RESEARCH INSTITUTE OF
INDUSTRIAL ECONOMICS

Price Instability in Multi-Unit Auctions

Edward Anderson and Pär Holmberg

This is an author-produced version of a paper accepted for publication in the Journal of Economic Theory*. The paper has been peer-reviewed but does not include the final proof corrections or pagination.* License information.

DOI/Link: https://doi.org/10.1016/j.jet.2018.01.008

Reference: Anderson, Edward and Pär Holmberg (2018), "Price Instability in Multi-Unit Auctions", Journal of Economic Theory 175(May), 318–341.

Research Institute of Industrial Economics
P.O. Box 55665
SE-102 15 Stockholm, Sweden
info@ifn.se
www.ifn.se

Price instability in multi-unit auctions

Edward Anderson*and Pär Holmberg†

January 26, 2018

Abstract

We consider a uniform-price procurement auction with indivisible units and private independent costs. We find an explicit solution for a Bayesian Nash equilibrium, which is unique if demand shocks are sufficiently evenly distributed. The equilibrium has a price instability in the sense that a minor change in a supplier's realized cost can result in a drastic change in the market price. We quantify the resulting volatility and show that it is reduced as the size of indivisible units decreases. In the limit, the equilibrium converges to the Supply Function Equilibrium (SFE) for divisible goods if costs are common knowledge.

Key words: Multi-unit auctions, indivisible unit, price instability, supply function equilibria, convergence of Nash equilibria, wholesale electricity markets

JEL Classification C62, C72, D43, D44, L94

*University of Sydney Business School, University of Sydney, NSW 2006, Australia. E-mail: edward.anderson@sydney.edu.au.

†Corresponding author: Research Institute of Industrial Economics (IFN), Stockholm. Associate Researcher of the Energy Policy Research Group (EPRG), University of Cambridge.

1 Introduction

Many procurement auctions trade indivisible units or have bid constraints that force suppliers to offer their goods with stepped supply functions. Such markets often have unpredictable and significant price variations even if demand is certain and costs are common knowledge (or have small variations). Similar to von der Fehr and Harbord (1993), we refer to this as a price instability. One illustrative example is the Bertrand-Edgeworth game, where capacity-constrained suppliers compete on prices and each supplier is paid its own price (Edgeworth, 1925; Kruse et al., 1994; Deneckere and Kovenock, 1996).[1] Another illustrative example is the related model of a uniform-price auction by von der Fehr and Harbord (1993). In the Nash equilibrium of both models, each supplier has positive mark-ups and chooses offers randomly to avoid a situation where the best response of a rival is to slightly undercut the supplier.[2] In both models, a supplier offers its entire production capacity at one price. In this paper we are the first to analyse price instability in markets where each supplier has multiple indivisible units and is allowed to offer each unit at a different price. We focus on procurement auctions, but results are analogous for sales auctions.

We generalize von der Fehr and Harbord's model by considering a uniform-price auction where each supplier offers a number of indivisible units at different prices. Another generalisation is that we allow suppliers to have private and independent costs. Producers are symmetric ex-ante, before they receive information about their cost. The sequence is such that each supplier receives a signal of its cost and then chooses an offer price, with higher signals leading to higher prices. As illustrated in Figure 1, this gives rise to an offer price range for each indivisible unit of the supplier. We solve for a pure-strategy Bayesian NE, where each supplier chooses offer prices for each of its units in order to maximise its expected profit given its private information. We show that if the cost uncertainty is sufficiently small and either indivisible units are sufficiently small or demand shocks are sufficiently evenly distributed, then there exists one symmetric pure-strategy Bayesian Nash equilibrium where the offer price ranges for the different units of a supplier will not overlap, as illustrated in Figure 1. We call this property *step separation.* We explicitly solve for the symmetric equilibrium and prove that it is the unique equilibrium if demand shocks are sufficiently evenly distributed. In the special case with two units per supplier, we also analyze NE where offer ranges of units overlap, as can occur with highly non-uniformly distributed demand shocks. Overlapping offer prices for non-overlapping marginal costs leads to welfare losses.

For circumstances with small indivisible units and small cost uncertainties, we show that the standard deviation of a supplier's equilibrium offer is approximately

[1] In Bertrand-Edgeworth games, price instability means that suppliers sell goods at different prices, which violates the law of one price. This phenomenon is sometimes called price dispersion (Varian, 1980).

[2] In case firms choose prices sequentially as in a dynamic Bertrand game, then they can undercut each other sequentially, which will give rise to Edgeworth price cycles (Maskin and Tirole, 1988; Noel, 2007).

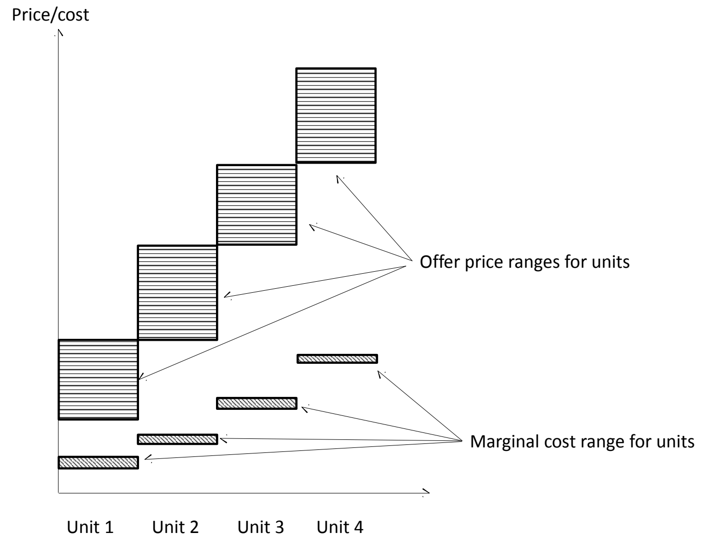

Figure 1: Illustration of price instability where small variations in costs introduces large changes in the offer prices of a supplier. Offer price and marginal cost ranges of its indivisible units are non-overlapping in this example. In addition, the maximum offer price of one unit equals the minimum offer price of the next unit. We refer to this as *step-separation without gaps.*

given by $(I-1)(p_n - c_n)/(\sqrt{12}n)$, where I is the number of symmetric suppliers and $p_n - c_n$ is the approximate mark-up for the n'th unit (the n'th cheapest unit of a supplier). For parameter values that are typical in a wholesale electricity market, this approximation would imply that the standard deviation in the offer for the most expensive production unit is in the range 0.2%-5% of the reservation price. Our analysis indicates that bid volatility should be less pronounced for less expensive production units. We also estimate the standard deviation of the market price (price instability) for each demand level. It can be 0.1%-3% of the reservation price for the highest demand level and tends to be smaller for lower demand levels.

In the limit where costs are common knowledge, our Bayesian NE with price instability corresponds to a mixed-strategy NE in accordance with the purification theorem (Harsanyi, 1973). We let the size of indivisible units decrease to show that the mixed-strategy NE converges to the supply function equilibrium (SFE); a pure-strategy NE in a market with divisible goods which was originally characterized by Klemperer and Meyer (1989). The convergence result confirms a conjecture made by Newbery (1998) and gives theoretical support to the use of SFE to approximate equilibria with bid constraints in wholesale electricity markets as in Green and Newbery (1992), Anderson and Hu (2008), Holmberg and Newbery (2010) and Vives (2011), and analogous approximations in sales auctions: Wilson (1979), Wang and Zender (2002), Hortaçsu and McAdams (2010), Rostek and Weretka (2012), and Ausubel et al. (2014).

A supplier's optimal offer prices for its indivisible units are determined from the characteristics of its residual demand curve, which is uncertain. This is due to uncertainties in both the auctioneer's demand and in the competitors' supply, with the latter arising from imperfect knowledge of competitors' costs, or because they randomize their offers. The standard approach in the literature when calculating the best response for such circumstances is to first characterize the stochastic residual demand curve of a supplier by a probability distribution function, the *market distribution function* (Anderson and Philpott, 2002b). Analogous probability distributions have been used by Wilson (1979) and in the empirical study of the Turkish treasury auction by Hortaçsu and McAdams (2010). A methodological contribution in this paper is that we develop a discrete version of the market distribution function, which is suitable when analysing markets with indivisible units. We derive necessary and sufficient conditions for the best response of a supplier facing a stochastic residual demand process. Wolak (2007) and Kastl (2012) present related necessary conditions, which are suitable for empirical studies of multi-unit auctions with various bid constraints. Our main contribution is that we establish a sufficient condition for global optimality that is crucial when Nash equilibria are constructed in multi-unit auctions with indivisible units.

A related paper by Holmberg et al. (2013) considers a divisible-good auction where costs are common knowledge among suppliers, permissible prices are given by a discrete set and suppliers choose quantities from a continuous set for each permissible price. Discrete prices imply that a supplier who wants to undercut a competitor needs to undercut it by a considerable amount and this means that NE can exist without price instability.[3] Holmberg et al. (2013) prove that pure-strategy NE in this setting converge to SFE as the tick-size (the distance between permissible price levels) shrinks to zero. This result is mainly relevant for markets with significant tick-sizes such as financial exchanges with continuous trading and related call markets, a single-round auction that opens or closes the exchange.

In this paper, we consider indivisible units and offer prices that are chosen from a continuum (i.e. no tick-size); a setting where price instability has been a concern. Our model is particularly relevant for the Colombian electricity market, where suppliers submit one offer price for the entire capacity of each production plant (Wolak, 2009). But our setting is also relevant for other electricity markets. Power generators often have constraints on both minimum and maximum output and often have an optimum output level in-between, where the efficiency is highest. Many market operators of deregulated electricity markets in U.S. take such constraints into account (Baldick et al., 2005). Several European countries – Austria, Benelux, France, Germany, Slovenia and the Nordic countries – allow producers to make indivisible block-orders for a plant, which must be completely accepted or rejected (Meeus et al., 2009). Our setting also has relevance for auctions with restrictions on the number of bids per supplier/plant, which are used in most single-round multi-unit auctions (Kastl, 2012; Holmberg et al., 2013). However, our setting is more restrictive as we constrain not only the number of steps, but also the length of each step. Moreover, our results could be useful when analysing bidding in experimental studies of multi-unit auctions, which often have

[3] Anderson and Xu (2004) studied a related problem with discrete prices, but they did not analyse equilibrium convergence.

a simple set-up similar to our setting (Brandts et al., 2014; Le Coq, 2017).

Our purpose, methodology and results significantly depart from convergence studies by Reny (1999), McAdams (2003) and Kastl (2012). It is likely that their arguments could be used to prove that our uniform-price auction with indivisible units has a pure-strategy Bayesian Nash equilibrium and that this equilibrium would converge to a related pure-strategy Bayesian NE in the limit game with divisible units. However, with such a non-constructive mathematical argument we wouldn't be able to characterize the price instability occurring for this pure-strategy Bayesian NE, or determine how price instability depends on the size of indivisible units. Unlike Reny, McAdams and Kastl, we also analyse the reverse problem. For costs that are common knowledge among suppliers, we show that for every symmetric SFE in the limit game with divisible-goods there is one mixed-strategy NE in a corresponding auction with indivisible units that converges to the symmetric SFE as the size of indivisible units decreases. This proves that an SFE is a robust approximation of the equilibrium in an auction with many indivisible units; there are no drastic changes in the equilibrium if the size of indivisible units increases from zero to a small number.

Wolak (2007) and Kastl (2011,2012) show that smooth bid-function approximations may not be accurate in some circumstances, and that it is sometimes preferable to apply empirical models that consider details in the bidding format. Empirical studies of the wholesale electricity market in Texas (ERCOT) show that offers of the two to three largest producers in this market, who submit a large set of offer prices per producer, roughly match Klemperer and Meyer's first-order condition for continuous supply functions, while the fit is worse for small producers (Sioshansi and Oren, 2007; Hortaçsu and Puller, 2008). Hortaçsu et al. (2017) show that the prediction of producers' bidding behaviour in Texas can be improved if one takes into account that large and small producers have different strategic abilities.

Anwar (2006) analyses Nash equilibria of auctions with indivisible units that all have the same marginal cost, which is common knowledge among suppliers. However, our main contributions do not overlap with his results, because he does not explicitly solve for Nash equilibria, quantify the price instability nor prove equilibrium convergence. Ausubel et al. (2014) present examples where an auctioneer sells two indivisible units, but otherwise their analysis focuses on divisible goods.

We introduce the baseline model with a symmetric duopoly and sufficiently evenly distributed demand shocks in Section 2. In Section 3 we derive necessary and sufficient conditions for the best response, and we characterise Bayesian NE for our baseline model. We explicitly solve for a unique Bayesian NE, which is step-separated, and we prove equilibrium convergence. The extension in Section 4 considers weaker demand assumptions where uniqueness cannot be ensured. For cases with 2 indivisible units per supplier and costs that are common knowledge, this section provides a more complete equilibrium analysis, including NE where offer ranges overlap. Section 5 considers multiple suppliers for the special case when costs are common knowledge. Section 5.1 uses stylized facts to predict price instability in wholesale electricity markets. Section 6 concludes. All proofs are in the online Appendix.

2 The model

In our baseline model, $I = 2$ suppliers compete in a single-shot game by making offers in a uniform-price auction. Each supplier has N indivisible production units of equal size h with a total production capacity $\overline{q} = Nh$.[4] Producers have private and independent costs.[5] The cost of each production unit of supplier i is decided by its private signal α_i, which is chosen by nature and which is not observed by the competitor.[6] There is no loss in generality in assuming that the range of α_i values is uniformly distributed on $[0,1]$, so that the probability distribution of a signal is $G(\alpha_i) = \alpha_i$.[7] We assume that suppliers are symmetric ex-ante; the marginal cost for the n'th unit of supplier i is given by $c_n(\alpha_i)$. We suppose that $c_n(\alpha_i)$ is weakly and continuously increasing in α_i and strictly increasing in n. In the special case where costs do not depend on signals, i.e. costs are common knowledge among suppliers, independent signals effectively act as randomization devices that help suppliers to independently randomize their strategies in a mixed-strategy NE, in accordance with the purification theorem by Harsanyi (1973). We write $C_n(\alpha_i) = h\sum_{m=1}^{n} c_m(\alpha_i)$ for the total cost for supplier i of supplying an amount nh. We assume that the highest marginal cost, $c_N(1)$, is strictly smaller than the reservation price $\overline{p}$. We also require successive units of supplier $i \in \{1,2\}$ not to have overlapping ranges for their marginal costs, as illustrated in Figure 1, i.e.

$$c_{n-1}(\alpha) < c_n(0) \tag{1}$$

for all $n = 2...N$ and $\alpha \in (0,1)$.[8]

[4]Production capacities in our procurement setting corresponds to purchase constraints in sales auctions. As an example, the U.S. Treasury auction has a 35% rule, which prevents anyone from buying more than 35% of the auctioneer's supply. Spectrum auctions by the Federal Communications Commission (FCC) have similar rules. California has purchase limits in its auction of Greenhouse Gas emission allowances. The purchase constraints are used to avoid a situation where a single bidder can corner the secondary market.

[5]Note that suppliers' production costs can still be correlated over time if the auction were to be repeated. Marginal costs can for example consist of a fuel cost that is the same for all suppliers (common knowledge) and a part that is private information, and which is independent of the competitor's private information.

[6]Our results would change if suppliers' costs were dependent. It is less critical whether a producer receives a one-dimensional or multi-dimensional signal, such as a vector with individual cost information for each of its production units. The Bayesian NE with step separation that we solve for should be the same, as long as suppliers receive independent signals and the (marginal) probability distribution for a unit's cost does not depend on the dimensionality of the signal.

[7]Note that we are free to choose the cost parameterization to achieve this. Assume that there is some signal $\widetilde{\alpha}$ with the probability distribution $\widetilde{G}(\widetilde{\alpha})$ and cost function $\widetilde{c}_n(\widetilde{\alpha})$, for which this is not true. Then we can always define a new signal $\alpha = \widetilde{G}(\widetilde{\alpha})$ and define a new cost function $c_n(\alpha) = \widetilde{c}_n(\widetilde{G}^{-1}(\alpha))$, which would satisfy our assumptions.

[8]If, in practice, the auction was to be repeated a month later, it could very well be that the cost of unit $n-1$ at that later point is higher than the cost of unit n today. It could even be that the ordering of units with respect to costs could be different a month later.

We analyse Bayesian Nash equilibria, where each supplier i first observes its signal α_i and then chooses an optimal offer price $p_n^i(\alpha_i)$ for each unit $n \in \{1, \ldots, N\}$. We consider cases where $p_n^i(\alpha_i)$ is a continuous, piecewise smooth and strictly increasing function of its signal α_i. Thus outcomes where a sharing rule is needed to clear the auction can be neglected. Moreover, offers are monotonic, i.e. $p_n^i(\alpha_i)$ is strictly increasing with respect to the unit number n for a given α_i.

Given a value of α_i and a set of offer prices $\{p_n^i(\alpha_i)\}_{n=1}^N$, the stepped supply of supplier i as a function of price is:

$$s_i(p, \alpha_i) = h \sup\{n : p_n^i(\alpha_i) \leq p\}.$$

Note that $s_i(p, \alpha_i)$ is a weakly decreasing function of α_i and weakly increasing with respect to p.

Similar to von der Fehr and Harbord (1993), we assume that demand is uncertain and inelastic up to the reservation price. The demand shock β is realized after suppliers have submitted their offers and is independent of suppliers' signals. In wholesale electricity markets the shock could correspond to uncertainty in consumers' demand (including own production, e.g. solar power) and uncertainty in the output of renewable power (e.g. wind power) or must-run plants from non-strategic competitors.[9]

On the demand side we consider a similar bidding format, and similar discreteness, to that which occurs on the supply side. Hence, we assume that the demand shock β can take values on the set $\mathcal{Q}(h) = \{h, 2h, 3h..., 2Nh\}$, where each element in the set $\mathcal{Q}(h)$ occurs with a positive probability, implying that all the suppliers' capacity is required at the highest level of demand. We let $F(\beta)$ be the probability distribution of the demand shock β, i.e. $F(b) = \Pr(\beta \leq b)$ with $b \in \mathcal{Q}(h)$, and let $f(\beta)$ be the probability mass function $f(b) = \Pr(\beta = b) > 0$ for $b \in \mathcal{Q}(h)$. Note that in case $\beta = 0$ would occur with a positive probability, then we can always transform the problem to an equivalent problem with the same Bayesian NE and where $\beta = 0$ occurs with zero probability.[10]

In order to characterize the uncertainty of the demand, we find it useful to introduce:

$$\tau_m = \frac{f(mh) - f((m-1)h)}{f((m-1)h)} \tag{2}$$

[9] There is an analogous supply shock in many multi-unit sales auctions. In Mexico, Finland and Italy, the treasury sometimes reduce the quantity of issued bonds after the bids have been received (McAdams, 2007). In treasury auctions in U.S. there is often an uncertain amount of non-competitive bids from many small non-strategic investors (Wang and Zender, 2002; Rostek et al., 2010). IPOs sometimes incorporate the so-called "Greenshoe Option", which allow issuing firms to increase the amount of shares being offered by up to 15% after the bids have been submitted (McAdams, 2007).

[10] Given any strategy profile for the players we may write $\overline{\pi}_i(k)$ for the expectation over α values of the profit made by player i when demand is k, so that the expected profit for player i is $\overline{\Pi}_i = \sum_{k=1}^{IN} f(kh)\,\overline{\pi}_i(k)$, since $\overline{\pi}_i(0) = 0$. In the case that $f(0) > 0$ we can consider a new set of demand probabilities f' with $f'(0) = 0$ and $f'(kh) = f(kh)/(1 - f(0)), k = 1, 2, ..., IN$, so that the expected profit becomes $\overline{\Pi}_i' = \overline{\Pi}_i/(1 - f(0))$. Thus an equilibrium under f is still an equilibrium under f'.

for $m = 2..., 2N$. The parameter τ_m is the relative increase in probability mass as the demand outcome increases from $m-1$ to m units. Note that both $f(mh)$ and $f((m-1)h)$ are non-negative, so $\tau_m \geq -1$.

Assumption 1: *Demand is sufficiently evenly distributed so that*

$$-1 < 3k\tau_k < 1 \tag{3}$$

for $k \in \{2, 3, \ldots, 2N\}$.

Thus we limit the proportional variation in the probability mass function in moving from one demand level to the next. We allow for larger relative variations in the probability mass function for small demand levels (small k).

Following the occurrence of a demand shock β, the auctioneer clears the market at the lowest price where supply is weakly larger than demand.

$$p = \inf \left\{ r : \beta \leq \sum_{i=1}^{2} s_i(r, \alpha_i) \right\}.$$

We consider a uniform-price auction, so all accepted offers are paid the clearing price p. Thus, the payoff of a supplier i is a random variable depending on the realized demand and the other supplier offer; when n units are sold at price p the payoff is:

$$\pi_i = pnh - C_n(\alpha_i).$$

We denote the expected profit of supplier i with information α_i who submits a stack of offer prices $\{r_n(\alpha_i)\}_{n=1}^N$ by $\Pi_i(r_1, r_2, ...r_N, \alpha_i) = \mathbb{E}(\pi_i(r_1, r_2, ...r_N) | \alpha_i)$.

3 Analysis

3.1 Best response

We start our analysis by deriving the best response of a supplier, who is facing an uncertain residual demand. The uncertainty comes about as demand is uncertain and also because competitors' offers are uncertain. Generally, the uncertain residual demand in a multi-unit auction can be characterized by a market distribution function as in Anderson and Philpott (2002) or equivalently by Wilson's (1979) probability distribution of the market price. In our case, we use a discrete version of the market distribution function, $\Psi_i(n, p)$, which gives the probability that the offer of the nth unit of supplier i is rejected if offered at the price p. In our application, the rejection probability depends on properties of the random demand shock β and how the competitor j's stepped supply function changes with respect to its cost signal α_j. We have

$$\Psi_i(n, p) = \Pr(\beta - s_j(p, \alpha_j) < nh).$$

It follows from our assumptions that $\Psi_i(n, p)$ will be continuous and piecewise smooth as a function of p. We can show the following result (all proofs are in the online Appendix).

Lemma 1

$$\frac{\partial \Pi_i (r_1, r_2, ...r_N, \alpha_i)}{\partial r_n} = nh (\Psi_i (n+1, r_n) - \Psi_i (n, r_n)) \qquad (4)$$
$$-\frac{\partial \Psi_i (n, r_n)}{\partial r_n} h (r_n - c_n(\alpha_i)),$$

provided that $\partial \Psi_i (n, r_n) / \partial r_n$ *exists.*

In the case where the left and right derivatives of $\frac{\partial \Psi_i(n,r_n)}{\partial r_n}$ do not match, then it is easy to see that (4) will still hold provided we choose either left or right derivatives consistently. The result in (4) can be interpreted as follows. Assume that supplier i increases the offer price of its unit n, then there are two counteracting effects on the expected pay-off. The revenue increases for outcomes when the offer for unit n is price-setting, which occurs with the probability $\Psi_i (n+1, r_n) - \Psi_i (n, r_n)$. Thus the first term in (4) corresponds to a price-effect; the marginal gain from increasing the offer price of the nth unit if acceptance was unchanged. On the other hand, a higher offer price means that there is a higher risk that the offer of the nth unit is rejected. This is the quantity effect. The marginal loss in profit is given by the increased rejection probability $\frac{\partial \Psi_i(n,r_n)}{\partial r_n}$ for the nth unit times the pay-off from this unit when it is on the margin of being accepted. Thus $\frac{\partial \Pi_i(r_1,r_2,...r_N,\alpha_i)}{\partial r_n}$ equals the price effect minus the loss related to the quantity effect.

We can identify the right-hand side of (4) as being a discrete version of Anderson and Philpott's (2002b) Z function for uniform-price auctions.[11] Thus we define:

Definition 1

$$Z_i (n, r_n, \alpha_i) = nh (\Psi_i (n+1, r_n) - \Psi_i (n, r_n)) - \frac{\partial \Psi_i (n, r_n)}{\partial r_n} h (r_n - c_n(\alpha_i)), \quad (5)$$

where we take the right-hand derivative of Ψ_i *if left and right derivatives do not match.*

Hence, $Z_i (n, r_n, \alpha_i)$ is the right-hand derivative of Π_i with respect to r_n, which is independent of other offer prices r_m, where $m \neq n$. We use $Z_i^- (n, r_n, \alpha_i)$ to denote the left-hand derivative of $\Pi_i (r_1, r_2, ...r_N, \alpha_i)$ in the following result.

Lemma 2 *A set of offers* $\{r_n^*\}_{n=1}^N$ *is globally optimal for supplier i for signal* α_i *if, for each n:*

$$Z_i (n, r_n, \alpha_i) \begin{array}{ll} \leq 0 & \text{for } r_n \geq r_n^* \\ \geq 0 & \text{for } r_n < r_n^*. \end{array} \qquad (6)$$

If Π_i *is differentiable at* r_n^* *then a necessary condition for offer prices* $\{r_n^*\}_{n=1}^N$ *to be optimal for signal* α_i *is*

$$Z_i (n, r_n^*, \alpha_i) = 0.$$

[11] Note that we have chosen our Z function to have a sign opposite to Anderson and Philpott (2002b). Thus there is also a corresponding change in our optimality conditions.

In case the left and right derivatives differ at r_n^*, *the necessary condition generalizes to*

$$Z_i\left(n, r_n^*, \alpha_i\right) \leq 0 \text{ and } Z_i^-\left(n, r_n^*, \alpha_i\right) \geq 0.$$

Intuitively, we can interpret Lemma 2 as follows: an offer $r_n^*\left(\alpha_i\right)$ is optimal for unit n for signal α_i if the quantity effect dominates for all prices above $r_n^*\left(\alpha_i\right)$ and the price effect dominates for all prices below this price.

3.2 Step separation without gaps

In this subsection, we use the optimality conditions from the previous subsection to show that in many cases the equilibrium must have the property that the lowest offer of unit n is at the same price as the highest offer for the previous unit $n-1$. We refer to this as step separation without gaps, which we illustrated in Figure 1. We can show the following for a duopoly.

Lemma 3 *Under Assumption 1, the Bayesian NE must have the following properties:*

1. *Offer ranges for successive units of supplier* $i \in \{1,2\}$ *do not overlap and do not have any gaps between them, i.e.* $p_{n-1}^i(1) = p_n^i(0)$ *for* $n \in \{2, \ldots, N\}$.
2. *For the highest realized cost, the highest offer price of supplier* $i \in \{1,2\}$ *is at the reservation price, i.e.* $p_N^i(1) = \overline{p}$.
3. *The Bayesian NE must be symmetric, i.e.* $p_n^1(\alpha) = p_n^2(\alpha)$ *for* $\alpha \in [0,1]$ *and* $n \in \{1, \ldots, N\}$.

Assumption 1, i.e. that demand needs to be sufficiently evenly distributed, is needed to rule out overlap. In Section 4, we will give an example where overlap occurs for highly non-uniform demand distributions. As shown in the online Appendix (Lemma 8), gaps between the offer price ranges of successive units of a supplier can be ruled out in general, even without Assumption 1. This result is established by showing that if player i has a gap with $p_{n-1}^i(1) < p_n^i(0)$, i.e. the player does not have any offer in this range of prices, then player $j \neq i$ can always improve an offer in this range by increasing it. Thus, in equilibrium, there will also be a matching gap in the offer of player j. But this implies that player i will gain from increasing the offer price $p_{n-1}^i(1)$, which contradicts the optimality of player i's offers. Symmetry of the equilibrium offers for unit N follows from observing that the first-order conditions and the initial condition $p_N^1(1) = p_N^2(1) = \overline{p}$ are symmetric. The property that offers are step separated without gaps implies that $p_{N-1}^1(1) = p_{N-1}^2(1)$, which gives a symmetric initial condition for the next unit. Repetition of this argument gives symmetry for all units.

3.3 Uniqueness and existence

In this subsection we establish uniqueness and existence of an equilibrium with step separated offers.

Proposition 1 *Under Assumption 1, there is a unique Bayesian Nash equilibrium, which is symmetric and has strictly positive mark-ups, i.e.* $p_n(\alpha) > c_n(\alpha)$ *for* $\alpha \in [0,1]$*. The equilibrium is given by the set of solutions* $\{p_n(\alpha)\}_{n=1}^{N}$ *and can be computed from the end-conditions*

$$\begin{aligned} p_N(1) &= \overline{p}, \\ p_n(1) &= p_{n+1}(0), \ \forall n \in \{1,\ldots,N-1\}, \end{aligned}$$

and

$$p_n(\alpha) = p_n(1)\frac{(\alpha\tau_{2n}+1)^{1/(n\tau_{2n})}}{(\tau_{2n}+1)^{1/(n\tau_{2n})}} + (\alpha\tau_{2n}+1)^{1/(n\tau_{2n})}\int_\alpha^1 \frac{c_n(u)\,(u\tau_{2n}+1)^{-1/(n\tau_{2n})-1}}{n}du. \tag{7}$$

If demand is uniformly distributed then Assumption 1 is satisfied, and by noting what happens as τ_{2n} approaches zero, we see that (7) is replaced by:

$$p_n(\alpha) = p_n(1)\,e^{\frac{\alpha-1}{n}} + \int_\alpha^1 \frac{c_n(u)e^{\frac{\alpha-u}{n}}}{n}du. \tag{8}$$

Next we analyse how offers depend on the probability distribution of the demand shocks. According to (2), a high τ_{2n} value means that the probability is high that the auctioneer buys $2n$ units relative to the probability that the auctioneer buys $2n-1$ units. For the unique equilibrium, which is symmetric and step-separated, this relaxes competition for unit n of a supplier and makes it optimal for the supplier to increase the offer price of this unit. It follows from the proof of Proposition 2 that the sensitivity of offer prices to changes in τ_{2n} is larger when mark-ups are high.

Proposition 2 *If Assumption 1 is satisfied, then the offer price* $p_n(\alpha_i)$ *of the unique Bayesian NE weakly increases for every unit* $n \in \{1,\ldots,N\}$ *and* $\alpha_i \in [0,1]$ *if* τ_{2n} *is weakly increased for every* $n \in \{1,\ldots,N\}$*. Thus offer prices (weakly) increase if outcomes where the auctioneer buys an even number of units become more likely and outcomes where the auctioneer buys an odd number of units become less likely.*

Proposition 1 simplifies as follows when costs are common knowledge. In this limit of our model, the private signals do not influence costs; they are simply used as randomization devices by the suppliers when choosing their offers. Thus our symmetric Bayesian Nash equilibrium corresponds to a symmetric mixed-strategy Nash equilibrium.

Corollary 1 *If Assumption 1 is satisfied and costs are common knowledge, then there is a unique equilibrium. This is a symmetric mixed-strategy Nash equilibrium, which is defined by the end-conditions:*

$$\begin{aligned} p_N(1) &= \bar{p}, \\ p_n(1) &= p_{n+1}(0) \ \forall n \in \{1, \ldots, N-1\}, \end{aligned}$$

and

$$p_n(\alpha) = (p_n(1) - c_n) \left(\frac{\alpha \tau_{2n} + 1}{\tau_{2n} + 1} \right)^{1/(n\tau_{2n})} + c_n. \tag{9}$$

An α-value gives the probability that unit n is offered at a lower price than $p_n(\alpha)$, which corresponds to a probability distribution. Thus (9) can be rewritten in the following form,

$$\Theta_n(p) = \left(\frac{p - c_n}{p_n(1) - c_n} \right)^{n\tau_{2n}} \left(1 + \frac{1}{\tau_{2n}} \right) - \frac{1}{\tau_{2n}}, \ p \in [p_n(0), p_n(1)], \tag{10}$$

where $\Theta_n(p)$ is the probability distribution for offer prices of unit n for the mixed strategy NE in Corollary 1.

Proposition 1 can be used to estimate the variance of a supplier's offer for its n'th unit:

Proposition 3 *If the unit size h is sufficiently small and Assumption 1 is satisfied, then the variance of an offer $p_n(\alpha)$ can be approximated by*

$$\frac{(p_n(1) - \hat{c})^2 h^2}{12\gamma^2} + O(h^3),$$

where $\gamma = nh$ and $p_n(1)$ is determined by Proposition 1 and

$$\hat{c} = (\tau_{2n} + 1)^{1/(n\tau_{2n})} \int_0^1 c_n(u) \left(1 - (u\tau_{2n} + 1)^{-1/(n\tau_{2n}) - 1} \right) du, \tag{11}$$

which ensures that $\hat{c} \in [c_n(0), c_n(1)]$.

Thus when a unit is small and the cost uncertainty of the unit is small relative to its mark-up, then the variance of an offer can be approximated by $\frac{(p_n(1)-c_n)^2}{12n^2}$. This term captures bid volatility that is driven by the indivisibility of units, where a small change in the realized marginal cost has a significant effect on the offer price.

3.4 Equilibrium convergence

The result in Proposition 3 also applies to cases where costs are common knowledge. Thus for a given output of a supplier, volatility decreases as the size of units decrease (n increases). This suggests that the mixed-strategy NE in Corollary

1 converges to a pure-strategy NE. This subsection establishes that the mixed-strategy NE converges to a supply function equilibrium (SFE), i.e. a pure-strategy Nash equilibrium of smooth supply functions for divisible goods. We know from Holmberg (2008) and Anderson (2013) that there is a unique supply function equilibrium for production capacities $\overline{q}$ when inelastic demand has support in the range $[0, 2\overline{q}]$. We let $\tilde{C}'(Q)$ be the non-decreasing marginal cost of the divisible output Q. The unique SFE is symmetric. For duopoly markets it can be determined from Klemperer and Meyer's (1989) differential equation:

$$P'(Q) = \frac{P(Q) - \tilde{C}'(Q)}{Q} \tag{12}$$

and the boundary condition $P(\overline{q}) = \overline{p}$. The solution to this differential equation is presented by Rudkevich et al. (1998), Anderson and Philpott (2002a) and Holmberg (2008). They also verify that this continuous solution is an SFE, i.e. no producer has a profitable deviation.

We get a related difference equation from Corollary 1 and the condition $p_{n-1}(1) = p_n(0)$; equilibrium offers for the highest signal can be determined from:

$$p_{n-1}(1) = (\tau_{2n} + 1)^{-1/(n\tau_{2n})} (p_n(1) - c_n) + c_n. \tag{13}$$

A solution to this equation is referred to as a discrete solution.

Below we consider a sequence of auctions with successively larger integers N and with closely related exogenous demand shock distributions, such that the limit distribution has a well-defined continuous probability density $\overline{f}(b) = \lim_{N\to\infty} \frac{f(\overline{b})}{h}$, where $h = \frac{\overline{q}}{N}$ and $\overline{b} = h\lceil b/h \rceil$ is the smallest demand shock in the set $\mathcal{Q}(h) = \{h, 2h, 3h..., 2Nh\}$ that is not smaller than b. Moreover, the limiting distribution is such that $\overline{f}'(b)/\overline{f}(b)$ is bounded in the interval $[0, 2\overline{q}]$. Below we will show that the discrete solution of (13), and accordingly also mixed-strategy NE in the sequence of auctions, converges to the SFE of the continuous model in the limit as the size of the indivisible production units h decreases towards 0.

The market design that we consider and the resulting optimality conditions that we derive are quite different from Holmberg et al. (2013). For example there is no price instability in their model. However, a similarity is that the convergence arguments in both papers partly apply techniques used in the numerical analysis of differential equations (Le Veque, 2007). These techniques can be used to prove that the solution of a difference equation converges to the solution of an associated differential equation. The first step is to show consistency, i.e. that the difference equation converges to the associated differential equation.

Lemma 4 *The difference equation*

$$p_{n-1}(1) = (\tau_{2n} + 1)^{-1/(n\tau_{2n})} (p_n(1) - c_n) + c_n \tag{14}$$

can be approximated by

$$p_n(1) - p_{n-1}(1) = \frac{p_n(1) - c_n}{n} + O(h^2)$$

and is consistent with the differential equation in (12) if

$$\begin{aligned} nh &\to Q, \\ c_n &\to \tilde{C}'(Q) \end{aligned}$$

and

$$c_n < p_n(1)$$

when $h \to 0$.

We can use a continuous SFE to approximate the solution to a discrete problem, so that the error is the difference between the two solutions. Then the convergence of the differential and difference equations (consistency) ensures that the local error that is introduced over a short price interval is reduced as the size of the production units, h, becomes small. However, this does not ensure that the discrete solution will converge to the continuous solution when $h \to 0$, because accumulated errors may still grow at an unbounded rate along a fixed price interval as h becomes smaller and the number of production units increases. Hence the next step in the convergence analysis is to establish stability, i.e. that small changes in $p_n(1)$ does not drastically change $p_n(0)$. Convergence, is verified by the proposition below.

Proposition 4 *Let $P(Q)$ be the unique pure-strategy continuous supply function equilibrium for divisible units, then there exists a corresponding mixed-strategy NE for indivisible units with properties as in Corollary 1, which converges to the continuous supply function equilibrium in the sense that $p_n(\alpha_i) \to P(nh)$ when $h \to 0$ and $c_n \to \tilde{C}'(nh)$. If $\left|\overline{f}'(x)\right| < \frac{\overline{f}(x)}{3x}$, i.e. the slope of the probability density of the demand shock is relatively small in the limit, then this ensures that the mixed-strategy NE is the unique equilibrium in the auction with indivisible units when $h \to 0$.*

Thus an SFE is a robust approximation of the NE in an auction with many indivisible units; there are no drastic changes in the equilibrium as the size of indivisible units increases from zero to a small number.

It follows directly from (14) that offer prices of indivisible units increase with respect to costs and the reservation price, which is similar to the symmetric SFE for divisible units (Holmberg, 2008). A difference, however, is that the symmetric SFE is ex-post optimal and independent of the probability distribution of demand shocks (Klemperer and Meyer, 1989). In our model for indivisible units, equilibrium offers do depend on the demand uncertainty as shown in Proposition 2. However, as shown above this dependence will weaken as units get smaller and disappear in the limit where $h \to 0$.

We end this section with a simple example. When the demand is uniformly distributed (so $\tau_{2n} = 0$), we obtain from (9) that:

$$p_n(\alpha) = (p_n(1) - c_n)\, e^{\frac{\alpha - 1}{n}} + c_n. \tag{15}$$

We illustrate this formula in Figure 2 for 20 indivisible units per supplier. In Figure 2 we also compare our equilibrium for indivisible units with the supply function equilibrium for perfectly divisible goods. The comparison illustrates that the supply function equilibrium (SFE) approximation works well in this example.

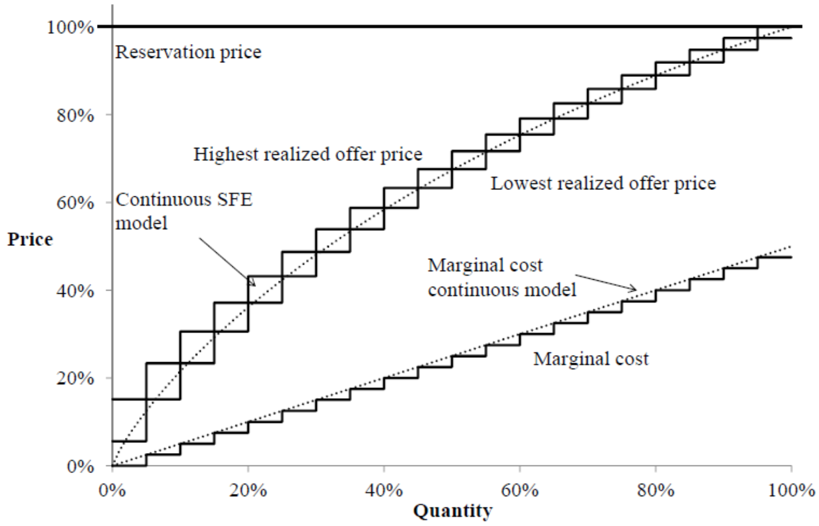

Figure 2: Offer prices for the unique, symmetric mixed-strategy NE (solid line) in a duopoly market with uniformly distributed demand where each producer has 20 indivisible units, each with the size 0.5. The equilibrium is compared with an SFE for divisible-goods (dotted line).

4 Extension 1: Alternative demand shock assumptions

4.1 Existence

In this extension, we will first show that existence of the step-separated Bayesian NE can be established for weaker conditions than Assumption 1.
Assumption $1'$**:** *Demand is such that:*

$$(k-1)\,\tau_k - (k-2)\,\tau_{k-1} \geq -1 \tag{16}$$

for all $k \in \{2, \ldots, 2N\}$.

This condition is for example satisfied for any demand distribution as long as indivisible units are sufficiently small. This is a property that we used in the proof of Proposition 4.

Proposition 5 *The Bayesian NE described in Proposition 1 exists, but is not necessarily unique, under Assumption* $1'$*.*

Later in this section, we will show that Assumption $1'$ is sufficient to ensure uniqueness in the special case where $N = 2$ and costs are common knowledge.

4.2 Highly non-uniform demand

So far, our equilibrium analysis has focused on Nash equilibria with step separation and no gaps. In this subsection, we will analyse alternative equilibria for the special case where $N = 2$ and costs are common knowledge, with $c_1(\alpha) = c_1$ and $c_2(\alpha) = c_2$. Without loss of generality, we take $h = 1$. We write $f_k = f(k)$ for $k = 1, 2, 3, 4$. The first step is to explore the conditions that are required for overlap to occur.

Lemma 5 *In a duopoly market where costs are common knowledge and each supplier has two units, ranges of equilibrium offers for units* 1 *and* 2 *can only overlap if*

$$(p_0 - c_1)(\tau_2 - 1 - 2\tau_3) > (c_2 - c_1)(\tau_2 + 1)^{1-\frac{1}{\tau_2}}, \qquad (17)$$

where $p_0 = p_1(0)$.

The condition (17) involves p_0 and this can only be determined when the solution with overlap is given. However we easily deduce from this condition that $2\tau_3 < \tau_2 - 1$ since the right hand side of the inequality is non-negative, and $p_0 > c_1$. Thus we see that the distribution of demand shocks needs to be rather non-uniform to get overlap. In particular, f_2 should be large relative to f_1 to satisfy this condition. This means that both producers can be fairly confident that they will sell at least one unit, which pushes up the equilibrium offer for the first unit in accordance with Proposition 2. Whether this is sufficient to get overlap depends on other parameters: costs, the reservation price and the probability mass for other demand levels.

One issue with overlap is that production costs will no longer be efficient. We have by assumption that $c_{n-1}(\alpha) < c_n(0)$, so if each producer has two units, the inefficiency occurs when the auctioneer has a total demand of two units and buys them from the same producer. We define p_H to be the highest price for the first unit and p_L to be the lowest price for the second unit, so $[p_L, p_H]$ is the range where overlap occurs. The expected welfare loss is given by:

$$W = 2f_2(c_2 - c_1)\int_{p_L}^{p_H} \theta_2(p)(1 - \Theta_1(p))\,dp, \qquad (18)$$

where, as before, we write $\Theta_n(p)$ for the probability distribution of offer prices for unit n and we write $\theta_n(p)$ for its derivative. The example below illustrates that even if demand is rather non-uniform and far from satisfying Assumption 1, the overlap may be negligible in the sense that welfare losses are small. The condition in (17) indicates that overlap and hence welfare losses can be avoided if p_0 (and $\bar{p}$) is decreased, so that mark-ups are lower.

Example 1 *We solve for a symmetric mixed-strategy NE with overlap. We will set* $c_1 = 1$ *and* $c_2 = 2$. *We consider values* $f_0 = 0$, $f_1 = 1/8$, $f_2 = f_3 = 3/8$, $f_4 = 1/8$. *Thus* $\tau_2 = 2$, $\tau_3 = 0$ *and* $\tau_4 = -2/3$. *It is easiest to solve the equilibrium backwards. We set* $p_0 = 3$, *solve for* $\Theta_1(p)$ *and* $\Theta_2(p)$ *and then set* $\overline{p}$ *equal to the price where* $\Theta_2(p) = 1$. *We note that condition (17) holds. We use results from the proof of Lemma 5 in the online Appendix when constructing the equilibrium, and equation references are to the online appendix. It follows from*

(83), (84) and $\rho_1 = f_2/f_1 = \tau_2 + 1$ *that the first-order condition in the range* p_0 *to* p_L *gives*

$$\Theta_1(p) = -\frac{1}{2} + \frac{1}{2}\left(\frac{p-1}{2}\right)^2,$$
$$\theta_1(p) = \frac{1}{2}\left(\frac{p-1}{2}\right).$$

Recall that $\tau_3 = (f_3 - f_2)/f_2 = 0$*. Hence, it follows from equation (82) in the online Appendix that the value for* p_L *is obtained from*

$$2 - \theta_1(p_L)(p_L - c_2) = 0$$

giving $p_L = 4.3723$ *and* $\Theta_1(p_L) = 0.92154$*. The solution in the overlap region can be determined numerically from the equations (77) and (78). We find that* $\Theta_1(p)$ *reaches* 1 *at* $p_H = 4.4689$ *when* $\Theta_2(p_H) = 0.001349$*. Now for the region* p_H *to* $\overline{p}$ *we revert to the differential equation which has solution*

$$\Theta_2(p) = \frac{1}{-\tau_4} + K(p - c_2)^{2\tau_4},$$

for constant K*. Hence*

$$0.001349 = \frac{3}{2} + K(2.4689)^{-4/3}$$

giving $K = -5.0007$*. Thus*

$$\Theta_2(p) = \frac{3}{2} - 5.0007(p-2)^{-4/3}$$

and

$$\theta_2(p) = 6.6676(p-2)^{-7/3}.$$

This will work with a value of $\overline{p} = 7.624$ *since at this point* $\Theta_2(\overline{p}) = 1$*. Using (18) and numerical results for the overlap region, we estimate the welfare loss to* 2.64×10^{-5}*, which is small in comparison to the total expected production cost* $1/8 + 3/8 \times 2 + 3/8 \times 4 + 1/8 \times 6 + 1.32 \times 10^{-5} = 3.125$*.*

As a last step, we verify that there are no profitable deviations for a supplier, if the competitor bids in accordance with the symmetric solution above. First, we realize that the supplier cannot gain by undercutting p_0*. We solve for a mixed-strategy NE in the range* $[p_0, p_H]$*, so the expected payoff is the same for any offer price in that range. We use properties of* $Z(1,p)$ *in (70) to verify that it is not profitable for the supplier to increase its offer for its first unit above* p_H*. The expected payoff from the second unit is the same for offers in the range* $[p_L, \overline{p}]$*. For lower prices we can use* $Z(2,p)$ *in Lemma 7 in the online Appendix to verify that it is not profitable to undercut* p_L *for unit 2.*

The next Proposition illustrates that if we are able to go through all possible types of equilibria in detail, then we can rule out overlap for a wider parameter range in comparison to Lemma 3, and in this case Assumption $1'$ ensures both existence and uniqueness of the equilibrium.

Proposition 6 *In a duopoly market where costs are common knowledge, each supplier has two units, and the demand shocks satisfies Assumption $1'$, the mixed-strategy equilibrium with step separation in Corollary 1 is a unique equilibrium.*

4.3 Narrow demand support

It is well-known from Klemperer and Meyer (1989), von der Fehr and Harbord (1993), Green and Newbery (1992), McAdams (2007), Holmberg (2008), Genc and Reynolds (2011) and Anderson (2013) that NE in uniform-price auctions are normally unique if the support of the demand shocks is sufficiently wide, so that any offer is price-setting with some probability. In Proposition 1 we proved this for our setting. On the other hand, if a producer knows with certainty that an offer will never be price-setting, then this gives the producer more flexibility for this offer. This means that the initial condition that is used to solve for SFE is undetermined, so that there is a range of NE. This also holds in our setting. If, contrary to our assumptions, each supplier would have a capacity to produce at least $2N$ units, so that it would be able to meet the maximum demand on its own, then unused production capacity will effectively work as a price cap if all of it is offered at a price p^* (or just above). Thus in this case, there would be a continuum of symmetric Bayesian NE of the type presented in Proposition 1, but with an initial condition $p_N(1) = p^* \in (c_N(1), \overline{p})$. Similar to the analysis of a divisible-good auction by Genc and Reynolds (2011), the size of the set of NE should decrease as the production capacity is reduced from $2N$ to N (where the equilibrium is typically unique).

Example 2 *Similar to Example 1, we will set $c_1 = 1$ and $c_2 = 2$. But we set $f_1 = f_2 = 1/2$, so that $\tau_2 = 0$, and $f_3 = f_4 = 0$. We also set $\overline{p} = 4$. Consider any $p^* \in (2, 4)$. It follows from (15) that there is a symmetric mixed-strategy NE with $p_1(\alpha) = (p^* - 1)e^{\alpha-1} + 1$ and $p_2(\alpha) \in \left[p^*, \min\left(4, \frac{p^*-\alpha}{1-\alpha}\right)\right]$. This equilibrium strategy gives an expected payoff of $\frac{p^*-1}{2}$. The offer for the second unit is always rejected in the symmetric equilibrium, which gives the producer some freedom. Still, $p_2(\alpha)$ should be in the range $[p^*, \overline{p}]$ and sufficiently close to p^* to ensure that it will not be possible for the competitor to increase its payoff by offering the first unit above p^*.*

Bayesian NE would still be symmetric when there is excess capacity in the market (see Lemma 14 in the online Appendix), if we disregard that the equilibrium could be asymmetric for offers that are never accepted. However, similar to SFE (Genc and Reynolds, 2011), we expect that there will be a continuum of asymmetric Bayesian NE when minimum demand is larger than h (the unit size).

In particular, if our setting were to allow for flat offers (i.e. a producer chooses the same offer price for each of its units) and if minimum demand is sufficiently high so that there is at least one player that cannot meet demand on its own, then the set of NE would include the asymmetric NE outlined by von der Fehr and Harbord (1993), where one producer offers at the price cap and the other makes a sufficiently low offer. Related NE have also been analysed by Fabra et al. (2006), Crawford et al. (2007), and Banal-Estañol and Micola (2009).

5 Extension 2: Multiple suppliers

In this section we generalize some results to multiple suppliers, $I \geq 2$. We use similar assumptions as in the duopoly case, but the demand shock β can now take values on the set $\mathcal{Q}(h) = \{h, 2h, 3h..., INh\}$. In a similar way we take the inequality of Assumption 1' to apply for $k \in \{2, \ldots, IN\}$. We know from the duopoly case that small cost uncertainties and the distribution of demand shocks have a negligible impact on step separated offers and the standard deviation of such offers, if units are sufficiently small. In this section, we will therefore focus on cases when costs are common knowledge and when estimating the standard deviation of offers we will assume that demand is uniformly distributed. This will significantly simplify our calculations. We use $K = I - 1$ to denote the number of competitors of a supplier.

Proposition 7 *In a multi-unit auction with $I = K + 1 \geq 2$ symmetric suppliers and costs that are common knowledge, the set of solutions $\{p_n(\alpha)\}_{n=1}^{N}$ as defined by the end-conditions*

$$\begin{aligned} p_N(1) &= \overline{p}, \\ p_n(1) &= p_{n+1}(0), \ \forall n \in \{1, \ldots, N-1\}, \end{aligned}$$

and

$$p_n(\alpha) = c_n + (p_n(1) - c_n) e^{-\int_\alpha^1 g(u)du/n}, \tag{19}$$

where α is a random variable that is uniformly distributed in the interval $[0,1]$ and

$$g(u) = \frac{\sum_{v=0}^{K-1} \frac{K!}{(K-1-v)!v!} u^v (1-u)^{K-1-v} f((n+v+K(n-1))h)}{\sum_{v=0}^{K} \frac{K!}{v!(K-v)!} u^v (1-u)^{K-v} f((n+v+K(n-1))h)}, \tag{20}$$

constitutes a symmetric mixed-strategy NE if Assumption 1′ is satisfied. The offer range of the mixed-strategy NE can be approximated from:

$$p_n(1) - p_{n-1}(1) = \frac{(p_n(1) - c_n) K}{n} + O(h^2). \tag{21}$$

We get the following results for the special case where demand is uniformly distributed.

Proposition 8 *In a multi-unit auction with uniformly distributed demand,* $I = K + 1 \geq 2$ *symmetric suppliers and costs that are common knowledge among suppliers, the set of solutions* $\{p_n(\alpha)\}_{n=1}^{N}$ *as defined by the end-conditions*

$$\begin{aligned} p_N(1) &= \bar{p}, \\ p_n(1) &= p_{n+1}(0), \ \forall n \in \{1, \ldots, N-1\}, \end{aligned}$$

and

$$p_n(\alpha) = c_n + (p_n(1) - c_n) e^{K(\alpha-1)/n}, \tag{22}$$

where α *is a random variable that is uniformly distributed in the interval* $[0,1]$*, constitutes a symmetric mixed-strategy NE. The equilibrium is unique. The variance of the offer* $p_n(\alpha)$ *is given by*

$$\frac{(p_n(1) - c_n)^2 K^2}{12n^2} + O\left(h^3\right). \tag{23}$$

Conditional on the demand level $(n-1)(K+1)+m$*, where* $m \in [1, K+1]$*, the variance of the market price can be approximated by:*

$$\frac{(p_n(1) - c_n)^2}{n^2} \frac{K^2 (K - m + 2) m}{(K+2)^2 (K+3)} + O\left(h^3\right). \tag{24}$$

Order statistics are used to calculate the variance of the market price for a given demand level $(n-1)(K+1)+m$, so this variance depends on m, the number of suppliers that sell n units. The remaining $K+1-m$ suppliers sell $n-1$ units. If we keep maximum demand and the market capacity $(K+1)\bar{q}$ fixed, N, the number of units per supplier is inversely proportional to $(K+1)$. We have $p_N(1) = \bar{p}$, so it is clear from (23) that the volatility in the offer price of a supplier's most expensive unit increases with more producers in the market. It follows from (24) that this is also true for the volatility in the market price. The explanation is that the offer range of the most expensive unit increases when there are more competitors in the market. This increases the volatility, but it also reduces the average mark-up of the most expensive unit and mark-ups for cheaper units.

Next, we will analyse equilibrium convergence for multiple suppliers and how the bid volatility changes with output. In the latter case, it will be useful to know under what conditions offers in a divisible-good auction are convex. By differentiating $P(Q)$ in Holmberg (2008), it can be shown that $P''(Q) > 0$ for $Q \in [0, Nh]$, if

$$\begin{aligned} \tilde{C}''(Q) &< (I-2) I^{I-1} Q^{I-2} \left(\frac{\bar{p}}{(I\bar{q})^{I-1}} - \frac{\tilde{C}'(Q)}{(IQ)^2} \right) + \\ & \quad (I-1)(I-2) I^{I-1} Q^{I-2} \int_{IQ}^{Nh} \frac{\tilde{C}'(x/I)\, dx}{x^I} \end{aligned} \tag{25}$$

for $Q \in [0, \bar{q}]$. This will be the case when the reservation price $\bar{p}$ is sufficiently high or the marginal cost of a supplier, $\tilde{C}'(Q)$, is sufficiently flat. Electricity markets often have very high reservation prices and convex inverse supply functions are consistent with hockey-stick bidding that has been observed in practice, i.e. observed offer prices and estimated mark-ups become drastically larger near the total production capacity of the market (Hurlbut et al., 2004; Holmberg and Newbery, 2010).

Proposition 9 *Let $P(Q)$ be the unique pure-strategy continuous supply function equilibrium for divisible units, then, if demand is uniformly distributed, there exists a unique mixed-strategy NE for indivisible units with properties as in Proposition 8, which converges to the continuous supply function equilibrium in the sense that $p_n(\alpha_i) \to P(nh)$ when $h \to 0$ and $c_n \to \tilde{C}'(nh)$. If, in addition, the divisible-good auction satisfies the condition in (25), then the volatility and offer price range of a supplier's offer increases with respect to n in the auction with indivisible units if h is sufficiently small.*

5.1 Price instability in wholesale electricity markets

In this subsection, we use our theoretical results to make a straightforward calculation to roughly estimate how large price instability would be for wholesale electricity markets in practice. We start by estimating price instability for the most expensive production unit and then we argue that the price instability is smaller for cheaper units.

In electricity markets, the production cost of a plant is primarily determined by fuel costs and its efficiency, which are fairly well-known and often assumed to be common knowledge (Green and Newbery, 1992; von der Fehr and Harbord, 1993; Rudkevich et al., 1998; Anderson and Hu, 2008; Holmberg and Newbery, 2010). Wolak (2007) uses a detailed empirical model to study bidding in the Australian electricity market, and he finds that a model where costs are common knowledge and producers choose offers to maximize their profits cannot be rejected. Here we assume that the unit size is small and that the cost uncertainty is sufficiently small, so that (1) is satisfied. Under these circumstances, the standard deviation of a supplier's most expansive unit can be estimated from (23).[12]

$$\frac{(p_N(1) - c_N)K}{\sqrt{12}N} \approx \frac{\overline{p}K}{\sqrt{12}N}, \tag{26}$$

because $p_N(1) = \overline{p}$. Moreover, wholesale electricity markets around the world would typically have reservation prices in the range \$1,000-\$20,000/MWh (Holmberg et al., 2013; Stoft, 2002), which often is much higher than the marginal cost of the most expensive unit. Market concentration in wholesale electricity markets as measured by the Herfindahl-Hirschman Index (HHI) is typically in the range 1000-2000, both in Europe (Newbery, 2009) and U.S. (Bushnell et al., 2008). This degree of market concentration can be represented by 5-10 symmetric suppliers,

[12] In a duopoly market this approximation holds for private costs and general demand distributions. We believe that this is the case also for multiple firms. However, for multiple firms, we only establish this approximation for cases with uniformly distributed demand and costs that are common knowledge.

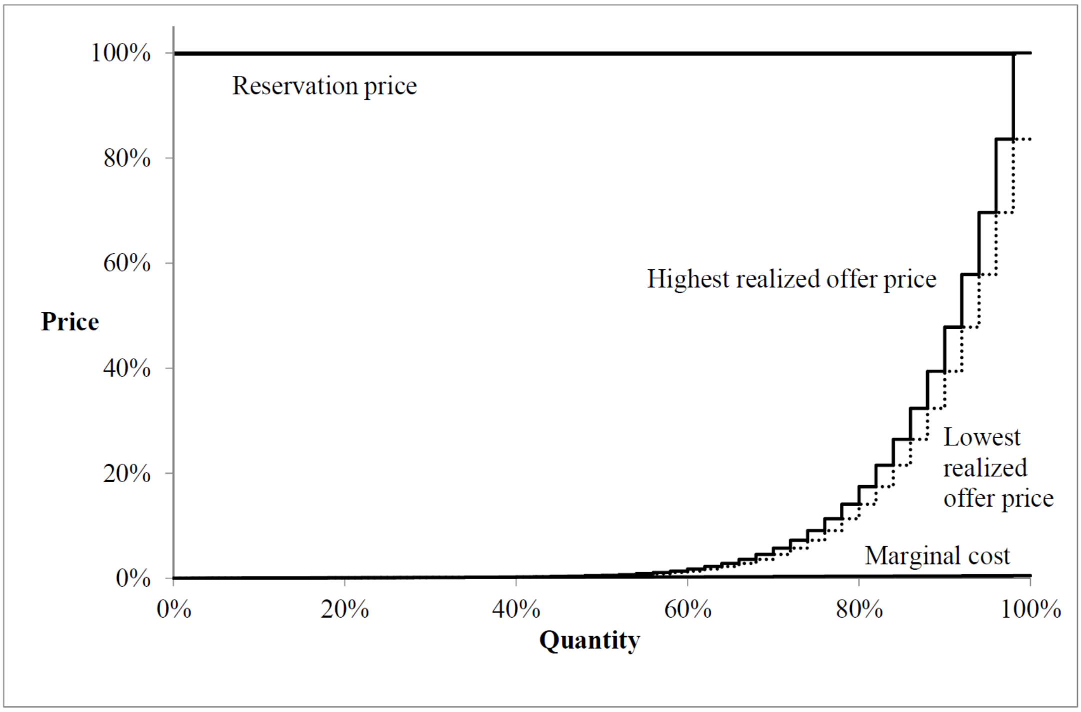

Figure 3: Offer prices for the unique, symmetric mixed-strategy NE in an oligopoly market with uniformly distributed demand and 500 units divided between 10 suppliers.

which gives an estimate of K. From the discussion in Holmberg et al. (2013) and Green and Newbery (1992), it is reasonable to assume that each representative supplier chooses 50-500 offer prices each, which gives an estimate of N. It follows from (26) that in a market with 5 suppliers and 500 offer prices per supplier, the volatility of a supplier's most expensive unit is small, as measured by the standard deviation, around 0.2% of the reservation price. According to Proposition 8, the volatility of the market price is 0.1%-0.2% of the reservation price for demand outcomes where all suppliers sell at least $N-1=499$ units. On the other hand, if the market instead has 10 suppliers and 50 offer prices per supplier, then the price volatility of a producer's most expensive offer would be fairly large, 5% of the reservation price. In this case, the volatility of the market price is 1%-3% of the reservation price for demand outcomes where all suppliers sell at least $N-1=49$ units. However, it is rare that offers from the most expensive units are accepted in wholesale electricity markets and it follows from Proposition 9 that offer price ranges and volatility of offers tends to decrease for cheaper units, at least if the reservation price is high and units are small. This is consistent with Figure 3, where we plot the mixed-strategy NE for 10 suppliers and 50 indivisible units per supplier, and with Figure 4, which plots the volatility (standard deviation as a percentage of the reservation price) as a function of the demand level for the same example.

Our simple and approximate calculation does not consider that in practice suppliers would normally sell a large part of their output in advance with forward

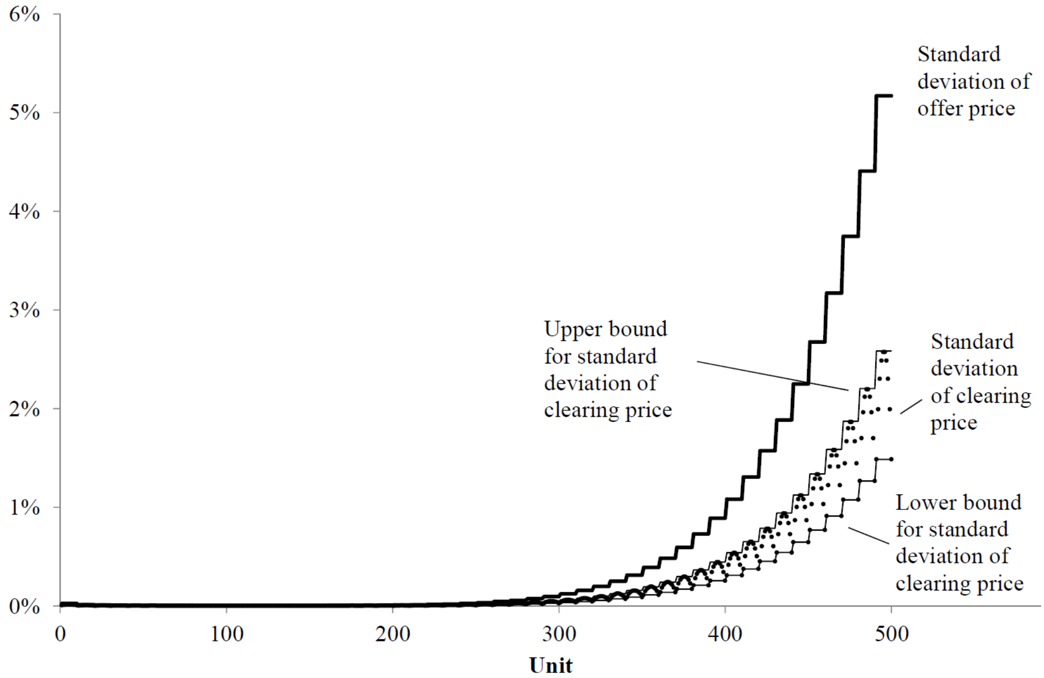

Figure 4: Standard deviation of the clearing price and standard deviation of a supplier's offer price as a percentage of the reservation price for different demand levels in an oligopoly market with uniformly distributed demand and 500 units divided between 10 suppliers.

contracts. As shown by Newbery (1998) and Holmberg (2011) this lowers mark-ups in electricity markets and (23) indicates that this should also mitigate price instability. Another factor that our analysis neglects is that producers' cost information (signals) could be positively correlated in practice. Vives (2011) shows that correlation of signals could have a large impact on equilibrium offers. There is less impact from correlated signals when the demand uncertainty dominates the cost uncertainty. Still, we believe that correlation of costs could influence the volatility of offers.

6 Conclusions

We consider a procurement multi-unit auction where each supplier makes an offer for each of its indivisible units, as in the Colombian electricity market. Related set-ups are often used in experimental studies of multi-unit auctions. Our design could also represent restrictions in the bidding format and restrictions in the production technology, such as production constraints or non-convex costs. Suppliers are symmetric ex-ante, before they receive private independent cost information. A supplier submits a higher offer for a unit when its costs are higher. This gives an offer price range for each indivisible unit of the supplier. We show that if the cost uncertainty is sufficiently small and units are sufficiently small, then there exists a Bayesian Nash Equilibrium in which the offer price ranges for the different units of a supplier do not overlap. We call this property *step separation*. In this case we can explicitly solve for equilibria in the multi-unit auction. We prove that this is the unique equilibrium if the auctioneer's demand is uncertain and sufficiently evenly distributed. In a duopoly market, offer prices increase if outcomes where the auctioneer buys an even number of units become more likely and outcomes where the auctioneer buys an odd number of units become less likely. In the special case with two units per supplier, we also analyze NE where offer ranges of units overlap, as can occur with highly non-uniformly distributed demand shocks. Overlapping offer prices for non-overlapping marginal costs lead to welfare losses, but they are very small in the examples that we have considered.

Indivisibility of units introduce a bid volatility: a small change in the realized cost of a unit has a much larger impact on the offer price of the unit. We show that the resulting standard deviation of a supplier's equilibrium offer is approximately given by $(I-1)(p_n - c_n)/(\sqrt{12}n)$, where I is the number of symmetric suppliers and $p_n - c_n$ is the approximate mark-up for the n'th unit (the n'th cheapest unit of a supplier). If the size of units are small, the influence on this volatility from small cost uncertainties and the probability distribution of demand shocks is negligible. For parameter values that are typical in a wholesale electricity market, this approximation would imply that the standard deviation in the offer for the most expensive production unit of a supplier could be substantial, in the range 0.2%-5% of the reservation price. The standard deviation of the market price (price instability) has a somewhat lower magnitude. Price instability is less pronounced for less expensive production units if the reservation price is sufficiently high or marginal costs are sufficiently flat. Our model predicts that the price instability decreases when the size of indivisible units decreases, or equivalently when suppliers are allowed to offer supply functions with more steps. In practice, we also believe that financial contracts would mitigate this volatility.

In the limit where costs are common knowledge, the Bayesian NE with unstable prices becomes a mixed-strategy NE, in accordance with Harsanyi's purification theorem. We prove that this mixed-strategy NE converges to a pure-strategy, supply-function equilibrium (SFE) without price instability when the unit size decreases towards zero. We also prove the reverse result that for every symmetric SFE for divisible units there exists a corresponding mixed-strategy NE for indivisible units, which converges to the SFE as the size of indivisible units decreases. These results give theoretical support to the use of smooth SFE to approximate stepped supply function offers in wholesale electricity markets, as in Green and Newbery (1992).

The paper also contributes by introducing a discrete version of Anderson and Philpott's (2002b) market distribution function and Wilson's (1979) probability distribution of the market price. We use this tool to characterize the uncertainty in the residual demand of a supplier. We also derive conditions for the globally best response of a supplier facing a given discrete market distribution function. These conditions can be used in both empirical and theoretical studies of auctions with multiple indivisible units.

Similar to SFE (Anderson, 2013; Holmberg, 2008) and related results for sales auctions (McAdams, 2007), our uniqueness result relies on the auctioneer's demand varying in a sufficiently wide range. A continuum of alternative equilibria would occur if maximum demand is lower than the total market capacity. If demand is certain, it has been popular in the literature that analyses multi-unit auctions with stepped bid functions to select the equilibrium that does not have price instability. When bidders' costs/values are common knowledge this corresponds to selecting a pure-strategy NE, as in Banal-Estañol and Micola (2009), Crawford et al. (2007), von der Fehr and Harbord (1993), Kremer and Nyborg (2004a; 2004b) and Fabra et al. (2006). These NE are rather extreme, since the market price is either at the marginal cost or reservation price.[13] Still, such bidding behaviour has been observed in the capacity market of New York State's electricity market, which is dominated by one supplier and where the demand variation is small (Schwenen, 2015). On the other hand, bidding in electricity spot markets (Sioshansi and Oren, 2007; Hortaçsu and Puller, 2008; Wolak, 2007) and experimental results by Brandts et al. (2014) are inconsistent with this type of extreme pure-strategy NE, and closer to the equilibria that we solve for in this paper.

7 Acknowledgements

We are grateful to referees, and to Mario Blázquez de Paz, Nils-Henrik von der Fehr, Chloé Le Coq, Erik Lundin, Andras Niedermayer and Frank Wolak for very helpful comments. We also want to acknowledge helpful comments from seminar participants at the Research Institute of Industrial Economics (IFN) and participants at the fifth Mannheim Energy conference 2016. Pär Holmberg has been financially supported by Jan Wallander and Tom Hedelius' Research Foundations, the Torsten Söderberg Foundation (E37/13), the Swedish Energy Agency (40653-1) and the Research Program the Economics of Electricity Markets.

References

Anderson, E.J., 2013. On the existence of supply function equilibria. *Mathematical Programming* **140**(2), 323-349.

Anderson, E.J., Philpott, A. B., 2002a. Using supply functions for offering generation into an electricity market. *Operations Research* **50**(3), 477-489.

Anderson, E. J., Philpott, A.B., 2002b. Optimal offer construction in electricity markets. *Mathematics of Operations Research* **27**(1), 82–100.

Anderson, E. J., Hu, X., 2008. Finding supply function equilibria with asymmetric Firms. *Operations Research* **56**(3), 697-711.

Anderson, E. J., Xu, H., 2004. Nash equilibria in electricity markets with discrete prices. *Mathematical Methods of Operations Research* **60**(2), 215–238.

[13] Fabra et al. (2006) show that other related pure-strategy NE could exist in sufficiently competitive markets where mark-ups are not larger than the difference in marginal cost for successive units in the merit order.

Ausubel, L.M., Cramton, P., Pycia, M., Rostek, M., and Weretka, M., 2014. Demand reduction and inefficiency in multi-unit auctions. *Review of Economic Studies* **81**(4), 1366-1400.

Baldick, R., Helman, U., Hobbs, B. F., and O'Neill, R. P., 2005. Design of efficient generation markets. *Proceedings of the IEEE*, **93**(11), 1998-2012.

Banal-Estañol, A., and Micola, A. R., 2009. Composition of electricity generation portfolios, pivotal dynamics, and market prices. *Management Science* **55**(11), 1813-1831.

Brandts, J., Reynolds, S. S., and Schram, A., 2014. Pivotal suppliers and market power in experimental supply function competition. *The Economic Journal* **124**(579), 887-916.

Bushnell, J., Mansur, E., and Saravia, C., 2008.Vertical arrangements, market structure and competition: An analysis of restructured U.S. electricity markets. *American Economic Review* **98**(1), 237-266.

Crawford, G. S., Crespo, J., and Tauchen, H., 2007. Bidding asymmetries in multi-unit auctions: implications of bid function equilibria in the British spot market for electricity. *International Journal of Industrial Organization* **25**(6), 1233-1268.

Deneckere, R. J., and Kovenock, D. 1996. Bertrand-Edgeworth duopoly with unit cost asymmetry. *Economic Theory* **8**(1), 1-25.

Edgeworth, F., *Papers Relating to Political Economy.* London: Macmillan, 1925.

Fabra, N., von der Fehr, N-H. M., Harbord, D., 2006. Designing electricity auctions. *RAND Journal of Economics* **37**(1), 23-46.

von der Fehr, N-H. M., Harbord, D., 1993. Spot market competition in the UK electricity industry. *Economic Journal* **103**(418), 531-46.

Genc, T., Reynolds, S., 2011. Supply Function Equilibria with Pivotal Electricity Suppliers. *International Journal of Industrial Organization* **29**(4), 432–442.

Green, R.J., Newbery, D.M., 1992. Competition in the British electricity spot market. *Journal of Political Economy* **100**(5), 929-53.

Harsanyi, J., 1973. Games with randomly distributed payoffs: A new rationale for mixed strategy equilibrium points. *International Journal of Game Theory* **2**(1), 1-23.

Holmberg, P., 2008. Unique supply function equilibrium with capacity constraints. *Energy Economics* **30**(1), 148–172.

Holmberg, P. 2011. Strategic forward contracting in the wholesale electricity market. *Energy Journal* **32**(1), 169-202.

Holmberg, P. and Newbery, D.M., 2010. The supply function equilibrium and its policy implications for wholesale electricity auctions. *Utilities Policy* **18**(4), 209–226.

Holmberg, P., Newbery, D., Ralph, D., 2013. Supply function equilibria: Step functions and continuous representations, *Journal of Economic Theory* **148**(4), 1509-1551.

Hortaçsu, A. and McAdams, D., 2010. Mechanism choice and strategic bidding in divisible good auctions: An empirical analysis of the Turkish treasury auction market. *Journal of Political Economy* **118**(5), 833-865.

Hortaçsu, A., Puller, S., 2008. Understanding strategic bidding in multi-unit auctions: A case study of the Texas electricity spot market. *Rand Journal of Economics* **39**(1), 86-114.

Hortaçsu, A., Luco, F., Puller, S. L., and Zhu, D., 2017. Does Strategic Ability Affect Efficiency? Evidence from Electricity Markets. Working Paper.

Kastl, J., 2011. Discrete bids and empirical inference in divisible good auctions. *Review of Economics Studies* **78**(3), 978-1014.

Kastl, J., 2012. On the properties of equilibria in private value divisible good auctions with constrained bidding. *Journal of Mathematical Economics* **48**(6), 339-352.

Klemperer, P. D., Meyer, M.A., 1989. Supply function equilibria in oligopoly under uncertainty. *Econometrica* **57**(6), 1243-1277.

Kremer, I., Nyborg, K.G., 2004a. Divisible good auctions: The role of allocation rules. *RAND Journal of Economics* **35**, 147–159.

Kremer, I., Nyborg, K.G., 2004b. Underpricing and market power in uniform price auctions. *The Review of Financial Studies* **17**(3), 849-877.

Kruse, J. B., Rassenti, S., Reynolds, S. S., and Smith, V. L. 1994. Bertrand-Edgeworth competition in experimental markets. *Econometrica* **62**, 343-371.

Le Coq, C., Orzen, H., and Schwenen, S., 2016. Pricing and Capacity Provision in Electricity Markets: an Experimental Study. *Journal of Regulatory Economics* **51**(2), 123-158.

LeVeque, R., 2007. *Finite Difference Methods for Ordinary and Partial Differential Equations*, Philadelphia: Society for Industrial and Applied Mathematics.

Maskin, E., and Tirole, J. 1988. A theory of dynamic oligopoly, II: Price competition, kinked demand curves, and Edgeworth cycles. *Econometrica*, 571-599.

McAdams, D., 2003. Isotone equilibrium in games of incomplete information. *Econometrica* **71**(4), 1191-1214.

McAdams, D., 2007. Adjustable supply in uniform price auctions: Non-commitment as a strategic tool. *Economics Letters*, **95**(1), 48-53.

Meeus, L., Verhaegen, K., Belmans, R., 2009. Block order restrictions in combinatorial electric energy auctions. *European Journal of Operational Research*, **196**(3), pp. 1202-1206.

Newbery, D. M. 1998. Competition, contracts, and entry in the electricity spot market, *RAND Journal of Economics* **29**(4), 726-749.

Newbery, D.M., 2009. Predicting market power in wholesale electricity markets. EUI Working Paper RSCAS 2009/03.

Noel, M. D. 2007. Edgeworth price cycles: Evidence from the Toronto retail gasoline market. *The Journal of Industrial Economics* **55**(1), 69-92.

Reny, P., 1999. On the existence of pure and mixed strategy Nash equilibria in discontinuous games. *Econometrica* **67**(5), 1029–1056.

Rostek, M., Weretka, M., and Pycia, M., 2010. Design of divisible good markets. Manuscript, University of Wisconsin–Madison.

Rostek, M. and Weretka, M., 2012. Price inference in small markets. *Econometrica* **80**(2), 687-711.

Rudkevich, A., Duckworth, M., Rosen, R., 1998. Modelling electricity pricing in a deregulated generation industry: The potential for oligopoly pricing in poolco. *The Energy Journal* **19**(3), 19-48.

Schwenen, S., 2015. Strategic bidding in multi-unit auctions with capacity constrained bidders: The New York capacity market. *The RAND Journal of Economics* **46**(4), 730-750.

Sioshansi, R., Oren, S., 2007. How good are supply function equilibrium models: An empirical analysis of the ERCOT balancing market. *Journal of Regulatory Economics* **31**(1), 1-35.

Stoft, S., 2002. *Power System Economics: Designing Markets for Electricity.* IEEE Press.

Wang, J. and Zender, J., 2002. Auctioning divisible goods. *Economic Theory* **19**(4), 673-705.

Varian, H. R., 1980. A model of sales. *The American Economic Review* **70**(4), 651-659.

Vives, X., 2011. Strategic supply function competition with private information. *Econometrica* **79**(6), 1919–1966.

Wilson, R., 1979. Auctions of shares. *Quarterly Journal of Economics* **93**(4), 675-689.

Wolak, F.A., 2007. Quantifying the supply-side benefits from forward contracting in wholesale electricity markets. *Journal of Applied Econometrics* **22**(7), 1179-1209.

Wolak, F.A., 2009. Report on Market Performance and Market Monitoring in the Colombian Electricity Supply Industry. Report prepared for Superintendencia de Servicios Públicos (SSPD).

Appendix. Supplementary material

Supplementary material with proofs to this article can be found at **...[Link to webpage]**

Price instability in multi-unit auctions ONLINE APPENDIX

Edward Anderson*and Pär Holmberg†

January 26, 2018

Abstract

We consider a uniform-price procurement auction with indivisible units and private independent costs. We find an explicit solution for a Bayesian Nash equilibrium, which is unique if demand shocks are sufficiently evenly distributed. The equilibrium has a price instability in the sense that a minor change in a supplier's realized cost can result in a drastic change in the market price. We quantify the resulting volatility and show that it is reduced as the size of indivisible units decreases. In the limit, the equilibrium converges to the Supply Function Equilibrium (SFE) for divisible goods if costs are common knowledge.

Key words: Multi-unit auctions, indivisible unit, price instability, supply function equilibria, convergence of Nash equilibria, wholesale electricity markets

JEL Classification C62, C72, D43, D44, L94

*University of Sydney Business School, University of Sydney, NSW 2006, Australia. E-mail: edward.anderson@sydney.edu.au.

†Corresponding author: Research Institute of Industrial Economics (IFN), Stockholm. Associate Researcher of the Energy Policy Research Group (EPRG), University of Cambridge.

Appendix A: Best response

In the proofs, we find it convenient to identify the values of α that give rise to offer prices for given units n. We define

$$\widehat{\alpha}_i(p,n) = \sup(\alpha_i : s_i(p,\alpha_i) \geq nh),\ \ n = 1, 2..., N,$$

with $\widehat{\alpha}_i(p,n) = 0$ if $s_i(p,0) < nh$. Thus $\widehat{\alpha}_i$ is the probability that supplier i will sell at least n units at the clearing price p. It follows from our assumptions for $p_n^i(\alpha_i)$ that $\widehat{\alpha}_i$ is increasing, continuous and piecewise smooth with respect to the price and decreasing with respect to n. To simplify our equations, we set $\widehat{\alpha}_i(p,0) = 1$ and $\widehat{\alpha}_i(p,N+1) = 0$. We let

$$\Delta\widehat{\alpha}_i(p,n) = \widehat{\alpha}_i(p,n) - \widehat{\alpha}_i(p,n+1) \geq 0.$$

Thus $\Delta\widehat{\alpha}_i(p,n)$ is the probability that an α_i value is chosen by nature such that an agent will sell exactly nh units at price p.

Lemma 6 *The expected profit of supplier i for a set of offer prices $\{r_n\}_{n=1}^N$ and a signal α_i is given by:*

$$\begin{aligned}\Pi_i\left(r_1, r_2, ...r_N, \alpha_i\right) = &\sum_{n=1}^{N}\left(\Psi_i\left(n+1, r_n\right) - \Psi_i\left(n, r_n\right)\right)\left(nhr_n - C_n(\alpha_i)\right) \\ &+ \sum_{n=1}^{N}\int_{r_n}^{r_{n+1}} \frac{\partial \Psi_i(n+1,p)}{\partial p}\left(nhp - C_n(\alpha_i)\right) dp,\end{aligned} \tag{27}$$

where we choose $\frac{\partial \Psi_i(n+1,p)}{\partial p}$ to equal the right-hand derivative of Ψ_i at any isolated point where left and right derivatives do not match.

Proof. We first calculate the probability that supplier i sells exactly n units. This can occur under two different circumstances. In the first case, r_n is price-setting. This occurs when the nth unit of supplier i is accepted and the competitor's last accepted unit has an offer price below r_n. The probability for this event is $\Psi_i\left(n+1, r_n\right) - \Psi_i\left(n, r_n\right)$.

Next we consider the case where supplier i sells exactly n units at a price $p \in (r_n, r_{n+1})$, which is set by the residual demand of supplier i. This occurs when the nth unit of supplier i is accepted and the competitor's last accepted unit has an offer price in the interval (r_n, r_{n+1}). The probability that the competitor has its last accepted offer price in an interval $[p, p+\Delta p]$ is given by $\Psi_i\left(n+1, p+\Delta p\right) - \Psi_i\left(n+1, p\right)$. When the derivative exists, this approaches $\frac{\partial \Psi_i(n+1,p)}{\partial p}\Delta p$ as $\Delta p \to 0$. Thus we can write the total expected profit of the supplier as in (27). By assumption $\Psi_i\left(n+1, p\right)$ has only isolated points where it is non-smooth, and consequently the choice of derivative value at these points will not affect the integral. ■

Proof. (Lemma 1) We have from (27) that

$$\begin{aligned}\frac{\partial \Pi_i\left(r_1,r_2,...r_N,\alpha_i\right)}{\partial r_n} &= \left(\frac{\partial \Psi_i\left(n+1,r_n\right)}{\partial r_n}-\frac{\partial \Psi_i\left(n,r_n\right)}{\partial r_n}\right)\left(nhr_n-C_n(\alpha_i)\right)\\ &+nh\left(\Psi_i\left(n+1,r_n\right)-\Psi_i\left(n,r_n\right)\right)\\ &+\frac{\partial \Psi_i\left(n,r_n\right)}{\partial r_n}\left(\left(n-1\right)hr_n-C_{n-1}(\alpha_i)\right)\\ &-\frac{\partial \Psi_i\left(n+1,r_n\right)}{\partial r_n}\left(nhr_n-C_n(\alpha_i)\right),\end{aligned}$$

which can be simplified to (4). Recall that $C_n(\alpha_i)=h\sum_{m=1}^{n}c_m(\alpha_i)$. ■

Proof. (Lemma 2) Suppose (6) is satisfied and that there is another set of monotonic offers $\{r_n\}_{n=1}^N$, such that $\Pi_i\left(r_1^*,r_2^*,...r_N^*,\alpha_i\right)<\Pi_i\left(r_1,r_2,...r_N,\alpha_i\right)$. However, this leads to a contradiction. $Z_i\left(n,r_n,\alpha_i\right)$ is the right-hand derivative of Π_i with respect to r_n whenever offers of supplier i are monotonic, so it follows from (6) that $\frac{\partial\Pi_i(r_1,r_2,...r_N,\alpha_i)}{\partial r_n}\leq 0$ for almost all $r_n\geq r_n^*$ and that $\frac{\partial\Pi_i(r_1,r_2,...r_N,\alpha_i)}{\partial r_n}\geq 0$ for almost all $r_n<r_n^*$. Thus the expected profit must (weakly) decrease as the set of offer prices is changed from $\{r_n^*\}_{n=1}^N$ to $\{r_n\}_{n=1}^N$. We realize that this change can be done in steps without violating the constraint that offer prices must be monotonic with respect to n. This gives the sufficient condition for a best response. The necessary conditions follow straightforwardly from $Z_i\left(n,r_n,\alpha_i\right)$ being the derivative of Π_i with respect to r_n. ■

Below we show how Ψ_i and Z_i can be expressed in terms of $\widehat{\alpha}_j$ values for the duopoly market that we are studying.

Lemma 7 *In a duopoly market*

$$\Psi_i\left(n,p\right)=\sum_{m=0}^{N}\Delta\widehat{\alpha}_j(p,m)F\left(\left(n+m-1\right)h\right) \tag{28}$$

and

$$\begin{aligned}Z_i\left(n,r_n,\alpha_i\right) &= nh\sum_{m=0}^{N}\Delta\widehat{\alpha}_j(r_n,m)f(\left(n+m\right)h)\\ &\quad -h\left(r_n-c_n(\alpha_i)\right)\sum_{m=0}^{N-1}\frac{\partial\widehat{\alpha}_j\left(r_n,m+1\right)}{\partial r_n}f(\left(n+m\right)h)\end{aligned} \tag{29}$$

where we interpret $\frac{\partial\widehat{\alpha}_j}{\partial r_n}$ *as a right derivative where left and right derivatives differ.*

Proof. An offer of n units at price p by supplier i is rejected if the competitor $j\neq i$ offers exactly m units at the price p, which occurs with probability $\Delta\widehat{\alpha}_j(p,m)$, when demand at this price is less than $n+m$ units, which has the probability $F\left(\left(n+m-1\right)h\right)$. We get $\Psi_i\left(n,p\right)$ in (28) by summing across all $m\in\{0,\ldots,N\}$. We have $f(\left(n+m\right)h)=F(\left(n+m\right)h)-F(\left(n+m-1\right)h)$, so it follows from

Definition 1 and (28) that

$$
\begin{aligned}
Z_i(n, r_n, \alpha_i) &= nh \sum_{m=0}^{N} \Delta\widehat{\alpha}_j(r_n, m) f((n+m)h) \\
&\quad - h(r_n - c_n(\alpha_i)) \sum_{m=0}^{N} \left(\frac{\partial \widehat{\alpha}_j(r_n, m)}{\partial r_n} - \frac{\partial \widehat{\alpha}_j(r_n, m+1)}{\partial r_n} \right) F((n+m-1)h) \\
&= nh \sum_{m=0}^{N} \Delta\widehat{\alpha}_j(r_n, m) f((n+m)h) \\
&\quad - h(r_n - c_n(\alpha_i)) \sum_{m=-1}^{N-1} \frac{\partial \widehat{\alpha}_j(r_n, m+1)}{\partial r_n} F((n+m)h) \\
&\quad + h(r_n - c_n(\alpha_i)) \sum_{m=0}^{N} \frac{\partial \widehat{\alpha}_j(r_n, m+1)}{\partial r_n} F((n+m-1)h),
\end{aligned}
$$

which can be simplified to (29), because $\frac{\partial \widehat{\alpha}_j(p,0)}{\partial r_n} = \frac{\partial \widehat{\alpha}_j(p,N+1)}{\partial r_n} = 0$. ■

Appendix B: Step separation without gaps

Lemma 8 *In a Bayesian Nash equilibrium, $p_{n-1}^i(1) \geq p_n^i(0)$ for each $n \in \{2, \ldots, N\}$, i.e. there are no gaps between the offer price ranges of successive units of a supplier $i \in \{1, 2\}$.*

Proof. Assume to the contrary that $p_{n-1}^i(1) < p_n^i(0)$ for some $n \in \{2, \ldots, N\}$. Thus for any price in the range $\left(p_{n-1}^i(1), p_n^i(0)\right)$ agent i sells exactly $(n-1)h$ units. In other words $\Delta\widehat{\alpha}_i(p, n-1) = 1$ and $\Delta\widehat{\alpha}_i(p, m) = 0$ for $p \in \left(p_{n-1}^i(1), p_n^i(0)\right)$ and $m \neq n-1$. In addition, $\frac{\partial \widehat{\alpha}_i(p,m)}{\partial p} = 0$ for $p \in \left(p_{n-1}^i(1), p_n^i(0)\right)$ and $m \in \{1, \ldots, N\}$.

Now, first suppose that supplier j makes some offer in this price range. Hence there is some $\widetilde{n} \in \{1, \ldots, N\}$, $\widetilde{\alpha}$ with $\widetilde{p} = p_{\widetilde{n}}^j(\widetilde{\alpha}) \in \left(p_{n-1}^i(1), p_n^i(0)\right)$. Then we have from Lemma 7 and Definition 1,

$$
\begin{aligned}
Z_j(\widetilde{n}, \widetilde{p}, \widetilde{\alpha}) &= \frac{\partial \Pi_j(r_1, r_2, \ldots r_N, \widetilde{\alpha})}{\partial r_{\widetilde{n}}} = \widetilde{n} h \sum_{m=0}^{N} \Delta\widehat{\alpha}_i(\widetilde{p}, m) f((\widetilde{n}+m)h) \\
&= \widetilde{n} h f((\widetilde{n}+n-1)h) > 0,
\end{aligned}
$$

because $f((\widetilde{n}+n-1)h) > 0$ (we assume that every value of demand from 0 up to $2N$ is possible). Hence supplier j would gain from increasing its offer price for unit $\widetilde{n}$ when observing signal $\widetilde{\alpha}$. This cannot occur in equilibrium, and so we deduce that there is no offer from supplier j in the range $\left(p_{n-1}^i(1), p_n^i(0)\right)$. This implies that $\frac{\partial \widehat{\alpha}_j(p,m)}{\partial p} = 0$ for p in this range.

With a similar argument as above, it now follows that

$$
\begin{aligned}
Z_i(n-1, p, 1) &= \frac{\partial \Pi_j(r_1, r_2, \ldots r_N, 1)}{\partial r_{n-1}} = (n-1) h \sum_{m=0}^{N} \Delta\widehat{\alpha}_j(p, m) f((n-1+m)h) \\
&= (n-1) h f((2n-2)h) > 0,
\end{aligned}
$$

for $p \in \left(p^i_{n-1}(1), p^i_n(0)\right)$. Hence, supplier i will gain by increasing its offer price for unit $n-1$ when observing signal $\alpha_i = 1$. Hence the strategy is not optimal for supplier i and again we have a contradiction. We can use the same argument to rule out that $p^i_N(1) < \overline{p}$. ■

Lemma 9 *In a duopoly market, offer ranges for successive units of the same supplier do not overlap in an equilibrium, i.e.* $p^i_{n-1}(1) \leq p^i_n(0)$ *for* $n \in \{2, \ldots, N\}$, *if*

$$f((n-2)h) < \Gamma_n f((n-1)h) \;\; \text{and} \;\; f((n-1)h) < \Gamma_n f(nh) \tag{30}$$

where

$$\Gamma_n = \left(\frac{n-1}{n-2}\right) \frac{\min\left[f((n-3)h), f((n-2)h)\right]}{\max\left[f((n-2)h), f((n-1)h)\right]}$$

for $n \in \{3, \ldots, N\}$.

Proof. We let p^i_Z be the highest price at which there is an overlap for supplier i, thus we have $p^i_Z = p^i_{n_i-1}(1) > p^i_{n_i}(0)$ for some n_i, and $p^i_{n-1}(1) \leq p^i_n(0)$ for $n > n_i$. Without loss of generality we can assume that $p^i_Z \geq p^j_Z$ and we will need to deal separately with the two cases $p^i_Z = p^j_Z$ and $p^i_Z > p^j_Z$.

First we take the case that they are equal. By assumption, offer prices are strictly increasing with respect to the number of units for a given signal, so $p^k_{n_k+1}(0) \geq p^k_{n_k}(1) > p^i_Z > p^k_{n_k-2}(1)$ for both firms. Thus we can find a p_0, such that $p^i_Z > p_0 > \max\{p^i_{n_i-2}(1), p^j_{n_j-2}(1), p^i_{n_i}(0), p^j_{n_j}(0)\}$. Next, we can identify signals α^j_X, α^j_Y, α^i_X, α^i_Y in the range $(0,1)$, such that $p_0 = p^j_{n_j-1}\left(\alpha^j_X\right) = p^j_{n_j}\left(\alpha^j_Y\right) = p^i_{n_i-1}\left(\alpha^i_X\right) = p^i_{n_i}\left(\alpha^i_Y\right)$ in the equilibrium. By assumption $p^i_n(\alpha)$ is strictly increasing with respect to n and α below the reservation price. Thus $\alpha^i_X > \alpha^i_Y$ and $\alpha^j_X > \alpha^j_Y$. Moreover,

$$\widehat{\alpha}_j(p_0, n) = \begin{cases} 1 & \text{for } n \leq n_j - 2 \\ \alpha^j_X & \text{for } n = n_j - 1 \\ \alpha^j_Y & \text{for } n = n_j \\ 0 & \text{for } n \geq n_j + 1, \end{cases}$$

and

$$\Delta\widehat{\alpha}_j(p_0, n) = \begin{cases} 0 & \text{for } n \leq n_j - 3 \\ 1 - \alpha^j_X & \text{for } n = n_j - 2 \\ \alpha^j_X - \alpha^j_Y & \text{for } n = n_j - 1 \\ \alpha^j_Y & \text{for } n = n_j \\ 0 & \text{for } n \geq n_j + 1. \end{cases}$$

It now follows from Lemma 7 that :

$$\begin{aligned} Z_i\left(n_i, p_0, \alpha^i_Y\right) &= n_i h(1-\alpha^j_X) f((n_i+n_j-2)h) \\ &\quad + n_i h\left(\alpha^j_X - \alpha^j_Y\right) f((n_i+n_j-1)h) \\ &\quad + n_i h \alpha^j_Y f((n_i+n_j)h) \\ &\quad - h\left(p_0 - c_{n_i}(\alpha^i_Y)\right) \sum_{m=n_j-1}^{n_j} \frac{\partial \widehat{\alpha}_j(p_0, m)}{\partial p} f((n_i+m-1)h), \end{aligned} \tag{31}$$

$$
\begin{aligned}
Z_i\left(n_i-1,p_0,\alpha_X^i\right) &= (n_i-1)h(1-\alpha_X^j)f((n_i+n_j-3)h) \\
&\quad +(n_i-1)h\left(\alpha_X^j-\alpha_Y^j\right)f\left(\left(n_i+n_j-2\right)h\right) \\
&\quad +(n_i-1)h\alpha_Y^j f\left((n_i+n_j-1)h\right) \\
&\quad -h\left(p_0-c_{n_i-1}(\alpha_X^i)\right)\sum_{m=n_j-1}^{n_j}\frac{\partial\widehat{\alpha}_j\left(p_0,m\right)}{\partial p}f\left(\left(n_i+m-2\right)h\right).
\end{aligned} \tag{32}
$$

From (30) we observe that

$$
\begin{aligned}
\eta &= \left(\alpha_X^j-\alpha_Y^j\right)\left(\Gamma_{n_i+n_j}f\left(\left(n_i+n_j-1\right)h\right)-f\left(\left(n_i+n_j-2\right)h\right)\right) \\
&\quad +\alpha_Y^j\left(\Gamma_{n_i+n_j}f\left((n_i+n_j)h\right)-f\left(\left(n_i+n_j-1\right)h\right)\right) \\
&> 0.
\end{aligned} \tag{33}
$$

We will write

$$
\begin{aligned}
f_{\max} &= \max\left[f\left(\left(n_i+n_j-2\right)h\right),f\left(\left(n_i+n_j-1\right)h\right)\right], \\
f_{\min} &= \min\left[f\left(\left(n_i+n_j-3\right)h\right),f\left(\left(n_i+n_j-2\right)h\right)\right].
\end{aligned}
$$

Then

$$
\begin{aligned}
&\Gamma_{n_i+n_j}\left(p_0-c_{n_i}(\alpha_Y^i)\right)\sum_{m=n_j-1}^{n_j}\frac{\partial\widehat{\alpha}_j\left(p_0,m\right)}{\partial p}f\left(\left(n_i+m-1\right)h\right) \\
&= \left(\frac{n_i+n_j-1}{n_i+n_j-2}\right)\left(f_{\min}\left(p_0-c_{n_i}(\alpha_Y^i)\right)\sum_{m=n_j-1}^{n_j}\frac{\partial\widehat{\alpha}_j\left(p_0,m\right)}{\partial p}\underbrace{\frac{f\left(\left(n_i+m-1\right)h\right)}{f_{\max}}}_{\leq 1}\right) \\
&\leq \left(\frac{n_i+n_j-1}{n_i+n_j-2}\right)\left(f_{\min}\left(p_0-c_{n_i-1}(\alpha_X^i)\right)\sum_{m=n_j-1}^{n_j}\frac{\partial\widehat{\alpha}_j\left(p_0,m\right)}{\partial p}\underbrace{\frac{f\left(\left(n_i+m-2\right)h\right)}{f_{\min}}}_{\geq 1}\right),
\end{aligned}
$$

because of (1). So we have from the above inequality, (31) and (33) that:

$$
\begin{aligned}
&\Gamma_{n_i+n_j}Z_i\left(n_i,p_0,\alpha_Y^i\right) \\
&\geq n_ih\eta+n_ih\Gamma_{n_i+n_j}(1-\alpha_X^j)f((n_i+n_j-2)h) \\
&\quad +n_ih\left(\left(\alpha_X^j-\alpha_Y^j\right)f\left(\left(n_i+n_j-2\right)h\right)+\alpha_Y^j f\left((n_i+n_j-1)h\right)\right) \\
&\quad -h\left(\frac{n_i+n_j-1}{n_i+n_j-2}\right)\left(\left(p_0-c_{n_i-1}(\alpha_X^i)\right)\sum_{m=n_j-1}^{n_j}\frac{\partial\widehat{\alpha}_j\left(p_0,m\right)}{\partial p}f\left(\left(n_i+m-2\right)h\right)\right).
\end{aligned}
$$

Now $\left(\frac{n_i+n_j-1}{n_i+n_j-2}\right)\leq\left(\frac{n_i}{n_i-1}\right)$ and so $n_i\geq\left(\frac{n_i+n_j-1}{n_i+n_j-2}\right)(n_i-1)$. We deduce from (32) that

$$
\begin{aligned}
\Gamma_{n_i+n_j}Z_i\left(n_i,p_0,\alpha_Y^i\right) &\geq n_ih\Gamma_{n_i+n_j}(1-\alpha_X^j)\left(f((n_i+n_j-2)h)-f((n_i+n_j-3)h)\right) \\
&\quad +\left(\frac{n_i+n_j-1}{n_i+n_j-2}\right)Z_i\left(n_i-1,p_0,\alpha_X^i\right)+n_ih\eta.
\end{aligned} \tag{34}
$$

It follows from our assumptions that $\widehat{\alpha}_j(p_0, m)$ is piecewise differentiable. We consider a presumed equilibrium. Hence, provided that we do not choose p_0 where some $\widehat{\alpha}_j(p_0, m)$ is non-smooth, we deduce from the necessary conditions in Lemma 2 that

$$Z_i\left(n_i - 1, p_0, \alpha_X^i\right) = 0 \tag{35}$$

and

$$Z_i\left(n_i, p_0, \alpha_Y^i\right) = 0. \tag{36}$$

By assumption $p_n^i(\alpha)$ is continuous with respect to α. Thus by choosing p_0 below and sufficiently close to p_Z^i, and thereby α_X close enough to 1, we can ensure that the right-hand side of (34) is strictly greater than zero, which would contradict (36). This leads to the conclusion that $p_Z^i = p_Z^j$ cannot occur in equilibrium.

The next step is to consider the case where $p_Z^i > p_Z^j$. We choose p_0 with $p_Z^i > p_0 > \max\{p_{n_i-2}^i(1), p_{n_i}^i(0), p_Z^j\}$. In this case, we can identify signals α_X^i, α_Y^i in the range $(0,1)$, such that $p_0 = p_{n_i-1}^i(\alpha_X^i) = p_{n_i}^i(\alpha_Y^i)$. Moreover from Lemma 8 we can deduce the existence of m_j and α^j for which $p_0 = p_{m_j}^j(\alpha^j)$. Thus

$$\widehat{\alpha}_j(p_0, n) = \begin{cases} 1 & \text{for } n \leq m_j - 1 \\ \alpha^j & \text{for } n = m_j \\ 0 & \text{for } n \geq m_j + 1, \end{cases}$$

and

$$\Delta\widehat{\alpha}_j(p_0, n) = \begin{cases} 0 & \text{for } n \leq m_j - 2 \\ 1 - \alpha^j & \text{for } n = m_j - 1 \\ \alpha^j & \text{for } n = m_j \\ 0 & \text{for } n \geq m_j + 1. \end{cases}$$

Now we can use Lemma 7 to show

$$\begin{aligned} Z_i\left(n_i, p_0, \alpha_Y^i\right) &= n_i h\left((1-\alpha^j) f((n_i + m_j - 1)h) + \alpha^j f\left((n_i + m_j)h\right)\right) \\ &\quad - h\left(p_0 - c_{n_i}(\alpha_Y^i)\right) \frac{\partial \widehat{\alpha}_j(p_0, m_j)}{\partial p} f\left((n_i + m_j - 1)h\right), \end{aligned}$$

$$\begin{aligned} Z_i\left(n_i - 1, p_0, \alpha_X^i\right) &= (n_i - 1) h\left((1-\alpha^j) f((n_i + m_j - 2)h) + \alpha^j f\left((n_i + m_j - 1)h\right)\right) \\ &\quad - h\left(p_0 - c_{n_i-1}(\alpha_X^i)\right) \frac{\partial \widehat{\alpha}_j(p_0, m_j)}{\partial p} f\left((n_i + m_j - 2)h\right). \end{aligned}$$

The rest of the proof follows from a contradiction achieved using the same argument as above, with m_j instead of n_j and a single term $\frac{\partial \widehat{\alpha}_j(p_0, m_j)}{\partial p}$. ■

Lemma 10 *Under Assumption 1, offer ranges for successive units of supplier* $i \in \{1, 2\}$ *do not overlap in an equilibrium, i.e.* $p_{n-1}^i(1) \leq p_n^i(0)$ *for* $n \in \{2, \ldots, N\}$.[14]

[14] According to Lemma 9 in the online Appendix this statement would also hold for a less stringent, but also more complex inequality than (3), which is used in Assumption 1.

Proof. Below we show that (3) is sufficient to satisfy the conditions for Lemma 9. We can deduce immediately from (2) that $|\tau_n| < 1/(3n)$ implies that:

$$f((n-1)h) > \left(\frac{3n}{3n+1}\right) f(nh), \tag{37}$$

and

$$f(nh) > \left(\frac{3n-1}{3n}\right) f((n-1)h). \tag{38}$$

There are two inequalities in (30) that we need to establish. The first inequality can be written

$$\begin{aligned} & (n-1)f\left((n-1)\,h\right)\min\left[f\left((n-3)\,h\right), f\left((n-2)\,h\right)\right] \\ > \;& (n-2)f\left((n-2)\,h\right)\max\left[f\left((n-2)\,h\right), f\left((n-1)\,h\right)\right]. \end{aligned}$$

We have to check four cases.
(a) $(n-1)f\left((n-1)\,h\right)f\left((n-3)\,h\right) > (n-2)f\left((n-2)\,h\right)f\left((n-2)\,h\right)$: From (37) applied at $n-2$ we have

$$f\left((n-3)\,h\right) > \left(\frac{3n-6}{3n-5}\right) f((n-2)h) \tag{39}$$

and with (38) applied at $n-1$ we have

$$f((n-1)h) > \left(\frac{3n-4}{3n-3}\right) f((n-2)h). \tag{40}$$

Since

$$(n-1)\left(\frac{3n-6}{3n-5}\right)\left(\frac{3n-4}{3n-3}\right) = \left(\frac{3n-4}{3n-5}\right)(n-2) > n-2,$$

we establish the inequality we require.
(b) $(n-1)f\left((n-1)\,h\right)f\left((n-2)\,h\right) > (n-2)f\left((n-2)\,h\right)f\left((n-2)\,h\right)$: From (40) we can deduce

$$(n-1)f\left((n-1)\,h\right) > \left(n-\frac{4}{3}\right) f((n-2)h) > (n-2)f((n-2)h),$$

which immediately implies the inequality.
(c) $(n-1)f\left((n-1)\,h\right)f\left((n-3)\,h\right) > (n-2)f\left((n-2)\,h\right)f\left((n-1)\,h\right)$: From (39) we obtain

$$(n-1)f((n-3)h) > (n-1)\left(\frac{3n-6}{3n-5}\right) f((n-2)h) = (n-2)\left(\frac{3n-3}{3n-5}\right) f((n-2)h),$$

which immediately implies the inequality.
(d) $(n-1)f\left((n-1)\,h\right)f\left((n-2)\,h\right) > (n-2)f\left((n-2)\,h\right)f\left((n-1)\,h\right)$: This is immediate.

Now we turn to the other inequality, we need to establish:

$$\begin{aligned} & (n-1)f\left(nh\right)\min\left[f\left((n-3)\,h\right), f\left((n-2)\,h\right)\right] \\ > \;& (n-2)f\left((n-1)\,h\right)\max\left[f\left((n-2)\,h\right), f\left((n-1)\,h\right)\right]. \end{aligned}$$

Again we have four cases to check.
(a) $(n-1)f(nh) f((n-3)h) > (n-2)f((n-1)h) f((n-2)h)$: Using (38) and (39) we have

$$\begin{aligned} & (n-1)f(nh) f((n-3)h) & (41) \\ > & (n-1)\left(\frac{3n-1}{3n}\right)\left(\frac{3n-6}{3n-5}\right) f((n-1)h)f((n-2)h). & (42) \end{aligned}$$

Since $(n-1)(3n-1) - n(3n-5) = n+1 > 0$ we have $(n-1)\left(\frac{3n-1}{n}\right)\left(\frac{1}{3n-5}\right) > 1$ which is enough to show the inequality we require.
(b) $(n-1)f(nh) f((n-2)h) > (n-2)f((n-1)h) f((n-2)h)$: Since $(n-1)(3n-1) - 3n(n-2) = 2n+1 > 0$, we deduce that $(n-1)\left(\frac{3n-1}{3n}\right) > n-2$. Then the inequality follows from (38).
(c) $(n-1)f(nh) f((n-3)h) > (n-2)f((n-1)h) f((n-1)h)$: From (37) at $n-1$ we see that

$$f((n-2)h) > \left(\frac{3n-3}{3n-2}\right) f((n-1)h). \quad (43)$$

Together with (41) we deduce that

$$\begin{aligned} & (n-1)f(nh) f((n-3)h) \\ > & \frac{(n-1)}{n}\left(\frac{3n-1}{3n-5}\right)\left(\frac{3n-3}{3n-2}\right)(n-2)f((n-1)h)f((n-1)h). \end{aligned}$$

Since $(n-1)(3n-1)(3n-3) - n(3n-5)(3n-2) = 5n-3 > 0$, we have $\frac{(n-1)}{n}\left(\frac{3n-1}{3n-5}\right)\left(\frac{3n-3}{3n-2}\right) > 1$ and the result is established.
(d)$(n-1)f(nh) f((n-2)h) > (n-2)f((n-1)h) f((n-1)h)$: From (38) and (43) we deduce

$$(n-1)f(nh) f((n-2)h) > (n-1)\left(\frac{3n-1}{3n}\right)\left(\frac{3n-3}{3n-2}\right) f((n-1)h)f((n-1)h).$$

However since $(n-1)(3n-1)(3n-3) - 3n(3n-2)(n-2) = 3n^2 + 3n - 3 > 0$ we have $(n-1)\left(\frac{3n-1}{3n}\right)\left(\frac{3n-3}{3n-2}\right) > n-2$, and the inequality follows. ■

Lemma 11 *Consider a duopoly market where each supplier has step separation without gaps in its offer strategy and consider a price p where there is a unique unit $\widehat{n}(p)$ such that $\widehat{\alpha}_j(p,\widehat{n}) \in (0,1)$, where supplier $j \neq i$ is the competitor of supplier i. In this case,*

$$\begin{aligned} Z_i(n, r_n, \alpha_i) &= nh\widehat{\alpha}_j(r_n, \widehat{n}(r_n))\left(f((n+\widehat{n}(r_n))h) - f((n+\widehat{n}(r_n)-1)h)\right) \\ &\quad + nhf((n+\widehat{n}(r_n)-1)h) \\ &\quad - h(r_n - c_n(\alpha_i))\frac{\partial \widehat{\alpha}_j(r_n, \widehat{n}(r_n))}{\partial r_n}(r_n) f((n+\widehat{n}(r_n)-1)h) \quad (44) \end{aligned}$$

The first-order condition for a symmetric Bayesian Nash equilibrium, so that $\widehat{n}(r_n) = n$ *and* $\widehat{\alpha}_i(r_n, n) = \widehat{\alpha}_j(r_n, n) = \widehat{\alpha}(r_n, n)$ *is given by:*

$$(\widehat{\alpha}(r_n, n)\tau_{2n} + 1)\, n = (r_n - c_n(\widehat{\alpha}(r_n, n))) \frac{\partial \widehat{\alpha}\,(r_n, n)}{\partial r_n}. \tag{45}$$

Proof. For $m < \widehat{n}(p)$ we have $\widehat{\alpha}_j(p, m) = 1$ and for $m > \widehat{n}(p)$ we have $\widehat{\alpha}_j(p, m) = 0$. Thus it follows from Lemma 7 that

$$\begin{aligned} Z_i\,(n, r_n, \alpha_i) \;=\;& nh\,(\widehat{\alpha}_j\,(r_n, \widehat{n}(r_n)) - 0)\, f\,((n + \widehat{n}(r_n))\, h) \\ & + nh\,(1 - \widehat{\alpha}_j\,(r_n, \widehat{n}(r_n)))\, f\,((n + \widehat{n}(r_n) - 1)\, h) \\ & - h\,(r_n - c_n(\alpha_i)) \frac{\partial \widehat{\alpha}_j\,(r_n, \widehat{n}(r_n))}{\partial r_n} f\,((n + \widehat{n}(r_n) - 1)\, h)\,, \end{aligned}$$

which gives (44). In a symmetric equilibrium $\widehat{n}(r_n) = n$, which yields (45), because the first-order condition is that $Z_i\,(n, r_n, \alpha_i) = 0$ and we can divide all terms by $hf\,((2n-1)\,h)$. ■

Lemma 12 *Under Assumption 1, the Bayesian NE must be symmetric, i.e.* $p_n^1\,(\alpha) = p_n^2\,(\alpha)$ *for* $\alpha \in [0,1]$ *and* $n \in \{1, \ldots, N\}$ *.*

Proof. It follows from the necessary first-order condition implied by (44) and $Z_i\,(n, r_n, \alpha_i) = 0$ that $p_n^i\,(\alpha_i) > c_n(\alpha_i)$ for $\alpha_i \in [0,1]$ and $n \in \{1, \ldots, N\}$. Using a similar argument as in the proof of Lemma 8, it can be shown that suppliers must submit identical offer prices for the lowest cost realization, i.e. $p_1^1\,(0) = p_1^2\,(0)$. The two suppliers also have identical costs ex-ante, before signals have been realized. Thus it follows from the Picard-Lindelöf theorem that the solution of the symmetric system of differential equations implied by (44) can only have a symmetric solution, such that $p_1^1\,(\alpha) = p_1^2\,(\alpha)$ for $\alpha \in [0,1]$. Under the stated assumptions, it follows from Lemma 8 and Lemma 10 that steps are necessarily separated without gaps, so that $p_n^i\,(1) = p_{n+1}^i\,(0)$. Thus, we can repeat this argument $N-1$ times to show that $p_1^1\,(\alpha) = p_1^2\,(\alpha)$ for $\alpha \in [0,1]$ and $n \in \{1, \ldots, N\}$. ■

Proof. (Lemma 3) The result follows from Lemma 8, Lemma 10 and Lemma 12. ■

Appendix C: Existence and uniqueness results

Lemma 13 *The first-order conditions for the symmetric Bayesian equilibrium when* $c_n\,(1) < p_n\,(1) = p_{n+1}\,(0)$ *has a unique symmetric solution for unit* n*:*

$$\begin{aligned} & p_n\,(\alpha) = p_n\,(1) \frac{(\alpha\tau_{2n} + 1)^{1/(n\tau_{2n})}}{(\tau_{2n} + 1)^{1/(n\tau_{2n})}} \\ & + (\alpha\tau_{2n} + 1)^{1/(n\tau_{2n})} \int_\alpha^1 \frac{c_n(u)\,(u\tau_{2n} + 1)^{-1/(n\tau_{2n}) - 1}}{n} du \\ > \;& c_n(\alpha), \end{aligned}$$

where

$$p_n'(\alpha) > 0. \tag{46}$$

Proof. In order to solve (45) we write the offer price as a function of the signal. We have $\frac{\partial \widehat{\alpha}(r_n,n)}{\partial r_n} = \frac{1}{p_n'(\alpha)}$ where $r_n = p_n(\alpha)$ and $\alpha = \widehat{\alpha}(r_n, n)$. Thus

$$\begin{aligned} (\alpha\tau_{2n}+1)\,n &= \frac{(p_n(\alpha) - c_n(\alpha))}{p_n'(\alpha)} \\ p_n'(\alpha) - \frac{p_n(\alpha)}{(\alpha\tau_{2n}+1)\,n} &= -\frac{c_n(\alpha)}{(\alpha\tau_{2n}+1)\,n}. \end{aligned} \tag{47}$$

Next we multiply both sides by the integrating factor $(\alpha + 1/\tau_{2n})^{-1/(n\tau_{2n})}$.

$$\begin{aligned} & p_n'(\alpha)(\alpha+1/\tau_{2n})^{-1/(n\tau_{2n})} - \frac{p_n(\alpha)(\alpha+1/\tau_{2n})^{-1/(n\tau_{2n})-1}}{n\tau_{2n}} \\ = & -\frac{c_n(\alpha)(\alpha+1/\tau_{2n})^{-1/(n\tau_{2n})}}{(\alpha+1/\tau_{2n})\,n\tau_{2n}}, \end{aligned}$$

so

$$\frac{d}{d\alpha}\left(p_n(\alpha)(\alpha+1/\tau_{2n})^{-1/(n\tau_{2n})}\right) = -\frac{c_n(\alpha)(\alpha+1/\tau_{2n})^{-1/(n\tau_{2n})-1}}{n\tau_{2n}}.$$

Integrating both sides from α to 1 yields:

$$\begin{aligned} & p_n(1)(1+1/\tau_{2n})^{-1/(n\tau_{2n})} - p_n(\alpha)(\alpha+1/\tau_{2n})^{-1/(n\tau_{2n})} \\ = & \int_\alpha^1 -\frac{c_n(u)(u+1/\tau_{2n})^{-1/(n\tau_{2n})-1}}{n\tau_{2n}}du, \end{aligned}$$

so

$$\begin{aligned} p_n(\alpha) &= p_n(1)\frac{(\alpha+1/\tau_{2n})^{1/(n\tau_{2n})}}{(1+1/\tau_{2n})^{1/(n\tau_{2n})}} \\ &+ (\alpha+1/\tau_{2n})^{1/(n\tau_{2n})}\int_\alpha^1 \frac{c_n(u)(u+1/\tau_{2n})^{-1/(n\tau_{2n})-1}}{n\tau_{2n}}du, \end{aligned}$$

which can be immediately written in the form of the Lemma statement. Thus

$$\begin{aligned} p_n(\alpha) &\geq p_n(1)\frac{(\alpha+1/\tau_{2n})^{1/(n\tau_{2n})}}{(1+1/\tau_{2n})^{1/(n\tau_{2n})}} + \\ & \quad (\alpha+1/\tau_{2n})^{1/(n\tau_{2n})} c_n(\alpha)\left[-(u+1/\tau_{2n})^{-1/(n\tau_{2n})}\right]_\alpha^1 \\ &= (p_n(1) - c_n(\alpha))\frac{(\alpha+1/\tau_{2n})^{1/(n\tau_{2n})}}{(1+1/\tau_{2n})^{1/(n\tau_{2n})}} + c_n(\alpha) > c_n(\alpha). \end{aligned}$$

Finally, it follows from (47) that $p_n'(\alpha) > 0$, because we know from **(2)** that $\alpha\tau_{2n} + 1 \geq 1 - \alpha \geq 0$. ■

Proof. (Proposition 1) The solution is given by the end-conditions and Lemma 13 above. We will now verify that this is an equilibrium using the optimality conditions in Lemma 2. When proving existence, we will work with Assumption 1′, which is weaker than Assumption 1. Assumption 1′ is introduced in Section 4. It is straightforward to show that it implies that:

$$(k-1)\,\tau_k - (k-2)\,\tau_{k-1} \geq -1 \tag{48}$$

for all $k \in \{2, \ldots, 2N\}$. By definition we have that $\tau_2 \geq -1$, so this is equivalent to:

$$m\tau_k - (m-1)\,\tau_{k-1} \geq -1 \tag{49}$$

for all $k \in \{2, \ldots, 2N\}$ and $m \in \{1, \ldots, k-1\}$, because the condition in (49) is linear with respect to m, so it is satisfied if and only if it is satisfied at the end points $m = 1$ and $m = k-1$. By setting $k = m + \widehat{n}$, the inequality above can be written in the following form:

$$m\tau_{m+\widehat{n}} - (m-1)\,\tau_{m-1+\widehat{n}} \geq -1 \tag{50}$$

for all $(m, \widehat{n}) \in \{1, \ldots, N\} \times \{1, \ldots, N\}$. Next we prove the following:

i) Prove that $Z_i\,(m, p, \alpha_i) \leq 0$ if $p \in (p_m\,(\alpha_i)\,, p_m\,(1))$. It is known from Lemma 13 that the first-order solutions are monotonic. Thus $\widehat{\alpha}_i\,(p, m) \geq \alpha_i$ and so $c_m(\widehat{\alpha}_i\,(p, m)) \geq c_m(\alpha_i)$. Thus, it follows from Lemma 11 that

$$\begin{aligned} Z_i\,(m, p, \alpha_i) &\leq mh\widehat{\alpha}_j\,(p, m)\,(f\,(2mh) - f\,((2m-1)\,h)) + mhf\,((2m-1)\,h) \\ &\quad -h\,(p - c_m(\widehat{\alpha}_i\,(p, m)))\,\tfrac{\partial \widehat{\alpha}_j(p,m)}{\partial p} f\,((2m-1)\,h) = Z_i\,(m, p, \widehat{\alpha}_i\,(p, m)) = 0. \end{aligned} \tag{51}$$

ii) Prove that $Z_i\,(m, p, \alpha_i) \leq 0$ if $p \in (p_n\,(0)\,, p_n\,(1))$ where $n > m$, so that $p_n\,(0) \geq p_m\,(1)$. For any price $p \in (p_n\,(0)\,, p_n\,(1))$, it follows from the argument above that

$$\frac{Z_i\,(n, p, 0)}{f\,((2n-1)\,h)\,h} = n\widehat{\alpha}_j\,(p, n))\,\tau_{2n} + n - (p - c_n(0))\,\frac{\partial \widehat{\alpha}_j\,(p, n)}{\partial p} \leq 0. \tag{52}$$

Now consider unit $\ell \in \{m, \ldots, n-1\}$ at the same price p. We have from the inequality in (50) that $\alpha\,((\ell+1)\,\tau_{n+\ell+1} - \ell\tau_{n+\ell}) + 1 \geq 0$ for $\alpha \in (0, 1)$, so

$$\begin{aligned} \tfrac{Z_i(\ell,p,\alpha_i)}{f((n+\ell-1)h)h} &\leq (\ell+1)\,\widehat{\alpha}_j\,(p, n)\,\tau_{n+\ell+1} + \ell + 1 \\ &\quad -(p - c_{\ell+1}(\widetilde{\alpha}_i))\,\tfrac{\partial \widehat{\alpha}_j(p,n)}{\partial p} = \tfrac{Z_i(\ell+1,p,\widetilde{\alpha}_i)}{f((n+\ell)h)h}, \end{aligned} \tag{53}$$

for any $\widetilde{\alpha}_i \in [0, 1]$. Starting with (52), we can use the expression above, to recursively prove that $Z_i\,(\ell, p, \alpha_i) \leq 0$ for all $\ell \in \{m, \ldots, n-1\}$.

iii) Prove that $Z_i\,(m, p, \alpha_i) \geq 0$ if $p \in (p_m\,(0)\,, p_m\,(\alpha_i))$. It is known from Lemma 13 that the first-order solutions are monotonic. Thus $\widehat{\alpha}_i\,(p, m) \leq \alpha_i$ and so $c_m(\widehat{\alpha}_i\,(p, m)) \leq c_m(\alpha_i)$. Thus, it follows from Lemma 11 that

$$\begin{aligned} Z_i\,(m, p, \alpha_i) &\geq mh\widehat{\alpha}_j\,(p, m)\,(f\,(2mh) - f\,((2m-1)\,h)) + mhf\,((2m-1)\,h) \\ &\quad -h\,(p - c_m(\widehat{\alpha}_i\,(p, m)))\,\tfrac{\partial \widehat{\alpha}_j(p,m)}{\partial p} f\,((2m-1)\,h) = Z_i\,(m, p, \widehat{\alpha}_i\,(p, m)) = 0. \end{aligned}$$

iiii) Prove that $Z_i(m,p,\alpha_i) \geq 0$ if $p \in (p_n(0), p_n(1))$ where $m > n$, so that $p_n(1) \leq p_m(0)$. For any price $p \in (p_n(0), p_n(1))$, it follows from an argument similar to step i) that

$$\frac{Z_i(n,p,1)}{f((2n-1)h)h} = n\widehat{\alpha}_j(p,n)\tau_{2n} + n - (p - c_n(1))\frac{\partial\widehat{\alpha}_j(r_n,n)}{\partial p} \geq 0. \tag{54}$$

Now consider unit $\ell \in \{n+1,\ldots,m\}$ at the same price p. We have from the inequality in (50) that $\alpha(\ell\tau_{n+\ell} - (\ell-1)\tau_{n+\ell-1}) + 1 \geq 0$ for $\alpha \in (0,1)$, so

$$\begin{aligned}\tfrac{Z_i(\ell,p,\alpha_i)}{f((n+\ell-1)h)h} &\geq (\ell-1)\widehat{\alpha}_j(p,n)\tau_{n+\ell-1} + \ell - 1 \\ &- (p - c_{\ell-1}(\widetilde{\alpha}_i))\tfrac{\partial\widehat{\alpha}_j(p,n)}{\partial p} = \tfrac{Z_i(\ell-1,p,\widetilde{\alpha}_i)}{f((n+\ell-2)h)h},\end{aligned} \tag{55}$$

for any $\widetilde{\alpha}_i \in [0,1]$. Starting with (54), we can use the expression above, to recursively prove that $Z_i(\ell,p,\alpha_i) \geq 0$ for all $\ell \in \{n+1,\ldots,m\}$.

Finally, the uniqueness result follows from the necessary conditions in Lemma 8, Lemma 10 and Lemma 12. ■

Proof. (Proposition 2) We have from Lemma 13 that

$$\begin{aligned}p_n(\alpha) = {} & p_n(1)\frac{(\alpha\tau_{2n}+1)^{1/(n\tau_{2n})}}{(\tau_{2n}+1)^{1/(n\tau_{2n})}} \\ & + (\alpha\tau_{2n}+1)^{1/(n\tau_{2n})}\int_\alpha^1 \frac{c_n(u)(u\tau_{2n}+1)^{-1/(n\tau_{2n})-1}}{n}du.\end{aligned}$$

We have

$$\int_\alpha^1 \frac{(u\tau_{2n}+1)^{-1/(n\tau_{2n})-1}}{n}du = (\alpha\tau_{2n}+1)^{-1/(n\tau_{2n})} - (\tau_{2n}+1)^{-1/(n\tau_{2n})},$$

so

$$p_n(\alpha) = \int_\alpha^1 \frac{(c_n(u) - p_n(1))}{n}\underbrace{(\alpha\tau_{2n}+1)^{1/(n\tau_{2n})}(u\tau_{2n}+1)^{-1/(n\tau_{2n})-1}}_{\Upsilon}du + p_n(1). \tag{56}$$

Hence,

$$\frac{\partial\Upsilon}{\partial\tau_{2n}} = \frac{-(\alpha\tau_{2n}+1)^{\frac{1}{n\tau_{2n}}}}{n^2\tau_{2n}^2(\tau_{2n}u+1)^{\frac{1}{n\tau_{2n}}(n\tau_{2n}+1)}}\left(\left(\ln\frac{(\alpha\tau_{2n}+1)}{(\tau_{2n}u+1)}\right) - \frac{\alpha\tau_{2n}}{(\alpha\tau_{2n}+1)} + \frac{\tau_{2n}u(1+n\tau_{2n})}{(\tau_{2n}u+1)}\right). \tag{57}$$

We have $\tau_{2n} \geq -1$ and $1 \geq u \geq \alpha \geq 0$, so $\frac{(\alpha-u)\tau_{2n}}{(\tau_{2n}u+1)} \geq -1$. Moreover, it follows from a standard inequality that $\frac{x}{1+x} \leq \ln(1+x)$ for $x \geq -1$. Thus

$$\ln\left(1 + \frac{(\alpha-u)\tau_{2n}}{(\tau_{2n}u+1)}\right) \geq \frac{(\alpha-u)\tau_{2n}}{(\tau_{2n}u+1) + (\alpha-u)\tau_{2n}} = \frac{(\alpha-u)\tau_{2n}}{1+\alpha\tau_{2n}}.$$

Hence, we have from (57) that

$$(c_n(u) - p_n(1)) \frac{\partial \Upsilon}{\partial \tau_{2n}} \geq \frac{(p_n(1) - c_n(u))(\alpha\tau_{2n}+1)^{\frac{1}{n\tau_{2n}}}}{n^2\tau_{2n}^2(\tau_{2n}u+1)^{\frac{1}{n\tau_{2n}}(n\tau_{2n}+1)}} \left(\frac{-u\tau_{2n}}{1+\alpha\tau_{2n}} + \frac{\tau_{2n}u(1+n\tau_{2n})}{(\tau_{2n}u+1)}\right) \geq 0. \quad (58)$$

Hence, we have from (56) and (58) that for the same or higher $p_n(1)$, a higher τ_{2n} will result in a higher $p_n(\alpha_i)$. We have $p_N(1) = \overline{p}$ irrespective of τ_{2n}. Thus for symmetric mixed-strategy equilibria with step separation without gaps, so that $p_{n-1}(1) = p_n(0)$, a weakly higher τ_{2n} for $n \in \{1, \ldots, N\}$ will weakly increase $p_n(\alpha_i)$ for $n \in \{1, \ldots, N\}$. Finally, it follows from (56) and (58) that $p_n(\alpha_i)$ is more sensitive to changes in τ_{2n} when mark-ups, $(p_n(1) - c_n)$, are high. ■

Proof. (Proposition 3) It will be convenient to set $\beta = \frac{1}{n\tau_{2n}}$. First we note that

$$(\alpha\tau_{2n}+1)^{\beta} \int_{\alpha}^{1} \frac{(u\tau_{2n}+1)^{-\beta-1}}{n} du = 1 - \frac{(\alpha\tau_{2n}+1)^{\beta}}{(\tau_{2n}+1)^{\beta}}$$

and

$$\int_{0}^{1} \left(1 - (u\tau_{2n}+1)^{-\beta-1}\right) du = (\tau_{2n}+1)^{-\beta}. \quad (59)$$

It follows from (7) and (59) that

$$p_n(\alpha) = A(\alpha) + B(\alpha) + \hat{c}$$

where

$$\begin{aligned} A(\alpha) &= (p_n(1) - \hat{c}) \frac{(\alpha\tau_{2n}+1)^{\beta}}{(\tau_{2n}+1)^{\beta}}, \\ B(\alpha) &= (\alpha\tau_{2n}+1)^{\beta} \int_{\alpha}^{1} \frac{(c_n(u) - \hat{c})(u\tau_{2n}+1)^{-\beta-1}}{n} du, \end{aligned}$$

and $\hat{c}$ is a fixed marginal cost, which we choose to define as in (11). It follows from (59) that $\hat{c} \in [c_n(0), c_n(1)]$.

Thus the variance of the offer price is given by:

$$\begin{aligned} & \mathbb{E}\left[p_n^2(\alpha)\right] - \mathbb{E}\left[p_n(\alpha)\right]\mathbb{E}\left[p_n(\alpha)\right] \\ = \; & \mathbb{E}\left[A^2(\alpha)\right] - (\mathbb{E}[A(\alpha)])^2 + \mathbb{E}\left[B^2(\alpha)\right] - (\mathbb{E}[B(\alpha)])^2 \\ & +2\mathbb{E}[A(\alpha)B(\alpha)] - 2\mathbb{E}[A(\alpha)]\mathbb{E}[B(\alpha)]. \end{aligned} \quad (60)$$

We have:

$$
\begin{aligned}
\mathbb{E}\left[B\left(\alpha\right)\right] &= \int_0^1 (\alpha\tau_{2n}+1)^{\beta} \int_\alpha^1 \frac{(c_n(u)-\hat{c})\,(u\tau_{2n}+1)^{-\beta-1}}{n} du d\alpha \\
&= \int_0^1 \frac{(c_n(u)-\hat{c})\,(u\tau_{2n}+1)^{-\beta-1}}{n} \int_0^u (\alpha\tau_{2n}+1)^{\beta}\, d\alpha du \\
&= \int_0^1 \frac{(c_n(u)-\hat{c})\,(u\tau_{2n}+1)^{-\beta-1}}{n} \frac{(u\tau_{2n}+1)^{\beta+1}-1}{1/n+\tau_{2n}} du \\
&= \int_0^1 (c_n(u)-\hat{c}) \frac{\left(1-(u\tau_{2n}+1)^{-\beta-1}\right)}{1+n\tau_{2n}} du = 0, \qquad (61)
\end{aligned}
$$

because of (11), which was the intention with this choice.

Now we may approximate

$$
\begin{aligned}
\tau_{2n} &= \frac{f(2nh)-f(2nh-h)}{f(2nh-h)} \\
&= hf'(2\gamma)+(1/2)f''(2\gamma)+O(h^3)
\end{aligned}
$$

where we fix $nh=\gamma$ so that as h gets smaller we increase n in order to consider the same point in the underlying demand distribution. So we have

$$
\tau_{2n} = h\rho + h^2\sigma + O(h^3)
$$

where $\rho = f'(2\gamma)$ and $\sigma = (1/2)f''(2\gamma)$ And

$$
\beta = \frac{1}{n\tau_{2n}} = \frac{1}{\gamma\rho + h\gamma\sigma} + O(h^2).
$$

Now we notice that when $\tau_{2n} = h\rho + h^2\sigma$ we have from a Taylor expansion

$$
\begin{aligned}
\frac{\left(1-(u\tau_{2n}+1)^{-\beta-1}\right)}{1+n\tau_{2n}} &= \frac{\left(1-(uh\rho+uh^2\sigma+1)^{-\beta-1}\right)}{1+\gamma\rho+h\gamma\sigma} \\
&= h\frac{u}{\gamma} + O(h^2).
\end{aligned}
$$

Thus, using the fact that $c_n(u)-\hat{c} = O\left(h\right)$ in order to satisfy (1)

$$
0 = \mathbb{E}\left[B\left(\alpha\right)\right] = \frac{h}{\gamma}\int_0^1 u\left(c_n(u)-\hat{c}\right)du + O\left(h^3\right),
$$

so

$$
\int_0^1 u\left(c_n(u)-\hat{c}\right)du = O\left(h^2\right). \qquad (62)
$$

Moreover we will show that the other terms involving $B(\alpha)$ are all either $O(h^3)$ or smaller. We have

$$\begin{aligned}
\mathbb{E}\left(B^2(\alpha)\right) &= \int_0^1 (\alpha\tau_{2n}+1)^{2\beta} \int_\alpha^1 \frac{(c_n(u)-\hat{c})^2 (u\tau_{2n}+1)^{-2\beta-2}}{n^2} du d\alpha \\
&= \int_0^1 \frac{(c_n(u)-\hat{c})^2 (u\tau_{2n}+1)^{-2\beta-2}}{n^2} \int_0^u (\alpha\tau_{2n}+1)^{2\beta} d\alpha du \\
&= \int_0^1 \frac{(c_n(u)-\hat{c})^2 (u\tau_{2n}+1)^{-2\beta-2}}{n^2} \frac{(u\tau_{2n}+1)^{2\beta+1}-1}{2/n+\tau_{2n}} du \\
&= \int_0^1 (c_n(u)-\hat{c})^2 \frac{\left((u\tau_{2n}+1)^{-1}-(u\tau_{2n}+1)^{-2\beta-2}\right)}{2n+n^2\tau_{2n}} du \\
&= \frac{h^2}{\gamma^2}\int_0^1 u\,(c_n(u)-\hat{c})^2 du + O\left(h^5\right) = O\left(h^4\right),
\end{aligned}$$

where the final step makes use of a Taylor expansion and that we have by necessity that $c_n(u)-\hat{c}=O(h)$. Similarly, it follows that

$$\begin{aligned}
&\mathbb{E}\left[A(\alpha)B(\alpha)\right] \\
&= \int_0^1 (\alpha\tau_{2n}+1)^\beta (p_n(1)-\hat{c}) \left(\frac{\alpha\tau_{2n}+1}{\tau_{2n}+1}\right)^\beta \int_\alpha^1 \frac{(c_n(u)-\hat{c})(u\tau_{2n}+1)^{-\beta-1}}{n} du d\alpha \\
&= \int_0^1 \frac{(c_n(u)-\hat{c})(p_n(1)-\hat{c})(u\tau_{2n}+1)^{-\beta-1}}{n(\tau_{2n}+1)^\beta} \int_0^u (\alpha\tau_{2n}+1)^{2\beta} d\alpha du \\
&= (p_n(1)-\hat{c}) \int_0^1 (c_n(u)-\hat{c}) \frac{(u\tau_{2n}+1)^\beta-(u\tau_{2n}+1)^{-\beta-1}}{(2+n\tau_{2n})(\tau_{2n}+1)^\beta} du \\
&= \frac{(p_n(1)-\hat{c})}{n}\frac{h}{\gamma}\int_0^1 u\,(c_n(u)-\hat{c})\, du + O\left(h^3\right) = O\left(h^3\right),
\end{aligned}$$

where the final step makes use of a Taylor expansion and (61).

We have,

$$\begin{aligned}
\mathbb{E}\left[A(\alpha)\right] &= \int_0^1 \left((p_n(1)-\hat{c})\left(\frac{\alpha\tau_{2n}+1}{\tau_{2n}+1}\right)^\beta\right) d\alpha \\
&= (p_n(1)-\hat{c}) \frac{(\tau_{2n}+1)^{1+\beta}-1}{\tau_{2n}(\tau_{2n}+1)^\beta(1+\beta)}.
\end{aligned}$$

And thus from a Taylor series expansion

$$\left(\mathbb{E}\left[A(\alpha)\right]\right)^2 = (p_n(1)-\hat{c})^2\left(1-\frac{h}{\gamma}+\frac{1}{12}\frac{h^2}{\gamma^2}(8\gamma\rho+7)\right)+O(h^3).$$

Similarly,

$$
\begin{aligned}
\mathbb{E}\left[A^2(\alpha)\right] &= \int_0^1 \left((p_n(1) - \hat{c})^2 \left(\frac{\alpha\tau_{2n} + 1}{\tau_{2n} + 1} \right)^{2\beta} \right) d\alpha \\
&= (p_n(1) - \hat{c})^2 \frac{(\tau_{2n} + 1)^{1+2\beta} - 1}{\tau_{2n}(\tau_{2n} + 1)^{2\beta}(1 + 2\beta)} \\
&= (p_n(1) - \hat{c})^2 \left(1 - \frac{h}{\gamma} + \frac{2}{3}\frac{h^2}{\gamma^2}(\gamma\rho + 1) \right) + O(h^3)
\end{aligned}
$$

Thus the variance of the offer is given by

$$
\begin{aligned}
& \mathbb{E}\left[p_n^2(\alpha)\right] - \mathbb{E}[p_n(\alpha)]\,\mathbb{E}[p_n(\alpha)] \\
= \;& \mathbb{E}\left[A^2(\alpha)\right] - (\mathbb{E}[A(\alpha)])^2 + O(h^3) \\
= \;& (p_n(1) - \hat{c})^2 \frac{1}{12}\frac{h^2}{\gamma^2} + O(h^3).
\end{aligned}
$$

■

Appendix D: Equilibrium convergence

Proof. (Lemma 4) A discrete approximation of an ordinary differential equation is consistent if the local truncation error is infinitesimally small when the step length is infinitesimally small (LeVeque, 2007). The local truncation error is the discrepancy between the continuous slope and its discrete approximation when values p_n in the discrete system are replaced with samples of the continuous solution $P(nh)$. The difference equation can be written as follows:

$$
p_{n-1} = (\tau_{2n} + 1)^{-1/(n\tau_{2n})}(p_n - c_n) + c_n
$$

We have $\tau_{2n} = \frac{f(2nh) - f((2n-1)h)}{f((2n-1)h)}$, so $\tau_{2n} \to \frac{f'(2nh)h}{f(2nh)}$ when $h \to 0$. By assumption $\frac{f'(2nh)}{f(2nh)}$ is bounded, so $\tau_{2n} = O(h)$. By means of a MacLaurin series expansion we obtain:

$$
(\tau_{2n} + 1)^{-1/(n\tau_{2n})} = e^{-1/n} + \frac{\tau_{2n}}{2n}e^{-1/n} + O\left(h^3\right) = 1 - \frac{1}{n} + O\left(h^2\right),
$$

so the difference equation can be written:

$$
\begin{aligned}
p_{n-1} &= c_n + (1 - 1/n)(p_n - c_n) + O\left(h^2\right) \\
\frac{p_n - p_{n-1}}{h} &= (p_n - c_n)/(nh) + O(h).
\end{aligned}
$$

This gives a discrete estimate of the slope of the continuous solution if we replace values in the discrete system are replaced with samples of the continuous solution

$P(nh)$. To calculate the local truncation error, υ^n, subtract this discrete estimate of the slope from the slope of the continuous solution which is given by (12), so

$$\upsilon^n = \frac{P(nh) - C'(nh)}{nh} - (p_n - c_n)/(nh) + O(h), \tag{63}$$

Hence, it follows from our assumptions that $\lim_{h\to 0} \upsilon^n = 0$. ■

Proof. (Proposition 4) Lemma 4 states that the discrete difference equation is a consistent approximation of the continuous differential equation. To show that the discrete solution converges to the continuous solution, it is necessary to prove that the stepped solution exists and is numerically stable, i.e., the error grows at a finite rate over the finite interval $[a, b]$, where $a = \tilde{C}'(0)$ and $b = \overline{p}$. The proof is inspired by LeVeque's (2007) convergence proof for general one-step methods. It follows from the difference equation in Lemma 4 that

$$\frac{dp_{n-1}(1)}{dp_n(1)} = (\tau_{2n} + 1)^{-1/(n\tau_{2n})} = \frac{1}{(\tau_{2n} + 1)^{1/(n\tau_{2n})}} \in [0, 1],$$

because $\tau_{2n} + 1 \geq 1$ when $\tau_{2n} \geq 0$ and $\tau_{2n} + 1 \leq 1$ when $\tau_{2n} \leq 0$. Thus we can introduce a Lipschitz constant $\lambda = 1$ (LeVeque, 2007) that uniformly bounds $\left|\frac{dp_{n-1}(1)}{dp_n(1)}\right|$ for each h. Define the global error at the quantity nh by $E^n = p_n(1) - P(nh)$. It follows that the global error satisfies the following inequality at the unit $N-1$:

$$\left|E^{N-1}\right| = |p_{N-1}(1) - P(nh)| \leq \lambda \left|E^N\right| + h\left|\upsilon^N\right|, \tag{64}$$

where υ^N is the local truncation error as defined by (63). Similarly

$$|E^n| = |p_n(1) - P(nh)| \leq \lambda \left|E^{n+1}\right| + h\left|\upsilon^{n+1}\right|. \tag{65}$$

Let $\upsilon_{\max} \geq |\upsilon^n|$ for $n \in \{1, \ldots, N\}$. From the inequality in (65) and $\lambda = 1$, we can show by induction:

$$\begin{aligned} |E^n| &\leq \left|E^N\right| + \sum_{m=n+1}^{N} h\,|\upsilon^m| \\ &\leq \left|E^N\right| + Nh\upsilon_{\max} = \left|E^N\right| + \upsilon_{\max}\overline{q}. \end{aligned} \tag{66}$$

The consistency property established in Lemma 4 ensures that the truncation error $\upsilon_{\max}$ can be made arbitrarily small by decreasing h. Moreover, $p_N(1) = P(\overline{q}) = \overline{p}$, so $\left|E^N\right| = 0$. Thus from (66), $|E^n| \to 0$ when $h \to 0$, proving that the discrete solution converges to the continuous one.

Note that (48), which corresponds to Assumption 1′ in Section 4, is satisfied when h is sufficiently small. Hence, it follows from Corollary 1 and Proposition 5 that the discrete solution corresponds to a mixed-strategy NE. The uniqueness condition corresponds to Assumption 1 when $h \to 0$. ■

Appendix E: Alternative demand assumptions

Proof. (Proposition 5) This follows from the proof of Proposition 1. ■

In the following proofs, it will be helpful to parameterise the space of f values using three positive parameters $\rho_1 = f_2/f_1 = \tau_2 + 1$, $\rho_2 = f_3/f_2 = \tau_3 + 1$, and $\rho_3 = f_4/f_3 = \tau_4 + 1$. We have from Lemma 7:

$$Z_i(1, r_1, \alpha) = f_1(1 - \Theta_1(r_1)) + f_2(\Theta_1(r_1) - \Theta_2(r_1)) + f_3\Theta_2(r_1) \tag{67}$$

$$- (f_1\theta_1(r_1) + f_2\theta_2(r_1))(r_1 - c_1) \tag{68}$$

$$Z_i(2, r_2, \alpha) = 2f_2(1 - \Theta_1(r_2)) + 2f_3(\Theta_1(r_2) - \Theta_2(r_2)) + 2f_4\Theta_2(r_2)$$

$$- (f_2\theta_1(r_2) + f_3\theta_2(r_2))(r_2 - c_2) \tag{69}$$

Lemma 14 *The equilibrium must be symmetric in a duopoly market where costs are common knowledge and each supplier has two units.*

Proof. The first step is to show that both firms make offers in the same range for each unit. It follows from Lemma 8 that both firms have their highest offer of the second unit at the price cap. A similar proof can be used to show that both firms make the lowest offer of the first unit at the same price p_0. If firms have different highest offer for the first unit, then we let firm i be the firm with the highest such offer. We denote the highest offers from the first unit by $p_H^{(1)}$ and $p_H^{(2)}$ with $p_H^{(i)} > p_H^{(j)}$. We use a similar notation for the lowest offer from the second unit, $p_L^{(i)}$ and $p_L^{(j)}$. First-order conditions and initial conditions are symmetric for $p \in \left(p_H^{(i)}, \bar{p}\right)$, so strategies must also be symmetric in that range. One implication of this is that $p_L^{(i)} < p_H^{(i)}$ (i.e. supplier i has overlapping offer ranges) is only possible if also $p_L^{(j)} < p_H^{(i)}$. Thus, in case offer ranges of firm i are overlapping, then there will be a range of prices $p \in \left(p^*, p_H^{(i)}\right)$ with $p^* \geq \max\left(p_L^{(i)}, p_H^{(j)}\right)$, where firm i has offers from its first and second units while $\theta_1(p) = 0$ and $\Theta_1(p) = 1$ for supplier j, so it follows from (67) that

$$Z_i(1, p, \alpha) = f_2(1 - \Theta_2(p)) + f_3\Theta_2(p) - f_2\theta_2(p)(p - c_1) = 0 \tag{70}$$

$$Z_i(2, p, \alpha) = 2f_3(1 - \Theta_2(p)) + 2f_4\Theta_2(p) - f_3\theta_2(p)(p - c_2) = 0. \tag{71}$$

or equivalently

$$(1 - \Theta_2(p)) + \rho_2\Theta_2(p) - \theta_2(p)(p - c_1) = 0$$

$$2(1 - \Theta_2(p)) + 2\rho_3\Theta_2(p) - \theta_2(p)(p - c_2) = 0.$$

The conditions above are identities for a range of prices. Differentiation of the two conditions yield:

$$\frac{(\rho_2 - 2)\,\theta_2(p)}{p - c_1} = \theta_2'(p)$$

$$\frac{(2\rho_3 - 3)\,\theta_2(p)}{p - c_2} = \theta_2'(p),$$

so we need $\frac{\rho_2-2}{p-c_1} = \frac{2\rho_3-3}{p-c_2}$, but this equality cannot be maintained for a range of prices when $c_2 > c_1$. Thus, we must have $p_L^{(i)} = p_H^{(i)}$ if $p_H^{(i)} > p_H^{(j)}$. Strategies are symmetric in the range $p \in \left(p_H^{(i)}, \bar{p}\right)$, so we must have $p_L^{(j)} = p_L^{(i)} = p_H^{(i)} > p_H^{(j)}$. However, such a gap would violate Lemma 8. Thus overlapping offer ranges can only occur when $p_H^{(i)} = p_H^{(j)}$. We can use a similar argument to prove that we also need $p_L^{(i)} = p_L^{(j)}$. Thus, in case of overlapping offers, firms must have identical offer ranges for each unit and they have symmetric conditions in each price range. Hence, we can use a similar argument as in the proof of Lemma 12 to show that the equilibrium must be symmetric, irrespective of overlap. ■

Proof. (Lemma 5) We consider an equilibrium solution in which offers are strictly monotonic for a given signal, so $\Theta_1(p) > \Theta_2(p)$ (corresponding to lower bids for the first unit) and $Z_i(1, r_1, \alpha) = Z_i(2, r_2, \alpha) = 0$. We will assume that there is an overlap, so that $\overline{p} > p_H > p_L > p_0$.

Consider the possibility that the highest price offer for unit 1 at p_H is increased. At equilibrium this cannot improve profit. So the right limit $Z_i(1, p_H^+, 1) \leq 0$. There is a possible discontinuity in Z_i at p_H in the case that there is a discontinuity of θ_1 or θ_2 at p_H. Looking at limits on the right hand side from $Z_i(2, p_H^+, \alpha) = 0$ we have

$$\begin{aligned} f_2(1-\Theta_2(p_H)) + f_3\Theta_2(p_H) &\leq f_2\theta_2(p_H^+)(p_H - c_1) \\ 2f_3(1-\Theta_2(p_H)) + 2f_4\Theta_2(p_H) &= f_3\theta_2(p_H^+)(p_H - c_2). \end{aligned}$$

On the left hand side we have

$$\begin{aligned} f_2(1-\Theta_2(p_H)) + f_3\Theta_2(p_H) &= (f_1\theta_1(p_H^-) + f_2\theta_2(p_H^-))(p_H - c_1), \\ 2f_3(1-\Theta_2(p_H)) + 2f_4\Theta_2(p_H) &= (f_2\theta_1(p_H^-) + f_3\theta_2(p_H^-))(p_H - c_2). \end{aligned}$$

Hence

$$\begin{aligned} f_2\theta_2(p_H^+) &\geq f_1\theta_1(p_H^-) + f_2\theta_2(p_H^-) \\ f_3\theta_2(p_H^+) &= f_2\theta_1(p_H^-) + f_3\theta_2(p_H^-). \end{aligned}$$

Thus either (1) $\theta_1(p_H^-) = 0$ and $\theta_2(p_H^+) = \theta_2(p_H^-)$ or (2) $(f_2/f_1) \geq (f_3/f_2)$ (equivalently $\rho_1 \geq \rho_2$) when it is possible that there is a discontinuity $\theta_2(p_H^+) > \theta_2(p_H^-)$.

We start by showing $\rho_1 \geq \rho_2$. We suppose that $\rho_1 < \rho_2$ and derive a contradiction. Thus we have case (1) and the equations at p_H^- become

$$\begin{aligned} f_2(1-\Theta_2(p_H)) + f_3\Theta_2(p_H) - f_2\theta_2(p_H)(p_H - c_1) &= 0, \\ 2f_3(1-\Theta_2(p_H)) + 2f_4\Theta_2(p_H) - f_3\theta_2(p_H)(p_H - c_2) &= 0. \end{aligned} \tag{72}$$

Eliminating $\theta_2(p_H)$ we obtain

$$(1 + (\rho_2 - 1)\Theta_2(p_H))(p_H - c_2) = (p_H - c_1)(2 + 2(\rho_3 - 1)\Theta_2(p_H)). \tag{73}$$

This determines an equation for p_H if the function Θ_2 is known But the value of p_H can also be found by solving the differential equation defining Θ_2 starting at $\overline{p}$ and setting p_H to be the p value where (72) holds, i.e.

$$(1-\rho_2)\Theta_2(p) + (p-c_1)\theta_2(p) - 1 = 0. \tag{74}$$

The differential equation for θ_2 is

$$2 + 2(\rho_3 - 1)\Theta_2(p) - (p - c_2)\theta_2(p) = 0$$

which has solution

$$\Theta_2(p) = \frac{1}{1-\rho_3} + K_0(p-c_2)^{2(\rho_3-1)},$$

for constant K_0. Using the boundary condition $\Theta_2(\overline{p}) = 1$ shows that

$$\begin{aligned} \Theta_2(p) &= \frac{1}{1-\rho_3} - \frac{\rho_3}{1-\rho_3}\left(\frac{p-c_2}{\overline{p}-c_2}\right)^{2\rho_3-2}, & (75)\\ \theta_2(p) &= \frac{2\rho_3}{(\overline{p}-c_2)}\left(\frac{p-c_2}{\overline{p}-c_2}\right)^{2\rho_3-3}. & (76) \end{aligned}$$

The assumption made in defining p_H from (74) is that in moving down from $\overline{p}$, $\Theta_2(p)$ does not become zero before p_H is reached (if this happened we would instead have a step separated solution). Hence there is an additional condition required. The p value at which Θ_2 becomes zero in the step separated case is, from (75),

$$\widetilde{p}_L = c_2 + (\overline{p} - c_2)(\rho_3)^{\frac{1}{2-2\rho_3}}.$$

At this point from (76) we have

$$\theta_2(\widetilde{p}_L) = \frac{2}{(\overline{p}-c_2)}(\rho_3)^{\frac{1}{2\rho_3-2}}.$$

To see whether there is a solution for p_H we can see whether there is a change of sign in the left hand side of (74) between $\overline{p}$ and $\widetilde{p}_L$. At $\widetilde{p}_L$ the left hand side takes the value

$$\begin{aligned} &(c_2 + (\overline{p}-c_2)(\rho_3)^{\frac{1}{2-2\rho_3}} - c_1)\frac{2}{(\overline{p}-c_2)}(\rho_3)^{\frac{1}{2\rho_3-2}} - 1 \\ =\ &\frac{2(c_2-c_1)}{(\overline{p}-c_2)}(\rho_3)^{\frac{1}{2\rho_3-2}} + 1 > 0. \end{aligned}$$

At $\overline{p}$ the left hand side of (74) is

$$-\rho_2 + (\overline{p}-c_1)(\frac{2\rho_3}{(\overline{p}-c_2)}),$$

which is negative as is required for this case (1) if

$$\rho_2(\overline{p}-c_2) > 2\rho_3(\overline{p}-c_1).$$

As $c_2 > c_1$ this implies $\rho_2 > 2\rho_3$.

Since $\theta_1(p_H^-) = 0$ and θ_1 must remain positive we must have $\theta_1'(p_H^-) < 0$. In the overlap region we have

$$(f_1\theta_1(p) + f_2\theta_2(p))(p - c_1) = f_1 + (f_2 - f_1)\Theta_1(p) + (f_3 - f_2)\Theta_2(p),$$
$$(f_2\theta_1(p) + f_3\theta_2(p))(p - c_2) = 2f_2 + 2(f_3 - f_2)\Theta_1(p) + 2(f_4 - f_3)\Theta_2(p),$$

which can be written

$$(\theta_1(p) + \rho_1\theta_2(p))(p - c_1) = 1 + (\rho_1 - 1)\Theta_1(p) + \rho_1(\rho_2 - 1)\Theta_2(p) \tag{77}$$
$$(\theta_1(p) + \rho_2\theta_2(p))(p - c_2) = 2 + 2(\rho_2 - 1)\Theta_1(p) + 2\rho_2(\rho_3 - 1)\Theta_2(p). \tag{78}$$

multiplying the first equation by $(p - c_2)\rho_2$ and the second by $(p - c_1)\rho_1$ allows us to eliminate $\theta_2(p)$ giving

$$\begin{aligned}(p - c_1)(p - c_2)(\rho_2 - \rho_1)\theta_1(p) \quad = \quad & (1 + (\rho_1 - 1)\Theta_1(p) + \rho_1(\rho_2 - 1)\Theta_2(p))\,(p - c_2)\rho_2 \\ & - (2 + 2(\rho_2 - 1)\Theta_1(p) + 2\rho_2(\rho_3 - 1)\Theta_2(p))\,(p - c_1)\rho_1.\end{aligned}$$

Hence

$$\begin{aligned}&(p - c_1)(p - c_2)(\rho_2 - \rho_1)\theta_1(p) = \\ &(p - c_2)\rho_2 - 2(p - c_1)\rho_1 + ((\rho_1 - 1)(p - c_2)\rho_2 - 2(\rho_2 - 1)(p - c_1)\rho_1)\,\Theta_1(p) \\ &+ (\rho_1(\rho_2 - 1)(p - c_2)\rho_2 - 2\rho_2(\rho_3 - 1)(p - c_1)\rho_1)\,\Theta_2(p).\end{aligned}$$

Taking derivatives we have

$$\begin{aligned}& (p - c_1)(p - c_2)(\rho_2 - \rho_1)\theta_1'(p) + (2p - c_2 - c_1 + c_1c_2)(\rho_2 - \rho_1)\theta_1(p) \\ = \; & (\rho_1(\rho_2 - 1)(p - c_2)\rho_2 - 2\rho_2(\rho_3 - 1)(p - c_1)\rho_1)\,\theta_2(p) \\ & + (\rho_1(\rho_2 - 1)\rho_2 - 2\rho_2(\rho_3 - 1)\rho_1)\,\Theta_2(p) - 2\rho_1 + \rho_2 \\ & + ((\rho_1 - 1)(p - c_2)\rho_2 - 2(\rho_2 - 1)(p - c_1)\rho_1)\,\theta_1(p) \\ & + ((\rho_1 - 1)\rho_2 - 2(\rho_2 - 1)\rho_1)\,\Theta_1(p).\end{aligned}$$

Now consider the values at p_H when $\Theta_1(p_H) = 1$ and $\theta_1(p_H) = 0$. This equation then reads

$$\begin{aligned}&2\rho_1 - \rho_2 - (\rho_1 - 1)\rho_2 - 2(\rho_2 - 1)\rho_1 + (p_H - c_1)(p_H - c_2)(\rho_2 - \rho_1)\theta_1'(p_H) = \\ &(\rho_1(\rho_2 - 1)(p_H - c_2)\rho_2 - 2\rho_2(\rho_3 - 1)(p_H - c_1)\rho_1)\,\theta_2(p_H) \\ &+ (\rho_1(\rho_2 - 1)\rho_2 - 2\rho_2(\rho_3 - 1)\rho_1)\,\Theta_2(p_H).\end{aligned}$$

Hence

$$\theta_1'(p) = \frac{\rho_1\rho_2}{(\rho_2 - \rho_1)(p_H - c_1)(p_H - c_2)}\left(\begin{array}{c}((\rho_2 - 1)(p_H - c_2) - 2(\rho_3 - 1)(p_H - c_1))\,\theta_2(p_H) \\ + (\rho_2 - 2\rho_3 + 1)\,\Theta_2(p_H) - 1\end{array}\right).$$

We need $\theta_1'(p) < 0$ and since we are assuming $\rho_1 < \rho_2$ we need

$$((\rho_2 - 1)(p_H - c_2) - 2(\rho_3 - 1)(p_H - c_1))\,\theta_2(p_H) + (\rho_2 - 2\rho_3 + 1)\,\Theta_2(p_H) < 1.$$

Moreover from the $Z = 0$ conditions

$$\begin{aligned} 1 + (\rho_2 - 1)\Theta_2(p_H) &= \theta_2(p_H)(p_H - c_1), \\ 2 + 2(\rho_3 - 1)\Theta_2(p_H) &= \theta_2(p_H)(p_H - c_2). \end{aligned}$$

So the condition becomes

$$2(\rho_2-1)+2(\rho_2-1)(\rho_3-1)\Theta_2(p_H)-2(\rho_3-1)-2(\rho_3-1)(\rho_2-1)\Theta_2(p_H)+(\rho_2 - 2\rho_3 + 1)\,\Theta_2(p_H) < 1,$$

which simplifies to

$$(\rho_2 - 2\rho_3 + 1)\,\Theta_2(p_H) < 2\rho_3 - 2\rho_2 + 1. \tag{79}$$

From this we deduce that $2\rho_3 - 2\rho_2 + 1 > 0$. So

$$\rho_2 < 2 - (\rho_2 - 2\rho_3 + 1) < 2.$$

From our previous condition (73) on p_H we know that

$$((p_H - c_2)(\rho_2 - 1) - 2(p_H - c_1)(\rho_3 - 1))\,\Theta_2(p_H) = p_H - 2c_1 + c_2.$$

As $p_H - 2c_1 + c_2 > 0$ we can deduce that $((p_H - c_2)(\rho_2 - 1) - 2(p_H - c_1)(\rho_3 - 1)) > 0$. Thus we can multiply both sides of (79) by this quantity and get

$$\begin{aligned} & (\rho_2 - 2\rho_3 + 1)\,((p_H - c_2)(\rho_2 - 1) - 2(p_H - c_1)(\rho_3 - 1))\,\Theta_2(p_H) \\ < \; & (2\rho_3 - 2\rho_2 + 1)\,((p_H - c_2)(\rho_2 - 1) - 2(p_H - c_1)(\rho_3 - 1))\,. \end{aligned}$$

Hence

$$(2\rho_3 - 2\rho_2 + 1)\,((p_H - c_2)(\rho_2 - 1) - 2(p_H - c_1)(\rho_3 - 1)) - (\rho_2 - 2\rho_3 + 1)\,(p_H - 2c_1 + c_2) > 0,$$

i.e.

$$2\,(\rho_2 - \rho_3)\,((2 - \rho_2)(c_1 - c_2) - (p_H - c_1)(\rho_2 - 2\rho_3 + 1)) > 0.$$

But now making use of a number of inequalities that we have derived we observe that $\rho_2 > \rho_3$, $2 - \rho_2 > 0$, $c_1 - c_2 < 0$ and $(p_H - c_1)(\rho_2 - 2\rho_3 + 1) > 0$. This gives a contradiction from our initial assumption that $\rho_1 < \rho_2$.

In the next step we rule out overlap when $\rho_1 = \rho_2$. Assume that this is the case, then subtracting (77) and (78) and evaluating the resulting expression at p_L^+ yields:

$$0 < (\theta_1(p) + \rho_1\theta_2(p))(c_2 - c_1) = -1 - (\rho_1 - 1)\Theta_1(p) < 0, \tag{80}$$

because $\Theta_2(p_L) = 0$. This is an obvious contradiction, so we must have $\rho_1 > \rho_2$.

Now we consider the other end of the overlap region, and make a similar argument at p_L. For an optimal choice we have, looking at derivatives to the left,

$$\begin{aligned} f_1(1 - \Theta_1(p_L)) + f_2\Theta_1(p_L) - f_1\theta_1(p_L^-)(p_L - c_1) = 0, \\ 2f_2(1 - \Theta_1(p_L)) + 2f_3\Theta_1(p_L) - f_2\theta_1(p_L^-)(p_L - c_2) \geq 0, \end{aligned} \tag{81}$$

and looking at derivatives to the right,

$$f_1(1-\Theta_1(p_L)) + f_2\Theta_1(p_L) - (f_1\theta_1(p_L^+) + f_2\theta_2(p_L^+))(p_L - c_1) = 0,$$
$$2f_2(1-\Theta_1(p_L)) + 2f_3\Theta_1(p_L) - (f_2\theta_1(p_L^+) + f_3\theta_2(p_L^+))(p_L - c_2) = 0.$$

Thus

$$\begin{aligned} f_1\theta_1(p_L^+) + f_2\theta_2(p_L^+) &= f_1\theta_1(p_L^-) \\ f_2\theta_1(p_L^+) + f_3\theta_2(p_L^+) &\geq f_2\theta_1(p_L^-). \end{aligned}$$

Now as we have shown $\rho_1 > \rho_2$ we can deduce that $\theta_2(p_L^+) = 0$, and hence there is no discontinuity in θ_1 at p_L. Thus

$$\begin{aligned} f_1(1-\Theta_1(p_L)) + f_2\Theta_1(p_L) - f_1\theta_1(p_L)(p_L - c_1) = 0, \\ 2f_2(1-\Theta_1(p_L)) + 2f_3\Theta_1(p_L) - f_2\theta_1(p_L)(p_L - c_2) = 0. \end{aligned} \tag{82}$$

Eliminating $\theta_1(p_L)$ we obtain

$$((1-\Theta_1(p_L)) + \rho_1\Theta_1(p_L))\,(p_L - c_2) = (p_L - c_1)\,(2(1-\Theta_1(p_L)) + 2\rho_2\Theta_1(p_L))\,.$$

This determines an equation for p_L if the function Θ_1 is known. The differential equation for the range (p_0, p_L) is

$$1 + (\rho_1 - 1)\Theta_1(p) - \theta_1(p)(p - c_1) = 0.$$

This has solution

$$\Theta_1(p) = \frac{1}{1-\rho_1} + K_1(p - c_1)^{(\rho_1 - 1)},$$

for constant K_1. At p_0 we have boundary condition $\Theta_1(p_0) = 0$. Hence

$$\Theta_1(p) = \frac{1}{1-\rho_1}\left(1 - \left(\frac{p - c_1}{p_0 - c_1}\right)^{(\rho_1 - 1)}\right), \tag{83}$$

$$\theta_1(p) = \frac{1}{p_0 - c_1}\left(\frac{p - c_1}{p_0 - c_1}\right)^{(\rho_1 - 2)}. \tag{84}$$

The p value at which Θ_1 becomes 1 if this differential equation describes the whole set of prices at which unit 1 is offered (the step separated case) is

$$\widetilde{p}_H = c_1 + (p_0 - c_1)\,\rho_1^{\frac{1}{(\rho_1 - 1)}}.$$

To see whether there is a solution for p_L we can see whether there is a change of sign in the left hand side of

$$2 + 2(\rho_2 - 1)\Theta_1(p) - \theta_1(p)(p - c_2) = 0$$

between p_0 and $\widetilde{p}_H$. At p_0 we have

$$\theta_1(p_0) = \frac{1}{p_0 - c_1}$$

and so the left hand side is

$$2 - \frac{(p_0 - c_2)}{p_0 - c_1} > 0.$$

At $\widetilde{p}_H$ the left hand side takes the value

$$\begin{aligned} & 2 + 2\frac{\rho_2 - 1}{1-\rho_1}\left(1 - \left(\frac{\widetilde{p}_H - c_1}{p_0 - c_1}\right)^{(\rho_1-1)}\right) - \left(\frac{1}{p_0-c_1}\left(\frac{\widetilde{p}_H - c_1}{p_0 - c_1}\right)^{(\rho_1-2)}\right)(\widetilde{p}_H - c_2) \\ = \ & \frac{2(\rho_2-\rho_1)}{1-\rho_1} - \left(\frac{2(\rho_2-1)}{1-\rho_1}\frac{\widetilde{p}_H - c_1}{p_0 - c_1} + \frac{\widetilde{p}_H - c_2}{p_0 - c_1}\right)\left(\frac{\widetilde{p}_H - c_1}{p_0 - c_1}\right)^{(\rho_1-2)} \\ = \ & \frac{2(\rho_2-\rho_1)}{1-\rho_1} - \frac{1}{(1-\rho_1)(p_0-c_1)}\left((2\rho_2 - 1 - \rho_1)(p_0 - c_1)\rho_1^{\frac{1}{(\rho_1-1)}} + (1-\rho_1)(c_1 - c_2)\right)\rho_1^{\frac{(\rho_1-2)}{(\rho_1-1)}} \\ = \ & (2\rho_2 - \rho_1) - \frac{(c_1-c_2)}{(p_0-c_1)}\rho_1^{\frac{(\rho_1-2)}{(\rho_1-1)}}. \end{aligned}$$

So the condition we require for a change of sign is

$$(\rho_1 - 2\rho_2)(p_0 - c_1) > (c_2 - c_1)\rho_1^{\frac{(\rho_1-2)}{(\rho_1-1)}}.$$

This condition, when rephrased in terms of τ_2 and τ_3, is exactly the inequality in the Lemma statement. ■

Proof. (Lemma 6) Assumption 1′ implies that $2\tau_3 \geq \tau_2 - 1$, so it follows from Lemma 5 that NE with overlap can be ruled out. Moreover, we know from Lemma 14 that the equilibrium must be symmetric. We have from Corollary 1 that an equilibrium exists if Assumption 1' is satisfied. This is the only NE with step separation that is symmetric, so it must be the unique NE. ■

Appendix F: Multiple suppliers

Consider a multi-unit auction with I suppliers. Let $\varphi_{-i}(p,m)$ be the probability that the $K = I - 1$ competitors of supplier i together offer at least m units at price p. To simplify our equations, we set $\varphi_{-i}(p,0) = 1$ and $\varphi_{-i}(p,N+1) = 0$. The function $\varphi_{-i}(p,m)$ is a generalised version of $\widehat{\alpha}_j(p,m)$. In the duopoly case, we would have $\varphi_{-i}(p,m) = \widehat{\alpha}_j(p,m)$, where $j \neq i$. Analogous to $\Delta\widehat{\alpha}_j$, we also introduce

$$\Delta\varphi_{-i}(p,m) = \varphi_{-i}(p,m) - \varphi_{-i}(p,m+1),$$

the probability that competitors of supplier i together offer exactly m units at price p.

Lemma 15 *In an oligopoly market*

$$\begin{aligned} Z_i(n,p,\alpha_i) = nh \sum_{m=0}^{N} \Delta\varphi_{-i}(p,m)\, f((n+m)h) \\ -h(p - c_n(\alpha_i)) \sum_{m=0}^{N} \frac{\partial \Delta\varphi_{-i}(p,m)}{\partial p} F((n+m-1)h), \end{aligned} \tag{85}$$

where $Z_i(n,p,\alpha_i)$ is defined.

Proof. (**Lemma 15**) An offer of n units at price p by supplier i is rejected if the competitors together offer exactly m units at the price p when demand is at most $n+m-1$ units. Thus

$$\Psi_i(n,p) = \sum_{m=0}^{N} \Delta\varphi_{-i}(p,m) F((n+m-1)h). \tag{86}$$

(85) now follows from Definition 1, Lemma 1 and (86). ∎

Consider an outcome with price p, where the K competitors of supplier i submit step-separated offers that are symmetric ex-ante (before private signals have been observed) and together sell m units. Let

$$\widehat{m} = \left\lceil \frac{m}{K} \right\rceil.$$

Thus $\widehat{m}$ is the smallest integer not smaller than $\frac{m}{K}$. For the considered outcome, each competitor will sell either $\widehat{m}$ units or $\widehat{m}-1$ units. The $m-K(\widehat{m}-1)$ competitors with the smallest signals will sell $\widehat{m}$ units and the remaining $K\widehat{m}-m$ competitors will sell $\widehat{m}-1$.

Lemma 16 *Consider step-separated offers that are symmetric ex-ante (before private signals have been observed) from K competitors, then $\Delta\varphi_{-i}(p,m)$ can be determined from a binomial distribution.*

$$\begin{aligned}&\Delta\varphi_{-i}(p,m)\\ =& \binom{K}{m-K(\widehat{m}-1)} (\widehat{\alpha}(p,\widehat{m}))^{m-K(\widehat{m}-1)} (1-\widehat{\alpha}(p,\widehat{m}))^{K\widehat{m}-m}\end{aligned} \tag{87}$$

if the price is such that $\widehat{m} = s_j(p,0)$ for $j \neq i$. Otherwise

$$\Delta\varphi_{-i}(p,m) = 0. \tag{88}$$

Proof. $\Delta\varphi_{-i}(p,m)$ is the probability that competitors together sell exactly m units at price p. Competitors' offers are symmetric ex-ante and have step separation. If the price is such that $\widehat{m} = s_j(p,0)$, then each supplier sells either $\widehat{m}$ units (with probability $\widehat{\alpha}(\widehat{m},p)$) or $\widehat{m}-1$ units at price p (with probability $1-\widehat{\alpha}(\widehat{m},p)$). This immediately gives (88). It also follows that $m-K(\widehat{m}-1)$ suppliers are selling exactly $\widehat{m}$ units and the other suppliers are selling exactly $\widehat{m}-1$ units. There are $\binom{K}{m-K(\widehat{m}-1)}$ such outcomes each occurring with a probability $(\widehat{\alpha}(\widehat{m},p))^{m-K(\widehat{m}-1)}(1-\widehat{\alpha}(p,\widehat{m}))^{K\widehat{m}-m}$, which gives (87). If $\widehat{m} > s_j(p,0)$, then competitors will sell at most $Ks_j(p,0) \leq K(\widehat{m}-1) < m$. Similarly, if $\widehat{m} < s_j(p,0)$, then competitors will sell at least $Ks_j(p,0) > K\widehat{m} \geq m$. Thus the probability is zero that competitors will together sell m units for such prices, which gives (88). ∎

Note that the binomial coefficient in Lemma 16 is defined as follows:

$$\left(\begin{array}{c} K \\ m - K(\widehat{m} - 1) \end{array} \right) = \frac{K!}{(m - K(\widehat{m} - 1))!\,(K\widehat{m} - m)!}. \tag{89}$$

In order to simplify our expressions, we introduce the following notation:

$$w\left(u, t\right) = \left(\begin{array}{c} K \\ t \end{array} \right) u^{t} \left(1 - u\right)^{K - t} \tag{90}$$

for the probability of t out of K to be chosen where each component has a probability u of being chosen. It follows from Lemma 15 and Lemma 16 that:

Corollary 2 *Consider a multi-unit auction where supplier i has $K = I - 1$ competitors that submit step-separated offers that are symmetric ex-ante (before private signals have been observed), then*

$$\begin{array}{l} Z_i\left(n, p, \alpha_i\right) = nh \sum\limits_{m=K(\widehat{m}-1)}^{K\widehat{m}} w\left(\widehat{\alpha}\left(p, \widehat{m}\right), m - K(\widehat{m} - 1)\right) f\left(\left(n + m\right) h\right) \\ -h\left(p - c_n(\alpha_i)\right) \sum\limits_{m=K(\widehat{m}-1)}^{K\widehat{m}} \frac{\partial w\left(\widehat{\alpha}\left(p, \widehat{m}\right), m - K(\widehat{m} - 1)\right)}{\partial p} F\left(\left(n + m - 1\right) h\right) \end{array}$$

if the price is such that $\widehat{m} = s_j\left(p, 0\right)$ for $j \neq i$.

Proof. (Proposition 7) In an equilibrium that is symmetric ex-ante, we have $n = \widehat{m}$. Hence, Corollary 2 gives the following symmetric first-order condition:

$$\begin{array}{rl} & n \sum\limits_{m=K(n-1)}^{Kn} w\left(\widehat{\alpha}, m - K(n - 1)\right) f\left(\left(n + m\right) h\right) \\ = & \left(p - c_n\right) \dfrac{\partial}{\partial p} \left(\sum\limits_{m=K(n-1)}^{Kn} w\left(\widehat{\alpha}, m - K(n - 1)\right) F\left(\left(n + m - 1\right) h\right) \right). \end{array}$$

So after differentiation

$$\begin{array}{rl} & \dfrac{n}{p - c_n} \sum\limits_{m=K(n-1)}^{Kn} w\left(\widehat{\alpha}, m - K(n - 1)\right) f\left(\left(n + m\right) h\right) \\ = & \dfrac{\partial \widehat{\alpha}}{\partial p} \sum\limits_{m=K(n-1)+1}^{Kn} \dfrac{w\left(\widehat{\alpha}, m - K(n - 1)\right)}{\widehat{\alpha}} \left(m - K(n - 1)\right) F\left(\left(n + m - 1\right) h\right) \\ & -\dfrac{\partial \widehat{\alpha}}{\partial p} \sum\limits_{m=K(n-1)}^{Kn-1} \dfrac{w\left(\widehat{\alpha}, m - K(n - 1)\right)}{1 - \widehat{\alpha}} \left(Kn - m\right) F\left(\left(n + m - 1\right) h\right). \end{array}$$

The differential equation above can be written in the following form:

$$\frac{n}{p - c_n} = g\left(\alpha\right) \frac{d\alpha}{dp}, \tag{91}$$

where

$$g(u) = \frac{\sum_{m=K(n-1)+1}^{Kn} w(u, m-K(n-1))(m-K(n-1)) F((n+m-1)h)}{u \sum_{m=K(n-1)}^{Kn} w(u, m-K(n-1)) f((n+m)h)} - \frac{\sum_{m=K(n-1)}^{Kn-1} w(u, m-K(n-1))(Kn-m) F((n+m-1)h)}{(1-u) \sum_{m=K(n-1)}^{Kn} w(u, m-K(n-1)) f((n+m)h)}.$$

We can simplify this to:

$$g(u) = \frac{\sum_{v=0}^{K-1} w(u, v+1)(v+1) F((n+v+K(n-1))h)}{u \sum_{v=0}^{K-1} w(u, v) f((n+v+K(n-1))h)} - \frac{\sum_{v=0}^{K-1} w(u, v)(K-v) F((n+v+K(n-1)-1)h)}{(1-u) \sum_{v=0}^{K-1} w(u, v) f((n+v+K(n-1))h)}.$$

Next, we use (89) and (90) to simplify this further.

$$g(u) = \frac{\sum_{v=0}^{K-1} \frac{K!}{v!(K-v-1)!} u^v (1-u)^{K-1-v} F((n+v+K(n-1))h)}{\sum_{v=0}^{K-1} w(u, v) f((n+v+K(n-1))h)} - \frac{\sum_{v=0}^{K-1} \frac{K!}{v!(K-v-1)!} u^v (1-u)^{K-1-v} F((n+v+K(n-1)-1)h)}{\sum_{v=0}^{K-1} w(u, v) f((n+v+K(n-1))h)},$$

which can be simplified to (20). The differential equation in (91) can be separated as follows:

$$\frac{ndp}{p-c_n} = g(u)\, du.$$

We integrate the left-hand side from $p_n(\alpha)$ to $p_n(1)$ and the right-hand side from α to 1. Hence,

$$n \ln\left(\frac{p_n(1)-c_n}{p_n(\alpha)-c_n}\right) = \int_{\alpha}^{1} g(u)\, du,$$

which gives (19). Using Corollary 2 above, it can be verified that this is an equilibrium with a similar argument as in the proof of Proposition 1. The approximation in (21) follows from a Taylor expansion of (19). ■

Proof. (Proposition 8) In case demand is uniformly distributed, we have from Proposition 7 and the binomial theorem that

$$\begin{aligned} g(u) &= \frac{K \sum_{v=0}^{K-1} \frac{(K-1)!}{(K-1-v)!v!} u^v (1-u)^{K-1-v}}{\sum_{v=0}^{K} \frac{K!}{v!(K-v)!} u^v (1-u)^{K-v}} \\ &= \frac{K (u+1-u)^{K-1}}{(u+1-u)^K} = K. \end{aligned}$$

Thus (22) follows from Proposition 7. Similar to the duopoly case, uniformly distributed demand shocks ensures that if a producer sells more units and has higher marginal costs, then it has incentives to increase its offer price, so that $p_{n-1}^i(1) \leq p_n^i(0)$. As for the duopoly case in Section 3.2, one can use this property and (1) to prove that the equilibrium must be symmetric and have step separation without gaps, i.e. the equilibrium is unique.

In the next step we calculate the variance of the n'th offer. The expected mark-up of the offer is given by:

$$\int_0^1 (p_n(1) - c_n) e^{K(\alpha-1)/n} d\alpha = (p_n(1) - c_n) \frac{1 - e^{-K/n}}{K/n}$$

and the expected value of the mark-up squared is given by:

$$\int_0^1 (p_n(1) - c_n)^2 e^{2K(\alpha-1)/n} d\alpha = (p_n(1) - c_n)^2 \frac{1 - e^{-2K/n}}{2K/n}.$$

We use the above results and a Taylor series expansion, for an offer made at a fixed quantity $\gamma = nh$, to get the approximate expression of the variance in (23).

We can also estimate the variance of the market price. Consider the demand level $(n-1)(K+1) + m$, where $m \in [1, K+1]$. Thus the auctioneer buys n units from m suppliers and $n-1$ units from $K+1-m$ suppliers. The probability density of the α value that sets the market price is given by order statistics: $(K+1) \begin{pmatrix} K \\ m-1 \end{pmatrix} \alpha^{m-1} (1-\alpha)^{K-m+1}$. Hence, the expected mark-up of the marginal offer is:

$$\begin{aligned} &\int_0^1 (p_n(1) - c_n) e^{K(\alpha-1)/n} (K+1) \begin{pmatrix} K \\ m-1 \end{pmatrix} \alpha^{m-1} (1-\alpha)^{K-m+1} d\alpha \qquad (92) \\ &= (p_n(1) - c_n) \left(1 - \frac{K}{n} \frac{(K-m+2)}{(K+2)} + \frac{K^2}{2n^2} \frac{(K-m+3)(K-m+2)}{(K+3)(K+2)}\right) + O(h^3) \end{aligned}$$

where we have used the fact that $e^{K(\alpha-1)/n} = 1 + K(\alpha-1)/n + \frac{1}{2} K^2 (\alpha-1)^2 / n^2 + O(h^3)$.

Similarly, the expected value of the marginal offer's mark-up squared is:

$$\begin{aligned} &\int_0^1 (p_n(1) - c_n)^2 e^{2K(\alpha-1)/n} (K+1) \begin{pmatrix} K \\ m-1 \end{pmatrix} \alpha^{m-1} (1-\alpha)^{K-m-1} d\alpha \qquad (93) \\ &= (p_n(1) - c_n) \left(1 - \frac{2K}{n} \frac{(K-m+2)}{(K+2)} + \frac{2K^2}{n^2} \frac{(K-m+3)(K-m+2)}{(K+3)(K+2)}\right) + O(h^3). \end{aligned}$$

The variance of the marginal offer is given by:

$$(p_n(1) - c_n)^2 \left(1 - \frac{2K}{n}\frac{(K-m+2)}{(K+2)} + \frac{2K^2}{n^2}\frac{(K-m+3)(K-m+2)}{(K+3)(K+2)}\right)$$
$$-\left((p_n(1) - c_n)\left(1 - \frac{K}{n}\frac{(K-m+2)}{(K+2)} + \frac{K^2}{2n^2}\frac{(K-m+3)(K-m+2)}{(K+3)(K+2)}\right)\right)^2 + O\left(h^3\right),$$

which using a Taylor series expansion gives (24). ■

Proof. (**Proposition 9)** It follows from Proposition 8 that when demand is uniform, equilibrium offers for the highest signal can be determined from the following difference equation:

$$p_{n-1}(1) = p_n(0) = c_n + (p_n(1) - c_n) e^{-K/n}. \tag{94}$$

Similar to the proof of Lemma 4, we can use (21) to prove that this difference equation is consistent with the first-order condition of an SFE for multiple suppliers in Rudkevich et al. (1998), Anderson and Philpott (2002a) and Holmberg (2008). It follows from Proposition 8 that

$$\frac{\partial p_{n-1}(1)}{\partial p_n(1)} = e^{-K/n} \in [0,1], \tag{95}$$

which ensures that the discrete solution is stable also for multiple suppliers, small changes in $p_n(1)$ give finite changes in $p_{n-1}(1)$. Thus we can use an argument similar to the proof of Proposition 4 to show that the mixed-strategy NE in Proposition 8 converges to a pure-strategy SFE also for $I \geq 2$ suppliers.

Next, a Taylor expansion of (22) and consistency of the first-order condition imply that

$$hP'(nh) = p_n(1) - p_{n-1}(1) + O\left(h^2\right) = \frac{(p_n(1) - c_n) K}{n} + O\left(h^2\right)$$

for all $n \in \{1, ..., N\}$. The left-hand side is strictly increasing with respect to n when (25) is satisfied. Thus it follows that the offer price range, $p_n(1) - p_{n-1}(1)$, and volatility, $\frac{(p_n(1)-c_n)^2 K^2}{12n^2} + O(h^3)$, must also be strictly increasing with respect to n if h is sufficiently small. ■