Some inequalities related to the analysis of electricity auctions

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Abstract

Most balancing markets of electric power are organized as uniform-price auctions. In 2001, the balancing market of England and Wales switched to a pay-as-bid auction with the intention of reducing wholesale electricity prices. Numerical simulations of an electricity auction model have indicated that this should lead to decreased average prices. In this article we prove two inequalities which give an analytic proof of this claim in the same model.

Keywords: supply function equilibrium, uniform-price auctions, pay-as-bid auctions, discriminatory auction, wholesale electricity markets, oligopoly, capacity constraint, inequalities.

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1 Introduction

In the balancing market, the system operator buys last-minute power from electric power producers. Most balancing markets are organized as uniform-price auctions (UPA), i.e. all accepted bids get the same price. The market price is set by the marginal bid, i.e. the highest accepted bid. Since the papers by Klemperer & Meyer [1], Bolle [2] and Green & Newbery [3], bidding behaviour in electricity UPA is often modeled by Supply Function Equilibria (SFE). The concept assumes that firms submit smooth supply functions simultaneously to a UPA in a one-shot game. In the non-cooperative Nash Equilibrium, each firm commits to the supply function that maximizes its expected profit given the bids of the competitors and the properties of uncertain demand.

In 2001, electricity trading in the balancing market of England and Wales switched from a UPA to a pay-as-bid auction (PABA). As the name suggests, all accepted bids in PABAs are paid their bid. It was the belief of the British regulatory authority (Ofgem) that the reform would decrease wholesale electricity prices. Before the collapse of the California Power Exchange, a similar switch was considered also for that market [4].

It is not straightforward to establish whether prices will be lower or higher in a PABA, as firms will change their bidding strategy after a switch from a UPA to a PABA [4]. Federico and Rahman [5] compare the two auction forms for two polar cases, perfect competition ($N \to \infty$) and monopoly (N=1), where N is the number of firms. In the competitive cases, average prices are lower in the PABA, if demand is elastic (price dependent). The same is true for the monopoly case, unless demand uncertainty is very high or costs increase steeply. Consumer surplus is higher in a PABA, but Federico and Rahman also show that total welfare tend to be lower in PABA, due to a reduced output. Fabra et al. derive a Nash equilibrium for an asymmetric duopoly model (N=2) with single units, i.e. marginal costs are constant and producers must submit a single price offer for their entire capacity [6]. For perfectly inelastic and certain demand, they show that average prices are lower in a PABA than in a UPA. Numerical examples suggest that the difference might be substantial. Son et al. [7] use a similar model as Fabra et al., but one of the two firms has two production units with different marginal costs. Son et al. also conclude that average prices are lower in the PABA than in a UPA if demand is certain and perfectly inelastic. Simulations suggest that the conclusion may hold also for elastic demand.

The second author has derived a unique SFE for a PABA with symmetric firms and uncertain perfectly inelastic demand [8]. It can be shown that the equilibrium always exists if demand follows the Pareto distribution of the second kind. Numerical calculations indicate that for this probability distribution, the average price is weakly lower in a PABA than in a UPA.[1]

In this paper we prove two inequalities which provide an analytic proof of this claim within the aforementioned model-framework. The inequalities are integral inequalities of rational functions and the proofs are based on investigating the derivatives of the functions involved. This robust method is often not fully appreciated, perhaps due to the influence of Hardy, Littlewood and Pólya [10], but has been used widely by researchers specializing in inequalities. As examples, we mention studies of inequalities of the gamma and poly-gamma functions, e.g. [11, 12]. Before the proofs we provide a more detailed description of the context in which the inequalities arose.

2 Comparing pay-as-bid and uniform-price auctions

With uncertain and perfectly inelastic demand that may exceed the production capacity, which are realistic assumptions for balancing markets, it can be shown that the SFE of the PABA and the UPA are unique [8,

[1] This paper focuses on the case when demand exceeds supply. In the opposite case, the system operator will sell power back to the producers [13]. Then the average price is weakly higher in a PABA if the supply of the system operator follows a Pareto distribution of the second kind. Thus the system operator prefers PABA both for positive and negative imbalances.

13]. For this, it is assumed that the cost function C of each firm is convex, increasing and twice continuously differentiable. We will be working with the derivative C' of this function, called the marginal cost, and assume additionally that $C'(\overline{\varepsilon}/N) < \overline{p}$, where N is the number of symmetric firms, $\overline{\varepsilon}$ is the total production capacity and $\overline{p}$ is the reservation price (price cap). The unique equilibrium price of a UPA with symmetric firms, p_U, is given by:

$$p_U(\varepsilon) = \frac{\overline{p}\varepsilon^{N-1}}{\overline{\varepsilon}^{N-1}} + (N-1)\varepsilon^{N-1}\int_{\varepsilon}^{\overline{\varepsilon}} \frac{C'(x/N)}{x^N}\,dx,$$

where $\varepsilon \geqslant 0$ is the realized demand outcome.

It was shown in [8] that if an SFE of a PABA exists, the marginal bid as a function of the demand is given by

$$p_P(\varepsilon) = \frac{N[1-F(\overline{\varepsilon})]^{\frac{N-1}{N}}\overline{p} + \int_{\varepsilon}^{\overline{\varepsilon}}(N-1)C'(u/N)f(u)\,[1-F(u)]^{\frac{N-1}{N}-1}\,du}{N[1-F(\varepsilon)]^{\frac{N-1}{N}}}$$

where $f = F'$ is the probability density of demand. In some cases a pure strategy equilibrium does not exist, e.g. if there is some interval $\varepsilon \in [\varepsilon_-, \varepsilon_+]$ where $C'(\varepsilon/N)$ is constant and $f'(\varepsilon) > 0$ [8]. A decreasing density function does not guarantee the existence of a pure strategy equilibrium but one can show that a pure strategy equilibrium always exists if f has the form [8]

$$\beta^{\frac{1}{\alpha}}(\alpha x+\beta)^{-\frac{1}{\alpha}-1} \quad \alpha \geqslant 0, \beta \geqslant 0$$

which implies that F is a *Pareto distribution of the second kind* [9] (the case $\alpha = 0$ is by continuous extension). Henceforth, only auctions in which demand follows a *Pareto distribution of the second kind* are considered.

The total expected revenue for symmetric firms bidding in a PAB auction is [8]:

$$R_P = \int_0^{\overline{\varepsilon}} \big(1-F(\varepsilon)\big)p_p(\varepsilon)\,d\varepsilon = \overline{p}\,\overline{\varepsilon}\,g_P\Big(\alpha, N, \frac{\alpha\overline{\varepsilon}}{\beta}\Big) + (N-1)\int_0^{\overline{\varepsilon}} C'\Big(\frac{u}{N}\Big)h_P\Big(\alpha, N, \frac{\alpha u}{\beta}\Big)\,du.$$

where we have denoted

$$g_P(\alpha,N,x) = \frac{(1+x)^{-\frac{1}{\alpha N}+1}-1}{(1-\frac{1}{\alpha N})x(1+x)^{\frac{N-1}{\alpha N}}} \quad \text{and} \quad h_P(\alpha,N,x) = (x+1)^{\frac{1-N}{\alpha N}-1}\frac{(x+1)^{1-\frac{1}{\alpha N}}-1}{\alpha N-1}.$$

It was further found in [8] that this can be simplified to

$$R_P = (\overline{p}-c)\overline{\varepsilon}\,g_P\Big(\alpha, N, \frac{\alpha\overline{\varepsilon}}{\beta}\Big) + \beta^{\frac{1}{\alpha}}c\int_0^{\overline{\varepsilon}}(\alpha\varepsilon+\beta)^{-\frac{1}{\alpha}}d\varepsilon$$

for constant marginal costs, $C' \equiv c$.

The total expected revenue for symmetric firms bidding in a UPA is [8]:

$$\begin{aligned} R_U &= \int_0^{\overline{\varepsilon}} f(\varepsilon)\varepsilon p_U(\varepsilon)\,d\varepsilon + (1-F(\overline{\varepsilon}))\overline{\varepsilon}\,\overline{p} \\ &= \overline{p}\,\overline{\varepsilon}\,g_U\Big(\alpha, N, \frac{\alpha\overline{\varepsilon}}{\beta}\Big) + (N-1)\int_0^{\overline{\varepsilon}} C'\Big(\frac{u}{N}\Big)h_U\Big(\alpha, N, \frac{\alpha u}{\beta}\Big)\,du, \end{aligned}$$

where we used the functions

$$g_U(\alpha,N,x) = \frac{N}{x^N}\int_0^x (1+t)^{-\frac{1}{\alpha}}t^{N-1}dt \quad \text{and} \quad h_U(\alpha,N,x) = \frac{1}{\alpha x^N}\int_0^x (1+t)^{-\frac{1}{\alpha}-1}t^N dt.$$

Continuing to follow [8], this simplifies to the following expression for the case of constant marginal costs:

$$R_U = (\overline{p} - c)\overline{\varepsilon}\, g_U\Big(\alpha, N, \frac{\alpha\overline{\varepsilon}}{\beta}\Big) + \beta^{\frac{1}{\alpha}} c \int_0^{\overline{\varepsilon}} (\alpha\varepsilon + \beta)^{-\frac{1}{\alpha}} d\varepsilon.$$

We conclude this section by stating the implications of the inequalities from next section in the model described so far.

Theorem 1. *For non-decreasing marginal costs we have $R_P \leqslant R_U$. Equality occurs for $N = 1$.*

Proof. We denote $G = g_U - g_P$ and $H = h_U - h_P$. It follows directly from Theorem 5 that H has profile $-$ or $-|+$ (as a function of x, for fixed parameters α and N). From the formulae for R_P and R_U we find that

$$R_U - R_P = \overline{p}\,\overline{\varepsilon}\, G\Big(\alpha, N, \frac{\alpha\overline{\varepsilon}}{\beta}\Big) + (N-1)\int_0^{\overline{\varepsilon}} C'\Big(\frac{u}{N}\Big) H\Big(\alpha, N, \frac{\alpha u}{\beta}\Big)\, du.$$

If H changes sign below $\overline{\varepsilon}$, then we define x^* to be the point where the sign-change occurs, otherwise we set $x^* = \overline{\varepsilon}$. Since C' is non-decreasing, we find that $C'(u/N) - C'(x^*/N)$ is non-negative when H is non-negative and non-positive when H is non-positive. Hence

$$\int_0^{\overline{\varepsilon}} C'\Big(\frac{u}{N}\Big) H\Big(\alpha, N, \frac{\alpha u}{\beta}\Big)\, du \geqslant C'\Big(\frac{x^*}{N}\Big) \int_0^{\overline{\varepsilon}} H\Big(\alpha, N, \frac{\alpha u}{\beta}\Big)\, du.$$

Suppose our initial data gave us R_P and R_U. Now we keep all the data fixed, except the marginal cost, which is set to the constant $C'(x^*/N)$, and leads to the revenues $\tilde{R}_P$ and $\tilde{R}_U$. Then we have shown that $R_U - R_P \geqslant \tilde{R}_U - \tilde{R}_P$. So it suffices to show that $\tilde{R}_U \geqslant \tilde{R}_P$, i.e. prove the claim for constant marginal costs. In this case we use the formulae for the constant marginal cost case to calculate

$$\tilde{R}_U - \tilde{R}_P = (\overline{p} - c)\overline{\varepsilon}\, G\Big(\alpha, N, \frac{\alpha\overline{\varepsilon}}{\beta}\Big).$$

It follows from Theorem 3 that G is a positive function, so that $\tilde{R}_U \geqslant \tilde{R}_P$. □

Recall that the demand is assumed to be perfectly inelastic and accordingly independent of the auction design. Thus Theorem 1 implies the following result:

Corollary 2. *Average prices are weakly lower in PABA than in UPA.*

3 The mathematical treatment of the inequalities

In order to state the proofs succinctly, we introduce some slightly different notation in this section. We use $a = 1/\alpha$, and we multiply the functions G and H (from the proof of Theorem 1) by suitable powers of x, as this does not affect their sign.

Theorem 3. *Let $x, a \in (0, \infty)$ and $N \geqslant 1$. The inequality*

$$\frac{1}{N-a} x^{N-1} (x+1)^{(1-N)a/N} \big[(1+x)^{1-a/N} - 1\big] \leqslant \int_0^x (1+t)^{-a} t^{N-1} dt,$$

holds when $N \neq a$. Corresponding to $N = a$, we also have

$$\frac{1}{N} \frac{x^{N-1}}{(1+x)^{N-1}} \log(x+1) \leqslant \int_0^x (1+t)^{-N} t^{N-1} dt.$$

Proof. The second inequality follows from the first as $a \to N$. We will use the short-hand notation $b = a/N$. To prove the first inequality for $b \neq 1$ we define

$$\begin{aligned} g(x) &= \int_0^x (1+t)^{-a} t^{N-1} dt - \frac{x^{N-1}}{N-a}(1+x)^{(1-N)b}\big[(1+x)^{1-b} - 1\big] \\ &= \int_0^x (1+t)^{-a} t^{N-1} dt - \frac{x^{N-1}}{N-a}\big[(1+x)^{1-a} - (1+x)^{(1-N)b}\big]. \end{aligned}$$

Then the claim is that $g \geqslant 0$. Since $g(0) = 0$, it suffices to show that $g'(x) \geqslant 0$ for all x. We find that

$$\begin{aligned} (N-a)g'(x) &= (N-a)(1+x)^{-a}x^{N-1} - (N-1)x^{N-2}\big[(1+x)^{1-a} - (1+x)^{(1-N)b}\big] \\ &\quad - x^{N-1}\big[(1-a)(1+x)^{-a} - (1-N)b(1+x)^{(1-N)b-1}\big]. \\ &= (1+x)^{-a}x^{N-2}\big\{(N-a)x - (N-1)\big[1+x-(1+x)^b\big] \\ &\quad - x\big[1-a+(N-1)b(1+x)^{b-1}\big]\big\} \\ &= (N-1)(1+x)^{-a}x^{N-2}\big\{-1+(1+x)^b - bx(1+x)^{b-1}\big\} \\ &= (N-1)b(1-b)(1+x)^{-a}x^{N-2}\int_0^x t(1+t)^{b-2}dt. \end{aligned} \tag{4}$$

By the definition of b, we have $\frac{(N-1)b(1-b)}{N-a} = (N-1)N^{-2}a \geqslant 0$, so our expression for g' directly implies that g is increasing. $\square$

We define the function $h\colon \mathbb{R}_+ \to \mathbb{R}$ by

$$h(x) = \int_0^x (1+t)^{-a-1} t^N dt - \frac{x^N}{N-a}\big[(1+x)^{-a} - (1+x)^{(1-N)a/N-1}\big]$$

for $N \geqslant 1$ and $a \in (0,\infty)$ with $N \neq a$. For $N = a$ we define h by the corresponding limit:

$$h(x) = \int_0^x (1+t)^{-N-1} t^N dt - \frac{x^N}{N}(1+x)^{-N}\log(1+x).$$

Theorem 5. *If $N > 1$, then there exists a value $x^* \in (0,\infty)$ such that $h \leqslant 0$ on $(0,x^*)$ and $h \geqslant 0$ on (x^*,∞). If $N = 1$, then $h \leqslant 0$ on $[0,\infty)$.*

Proof. We start by assuming that $N \neq a$ and again employ the notation $b = a/N$. Differentiating h we find that

$$\begin{aligned} (N-a)h'(x) &= (N-a)(1+x)^{-a-1}x^N - Nx^{N-1}\big[(1+x)^{-a} - (1+x)^{b-a-1}\big] \\ &\quad - x^N\big[-a(1+x)^{-a-1} - \big(b-a-1\big)(1+x)^{b-a-2}\big] \\ &= x^{N-1}(1+x)^{-a-1}\big\{(N-a)x - N\big[1+x-(1+x)^b\big] \\ &\quad + x\big[a + \big(b-a-1\big)(1+x)^{b-1}\big]\big\} \\ &= x^{N-1}(1+x)^{-a-1}\big\{-N + N(1+x)^b + \big(b-a-1\big)x(1+x)^{b-1}\big\}. \end{aligned}$$

We throw away the factor $x^{N-1}(1+x)^{-a-1}$ which is not relevant for the sign, so h' is positive if and only if

$$\begin{aligned} q(x) &= (N-a)^{-1}\big[-N + N(1+x)^b + \big((1-N)b-1\big)x(1+x)^{b-1}\big] \\ &= (N-a)^{-1}\big[-N + (1+x)^{b-1}\big(N + (N-1)(1-b)x\big)\big] \end{aligned}$$

is also positive. Using the second expression we derive the formula

$$q'(x) = (N-a)^{-1}(b-1)(1+x)^{b-2}\big[1-(N-1)bx\big] = -\tfrac{1}{N}(1+x)^{b-2}\big[1-(N-1)bx\big]$$

for the derivative. The claim regarding the case $N = 1$ follows directly from this, so from now on we assume that $N > 1$. Then q is initially decreasing and then increasing. Since $q(0) = 0$, this means that q and hence h' has the profile $-$ or $-|+$. By continuity, we see that this conclusion holds also for $N = a$.

Since $h(0) = 0$, this means that h has profile $-$ or $-|+$. Hence we need to investigate $\lim_{x\to\infty} h(x)$. For $N < a$ we go back to the definition of h and note that the second term tends to zero as $x \to \infty$. Therefore h is the integral of a positive function, hence positive. For $N > a$ we have the inequality $(1+t)^N - t^N \le N(1+t)^{N-1}$, which is derived by dividing by t^N, setting $s = 1/t$, and using the identity in the two last lines of (4). Using this inequality, we derive

$$\begin{aligned}\int_0^x (1+t)^{-a-1}t^N dt &= \int_0^x (1+t)^{N-a-1}dt + O\Big(\int_0^x (1+t)^{N-a-2}dt\Big)\\ &= \frac{(1+x)^{N-a}}{N-a} + O\Big(\frac{(1+x)^{N-a-1}-1}{N-a-1}\Big)\end{aligned} \tag{6}$$

provided $N \neq a+1$. Using this we conclude that

$$\begin{aligned}(N-a)h(x) &= (1+x)^{N-a} + O\big(x^{N-a-1}-1\big) - x^N\big[(1+x)^{-a} - (1+x)^{(1-N)b-1}\big]\\ &= ((1+x)^N - x^N)(1+x)^{-a} + O\big(x^{N-a-1}-1\big) + x^N(1+x)^{(1-N)b-1}\\ &= O\big(x^{N-a-1}-1\big) + x^N(1+x)^{(1-N)b-1}.\end{aligned}$$

Since $N+(1-N)b-1 > \max\{0, N-a-1\}$, we see that the second term is dominant. In the case $N-a = 1$, we have to adjust (6) accordingly with a logarithmic term, but the conclusion still follows. Thus we see that $h(\infty) = \infty$ also if $N > a$.

The borderline case, $N = a$, remains to be investigated. We have from (6) that

$$\int_0^x (1+t)^{-N-1}t^N dt = \log(1+x) + O(1).$$

So it follows that

$$h(x) = \log(1+x) + O(1) - \frac{1}{N}\frac{x^N}{(1+x)^N}\log(1+x) = \Big(1 - \frac{1}{N}\frac{x^N}{(1+x)^N}\Big)\log(1+x) + O(1).$$

Since $N > 1$ the first term is unbounded and hence dominant. □

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