

Numerical Calculation of an Asymmetric Supply Function Equilibrium with Capacity Constraints¹

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Abstract

Producers submit offer curves to a procurement auction, e.g. an electricity auction, before uncertain demand has been realised. In the Supply Function Equilibrium (SFE), every firm commits to the offer curve that maximises its expected profit, given the offer curves of competitors. The equilibrium is given by a system of differential equations. In practice, it has been very difficult to find valid SFE, i.e. non-decreasing solutions, from this system, especially for asymmetric producers. This paper shows that valid SFE can be calculated by means of a shooting algorithm that combines numerical integration with an optimisation procedure that searches for an end-condition. Multiple/parallel shooting is used for ill-conditioned cases.

Keywords: Auctions/bidding, Game theory, OR in energy, Supply Function Equilibrium, Numerical integration, Shooting

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1. INTRODUCTION

Deregulated electricity markets are monitored by regulatory authorities, competition authorities and transmission system operators to detect and prevent abuse. Moreover, they examine proposed rule changes, mergers and acquisitions. Most of these oversight or regulatory functions require an ability to predict electricity prices, considering market behaviour of strategic electricity producers, under various counterfactuals – what might happen if the market rules are changed, or a merger waived through. Often authorities are content with using concentration measures, such as the Herfindahl-Hirschman index (HHI), to assess the degree of competition in the market. However, these measures work poorly for electricity markets, because electricity is expensive to store [7]. Thus given installed production capacities, it depends very much on the instantaneous demand level whether the market will suffer from the exercise of market power; the problem starts when demand rises beyond levels for which several producers can contest to supply that load [7].

Producers' profit maximizing mark-ups can be determined from the elasticity of their residual demand. As the elasticity of demand is approximately perfectly inelastic, mark-ups depend very much on the elasticity of competitors' supply curves. They can be calculated by means of the Supply Function Equilibrium (SFE), in which strategic producers submit bids to a uniform-price auction in a one-shot game [17]. In the non-cooperative Nash Equilibrium, each producer commits to the supply function that maximises his expected profit given the bids of competitors and the properties of uncertain demand. Bolle [6] and Green & Newbery [12] observed that the framework is similar to the organisation of most electricity markets, and the equilibrium is now an established model of bidding behaviour in electric power auctions.

Klemperer & Meyer [17] showed that all smooth SFE are characterised by a differential equation, in this paper labelled the KM first-order condition. For markets with perfectly inelastic demand, closed-form solutions to the first-order condition have been found for symmetric firms with general cost functions [1,21] and for asymmetric firms with constant marginal costs [11,15,20]. For markets with linear demand and linear marginal costs (and no capacity constraints) there is also a simple equilibrium [3,13]. However, capacity constraints are important when assessing market-power in electricity markets and symmetric representations of the market can, depending on the demand level, grossly overstate or understate the degree of competition [15]. Moreover, symmetric representations cannot be used to estimate welfare losses due to production inefficiencies arising from asymmetric mark-ups by producers. Thus it would be very valuable for the monitoring authorities if methods were available to let them solve the general case, in which producers have non-constant, non-linear

marginal costs and asymmetric production capacities. In this case, the KM first-order conditions — one for each firm — constitute a system of non-autonomous ordinary differential equations. To solve this system analytically is not only very difficult, but likely to be impossible. Baldick & Hogan [4] calculate approximate asymmetric SFE by numerically integrating the system of ordinary differential equations. They note that it is difficult to find valid solutions that do not violate the requirement that supply functions must be non-decreasing.³ Three exceptions to this are as follows: symmetric firms with identical cost functions, cases with affine solutions — i.e. affine marginal costs and no capacity constraints — and small variations in demand.

In this paper, I outline a shooting algorithm that can be used to find a valid SFE. The algorithm is intended for $N \geq 2$ asymmetric firms and cost functions more general than the three special cases identified by Baldick & Hogan [4]. With a shooting algorithm I mean that the system of differential equations is solved for some set of initial values and that these initial values are iteratively updated until the solution becomes valid. The equilibrium consists of piece-wise smooth supply functions and is inspired by an equilibrium derived for asymmetric producers with constant marginal costs and perfectly inelastic demand [15]. Several of the properties are reasonable also for increasing marginal costs and sufficiently inelastic demand. First, large firms have more market power and higher mark-ups. Hence, capacity constraints of smaller firms bind at lower prices. Second, the capacity constraint of the second largest firm starts to bind at the price cap, \bar{p} . Let p_i be the price at which the capacity constraint of firm i starts to bind. Arranging the producers according to size, starting with the smallest firm, these two properties can be stated as $C'(0) < p_1 \leq \dots \leq p_{N-1} = \bar{p}$, where $C()$ is the aggregate cost function. Third, the largest producer offers its remaining capacity ΔS_N with perfectly elastic supply at the price cap. Fourth, all firms offer their first unit of power at marginal cost, which is in agreement with general results for uniform-price auctions [18]. Lastly, the equilibrium is expected to be unique if capacity constraints of all firms, but possibly the largest, bind at maximum demand.

The constants $\Delta S_N, p_1, p_2, \dots, p_{N-2}$ are unknown a priori. But given initial or updated guesses of these values, the end-conditions of the system of KM first-order conditions are known and the supply functions of all firms can be solved by numerical integration. The numerical integration starts at the price cap and proceeds in the direction of decreasing prices.

³ Non-decreasing supply functions are required by most electricity auctions.

The integration is terminated as soon as any supply function violates the requirement that a supply function must be non-decreasing. The criterion function $\Gamma(p_1, \dots, p_{N-2}, \Delta S_N)$ is equal to the terminated price. In theory, all considered SFE candidates should fulfil

$\Gamma(p_1 \dots p_{N-2}, \Delta S_N) = C'(0)$. But in practice $\Gamma(p_1 \dots p_{N-2}, \Delta S_N) > C'(0)$ due to numerical errors. A unique equilibrium can be found by an optimisation algorithm minimising Γ .

The paper discusses extensions of the algorithm that are useful when SFE are calculated for real electricity markets. First, as considered by Anderson & Hu [2], firms may have different marginal costs at zero supply, then firms' minimum capacity will bind at different prices. Second, with sufficiently elastic demand, the price cap may not bind at the highest realized price. Third, it is noted that the ordinary shooting algorithm is very ill-conditioned for low mark-ups, such as when demand is very elastic. It is illustrated that this problem can be solved using multiple shooting.

Instead of numerically integrating the KM first-order condition, asymmetric SFE have also been calculated using piece-wise linear or polynomial approximations of supply functions [3-5, 9,22]. Often it has been problematic to find valid SFE with these approaches [2]. However, recently Anderson & Hu [2] have presented a very promising algorithm. They use a piece-wise linear approximation of supply functions, but by discretizing over demand shocks instead of over prices, they manage to calculate asymmetric SFE more robustly compared to earlier piece-wise linear algorithms. The algorithm of Anderson & Hu is limited to what they refer to as *strong* SFE. In such equilibria firms' profits are, given competitors' strategies, maximized for every demand outcome that occurs with a positive probability. Strong supply function equilibria depend on the support of the probability distribution of demand but are otherwise independent of the demand distribution. This is also a property of the SFE conjectured in this paper. Thus for the same market assumptions the two methods should return the same SFE.

The structure of the paper is as follows. Section 2 introduces the notation and assumptions used in the analysis. Section 3 presents a set of systems of differential equations and a numerical algorithm that can be employed to calculate the conjectured SFE. In Section 4, the numerical algorithm is applied to an example with three firms. Section 5 presents extensions of the algorithm, which are useful when modelling bidding in real electricity markets, and applies the algorithm to two examples from Anderson and Hu [2]. The paper is concluded in Section 6.

2. NOTATION AND ASSUMPTIONS

Except for firms' capacities and costs, the notation and market assumptions are the same as in previous papers by Holmberg [14,15]. There are N asymmetric producers. Forward contracts are neglected in the calculations, but they could be considered by interpreting available capacities and demand as being net of forward contracts. The offer of each firm i consists of a piece-wise smooth — i.e. piece-wise twice continuously differentiable — non-decreasing left continuous supply function $S_i(p)$.⁴ The aggregate supply of firm i 's competitors is denoted $S_{-i}(p)$ and total supply is denoted $S(p)$.

Let $\bar{\varepsilon}_i$ be the capacity constraint of producer i . Total capacity is designated by $\bar{\varepsilon}$, i.e.

$$\bar{\varepsilon} = \sum_{i=1}^N \bar{\varepsilon}_i. \text{ Let } p_i \text{ denote the price at which firm } i \text{ chooses to offer its last unit, i.e.}$$

$S_i(p_i +) = \bar{\varepsilon}_i$. For presentational convenience assume that it is possible to order firms a priori such that $p_1 \leq \dots \leq p_{N-1}$. However, this assumption is not crucial; the algorithm can be implemented such that the order is allowed to vary during the calculation.

Denote the perfectly inelastic demand by ε and its probability density function by $f(\varepsilon)$. The density function is continuously differentiable and has a convex support set, which includes zero demand. Moreover, extreme demand outcomes are permitted, i.e. ε such that $\varepsilon > S(\bar{p})$ occurs with positive probability.⁵ In equilibrium, this implies that the capacity constraints of the $N-1$ smallest firms bind with a positive probability. The assumed support of $f(\varepsilon)$ ensures that there is a unique equilibrium. However, as illustrated in Section 5, the algorithm can also be used for markets where several firms have non-binding capacity constraints at maximum demand, in which case there are multiple equilibria.

All firms have increasing, strictly convex and twice continuously differentiable cost functions. Denote the aggregated cost function of all firms by $C(S)$. In Section 3 and 4, it is assumed that $C_i'(0) = C_j'(0)$. In Section 5, the algorithm is extended to cases with

$$C_i'(0) \neq C_j'(0).$$

⁴ Consider a supply function $S_i(p)$ with a discontinuity at p_0 . It is then assumed that firm i is willing to produce any supply in the range $[S_i(p_0), S_i(p_0 +)]$, if the price is p_0 . Thus the left continuous supply function is actually just a representation of a *correspondence*.

⁵ Note that $S(\bar{p})$ does not include ΔS_N , as supply functions are left continuous.

Residual demand of an arbitrary producer i is denoted by $q_i(p, \varepsilon)$. As long as the supply functions of his competitors are not perfectly elastic at p , residual demand is

$$q_i(\varepsilon, p) = \varepsilon - S_{-i}(p). \quad (1)$$

3. THE CONJECTURED SFE

For symmetric producers and producers with asymmetric capacities and identical constant marginal costs, it has been shown that there exists a unique equilibrium with the following properties if extreme demand outcomes occur with a positive probability [14,15] and if demand is perfectly inelastic:

- 1) All producers offer their first units of power at the price $C'(0)$.
- 2) All supply functions are twice continuously differentiable, except at points where the capacity constraint of at least one producer starts to bind.
- 3) There are no supply functions with perfectly elastic segments below the price cap and only firm N (the largest firm) can have a perfectly elastic segment at the price cap. This implies that all supply functions $S_i(p)$ are continuous below the price cap.
- 4) A firm's supply function does not have perfectly inelastic segments below its critical price p_i , where the firm's capacity constraint starts to bind.
- 5) Below the price cap, all supply functions with non-binding capacity constraints fulfil the KM first-order condition.
- 6) $C'(0) < p_1 \leq p_2 \leq \dots \leq p_{N-1} = \bar{p}$.

In the general case, existence of asymmetric SFE cannot be assured. For example a recent paper by Edin [10] indicates that existence may be problematic if some firms have locally very convex marginal cost functions. However, assuming that an asymmetric SFE exists, and that the market is well-behaved in other aspects as well - e.g. demand is sufficiently inelastic, all firms have the same marginal cost at zero capacity, and cost functions are smooth up to the capacity constraints - then it is conjectured that the SFE has the listed properties. Indeed if one limits attention to *strong SFE*, for which SFE are independent of the demand distribution, then proofs by Anderson & Hu [2] indicate that any equilibrium must have several of these properties: Property 1) follows from Lemma 1 in Anderson & Hu [2]. Property 2 follows from Lemma 3 in Anderson & Hu [2]. Property 3 is related to Theorem 2 in Anderson & Hu [2], but there are differences. Because Anderson & Hu [2] assume that demand is sufficiently elastic, so that the price cap never binds in equilibrium, i.e. in their analysis the largest firm will not have

a perfectly elastic segment along the price cap. Another difference in Theorem 2 is that it considers cases with $C_i'(0) \neq C_j'(0)$, which could introduce additional perfectly elastic segments. We will come back to these differences when extensions of the presented algorithm are presented in Section 5. Theorem 12 in Baldick and Hogan (2002) is also related to property 3, but their proof is restricted to producers and price ranges, for which the residual demand is smooth. Property 4-5 are conjectures that are not verified in Anderson & Hu [2], because they do not rule out equilibria with perfectly inelastic segments at prices, for which a firm's capacity constraint does not bind. Property 6 can be interpreted as a convention that is always satisfied if firms are reordered as the calculation proceeds.

3.1. Necessary conditions

Assuming that competitors do not have perfectly elastic supply functions below the price cap, the residual demand of an arbitrary producer i is given by (1). Hence, for given demand and price, the profit of producer i is

$$\pi_i(\varepsilon, p) = [\varepsilon - S_{-i}(p)]p - C_i[\varepsilon - S_{-i}(p)] \text{ if } S_i \leq \bar{\varepsilon}_i \text{ and } p < \bar{p}, \quad (2)$$

where $S_i = \varepsilon - S_{-i}(p)$. In the traditional SFE literature, see Klemperer & Meyer [17] for example, the KM first-order condition is derived by simply differentiating (2) with respect to p . That is,

$$S_i(p) - S_{-i}'(p)[p - C_i'(S_i(p))] = 0. \quad (3)$$

Below the price cap, all equilibrium supply functions with non-binding capacity constraints fulfil the KM first-order condition whenever the supply functions are differentiable. It has been conjectured that points of non-differentiability only occur at prices where capacity constraints begin to bind. This implies that all SFE candidates are given by $N-1$ systems of differential equations. The first system has N differential equations and is valid for the price interval $(C'(0), p_1)$. The second system has $N-1$ differential equations and is valid for the price interval (p_1, p_2) and so on. The continuity assumption links the end-conditions of the systems of differential equations. Including the end-conditions, the $N-1$ systems of differential equations are as follows:

$$\begin{array}{l}
p \in (C'(0), p_1) \\
\vdots \\
p \in (p_{N-3}, p_{N-2}) \\
p \in (p_{N-2}, \bar{p})
\end{array}
\begin{array}{l}
\left\{ \begin{array}{l} S_1(p) - S'_1[p - C'_1(S_1(p))] = 0 \\ \vdots \\ S_N(p) - S'_N[p - C'_N(S_N(p))] = 0 \end{array} \right. \\
\left\{ \begin{array}{l} S_{N-2}(p) - S'_{-(N-2)}[p - C'_{N-2}(\cdot)] = 0 \\ S_{N-1}(p) - S'_{-(N-1)}[p - C'_{N-1}(\cdot)] = 0 \\ S_N(p) - S'_N[p - C'_N(S_N(p))] = 0 \end{array} \right. \\
\left\{ \begin{array}{l} S_{N-1}(p) - S'_{-(N-1)}[p - C'_{N-1}(\cdot)] = 0 \\ S_N(p) - S'_N[p - C'_N(S_N(p))] = 0 \end{array} \right.
\end{array}
\begin{array}{l}
\left\{ \begin{array}{l} S_1(p_1) = \bar{\varepsilon}_1 \\ \vdots \\ S_N(p_1 -) = S_N(p_1 +) \end{array} \right. \\
\left\{ \begin{array}{l} S_{N-2}(p_{N-2}) = \bar{\varepsilon}_{N-2} \\ S_{N-1}(p_{N-2} -) = S_{N-1}(p_{N-2} +) \\ S_N(p_{N-2} -) = S_N(p_{N-2} +) \end{array} \right. \\
\left\{ \begin{array}{l} S_{N-1}(\bar{p}) = \bar{\varepsilon}_{N-1} \\ S_N(\bar{p}) = \bar{\varepsilon}_N - \Delta S_N \end{array} \right.
\end{array} \quad (4)$$

Given a set of values $\{p_1, p_2, \dots, p_{N-2}, \Delta S_N\}$, the $N-1$ systems of differential equations can be solved backwards. One must start with the price interval (p_{N-2}, \bar{p}) , for which all end-conditions are known, i.e. $S_{N-1}(\bar{p}) = \bar{\varepsilon}_{N-1}$ and $S_N(\bar{p}) = \bar{\varepsilon}_N - \Delta S_N$. Thus $S_N(p)$ and $S_{N-1}(p)$ can be calculated for (p_{N-2}, \bar{p}) . This solution can then be used to determine the end-conditions for the price interval (p_{N-3}, p_{N-2}) . After solving the system of differential equations associated with this price interval, one can proceed recursively to the interval (p_{N-4}, p_{N-3}) and so on.

Let the integration of the systems of ordinary differential equations start at the price cap and proceed in the direction of decreasing prices. Assume that the integration terminates as soon as any supply function violates the non-decreasing constraint. Let the criterion function $\Gamma(p_1, p_2, \dots, p_{N-2}, \Delta S_N)$ return the terminating price. According to the conjecture, all producers will, in equilibrium, offer their first unit of power at $C'(0)$. Thus, theoretically all SFE candidates must fulfil $\Gamma[p_1, p_2, \dots, p_{N-2}, \Delta S_N] = C'(0)$. Fig. 1 shows an example where the integration terminates when the supply function of firm 3 becomes invalid. In the example the capacity constraint of firm 1 is assumed to bind above Γ .

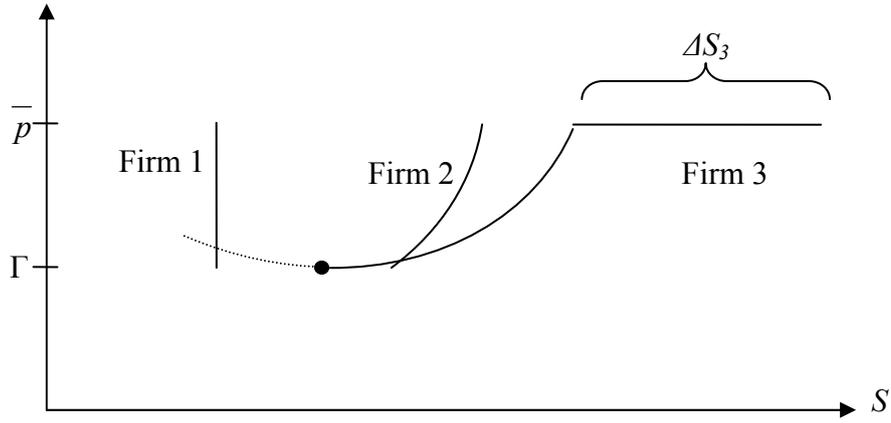


Figure 1. *The integration starts at the price cap, proceeds in the direction of decreasing prices and is terminated as soon as any supply function becomes invalid (in this case the supply function of firm 3).*

3.2. A sufficient condition

Equilibrium candidates must necessarily satisfy the KM first-order condition and

$$\Gamma[p_1, p_2, \dots, p_{N-2}, \Delta S_N] = C'(0) \quad \text{as these conditions ensure that firms' profits are at a local}$$

extremum and that their supply functions are non-decreasing. However, when competitors follow strategies implied by the candidate, it is still not clear that the globally best response for a producer is to follow the SFE candidate. A sufficiently strong second-order condition is that the market price of the equilibrium candidate globally maximises $\pi_i(\varepsilon, p)$ for every ε , given that competitors follow the candidate. If the equilibrium is a strong SFE, then it is also a SFE.

3.3. The numerical algorithm

For asymmetric producers with general cost functions, it is difficult and likely impossible to calculate SFE analytically. Nevertheless, the system of differential equations in (4) can be solved by numerical integration, given $\{p_1, p_2, \dots, p_{N-2}, \Delta S_N\}$. By gridding the space and/or employing optimisation algorithms, values $\{p_1, p_2, \dots, p_{N-2}, \Delta S_N\}$ that (nearly) fulfil $\Gamma = C'(0)$ can be found. In practice, because of numerical errors, SFE candidates that almost

fulfil $\Gamma[p_1, p_2, \dots, p_{N-2}, \Delta S_N] = C'(0)$ cannot be ruled out. The second-order condition can be checked graphically or numerically.

4. AN EXAMPLE WITH THREE ASYMMETRIC FIRMS

The numerical procedure to find valid SFE is illustrated by an example with three firms. Their production capacities are: $\bar{\varepsilon}_1 = \frac{\bar{\varepsilon}}{7}$, $\bar{\varepsilon}_2 = \frac{2\bar{\varepsilon}}{7}$ and $\bar{\varepsilon}_3 = \frac{4\bar{\varepsilon}}{7}$. The marginal cost function of all

firms is linear, $C'_i = c \left(1 + \frac{S_i}{\varepsilon_i} \right)$, up to the capacity constraint. Assume further that the price cap

is $\bar{p} = 4c$.

4.1. Necessary conditions

The KM first-order conditions of the SFE candidates corresponding to (4) are given by the following set of two systems of differential equations:

$$\begin{aligned}
 p \in (c, p_1) & \begin{cases} S_1(p) - S'_{-1}(p) \left[p - c \left(1 + \frac{7S_1}{\varepsilon} \right) \right] = 0 \\ S_2(p) - S'_{-2}(p) \left[p - c \left(1 + \frac{7S_2}{2\varepsilon} \right) \right] = 0 \\ S_3(p) - S'_{-3}(p) \left[p - c \left(1 + \frac{7S_3}{4\varepsilon} \right) \right] = 0 \end{cases} & \begin{cases} S_1(p_1) = \frac{\bar{\varepsilon}}{7} \\ S_2(p_1^-) = S_2(p_1^+) \\ S_3(p_1^-) = S_3(p_1^+) \end{cases} \\
 p \in (p_1, 4c) & \begin{cases} S_2(p) - S'_{-2}(p) \left[p - c \left(1 + \frac{7S_2}{2\varepsilon} \right) \right] = 0 \\ S_3(p) - S'_{-3}(p) \left[p - c \left(1 + \frac{7S_3}{4\varepsilon} \right) \right] = 0 \end{cases} & \begin{cases} S_2(4c) = \frac{2\bar{\varepsilon}}{7} \\ S_3(4c) = \frac{4\bar{\varepsilon}}{7} - \Delta S_3 \end{cases}
 \end{aligned}$$

The variables can be normalised such that $p = c\tilde{p}$ and $S_i(p) = \bar{\varepsilon}\tilde{S}_i(\tilde{p})$, so that

$$\begin{aligned}
\tilde{p} \in (1, \tilde{p}_1) & \left\{ \begin{aligned} & \tilde{S}_1(\tilde{p}) - \tilde{S}'_{-1}(\tilde{p}) \left[\tilde{p} - \left(1 + 7\tilde{S}_1 \right) \right] = 0 \\ & \tilde{S}_2(\tilde{p}) - \tilde{S}'_{-2}(\tilde{p}) \left[\tilde{p} - \left(1 + \frac{7\tilde{S}_2}{2} \right) \right] = 0 \\ & \tilde{S}_3(\tilde{p}) - \tilde{S}'_{-3}(\tilde{p}) \left[\tilde{p} - \left(1 + \frac{7\tilde{S}_3}{4} \right) \right] = 0 \end{aligned} \right. & \left\{ \begin{aligned} & \tilde{S}_1(\tilde{p}_1) = \frac{1}{7} \\ & \tilde{S}_2(\tilde{p}_1 -) = \tilde{S}_2(\tilde{p}_1 +) \\ & \tilde{S}_3(\tilde{p}_1 -) = \tilde{S}_3(\tilde{p}_1 +) \end{aligned} \right. \\
\tilde{p} \in (\tilde{p}_1, 4) & \left\{ \begin{aligned} & \tilde{S}_2(\tilde{p}) - \tilde{S}'_{-2}(\tilde{p}) \left[\tilde{p} - \left(1 + \frac{7\tilde{S}_2}{2} \right) \right] = 0 \\ & \tilde{S}_3(\tilde{p}) - \tilde{S}'_{-3}(\tilde{p}) \left[\tilde{p} - \left(1 + \frac{7\tilde{S}_3}{4} \right) \right] = 0 \end{aligned} \right. & \left\{ \begin{aligned} & \tilde{S}_2(4) = \frac{2}{7} \\ & \tilde{S}_3(4) = \frac{4}{7} - \Delta\tilde{S}_3 \end{aligned} \right.
\end{aligned} \tag{5}$$

Given a set of values $\{\tilde{p}_1, \Delta\tilde{S}_3\}$, the system representing the price interval $(\tilde{p}_1, 4)$ can be solved by numerical integration (see Appendix for technical details regarding the numerical integration). The integration starts at 4, the normalized price cap, and proceeds in the direction of decreasing prices until $p = \tilde{p}_1$. This solution yields end-conditions for the system of differential equations valid for $\tilde{p} \in (1, \tilde{p}_1)$, which in turn can be solved. The integration stops as soon as one of the calculated supply functions violates the non-decreasing requirement. The criterion function $\tilde{\Gamma}(\tilde{p}_1, \Delta\tilde{S}_3)$ returns the terminated price.

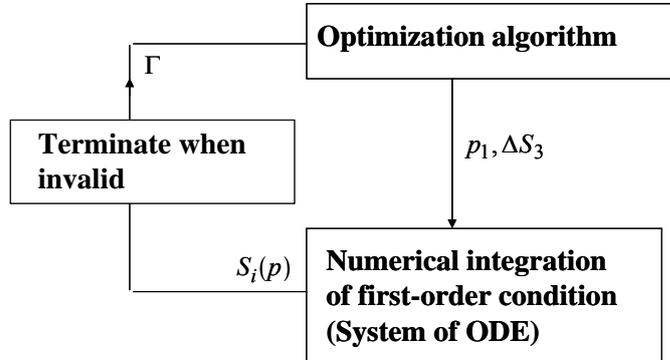


Figure 2. The numerical procedure to find valid SFE candidates.

An example of a parameter set that generates a non-valid SFE is $\tilde{p}_1 = 4$ and $\Delta\tilde{S}_3 = 0$. This is the boundary condition if the price cap is viewed as a public signal that coordinates bids.⁶ The equilibrium was suggested by Baldick & Hogan, as it yields a unique SFE for symmetric producers [4]. They observe, however, that for asymmetric producers the public signal assumption often leads to invalid equilibria as in Figure 3.

⁶ All supply functions are smooth up to the price cap, at which point all capacity constraints bind.

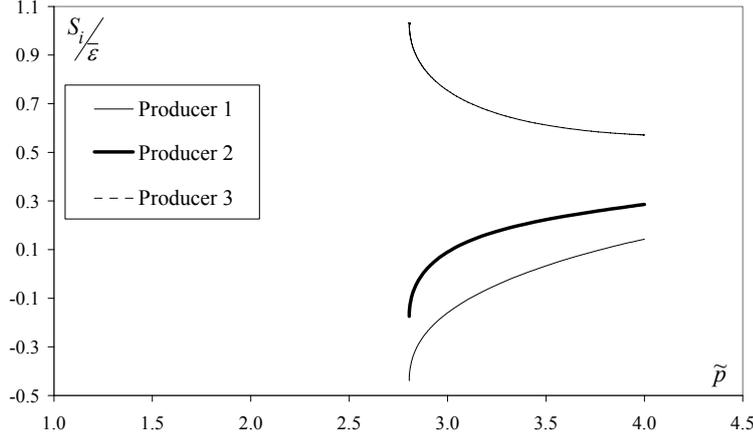


Figure 3. The parameter set $\tilde{p}_1 = 4$ and $\Delta\tilde{S}_3 = 0$ generates invalid supply functions.

To get an idea of the parameter space for which $\tilde{\Gamma}(\tilde{p}_1, \Delta\tilde{S}_3) = 1$, $\tilde{\Gamma}$ is calculated for a grid with 400x400 points in the space $(\tilde{p}_1, \Delta\tilde{S}_3) \in [2, 4] \times \left[0, \frac{4}{7}\right]$. The result is presented as a contour plot in Figure 4, which indicates that $\tilde{\Gamma}(\tilde{p}_1, \Delta\tilde{S}_3)$ has a minimum around $\tilde{p}_1 \approx 3$ and $\Delta\tilde{S}_3 \approx 0.25$. By means of an optimisation algorithm, the minimum of $\tilde{\Gamma}(\tilde{p}_1, \Delta\tilde{S}_3)$ is located at $\tilde{p}_1 \approx 3.117$ and $\Delta\tilde{S}_3 \approx 0.2541$, and the min-value is approximately 1.005.⁷ $\tilde{\Gamma} \approx 1.005$ is close to, but still above, $\tilde{\Gamma} = 1$, which is theoretically necessary for the conjectured SFE. The difference can be explained by the numerical sensitivity of the solution. If a unique SFE exists, as should intuitively be expected based on previous SFE studies [14,15], then a unique set of $\{\tilde{p}_1, \Delta\tilde{S}_3\}$ exists that yields valid supply functions. However, the slightest deviation from the unique set of valid SFE, due to a small numerical error, will lead to $\tilde{\Gamma}[\tilde{p}_1, \Delta\tilde{S}_3] > 1$. The calculated supply functions for the set $\{\tilde{p}_1 \approx 3.117, \Delta\tilde{S}_3 \approx 0.2541\}$ are plotted in Figure 5.

⁷ The optimisation calculation employs the fminsearch algorithm in Matlab, which uses the Nelder-Mead simplex (direct search) method [19]. Estimation of the min-value depends on tolerances used in the numerical integration.

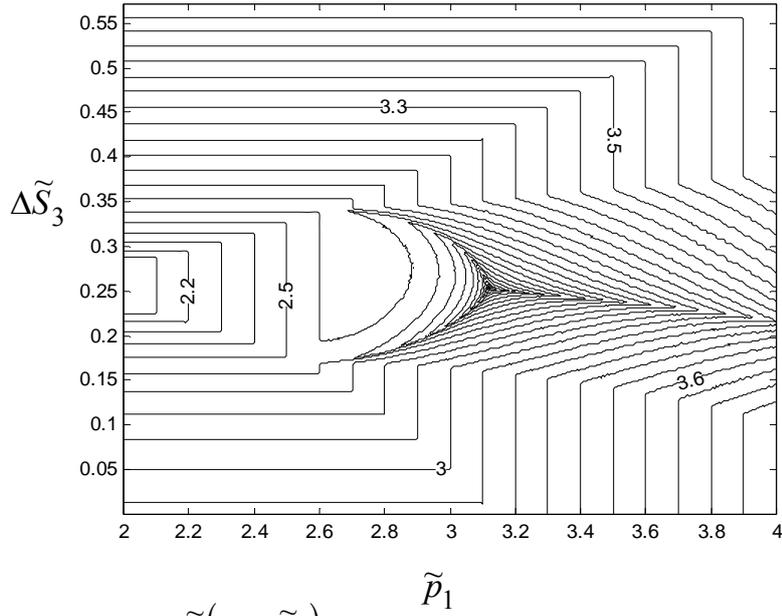


Figure 4. Contour plot of $\tilde{\Gamma}(\tilde{p}_1, \Delta\tilde{S}_3)$. A minimum exists around $\tilde{p}_1 \approx 3.1$ and $\Delta\tilde{S}_3 \approx 0.25$.

Note that firms 2 and 3 have a kink in their supply functions at \tilde{p}_1 , which compensates firm 1's switch from price-responsive supply to perfectly inelastic supply. It might seem counter-intuitive that the two firms' supply becomes more elastic above \tilde{p}_1 , as there are less competing firms in this price range. However, a discontinuous increase in S_2' and S_3' ensures that the slopes of firm 2's and firm 3's residual demands are continuous at \tilde{p}_1 , and it follows from the KM first-order condition in (3) that this is necessary if the supply functions of firms 2 and 3 are to be continuous at p_l . Actually, also the elasticity of the industry supply as a whole increases discontinuously at \tilde{p}_1 , because the slope of the residual demand of firm 2 (and firm 3) is continuous at \tilde{p}_1 , while S_2' (and S_3') increases discontinuously at this price.

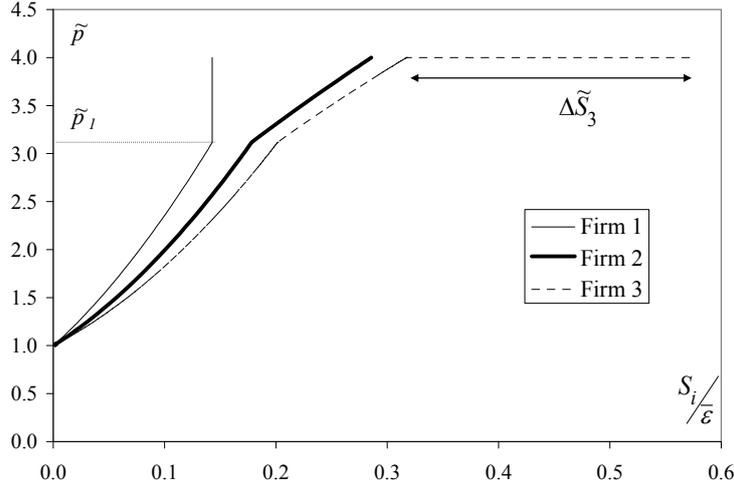


Figure 5. *The equilibrium candidate.*

As shown in a previous paper, the unique asymmetric equilibrium for constant marginal costs is piece-wise symmetric [15]. Two arbitrary producers have the same supply function unless the capacity constraint of one binds. With the strictly convex cost functions assumed in this paper, it will be more expensive for a smaller firm to produce a given supply compared to a larger firm. Thus it is expected that producers with more capacity sell more at every price, as in Figure 5. Still it is apparent that the largest firm uses its market power extensively. More than 40 percent of the capacity of producer 3 is not offered below the price cap.

4.2. The second-order condition

Does the candidate fulfil the sufficient second-order condition? Denote the supply functions of the SFE candidate in Figure 5 by $S_i^X(p)$ and denote its market price by $p^X(\varepsilon)$. Given $S_i^X(p)$, does $p^X(\varepsilon)$ globally maximise $\pi_i(\varepsilon, p)$ for every ε ? To check this, the isoprofit lines of all producers are plotted in Figures 6 to 8 together with $p^X(\varepsilon)$. For a local extremum, a vertical line, corresponding to a constant ε , should have a tangency point with a isoprofit line at $p^X(\varepsilon)$. This corresponds to the KM first-order condition and appears to be true for every demand level for all producers with non-binding capacity constraints. For such firms, one can also deduce from the shape of the isoprofit lines that profit is globally maximised at $p^X(\varepsilon)$.

In regions where producers cannot control the price, either due to a binding capacity constraint or a binding price cap, the tangency condition is not necessarily fulfilled. For example, due to its capacity constraint, firm 1 cannot unilaterally push the price below $p^X(\varepsilon)$ for

$\varepsilon > S^X(p_1)$. By increasing mark-ups, the firm is still able to increase the market price. However, according to Figure 6, such deviations decrease profits. Neither firm 1 nor 2 can control the price for $\varepsilon > S^X(\bar{p})$.⁸ Their capacity constraints prevent them from reducing the price and the price cap prevents them from increasing the price. Firm 3 could reduce the price for $\varepsilon > S^X(\bar{p})$, but according to Figure 8 it would not be profitable. Thus it appears that $p^X(\varepsilon)$ globally maximises $\pi_i(\varepsilon, p)$ for every ε when competitors' aggregate supply is given by $S_{-i}^X(p)$.

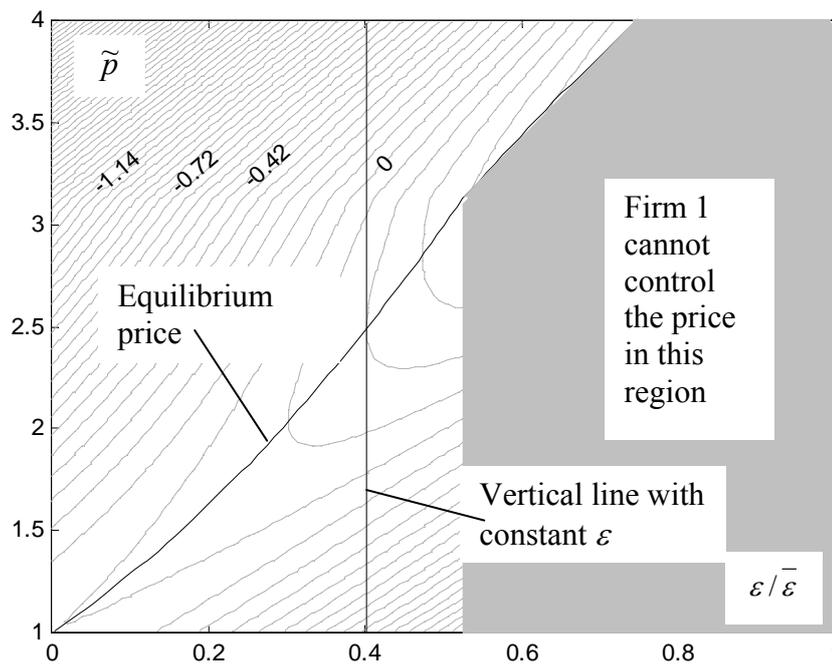


Figure 6. *Isoprofit lines of firm 1.*

⁸ Recall that $S(\bar{p})$ does not include ΔS_N , as supply functions are left continuous.

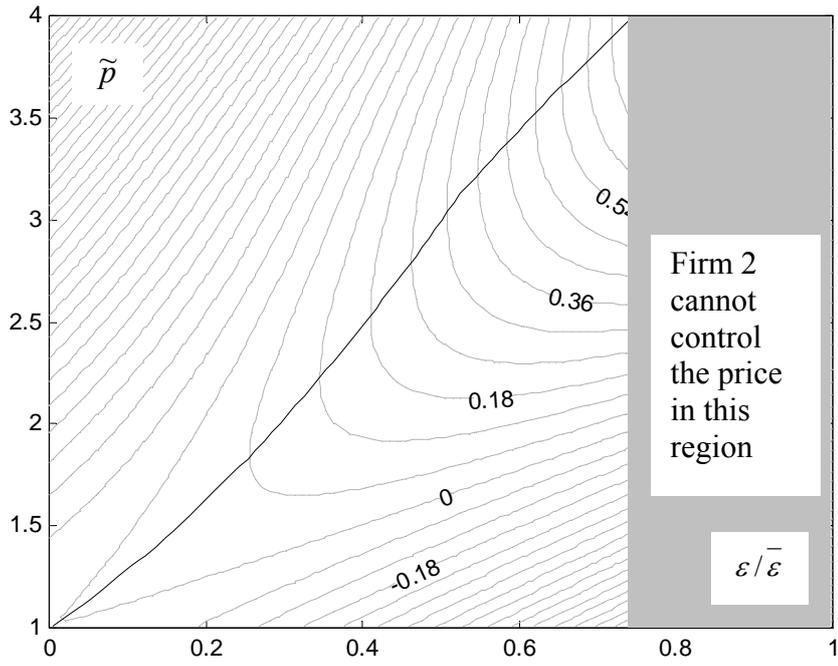


Figure 7. *Isoprofit lines of firm 2.*

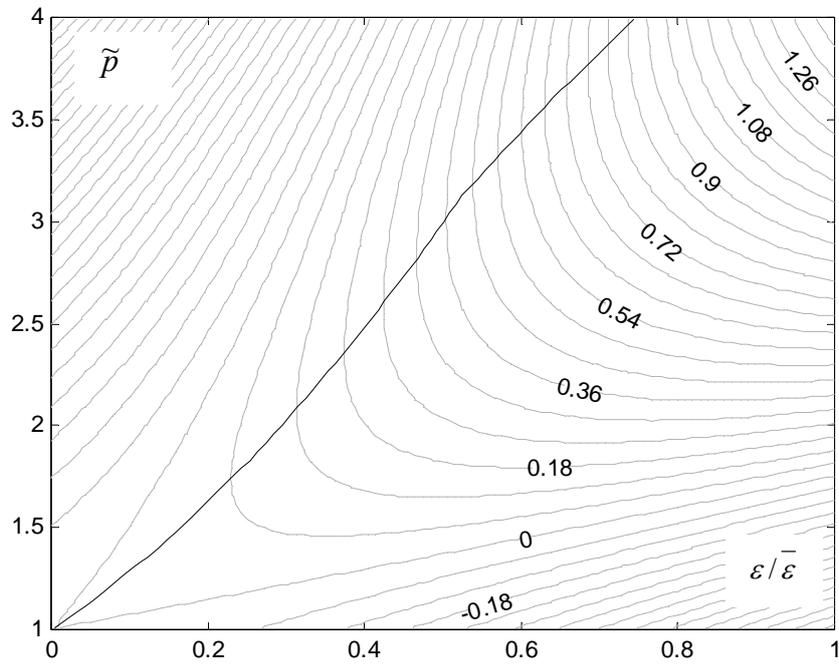


Figure 8. *Isoprofit lines of firm 3.*

5. EXTENSIONS OF THE ALGORITHM

When applied to real electricity markets, extensions of the presented algorithm might be necessary. If firms have different marginal costs at zero supply, then the set of systems of differential equations will change at each price for which $p=C'_i(0)$; only firms with $C'_j(0)<p$ will take part in the system of differential equations. Moreover, if one firm, which we can call firm k , has lower marginal cost at zero supply than all of its competitors, then results by Anderson & Hu [2] and Baldick et al. [3] indicate that firm k can have a perfectly elastic segment at $C'_k(0)$, where $C'_{-k}(S_{-k})$ is the competitors' aggregated cost function. In this case, the necessary condition for a valid equilibrium changes from $\Gamma(p_1 \dots p_{N-2}, \Delta S_N) = C'(0)$ to $\Gamma(p_1 \dots p_{N-2}, \Delta S_N) = C'_{-k}(0)$.

In this paper it has been assumed that demand is perfectly inelastic, which is a reasonable approximation for electricity markets. In principle, it is straightforward to apply the numerical procedure to price-responsive demand as well. Elastic demand adds a variable to the first-order condition in (3) [2,4,17]. Moreover, if demand is sufficiently elastic or the price cap sufficiently high, then the equilibrium will qualitatively change character in that when capacity constraints of smaller firms bind, the largest firm will offer its remaining capacity at a monopoly price below the price cap, as in the asymmetric SFE studied by Anderson & Hu [2] and Newbery [20].

Equilibrium bids are only required to maximize firms' profits for demand outcomes that occur with a positive probability. If there are several firms with non-binding capacity constraints at maximum demand, then there are multiple equilibria [2,4,11,12]. By adjusting the end-conditions of the presented algorithm it can be used also for this case.

5.1. Instabilities in the numerical integration can be solved by multiple shooting

It has been observed by Newbery that the coupled differential equations associated with SFE are highly sensitive to the starting point chosen for the numerical integration [20]. Similar instability problems have been noted by Baldick & Hogan [4]. In the example in Section 4, in which case demand is perfectly inelastic, the instability started near the marginal cost, when the numerical integration was almost done, and it did not cause any major problems. But my experience is that the stability problems become significantly worse when demand is more elastic, even if one uses a robust numerical method, such as Euler backward. Indeed, one can

prove (at least for the duopoly case) that it is the initial value problem itself which is ill-conditioned.

Let $t_1(p)$ and $t_2(p)$ be perturbation functions that returns the change in the solution of the Klemperer & Meyer equation after a infinitesimally small change in the boundary conditions. It can be proven that these perturbation functions can be determined from the following differential equation, which is valid also for elastic demand [16]:

$$\begin{cases} t_1'(p) = \frac{[p - C_2'(s_2(p))]t_2(p) + s_2(p)C_2''(s_2(p))t_2(p)}{[p - C_2'(s_2(p))]^2} \\ t_2'(p) = \frac{[p - C_1'(s_1(p))]t_1(p) + s_1(p)C_1''(s_1(p))t_1(p)}{[p - C_1'(s_1(p))]^2} \end{cases} \quad (6)$$

Stability-wise, this corresponds to a linear system of differential equations with one positive and one negative eigenvalue. It follows from (6) that errors that go in the same direction, i.e. $t_1(p) \cdot t_2(p) > 0$, are weakened as the integration proceeds downwards. On the other hand, opposite errors, i.e. $t_1(p) \cdot t_2(p) < 0$, are amplified if one integrates from the price cap and downwards. Due to the singularity, the amplification factor is extremely large near the marginal cost. Thus small changes in the initial condition can result in large changes in the solution near the marginal cost. Thus one would expect the problem to be ill-conditioned if mark-ups are small. In practice, reversing the integration direction can significantly improve the numerical instability as illustrated by Edin [10]. However, this may depend on the circumstances, because according to (6) the problem could still be ill-conditioned for low mark-ups. But in this case, opposite errors would weaken and same errors would increase as the integration proceeds upwards.

In order to increase the robustness of the numerical method, multiple/parallel shooting can be applied to split up each integrated interval into several smaller ones for numerical integration [8]. The standard parallel shooting algorithm is used for all intervals except for the last one, i.e. initial values are adjusted until the solution at the end-point matches the boundary condition, the initial value of the next interval [8]. The last interval is handled as in the shooting algorithm described in Sections 3 and 4, i.e. the initial value is up-dated until the criterion function F is minimized. Rather than updating initial values of each interval subsequently, they are updated simultaneously, in parallel. At each multiple shooting node p_m , the values of the state variables $\{s_i(p_m)\}_{i=1}^N$ become unknowns that have to be determined by

the optimization procedure, which minimizes the sum of the criterion function I and the absolute value of the discrepancies between the joined integrated solutions at each multiple shooting node. The multiple shooting algorithm allows the numerical solution to be cleared from amplified numerical errors at each shooting node. On the other hand, the number of parameters that are to be determined by the optimization procedure increases significantly, and this can slow down the calculations. Below the multiple shooting algorithm is applied to two examples with elastic demand, for which the ordinary shooting method failed to provide a valid solution, because of numerical instabilities.

The first example is a replica of Example 12 in Anderson & Hu [2]. The cost functions of the three firms in the example are $C_1(q) = 5q + 0.8q^2$, $C_2(q) = 8q + 1.2q^2$ and $C_3(q) = 12q + 2.3q^2$ with capacities $\bar{\varepsilon}_1 = 11$, $\bar{\varepsilon}_2 = 8$, and $\bar{\varepsilon}_3 = 8$. The linear demand curve has a slope of -0.5 . The maximum demand curve that occurs with a positive probability is $D(p) = 52.5 - 0.5p$ and the minimum demand curve occurring with a positive probability is $D(p) = 2.5 - 0.5p$. This ensures a unique SFE, because the capacity constraints of all firms but the largest bind with a positive probability. Demand is sufficiently elastic so that the price cap does not bind. The calculated result is presented in Fig. 9. By and large it is very similar to Anderson & Hu [2], which verifies both my and their algorithm. Differences can be explained by that the relative error in the numerical integration is set to 10^{-3} and corresponding numerical errors in Anderson & Hu [2]. The highest realized price is 54.21. The three firms have non-binding capacity constraints in the price intervals $[5, 42.27]$, $[8, 41.74]$ and $[12, 54.21]$, respectively. Thus firm 3 is the only firm with a non-binding capacity constraint in the price range $[42.47, 54.21]$ and in this range it charges the monopoly price, i.e. its mark-up is inversely proportional to the elasticity of demand [23]. Similarly, firm 1 charges a monopoly price in the range $[5, 8)$. Moreover, firm 1 has a perfectly elastic segment at the price 8, where the non-negativity constraint of firm 2 starts to bind.

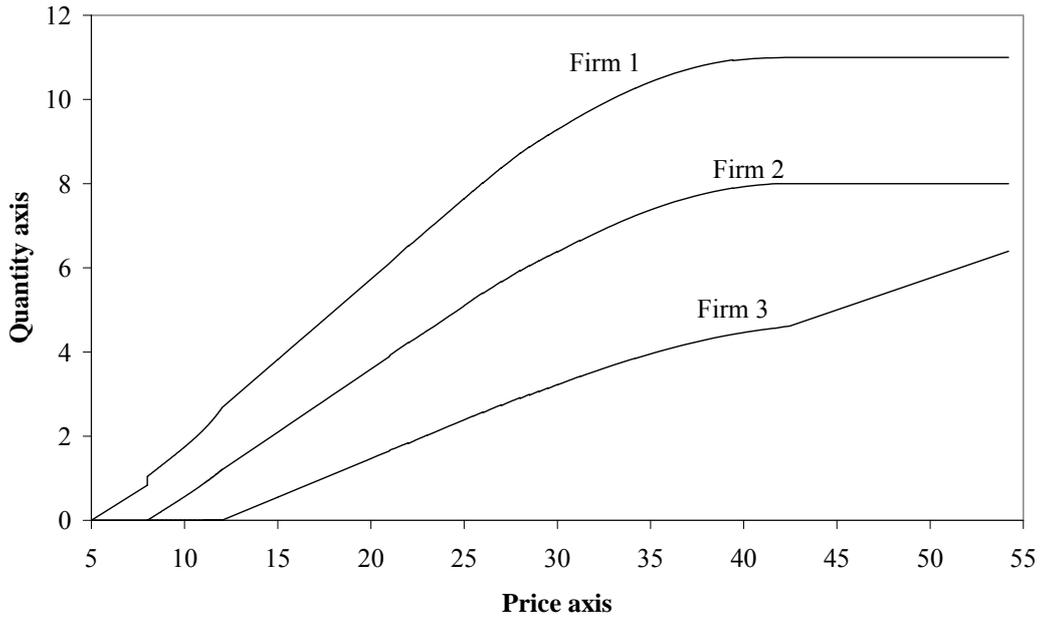


Figure 9. Calculated SFE for a market with three firms and elastic demand, corresponding to Example 12 in Anderson & Hu [2].

The second example is a replica of Example 13 in Anderson & Hu [2], who in their turn based their example on a piece-wise linear model of bidding in the electricity market in England and Wales [3]. This example has five firms. The cost functions of the five firms are $C_1(q)=8q+0.8945q^2$, $C_2(q)=8q+0.965q^2$, $C_3(q) = C_4 (q) =12q+2.3075q^2$, and $C_5(q)=12q+1.34355q^2$ with capacities $\bar{\varepsilon}_1 = 10.4482$, $\bar{\varepsilon}_2 = 9.70785$, $\bar{\varepsilon}_3 = 3.35325$, $\bar{\varepsilon}_4 = 3.3609$, and $\bar{\varepsilon}_5 = 5.70945$. As in Anderson & Hu [2], the least competitive equilibrium is calculated under the assumption that the linear demand curve for the maximum shock is $D(p)=36-0.1p$. In the least competitive equilibrium, the supply curves of the two largest firms will have locally vertical supply at the highest realized price. This vertical property and the first-order conditions for the two largest firms can be used to calculate the highest realized price, 89.059, and the two largest firms' supplies at this price, $S_1=6.876$ and $S_2= 6.795$. Again the calculated SFE is very similar to Anderson & Hu [2]. Applying the multiple shooting method, I find that the capacity constraint of firm 5 starts to bind at the price 83.440. The capacity constraints of firm 3 and 4 starts to bind at 42.898 and 43.127, respectively. Both of these firms have identical cost functions up to the capacity of firm 3, and their supply functions are identical in the interval [12, 42.898], where all of the five firms have non-binding capacity constraints. The non-negativity constraint of the three smallest firms bind and they have zero supply below the price 12. The two largest firms have elastic supply down to the price 8.

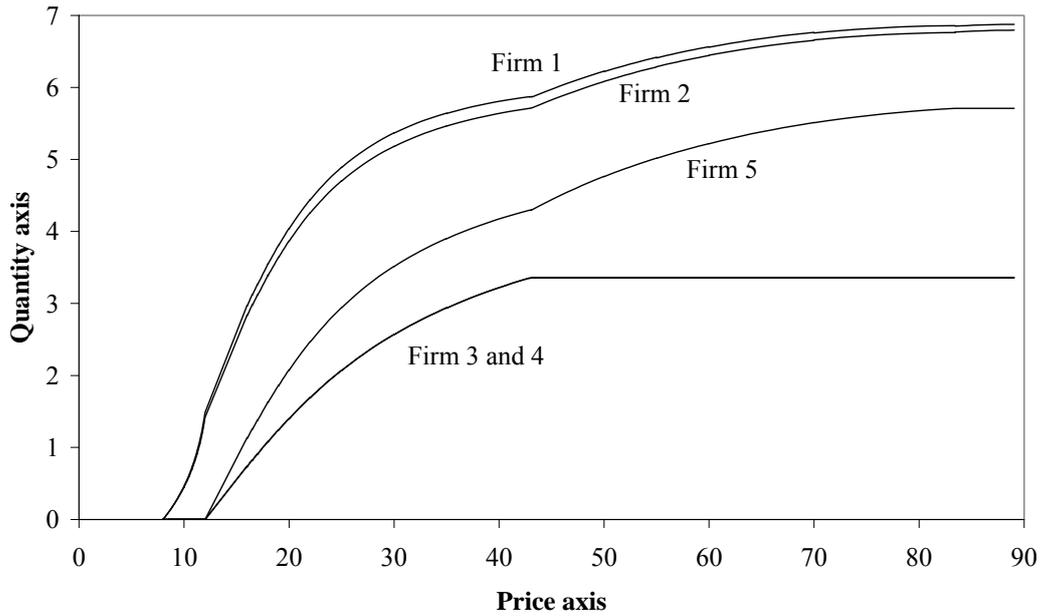


Figure 10. *Calculated SFE for a market with five firms and elastic demand, corresponding to Example 13 in Anderson & Hu [2].*

Figure 11 compares the market price with the system marginal cost as a function of total output for the second example with five firms. Mark-ups imply that surplus will be allocated from consumers to producers. Moreover, asymmetric mark-ups imply that production becomes inefficient; some units with a high marginal cost are accepted from small firms with small mark-ups instead of cheaper production from larger firms with high mark-ups. Thus the system marginal cost will differ from the optimal system marginal cost. The area between the system marginal cost and the optimal system marginal cost gives a welfare loss. It is largest at the total output 22.61, where the capacity constraint of firm 4 starts to bind (the capacity constraint of firm 3 binds at the total output 22.58). Beyond this point, increased output must be supplied by large firms with low marginal costs and high mark-ups. The welfare loss also has a local maximum at the total output 26.09, where the capacity constraint of firm 5 starts to bind.

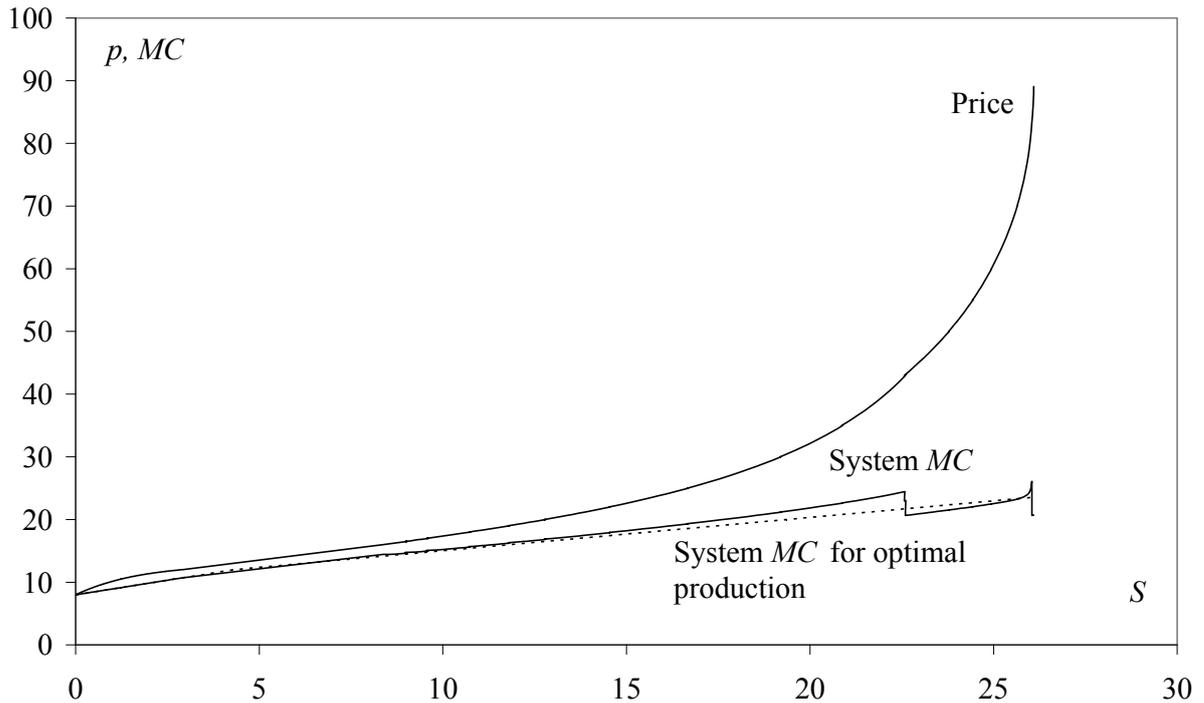


Figure 11. *Calculated market price and system marginal cost as a function of total output for example with five firms.*

6. CONCLUSIONS

Electricity producers typically have non-constant marginal costs and asymmetric production capacities. In this general case, the first-order conditions of a Supply Function Equilibrium (SFE) constitute a system of non-autonomous ordinary differential equations. Solving such a system analytically is very difficult and likely to be impossible. Nevertheless, it can be solved by numerical integration. One problem, however, is that electricity auctions normally require non-decreasing supply functions and Baldick & Hogan [4] have observed that numerically calculated asymmetric supply function equilibria tend to violate this restriction. The three exceptions are: symmetric firms with identical cost functions, cases with affine solutions — i.e. affine marginal costs and no capacity constraints — and when there are small variations in demand.

This paper presents a shooting algorithm that can solve the problem of invalid asymmetric supply function equilibria by means of numerical integration. It is conjectured that asymmetric SFE in markets with perfectly inelastic demand has properties similar to those found in the case of constant marginal costs, which are analysed in [15]. The equilibrium is unique if capacities of all firms but the largest bind at maximum demand. All supply functions fulfil the first-order

condition from the lowest marginal cost up to the price at which either the capacity constraint or price cap binds. The capacity constraints of small firms bind at lower prices compared to firms with larger capacity. The capacity constraint of the second largest firm starts to bind at the price cap. In turn, the largest firm has a perfectly elastic supply ΔS_N at the price cap.

The first-order conditions of the conjectured equilibrium yield $N-1$ systems of non-autonomous ordinary differential equations. Except for the two largest firms, the prices p_i at which the capacity constraints of firms bind are unknown a priori, as is ΔS_N . But given initial or updated guesses $\{p_1, p_2, \dots, p_{N-2}, \Delta S_N\}$, the set of systems can be solved by means of numerical integration starting at the price cap and proceeding in the direction of a decreasing price. When any of the supply functions violate the restrictions that a supply function must be increasing, the integration is terminated. The criterion function $\Gamma(p_1, \dots, p_{N-2}, \Delta S_N)$ returns the price at which the integration terminates. For a valid SFE candidate, Γ must in theory equal the marginal cost at zero supply. However, in practice Γ will be larger than $C'(0)$ due to numerical errors. The unique SFE can then be found by an optimisation algorithm minimising Γ .

The procedure for finding asymmetric SFE candidates is illustrated by an example with three firms and linear marginal costs. A contour plot of Γ and an optimisation algorithm indicate that it has a unique minimum just above the marginal cost at zero supply. Numerically calculated isoprofit lines indicate that no producer will find it profitable to unilaterally deviate from the SFE candidate.

At the price p_i , for which the capacity constraint of firm i starts to bind, the elasticity of supply will increase discontinuously for each firm with a non-binding capacity constraint. This ensures that the slope of residual demand of all firms with non-binding capacity constraints is continuous at p_i . Thus in equilibrium, all but the smallest firm have kinks in their supply functions below their capacity constraint.

Extensions of the algorithm are presented that can consider price-responsive demand and markets in which firms have different marginal costs at zero supply. It is shown that the algorithm is ill-conditioned when mark-ups are small, i.e. when demand is very elastic and/or there are many firms in the market. Two examples illustrate that multiple/parallel shooting can overcome this problem.

For asymmetric firms with increasing marginal costs, asymmetric mark-ups imply inefficient production, as some output from costly generators of small firms will be accepted instead of cheaper production from larger firms.

7. REFERENCES

- [1] E. J. Anderson and A. B. Philpott, “Using supply functions for offering generation into an electricity market”, *Operations Research*, Vol. 50, No. 3, pp. 477-489, 2002.
- [2] E.J. Anderson and X. Hu, “Finding Supply Function Equilibria with Asymmetric Firms”, *Operations Research* 56(3), pp. 697-711, 2008.
- [3] R. Baldick, R. Grant and E. Kahn, “Theory and Application of Linear Supply Function Equilibrium in Electricity Markets”, *Journal of Regulatory Economics*, Vol. 25, No. 2, pp. 143 – 167, 2004.
- [4] R. Baldick and W. Hogan, *Capacity constrained supply function equilibrium models for electricity markets: Stability, non-decreasing constraints, and function space iterations*, POWER Paper PWP-089, University of California Energy Institute, August 2002.
- [5] R. Baldick, R. and W. Hogan, *Polynomial approximations and supply function equilibrium stability*, Center for Business and Government, Harvard University, Cambridge, Massachusetts 02138, 2004.
- [6] F. Bolle, “Supply function equilibria and the danger of tacit collusion. The case of spot markets for electricity”, *Energy Economics*, Vol. 14, pp. 94-102, 1992.
- [7] S. Borenstein, J. Bushnell, and C. Knittel, “Market Power in Electricity Markets, Beyond Concentration Measures”, *The Energy Journal*, Vol. 20, No. 4, pp. 65-88, 1999.
- [8] G. Dahlquist and Å. Björck, *Numerical Methods*, Prentice-Hall, Inc., London, 1974.
- [9] C. J. Day. and D. W. Bunn, “Divestiture of generation assets in the electricity pool of England and Wales: A computational approach to analyzing market power”, *Journal of Regulatory Economics*, Vol. 19, pp. 123–141, 2001.
- [10] K-A Edin, *Hydro-Thermal Auction Markets*, Mimeo, Tentum, Stockholm, 2007.

- [11] T. Genc & S. Reynolds, *Supply Function Equilibria with Pivotal Electricity Suppliers*, Eller College Working Paper No.1001-04, University of Arizona, July 2004.
- [12] R.J. Green and D.M. Newbery, “Competition in the British Electricity Spot Market”, *Journal of Political Economy*, Vol. 100, no. 5, pp. 929-953, 1992.
- [13] R. Green, “Increasing Competition in the British Electricity Spot Market”, *Journal of Industrial Economics*, Vol. XLIV, No 2, pp. 205-16, 1996.
- [14] P. Holmberg, “Unique supply function equilibrium with capacity constraints”, *Energy Economics*, Vol. 30, No. 1, pp. 148-172, 2008.
- [15] P. Holmberg, “Asymmetric supply function equilibrium with constant marginal costs”, *Energy Journal*, Vol. 28, No. 2, pp. 55-82, 2007.
- [16] P. Holmberg, *Game-theoretical, strategic forward contracting in the electricity market*, Working Paper 756, Research Institute of Industrial Economics, Stockholm, 2008.
- [17] P.D. Klemperer and M.A. Meyer, “Supply function equilibria in an oligopoly under uncertainty”, *Econometrica*, Vol. 57, pp. 1243-1277, 1989.
- [18] V. Krishna, *Auction theory*, Academic Press, London, 2002.
- [19] Mathworks, *Matlab: Function Reference*, Version 7, www.mathworks.com.
- [20] D. M. Newbery, “Capacity-constrained Supply Function Equilibria: Competition and Entry in the Electricity Spot Market”, Unpublished manuscript. University of Cambridge, Department of Applied Economics, 1991.
- [21] A. Rudkevich, M. Duckworth and R. Rosen, “Modeling electricity pricing in a deregulated generation industry: The potential for oligopoly pricing in poolco”, *The Energy Journal*, Vol. 19, No. 3, pp. 19-48, 1998.
- [22] A. Rudkevich, *Supply function equilibrium: theory and applications*, Proceedings

of the 36th Hawaii International Conference on Systems Science, 2003.

[23] J. Tirole, *The Theory of Industrial Organization*, MIT Press: Cambridge, US; 2003.

[24] J. Wang and J. Zender, “Auctioning divisible goods”, *Economic Theory*, Vol. 19, pp. 673-705, 2002.

[25] R. Wilson, “Auctions of shares”, *Quarterly Journal of Economics*, Vol. 93, pp. 675-689, 1979.

APPENDIX

The numerical integration is performed in Matlab. A robust solver is employed, the ode15s of Matlab with the backward differentiation option [19]. The *Event Location Property* of the Matlab function is used to detect invalid supply functions [19].

When using numerical integration algorithms, it is often necessary to rewrite the system of differential equations in the standard form $x'(p) = f(x(p))$. This transformation is illustrated for the system of differential equations below. The first-order condition is

$$\begin{cases} S_1(p) - S'_{-1}(p)[p - C_1'(S_1(p))] = 0 \\ \vdots \\ S_N(p) - S'_{-N}(p)[p - C_N'(S_N(p))] = 0. \end{cases}$$

The system can be rewritten in the following form:

$$\begin{cases} \frac{S_1(p)}{p - C_1'(S_1(p))} = S'_{-1}(p) \\ \vdots \\ \frac{S_N(p)}{p - C_N'(S_N(p))} = S'_{-N}(p). \end{cases} \quad (7)$$

Summing over all equalities yields

$$\sum_{j=1}^N \frac{S_j(p)}{p - C_j'(S_j(p))} = (N-1)S'(p).$$

As $S'_{-i}(p) = S'(p) - S'_i(p)$, the system in (7) can now be rewritten as

$$\left\{ \begin{array}{l} S'_1(p) = \frac{1}{N-1} \sum_{j=1}^N \frac{S_j(p)}{p - C_j'(S_j(p))} - \frac{S_1(p)}{p - C_1'(S_1(p))} \\ S'_N(p) = \frac{1}{N-1} \sum_{j=1}^N \frac{S_j(p)}{p - C_j'(S_j(p))} - \frac{S_N(p)}{p - C_N'(S_N(p))} \end{array} \right.$$

which has the standard form $x'(p) = f(x(p))$. A more general expression, which also considers elastic demand, has been derived by Baldick & Hogan [4].