}_{\mathbf{s}(\mathbf{p})} + \underbrace{\begin{bmatrix} -1 & 0 & 1\\ 1 & -1 & 0\\ 0 & 1 & -1\end{bmatrix}}_{\mathbf{A}}\underbrace{\begin{bmatrix} -\overline{t}\\ 0\\ \overline{t}\end{bmatrix}}_{\mathbf{t}} \\
&= \begin{bmatrix} S_1\left(p\right)+2\overline{t}\\ S_2\left(p_2\right)-\overline{t}\\ S_3\left(p_3\right)-\overline{t}\end{bmatrix}. \qquad (135)
\end{aligned}$$

It follows from (75) that

$$\begin{bmatrix} p_2\\ p_3\end{bmatrix} = \begin{bmatrix} p-\frac{2}{3}\sigma_1-\frac{1}{3}\rho_3\\ p-\frac{1}{3}\sigma_1-\frac{2}{3}\rho_3\end{bmatrix}. \qquad (136)$$

We now have from (135) and (136) that

$$\frac{\partial}{\partial\sigma_1}\begin{bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\end{bmatrix} = \begin{bmatrix}\frac{\partial\varepsilon_1}{\partial p_1}\frac{\partial p_1}{\partial\sigma_1}\\ \frac{\partial\varepsilon_2}{\partial p_2}\frac{\partial p_2}{\partial\sigma_1}\\ \frac{\partial\varepsilon_3}{\partial p_3}\frac{\partial p_3}{\partial\sigma_1}\end{bmatrix} = \begin{bmatrix} 0\\ -\frac{2}{3}S_2'\left(p_2\right)\\ -\frac{1}{3}S_3'\left(p_3\right)\end{bmatrix} \qquad (137)$$

and

$$\frac{\partial}{\partial\rho_3}\begin{bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\end{bmatrix} = \begin{bmatrix}\frac{\partial\varepsilon_1}{\partial p_1}\frac{\partial p_1}{\partial\rho_3}\\ \frac{\partial\varepsilon_2}{\partial p_2}\frac{\partial p_2}{\partial\rho_3}\\ \frac{\partial\varepsilon_3}{\partial p_3}\frac{\partial p_3}{\partial\rho_3}\end{bmatrix} = \begin{bmatrix} 0\\ -\frac{1}{3}S_2'\left(p_2\right)\\ -\frac{2}{3}S_3'\left(p_3\right)\end{bmatrix}. \qquad (138)$$

We use (137) and (138) to calculate

$$\begin{aligned}
J_V(\omega_{11}) &= \left|\frac{\partial\boldsymbol{\varepsilon}}{\partial(\mathbf{t}'_{B(\omega)},\boldsymbol{\rho}_{U(\omega)},\boldsymbol{\sigma}_{L(\omega)},\varepsilon_m)}\right| = \left|\det\begin{pmatrix}\frac{\partial\varepsilon_1}{\partial\sigma_1} & \frac{\partial\varepsilon_1}{\partial\rho_3} & \frac{\partial\varepsilon_1}{\partial\varepsilon_1}\\ \frac{\partial\varepsilon_2}{\partial\sigma_1} & \frac{\partial\varepsilon_2}{\partial\rho_3} & \frac{\partial\varepsilon_2}{\partial\varepsilon_1}\\ \frac{\partial\varepsilon_3}{\partial\sigma_1} & \frac{\partial\varepsilon_3}{\partial\rho_3} & \frac{\partial\varepsilon_3}{\partial\varepsilon_1}\end{pmatrix}\right| \\
&= \left|\det\begin{pmatrix} 0 & 0 & 1\\ -\frac{2S_2'(p_2)}{3} & -\frac{S_2'(p_2)}{3} & 0\\ -\frac{S_3'(p_3)}{3} & -\frac{2S_3'(p_3)}{3} & 0\end{pmatrix}\right| \\
&= \frac{S_2'\left(p_2\right)S_3'\left(p_3\right)}{3}.
\end{aligned}$$

Thus we have from Lemma 10 that

$$P\left(p,q,\boldsymbol{\rho},\boldsymbol{\sigma},\omega_{11}\right)=\frac{S_2'\left(p_2\right)S_3'\left(p_3\right)}{3V_3}.$$

The residual demand slope in this state is

$$-D_{mn}'(p,\omega_{11},\boldsymbol{\rho},\boldsymbol{\sigma})=-(N-1)Q'(p),$$

because the flows in all lines into node 1 are fixed.

We make the substitution $u=p-\frac{2}{3}\sigma_1-\frac{1}{3}\rho_3$ and $v=p-\frac{1}{3}\sigma_1-\frac{2}{3}\rho_3$. It can be shown that $\frac{\partial(\sigma_1,\rho_3)}{\partial(u,v)}=3$. We now have from (133) and (134) that the contribution from this state to Z_{mn} in Proposition 5 is:

$$\begin{aligned}
&\int_{\boldsymbol{\rho}_{U(\omega_{11})}}\int_{\boldsymbol{\sigma}_{L(\omega_{11})}}-D_{mn}'(p,\omega_{11},\boldsymbol{\rho},\boldsymbol{\sigma})P\left(p,q,\boldsymbol{\rho},\boldsymbol{\sigma},\omega_{11}\right)d\boldsymbol{\sigma}_{L(\omega_{11})}d\boldsymbol{\rho}_{U(\omega_{11})}\\
=&\frac{(N-1)Q'(p)}{3V_3}\int_0^p\int_0^p S_2'\left(u\right)S_3'\left(v\right)\frac{\partial\left(\sigma_1,\rho_3\right)}{\partial\left(u,v\right)}dudv\\
=&\frac{(N-1)N^2Q'(p)}{V_3}\int_0^p\int_0^p Q'\left(u\right)Q'\left(v\right)dudv\\
=&\frac{(N-1)N^2Q'(p)}{V_3}Q^2\left(p\right).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int_{\boldsymbol{\rho}_{U(\omega_{11})}}\int_{\boldsymbol{\sigma}_{L(\omega_{11})}}P\left(p,q,\boldsymbol{\rho},\boldsymbol{\sigma},\omega_{11}\right)d\boldsymbol{\sigma}_{L(\omega_{11})}d\boldsymbol{\rho}_{U(\omega_{11})}\\
=&\frac{1}{3V_3}\int_0^p\int_0^p S_2'\left(u\right)S_3'\left(v\right)\frac{\partial\left(\sigma_1,\rho_3\right)}{\partial\left(u,v\right)}dudv=\frac{N^2}{V_3}Q^2\left(p\right).
\end{aligned}$$

State ω_{12}

State	$t_1(\omega)$	$t_2(\omega)$	$t_3(\omega)$	$L(\omega)$	$B(\omega)$	$U(\omega)$	$\Xi(\omega)$	$\digamma(\omega)$	$\Upsilon(\omega)$
ω_{12}	0	$\bar{t}$	$-\bar{t}$	$\{3\}$	$\emptyset$	$\{2\}$	$\{1\}$	$\{2,3\}$	$\emptyset$

It follows from (75) that

$$\begin{bmatrix}p_2\\p_3\end{bmatrix}=\begin{bmatrix}p-\frac{1}{3}\rho_2+\frac{1}{3}\sigma_3\\p+\frac{1}{3}\rho_2+\frac{2}{3}\sigma_3\end{bmatrix}.$$

From the perspective of a producer in node 1, state ω_{12} is symmetric with ω_8, so

$$P\left(p,q,\boldsymbol{\rho},\boldsymbol{\sigma},\omega_{12}\right)=\frac{S_2'\left(p_2\right)S_3'\left(p_3\right)}{3V_3}.$$

and

$$-D_{mn}'(p,\omega_{12},\boldsymbol{\rho},\boldsymbol{\sigma})=-(N-1)Q'(p).$$

Moreover, the contribution from this state to Z_{mn} in Proposition 5 is:

$$\int_{\boldsymbol{\rho}_{U(\omega_{12})}} \int_{\boldsymbol{\sigma}_{L(\omega_{12})}} -D'_{mn}(p,\omega_{12},\boldsymbol{\rho},\boldsymbol{\sigma}) P\left(p,q,\boldsymbol{\rho},\boldsymbol{\sigma},\omega_{12}\right) d\boldsymbol{\sigma}_{L(\omega_{12})} d\boldsymbol{\rho}_{U(\omega_{12})}$$
$$= \frac{(N-1)N^2 Q'(p)}{V_3}\left(Q\left(\overline{p}\right)-Q\left(p\right)\right)Q\left(\overline{p}\right)$$

and

$$\int_{\boldsymbol{\rho}_{U(\omega_{12})}} \int_{\boldsymbol{\sigma}_{L(\omega_{12})}} P\left(p,q,\boldsymbol{\rho},\boldsymbol{\sigma},\omega_{12}\right) d\boldsymbol{\sigma}_{L(\omega_{12})} d\boldsymbol{\rho}_{U(\omega_{12})}$$
$$= \frac{N^2}{V_3}\left(Q\left(\overline{p}\right)-Q\left(p\right)\right)Q\left(\overline{p}\right).$$

State ω_{13}

State	$t_1(\omega)$	$t_2(\omega)$	$t_3(\omega)$	$L(\omega)$	$B(\omega)$	$U(\omega)$	$\Xi(\omega)$	$F(\omega)$	$\Upsilon(\omega)$
ω_{13}	0	$-\overline{t}$	$\overline{t}$	$\{2\}$	$\emptyset$	$\{3\}$	$\{1\}$	$\{2,3\}$	$\emptyset$

It follows from (75) that

$$\begin{bmatrix} p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} p+\frac{1}{3}\sigma_2-\frac{1}{3}\rho_3 \\ p-\frac{1}{3}\sigma_2-\frac{2}{3}\rho_3 \end{bmatrix}.$$

From the perspective of a producer in node 1, state ω_{13} is symmetric with ω_{10}, so

$$P\left(p,q,\boldsymbol{\rho},\boldsymbol{\sigma},\omega_{13}\right) = \frac{S_2'\left(p_2\right)S_3'\left(p_3\right)}{3V_3}$$

and

$$-D'_{mn}(p,\omega_{13},\boldsymbol{\rho},\boldsymbol{\sigma}) = -(N-1)Q'(p).$$

Moreover, the contribution from this state to Z_{mn} in Proposition 5 is:

$$\int_{\boldsymbol{\rho}_{U(\omega_{13})}} \int_{\boldsymbol{\sigma}_{L(\omega_{13})}} -D'_{mn}(p,\omega_{13},\boldsymbol{\rho},\boldsymbol{\sigma}) P\left(p,q,\boldsymbol{\rho},\boldsymbol{\sigma},\omega_{13}\right) d\boldsymbol{\sigma}_{L(\omega_{13})} d\boldsymbol{\rho}_{U(\omega_{13})}$$
$$= \frac{(N-1)N^2 Q'(p)}{V_3}Q\left(p\right)Q\left(\overline{p}\right)$$

and

$$\int_{\boldsymbol{\rho}_{U(\omega_{13})}} \int_{\boldsymbol{\sigma}_{L(\omega_{13})}} P\left(p,q,\boldsymbol{\rho},\boldsymbol{\sigma},\omega_{13}\right) d\boldsymbol{\sigma}_{L(\omega_{13})} d\boldsymbol{\rho}_{U(\omega_{13})}$$
$$= \frac{N^2}{V_3}Q\left(p\right)Q\left(\overline{p}\right)$$