RESEARCH INSTITUTE OF INDUSTRIAL ECONOMICS

IFN Working Paper No. 1328, 2020

Homogeneity, Returns to Scale and (Log)Concavity

Per Hjertstrand

Research Institute of Industrial Economics
P.O. Box 55665
SE-102 15 Stockholm, Sweden
info@ifn.se
www.ifn.se

Homogeneity, Returns to Scale and (Log)Concavity

Per Hjertstrand*

Research Institute of Industrial Economics (IFN)

Stockholm

March 2020

Abstract

This brief note shows that if a production function, f, is quasiconcave, increasing and homogeneous, then f is concave if it displays nonincreasing returns to scale, and f is logconcave if it displays increasing returns to scale.

JEL Codes: D21

Keywords: Homogeneity; Returns to scale; Rho-concavity

1 The result

It is a standard economics textbook exercise to show that a quasiconcave and increasing production function displaying nonincreasing (i.e., decreasing and constant) returns to scale is, in fact, concave on its entire support; See e.g., Theorem 3.1 in Jehle and Reny (2011, p.131).[1] However, I have been unable to find any analogous result in the literature for a quasiconcave and increasing production function displaying *increasing* returns to scale. That is, can the shape restriction of such a production function be further strengthened? In this brief note, I show that this question has a positive answer: A quasiconcave and increasing production function displaying *increasing* returns to scale is always *logconcave*. In particular, I prove the following general theorem, which I relate to the concept of returns to scale in Corollary 1 below:

Theorem 1 *If a function $f : B \subset \mathbb{R}^K_+ \to \mathbb{R}_{++}$ is quasiconcave, increasing and homogeneous of degree γ, then f is concave if $0 < \gamma \leq 1$ and logconcave if $1 < \gamma < \infty$.*

Before providing a simple proof of this result, we need some preliminary definitions:

Definition 1 *A function $f : B \subset \mathbb{R}^K_+ \to \mathbb{R}_{++}$ is homogeneous of degree γ if it can be written as:*

$$f(tx) = t^\gamma f(x),$$

for any number $t > 0$.

*Research Institute of Industrial Economics (IFN), P.O. Box 55665, SE-102 15 Stockholm, Sweden. E-mail: Per.Hjertstrand@ifn.se. I thank Jan Wallander och Tom Hedelius stiftelse and Marianne och Marcus Wallenberg stiftelse for funding.

[1] Simon and Blume (1994, Theorem 21.15) show that a quasiconcave and increasing production function displaying constant returns to scale is concave. See also Dalal (2000) and Prada (2011) who gives short proofs of this result in the case of non-increasing returns to scale.

Definition 2 (Caplin and Nalebuff 1991) *Consider* $\rho \in [-\infty, \infty]$. *For* $\rho > 0$, *a function,* $f : B \subset \mathbb{R}_{+}^{K} \rightarrow \mathbb{R}_{++}$, *where* B *is convex, is called* $\rho-$*concave if for all* $x_1, x_2 \in B$ *and any* $\lambda \in [0,1]$:

$$f\left(\lambda x_1 + (1-\lambda) x_2\right) \geq \left[\lambda f\left(x_1\right)^{\rho} + (1-\lambda) f\left(x_2\right)^{\rho}\right]^{\frac{1}{\rho}}.$$

For $\rho < 0$ *the condition is exactly as above except when* $f(x_1) f(x_2) = 0$, *in which case there is no restriction other than* $f(\lambda x_1 + (1-\lambda) x_2) \geq 0$. *Finally, the definition is extended to include* $\rho = \infty, 0, -\infty$ *through continuity arguments.*

Caplin and Nalebuff (1991) discuss implications and limiting cases. For $\rho > 0$, Definition 2 states that f^{ρ} is concave, while for $\rho < 0$, $-f^{\rho}$ is concave. Higher values of ρ correspond to more stringent variants of concavity; that is, a $\rho-$concave function is also $\rho'-$concave for all $\rho' < \rho$. We have the following limiting cases:

- If $\rho = \infty$ then f is uniform on its support. Specifically, $\lim_{\rho \rightarrow \infty} \left[\lambda f(x_1)^{\rho} + (1-\lambda) f(x_2)^{\rho}\right]^{\frac{1}{\rho}} = \max\{f(x_1), f(x_2)\}$.
- If $\rho = 1$ then we obtain the standard definition of concavity.
- If $\rho = 0$ then f is logconcave. Using L'Hospital's rule we have: $\lim_{\rho \rightarrow 0} \left[\lambda f(x_1)^{\rho} + (1-\lambda) f(x_2)^{\rho}\right]^{\frac{1}{\rho}} = f(x_1)^{\lambda} f(x_2)^{(1-\lambda)}$. Thus, by logtransformation, f is logconcave, i.e., $\log f(\lambda x_1 + (1-\lambda) x_2) \geq \lambda \log f(x_1) + (1-\lambda) \log f(x_2)$.
- If $\rho = -\infty$ then Definition 2 takes the weakest form of quasiconcavity. Specifically, $\lim_{\rho \rightarrow -\infty} \left[\lambda f(x_1)^{\rho} + (1-\lambda) f(x_2)^{\rho}\right]^{\frac{1}{\rho}} = \min\{f(x_1), f(x_2)\}$.

Proof of Theorem 1. Consider any $x_1, x_2 \in B$, with B convex, where $L \geq 1$, and a positive, increasing, quasiconcave and homogeneous of degree γ function $f : B \subset \mathbb{R}_{+}^{K} \rightarrow \mathbb{R}_{++}$. Define $y_1^{\gamma} = f(x_1)$ and $y_2^{\gamma} = f(x_2)$ such that $y_i = f(x_i)^{\frac{1}{\gamma}}$ for $i = 1, 2$. Homogeneity of degree γ implies:

$$f\left(\frac{x_i}{y_i}\right) = \frac{1}{y_i^{\gamma}} f(x_i) = \frac{1}{f(x_i)} f(x_i) = 1,$$

for $i = 1, 2$. By quasiconcavity, we have, for any $\alpha \in [0,1]$:

$$f\left(\alpha \frac{x_1}{y_1} + (1-\alpha) \frac{x_2}{y_2}\right) \geq \min\left\{f\left(\frac{x_1}{y_1}\right), f\left(\frac{x_2}{y_2}\right)\right\} = 1.$$

Set:

$$\alpha = \frac{\lambda y_1}{\lambda y_1 + (1-\lambda) y_2}.$$

Substituting α and by homogeneity of degree γ, we have:

$$\begin{aligned}
1 &\leq f\left(\alpha \frac{x_1}{y_1} + (1-\alpha) \frac{x_2}{y_2}\right) \\
&= f\left(\left(\frac{\lambda y_1}{\lambda y_1 + (1-\lambda) y_2}\right) \frac{x_1}{y_1} + \left(1 - \left(\frac{\lambda y_1}{\lambda y_1 + (1-\lambda) y_2}\right)\right) \frac{x_2}{y_2}\right) \\
&= f\left(\frac{\lambda x_1 + (1-\lambda) x_2}{\lambda y_1 + (1-\lambda) y_2}\right) \\
&= \frac{1}{\left[\lambda y_1 + (1-\lambda) y_2\right]^{\gamma}} f\left(\lambda x_1 + (1-\lambda) x_2\right).
\end{aligned}$$

Thus, since $y_1 > 0$ and $y_2 > 0$, we get:

$$f\left(\lambda x_1 + (1-\lambda) x_2\right) \geq \left[\lambda y_1 + (1-\lambda) y_2\right]^{\gamma}.$$

Substituting $y_i = f(x_i)^{\frac{1}{\gamma}}$ for $i = 1, 2$, we obtain:

$$f(\lambda x_1 + (1-\lambda) x_2) \geq \left[\lambda f(x_1)^{\frac{1}{\gamma}} + (1-\lambda) f(x_2)^{\frac{1}{\gamma}}\right]^{\gamma}.$$

Hence, it immediately follows from Definition 2 that f is $\rho-$concave with $\rho = \frac{1}{\gamma}$. We distinguish between the following cases:

- If $-\infty < \gamma < 0$, then $-\infty < \rho < 0$, i.e., f is quasiconcave.
- If $0 < \gamma < 1$, then $1 < \rho < \infty$, i.e., f is concave.
- If $\gamma = 1$, then $\rho = 1$, i.e., f is concave.
- If $1 < \gamma < \infty$, then $0 < \rho < 1$, i.e., f is logconcave

This completes the proof. ■

Consider next the following standard definition of returns to scale:

Definition 3 *Consider Definition 1. A production function, f, displays:*

- *decreasing returns to scale when f is homogeneous of degree $0 < \gamma < 1$ for any number $t \geq 1$;*
- *constant returns to scale when f is homogeneous of degree $\gamma = 1$ for any number $t > 0$;*
- *increasing returns to scale when f is homogeneous of degree $1 < \gamma < \infty$ for any number $t \geq 1$.*

The following result then immediately follows from Theorem 1 and Definition 3:

Corollary 1 *Suppose that f is a quasiconcave and increasing production function. Then:*

1. *If f displays nonincreasing (i.e., decreasing or constant) returns to scale then f is concave.*
2. *If f displays increasing returns to scale then f is logconcave.*

Some concluding remarks:

- Case 1 reproduce the well-known result that a production function displaying nonincreasing returns to scale is always concave; See e.g., Jehle and Reny (2011, Theorem 3.1).
- Case 2 is new, and states that a quasiconcave and increasing production function displaying increasing returns to scale is always logconcave.
- Since $\rho < 1$, the shape restriction in case 2 cannot be further strenghened to obtain concavity.
- Given the discussion of the limiting cases following Definition 2, it is clear that logconcavity is a significantly stronger shape restriction than quasiconcavity.
- The proof of Theorem 1 shows that, without loss of generality, the degree of concavity, ρ, can be taken to be inversely related to the degree of homogeneity, i.e., $\rho = \frac{1}{\gamma}$.
- Corollary 1 is exhaustive in the sense that it covers all relevant degrees of homogeneity in production, and consequently, all various forms of returns to scale.

References

Caplin, A., and B. Nalebuff (1991). Aggregation and social choice: a mean voter theorem. *Econometrica* **59**, 1–24.

Dalal, A. (2000). Strict concavity with homogeneity and decreasing returns to scale. *Atlantic Economic Journal* **28**, 381–382.

Jehle, G.A., and P.J. Reny (2011). *Advanced Microeconomic Theory*. Prentice Hall, London.

Prada, J.P. (2011). A note on concavity, homogeneity and nonincreasing returns to scale. *Economics Bulletin* **31**, 100-105.

Simon, C.P., and L. Blume (1994). *Mathematics for economists*. W.W. Norton & Company, Inc. New York.