

Unique supply function equilibrium with capacity constraints¹

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Abstract

Consider a market where producers submit supply functions to a procurement auction with uncertain demand, e.g. an electricity auction. In the Supply Function Equilibrium (SFE), every firm commits to the supply function that maximises expected profit in the one-shot game given the supply functions of competitors. A basic weakness of the SFE is the presence of multiple equilibria. This paper shows that with (i) symmetric producers, (ii) perfectly inelastic demand, (iii) a price cap, and (iv) capacity constraints that bind with a positive probability, there exists a unique, symmetric SFE.

Keywords: supply function equilibrium, uniform-price auction, uniqueness, oligopoly, capacity constraint, wholesale electricity market

JEL codes: C62, D43, D44, L11, L13, L94

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1. INTRODUCTION

The Supply Function Equilibrium (SFE) under uncertainty was introduced by Klemperer and Meyer (1989). The concept assumes that producers submit supply functions simultaneously to a uniform-price auction in a one-shot game. In the non-cooperative Nash Equilibrium (NE), each producer commits to the supply function that maximises expected profit given the bids of competitors and the properties of uncertain demand. The set-up of the model is similar to the organisation of most electricity auctions and the equilibrium is often used when modelling bidding behaviour in such auctions. This application was first observed by Bolle (1992) and Green and Newbery (1992). More broadly, the SFE can be applied to any uniform-price auction where bidders have common knowledge, quantity discreteness is negligible — objects are divisible (Wilson, 1979) — and the demand/supply of the auctioneer is uncertain. An example of an alternative SFE application is the treasury auction with random non-competitive bids analysed by Wang and Zender (2002). Multiplicity of equilibria is a basic weakness of SFE. This paper demonstrates that under certain conditions that are reasonable for electric power markets, especially balancing markets, a unique symmetric SFE exists if producers are symmetric.

Supply Function Equilibria are traditionally found by making the following observation: each producer submits a supply function such that for each demand outcome, the market price is optimised with respect to his residual demand. Each producer acts as a monopolist with respect to his residual demand and the optimal price of a producer is given by the inverse elasticity rule (Tirole, 2003). Hence, the mark-up percentage is inversely proportional to the elasticity of the residual demand curve for every outcome. The elasticity of residual demand is comprised of derivatives of competitors' supply functions. Thus the SFE is given by the solution to a system of differential equations. For symmetric producers with smooth supply functions and non-positive minimum demand, one can show that only symmetric equilibria exist (Klemperer and Meyer, 1989) and the system can be reduced to a single differential equation. However, there is no end-point condition so the solution includes an integration constant.

The integration constant allows for a continuum of symmetric equilibria, bounded by the Cournot and Bertrand equilibria. The continuum can intuitively be understood by means of the inverse elasticity rule. When competitors' supply functions are highly elastic, i.e. they have low mark-ups at every supply, the best response is to have a low mark-up at every

supply. When competitors' supply is inelastic, i.e. they have large mark-ups at every supply, the best response is to have a large mark-up at every supply. Multiple equilibria make it difficult to predict outcomes with SFE. Furthermore, it complicates comparative statics and comparisons of different auction designs. How can one be sure that the integration constant associated with an equilibrium does not change when the organisation of the market is changed? Thus, multiplicity of equilibria represents a considerable drawback for SFE.

As in the original model by Klemperer and Meyer (1989) a one-shot game is analysed, i.e. entry is not taken into account and forward contracting is assumed to be exogenously determined. I consider a market with symmetric producers, *perfectly inelastic demand*, a *price cap* and *capacity constraints* that bind with a positive probability. I show that under these conditions there is a unique symmetric SFE, i.e. there is only one SFE with symmetric supply functions.² The symmetric equilibrium price reaches the price cap precisely when the capacity constraints bind. Hence, it turns out that the integration constant in the solution of the differential equation is pinned down by the price cap and the total production capacity. The assumptions leading to uniqueness and existence are reasonable for electric power markets. In particular, *short-run demand is very inelastic* in the electric power market, and perfectly inelastic demand is often assumed in models of real-time and spot markets (von der Fehr and Harbord, 1993; Rudkevich et al., 1998; Anderson and Philpott, 2002; Genc and Reynolds, 2004).

Capacity constraints reduce the set of SFE in the electric power market, as has been shown in previous research (Green and Newbery, 1992; Newbery, 1998; Baldick and Hogan, 2002). Genc and Reynolds (2004) have recently shown that the range of SFE can be reduced even further by considering pivotal suppliers. Specifically, they observe that the concavity of firms' profit functions, originally proven by Klemperer and Meyer (1989), does not automatically apply to markets with capacity constraints.³ Thus some symmetric candidates that were previously thought to be SFE in markets with capacity constraints can be ruled out. The current paper goes one step further, it is shown that a positive Loss-Of-Load-Probability (LOLP) implies a unique symmetric equilibrium. A power shortage can occur in any delivery-period, either because of demand shocks — temperature shocks or that many consumers by

² Perfectly inelastic demand and symmetry simplify the analysis, but intuitively these assumptions are not critical to rule out multiple equilibria. As illustrated by Klemperer and Meyer (1989), however, with elastic demand there are not valid non-decreasing supply function equilibria for all end-point conditions. Thus there is a risk that introducing capacity constraints that bind with a positive probability leads to non-existing SFE for some parameter values.

³ As noted in Section 3.6 it seems that Klemperer and Meyer (1989) prove local concavity rather than global concavity.

coincidence switch on machines and appliances simultaneously—or unexpected failures in one or several power plants. Even if power shortages are infrequent and may occur years or even decades apart, they are not zero-probability events. The support of the probability density of demand determines the set of SFE, but otherwise SFE do not depend on how likely an outcome is (Klemperer and Meyer, 1989). For this reason, even an arbitrarily small risk of power shortage is enough to yield a unique symmetric SFE. To avoid inconsistencies in the model, it is suggested that the risk of power failures is only considered for generators who do not bid strategically in real-time.⁴ As the power rating of equipment on the consumer side is typically much lower than on the production side, demand tends to be averaged out by the law of large numbers. Thus shocks on the demand side are less likely than on the production side, but they are not zero-probability events.

Price caps are employed in most deregulated power markets and are considered in some previous models of electric power markets (von der Fehr and Harbord, 1993; Baldick and Hogan, 2002; Genc and Reynolds, 2004). One argument for price caps is that consumers who do not switch off their equipment when electricity prices become extremely high do not necessarily have a high marginal benefit of power. Instead, they may not have the option to switch off or they do not face the real-time price, e.g. household consumers with fixed retail rates. Thus at some sufficiently high price, which is often called Value of Lost Load (VOLL), social welfare is maximized by rationing demand.

With perfectly inelastic demand, the uniqueness of the symmetric equilibrium can intuitively be understood from the following reasoning (see Figure 1). When demand is sufficiently high to make the capacity constraints of competitors bind, a producer faces perfectly inelastic residual demand. If such an outcome occurs with a positive probability, the producer's optimal price for this outcome should, following the inverse elasticity rule, be as high as possible, i.e. equal to the price cap. Thus the equilibrium price must reach the price cap. Furthermore, any firm would find it profitable to unilaterally deviate from equilibrium candidates hitting the price cap before the capacity constraints bind. The reason being that it is profitable to slightly undercut competitors' horizontal supply à la Bertrand.

⁴ Two examples of such generators in Britain are British nuclear group and British Energy, both of whom exclusively produce nuclear power. Another example is that generators exporting power to NordPool are not allowed to bid in the balancing markets of the Nordic countries.

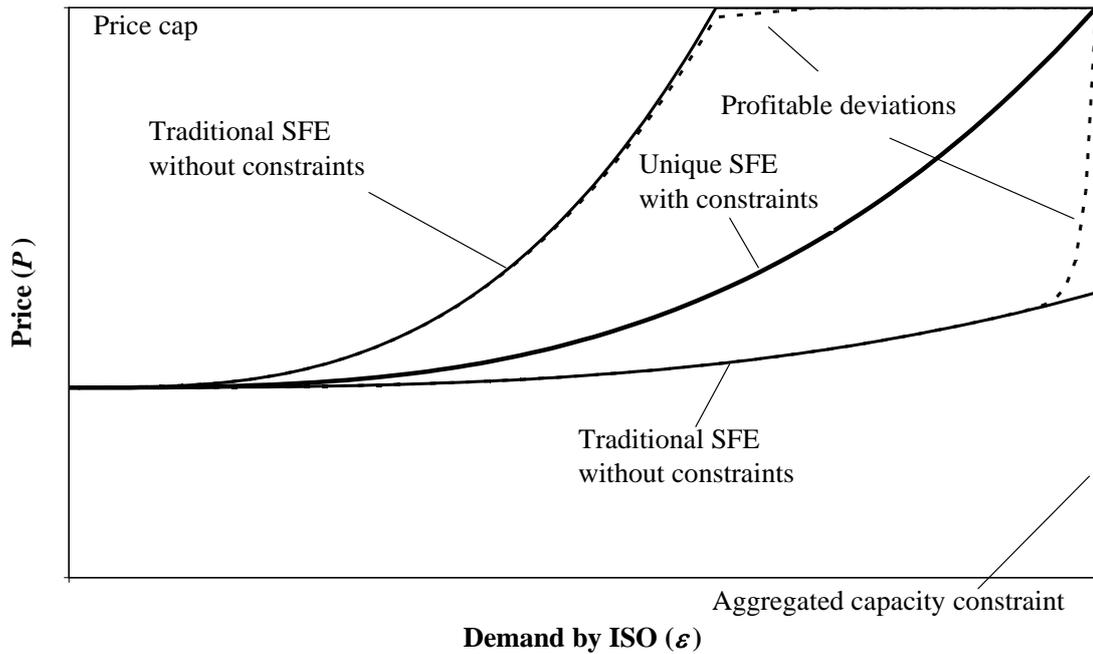


Figure 1. *Capacity constraints and a price cap rule out all but one traditional SFE.*

Many papers in the SFE literature try to single out a unique equilibrium. Klemperer and Meyer (1989) show that if outcomes with infinite demand occur with positive probability, and if an infinite demand can be met with non-binding capacity constraints — not realistic for the electric power market — then a unique SFE exists. With a price cap and capacity constraints, Baldick and Hogan (2002) single out the same equilibrium as in this paper, but provide a weaker motivation for their result. In their analysis, price caps are seen as a public signal that coordinate the bids of producers. Green and Newbery (1992) consider a model with linear demand and use the equilibrium in which firms have the highest profit; the worst case for consumers. This equilibrium is unique if maximum demand could just be met at the Cournot price at full capacity. Newbery (1998) finds a unique SFE by considering entry and assuming bid-coordination; incumbent firms coordinate their bids to the most profitable equilibrium that deters entry. Rudkevich et al. (1998) assume that the least profitable equilibrium is most likely to approximate reality. Anderson and Xu (2002) and Baldick and Hogan (2002) find a unique equilibrium in some cases by ruling out unstable equilibria. Stability is tested assuming an infinite speed of adjustment when there are small deviations from best-response bids. It is possible that with a sufficiently slow speed of adjustment, other equilibria might also be stable.

In addition to considering a positive Loss-of-Load-Probability, which ensures a unique symmetric SFE, this paper makes further contributions beyond the recent work of Genc and

Reynolds (2004). First, their results are proven for the case of constant marginal costs and a specific load function, which corresponds to a specific probability density of demand, whereas a general cost function and a general probability density of demand is allowed in this paper. Second, this paper is the first to rule out symmetric SFE with vertical and horizontal segments. Thus extending the space of allowed strategies, as in this paper and in Genc and Reynolds' paper, does not generate any new symmetric SFE. This is true for all bids that can be accepted in equilibrium, even if there is no risk of power shortage. This is a relevant contribution, as two recent papers have demonstrated that asymmetric SFE will generally include horizontal and vertical segments (Holmberg, 2005a, 2005b). Third, because the equilibrium is unique it can be analysed with comparative statics.

The structure of the paper is as follows. Section 2 presents the notation and assumptions used in the analysis. In Section 3, the unique symmetric SFE is derived in several steps. A first-order condition is derived for smooth and monotonically increasing segments of a symmetric SFE by means of optimal control theory. The result is the first-order condition derived for unconstrained production by Klemperer and Meyer (1989). Next, symmetric equilibria with vertical or horizontal segments are ruled out by using optimal control theory with final values and their associated transversality conditions. To avoid horizontal and vertical segments in the supply, the equilibrium price must reach the price cap exactly when the capacity constraint binds. It is shown that there is exactly one symmetric SFE candidate that fulfils this end-condition and the first-order condition. It is verified that the unique candidate is an equilibrium, i.e. there are no unilateral profitable deviations.

Section 4 characterises the unique symmetric SFE. Comparative statics show that the equilibrium has intuitive properties, e.g. mark-ups are reduced if there are more competitors. Another important implication of the analysis is that the price cap and capacity constraints also affect the equilibrium price for outcomes when the constraints do not bind. The assumptions leading to the unique symmetric SFE are realistic for electric power auctions, but even more so for balancing markets. Such a market is considered in Section 5. In Section 6, the unique symmetric equilibrium is illustrated with an example of a quadratic cost function and Section 7 concludes.

2. NOTATION AND ASSUMPTIONS

Assume that there are N symmetric producers. The bid of each producer i consists of a supply function $S_i(p)$, where p is the price. $S_i(p)$ is required to be non-decreasing. Aggregate supply of the competitors of producer i is denoted $S_{-i}(p)$ and total supply is denoted $S(p)$.

In Klemperer and Meyer's (1989) original model, the analysis was confined to twice continuously differentiable supply functions. In this paper the set of admissible bids is extended to include piece-wise twice continuously differentiable supply functions (see Figure 2). The extension allows for supply functions with vertical and horizontal segments, i.e. binding slope constraints. $S_i(p)$ is not necessarily differentiable at every price, but it is required that it is differentiable on the left and right at every price. Furthermore, all supply functions are assumed to be left continuous.⁵ From the requirements of the supply functions, it follows that all supply functions are twice continuously differentiable in the interval $[p_-, p]$ if p_- is sufficiently close to p .

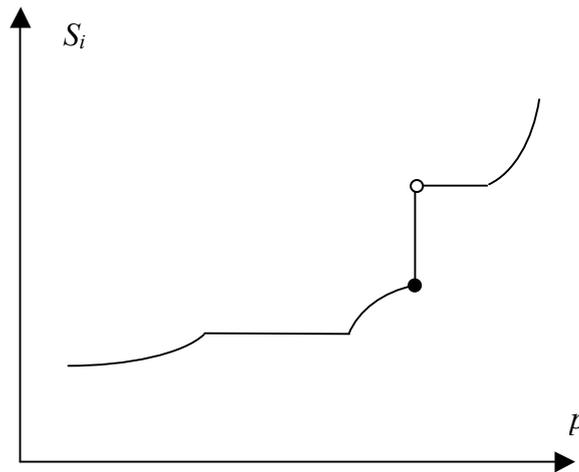


Figure 2. Admissible supply functions are left-continuous and piece-wise twice continuously differentiable.

Denote the perfectly inelastic demand by ε and its probability density function by $f(\varepsilon)$. The density function is continuously differentiable and has a convex support set which includes $\varepsilon=0$. Let the capacity constraint of each producer be $\bar{\varepsilon} / N$, so that $\bar{\varepsilon}$ is the total

⁵ Consider a supply function $S_i(p)$ with a discontinuity at p_0 . It is then assumed that firm i is willing to produce any supply in the range $[S_i(p_0), S_i(p_0 +)]$ if the price is p_0 . Thus the left continuous supply function is actually just a representation of a *supply correspondence* (see Mas-Colell et al., 1995). The same is true for the left-continuous demand function used by Kremer and Nyborg (2004) and the right continuous supply function of Genc and Reynolds (2004).

capacity of all producers. A key assumption is that the capacity constraints of all producers will bind with a positive probability, i.e. there are extreme outcomes for which $\varepsilon > \bar{\varepsilon}$.

Above the reservation price, demand is zero. In the electricity market this is achieved by means of forced disconnection of consumers when the price threatens to rise above the price cap. Thus the market price for extreme outcomes equals the price cap. Allowing for extreme outcomes and rationing is realistic, especially for real-time and balancing markets. However, it differs from the traditional SFE models which ensure market clearing by assuming that firms receive nothing if the market does not clear (Klemperer and Meyer, 1989; Genc and Reynolds, 2004).

In the case where total supply has a perfectly inelastic segment — i.e. there is some price interval $p \in (a, b)$ for which $S'(p) = 0$ — that coincides with perfectly inelastic demand, it is assumed that the market design is such that the lowest price is chosen.⁶ This implies that the equilibrium price as a function of demand is left continuous.

Let $q_i(\varepsilon, p)$ be the residual demand that producer i faces for $p < \bar{p}$. Provided that the supply functions of his competitors are non-horizontal at p , his residual demand is given by

$$q_i(\varepsilon, p) = \varepsilon - S_{-i}(p) \text{ if } p < \bar{p}. \quad (1)$$

All firms have identical cost functions $C(q_i)$, which are increasing, strictly convex, twice continuously differentiable, and fulfil $C'(\bar{\varepsilon}/N) < \bar{p}$. Thus marginal costs are monotonically increasing.

Producer i has a perfectly elastic segment at p_0 if $\Delta S_i(p_0) \equiv S_i(p_0+) - S_i(p_0) > 0$, where $S_i(p_0+) \equiv \lim_{p \downarrow p_0} S_i(p)$.⁷ Similarly, the total perfectly elastic supply of his competitors at p_0 is

$\Delta S_{-i}(p_0) \equiv S_{-i}(p_0+) - S_{-i}(p_0)$. If more than one producer has a supply function with a perfectly elastic segment at some price p_0 , supply rationing at this price is necessary for some demand outcomes. I assume that the rationed supply of producer i at p_0 is given by $S_i(p_0) + R(\varepsilon - S(p_0), \Delta S_i(p_0), \Delta S_{-i}(p_0))$. In addition, it is assumed that the rationing mechanism has the following properties: $R_1 \geq 0$, $R_2 \geq 0$, $R_3 \leq 0$, and $R(0, \Delta S_i(p_0), \Delta S_{-i}(p_0)) = 0$.

Furthermore, if $\Delta S_{-i}(p_0) > 0$, then

$$\begin{aligned} 0 \leq R_1 + R_2 < 1 & \text{ if } S(p_0) \leq \varepsilon < S(p_0+) \\ R_1 + R_2 = 1 & \text{ if } \varepsilon = S(p_0+). \end{aligned} \quad (2)$$

⁶ The same assumption is used by Baldick and Hogan (2002).

⁷ Recall that supply functions are left continuous.

The intuition for this assumption is as follows. Consider a case where rationing is needed at p_0 . Assume that producer i increases the price up to p_0 for one unit that was previously offered below p_0 . Then firms' accepted supply should decrease. The assumed properties can be verified for a rationing mechanism, for example, where all producers receive a ration proportional to their perfectly elastic supply at p_0 . This mechanism is called pro-rata on the margin and is used in most uniform-price auctions (Genc and Reynolds, 2004; Kremer and Nyborg, 2004).

I also assume that if total supply has perfectly elastic segments at the price cap, all of these bids are accepted before demand is rationed.

3. THE UNIQUE SYMMETRIC SFE

As in the recent paper by Genc and Reynolds (2004), optimal control theory is used in the derivation of Supply Function Equilibria. Allowing for vertical and horizontal segments complicates the analysis, as it requires ruling out SFE with vertical and horizontal segments to achieve a unique symmetric equilibrium. Furthermore, to ensure that optimal control theory is applicable when testing whether a supply function of a producer is the best response, one needs to ensure that the supply functions of his competitors are continuously differentiable in the integrated price range. In addition, the control variable needs to be finite. These technicalities imply that supply functions of a potential equilibrium have to be studied piece-by-piece.

In Section 3.1, optimal control theory is used to derive the conditions that must be fulfilled for all *smooth and monotonically increasing* segments of a symmetric supply function equilibrium. These conditions are simplified to a differential equation which yields the standard first-order condition used in the SFE literature, for which an analytic solution exists for perfectly inelastic demand.

In Sections 3.2 to 3.4, irregular symmetric SFE are ruled out. In Section 3.2, it is proven that there are no symmetric supply function equilibria with *perfectly elastic segments*. This can be shown by means of optimal control theory with a final value.⁸ The result of Section 3.2 also rules out perfectly elastic segments at the price cap. In Section 3.3, equilibria with *discontinuities in the equilibrium price* are also ruled out using optimal control theory with a final value. To avoid a discontinuity in the price when all bids have been accepted, the total

⁸ In the special case when the total supply is inelastic just below the perfectly elastic segment, this is proven with a profitable deviation.

supply must be elastic up to the price cap. Section 3.4 shows that no capacity is withheld in equilibrium.

The conclusion is that all supply functions of a symmetric equilibrium must fulfil the first-order condition over the whole price range. The end-condition is that the symmetric supply function must reach the price cap exactly when all capacity constraints bind. In Section 3.5 it is observed that a unique symmetric SFE candidate exists that fulfils the first-order condition and the end-condition. In Section 3.6 it is shown that it is a globally best response for a firm to follow the strategy implied by the unique candidate, given that competitors also follow the unique candidate. Thus the only remaining symmetric equilibrium candidate is a Nash-equilibrium and a SFE.

3.1. The optimal control problem for smooth segments of a SFE

In equilibrium, an arbitrary producer i submits his best supply function out of the class of allowed supply functions, given the bids of his competitors. Now consider *a segment* of a symmetric equilibrium candidate $\widehat{S}_i(p)$, for which supply functions are monotonically increasing and twice continuously differentiable in the range $p_- \leq p \leq p_+$. Assume that the competitors of producer i follow the equilibrium candidate. Will it be a best response of producer i to follow as well? In this section, only local deviations in the range $p_- \leq p \leq p_+$ are considered. Firm i 's bid outside this range is unchanged, i.e. $S_i(p_-) = \widehat{S}_i(p_-)$ and $S_i(p_+) = \widehat{S}_i(p_+)$. Considering such deviations yields a necessary, but not sufficient, condition for SFE. Let the set \mathcal{S} be defined by $\widehat{S}_i(p)$ and all considered deviations of firm i . Note that by choosing his supply function, producer i can control the total supply function, $S(p)$, where

$$S(p) \equiv S_i(p) + \widehat{S}_{-i}(p) = \varepsilon. \quad (3)$$

As competitors follow the equilibrium candidate and $S_i(p)$ is required to be non-decreasing, the total supply, $S(p)$, is monotonically increasing in the interval $p_- \leq p \leq p_+$. Hence, the inverse function of $S(p)$ exists for this range. It is denoted $p(\varepsilon)$:

$$p(\varepsilon) \equiv S^{-1}(\varepsilon). \quad (4)$$

In terms of demand, the studied range is given by $\varepsilon_- \leq \varepsilon \leq \varepsilon_+$, where $\varepsilon_- = S(p_-) = \widehat{S}(p_-)$ and $\varepsilon_+ = S(p_+) = \widehat{S}(p_+)$. Hence, controlling the aggregated supply function, producer i

effectively determines the price for each outcome in the range $\varepsilon_- \leq \varepsilon \leq \varepsilon_+$ under the constraints $p_- = p(\varepsilon_-)$ and $p_+ = p(\varepsilon_+)$. The optimal $p(\varepsilon)$ for this range can be calculated by solving an optimal control problem. The control variable is defined as $u = p'(\varepsilon)$, i.e. the rate of change in the price. The derivative of the inverse function in (4) can be shown to be

$$u = p'(\varepsilon) = \frac{dp}{d\varepsilon} = \left(\frac{d\varepsilon}{dp} \right)^{-1} = \frac{1}{S'(p(\varepsilon))}. \quad (5)$$

Depending on $S_i(p)$, $S(p)$ is not necessarily differentiable at every price, but it is differentiable on the left and right at every price.⁹ Thus u is piece-wise continuous. It is required that all supply functions fulfil $0 \leq S'_i(\cdot)$, so the control variable is constrained by

$$0 \leq u \leq \frac{1}{\widehat{S}_{-i}'(p(\varepsilon))}. \quad (6)$$

In equilibrium, producer i submits his best allowed supply function given $\widehat{S}_{-i}(p)$, the aggregate bid of the competitors. $\widehat{S}_i(p)$ belongs to the set \mathcal{S} , which also includes all considered local deviations. Accordingly, if $\widehat{S}_i(p)$ is the globally best response, it must also be the best response in \mathcal{S} . Thus, given the bids of the competitors, it is necessary, but not sufficient, that the following optimal control problem returns the equilibrium candidate.

$$\begin{aligned} \text{Max}_{p(\varepsilon)} \int_{\varepsilon_-}^{\varepsilon_+} \{ \varepsilon - \widehat{S}_{-i}(p(\varepsilon)) \} p(\varepsilon) - C(\varepsilon - \widehat{S}_{-i}(p(\varepsilon))) f(\varepsilon) d\varepsilon \\ \text{s.t. } u = p'(\varepsilon) \quad 0 \leq u \leq \frac{1}{\widehat{S}_{-i}'(p(\varepsilon))} \quad p(\varepsilon_-) = p_- \quad p(\varepsilon_+) = p_+ \end{aligned} \quad (7)$$

As a result, it is necessary that the contribution to expected profit from the demand interval $[\varepsilon_-, \varepsilon_+]$ is maximised in equilibrium, given $p(\varepsilon_+) = p_+$, $p(\varepsilon_-) = p_-$, and the bids of the competitors. The integrand of an optimal control problem should be continuously differentiable in the state variable (Seierstad and Sydsaeter, 1987), in this case p . This condition is fulfilled as all competitors' supply functions are twice continuously differentiable in $[p_-, p_+]$ and the cost function itself is twice continuously differentiable.

The slope constraint $0 \leq u \leq \frac{1}{\widehat{S}_{-i}'(p(\varepsilon))}$ may bind if there is a profitable deviation from

$\widehat{S}_i(p)$, i.e. $\widehat{S}_i(p)$ is not an equilibrium. However, if, as assumed, $\widehat{S}_i(p)$ is to be a symmetric

⁹ Recall that all supply functions are piece-wise twice differentiable and always have left and right derivatives.

equilibrium with a monotonically increasing and smooth segment, i.e. $0 < \widehat{S}_i'(p) < \infty$ for $p \in [p_-, p_+]$, then the slope constraints cannot bind in this interval. Hence, the Hamiltonian of the problem in (7) is

$$H(u, p, \lambda, \varepsilon) = \left\{ \varepsilon - \widehat{S}_{-i}(p(\varepsilon)) \right\} p(\varepsilon) - C(\varepsilon - \widehat{S}_{-i}(p(\varepsilon))) \left\{ f(\varepsilon) + \lambda(\varepsilon) u(\varepsilon) \right\}, \quad (8)$$

where λ is a co-state or auxiliary variable of the optimal control problem (Chiang, 2001). The control variable u should be chosen such that the Hamiltonian is maximised for every ε (Chiang, 2001). Hence,

$$\frac{\partial H}{\partial u} = 0 = \lambda(\varepsilon) \quad (9)$$

and

$$\lambda(\varepsilon) \equiv \lambda'(\varepsilon) \equiv 0 \text{ for } \varepsilon \in [\varepsilon_-, \varepsilon_+]. \quad (10)$$

The following equations of motion conditions are also necessary for the optimal solution (Chiang, 2001):

$$p'(\varepsilon) = \frac{\partial H}{\partial \lambda} = u(\varepsilon) \quad (11)$$

and

$$\lambda'(\varepsilon) = -\frac{\partial H}{\partial p} = -\left\{ \varepsilon - \widehat{S}_{-i}(p(\varepsilon)) \right\} + \left[C'(\varepsilon - \widehat{S}_{-i}(p(\varepsilon))) - p(\varepsilon) \right] \widehat{S}_{-i}'(p(\varepsilon)) \left\{ f(\varepsilon) \right\}. \quad (12)$$

Combining (10) and (12) yields

$$0 = \left[\varepsilon - \widehat{S}_{-i}(p(\varepsilon)) \right] + \left[C'(\varepsilon - \widehat{S}_{-i}(p(\varepsilon))) - p(\varepsilon) \right] \widehat{S}_{-i}'(p(\varepsilon)) \quad \forall \varepsilon \in (\varepsilon_-, \varepsilon_+).$$

We can now use (3) to simplify the above equation,

$$S_i(p(\varepsilon)) - \widehat{S}_{-i}'(p(\varepsilon)) [p(\varepsilon) - C'(S_i(p(\varepsilon)))] = 0 \quad \forall \varepsilon \in (\varepsilon_-, \varepsilon_+). \quad (13)$$

Before continuing with the analysis of this differential equation, note that by means of (1), (13) can be rewritten as

$$\frac{p(\varepsilon) - C'[S_i(p(\varepsilon))]}{p(\varepsilon)} = \frac{S_i(p(\varepsilon)) / p(\varepsilon)}{\widehat{S}_{-i}'(p(\varepsilon))} = \frac{q_i(\varepsilon, p(\varepsilon)) / p(\varepsilon)}{-\frac{\partial q_i(\varepsilon, p(\varepsilon))}{\partial p}} = -\frac{1}{\gamma_i^{res}}. \quad (14)$$

Hence, a producer maximises his profit for every outcome ε by observing the elasticity of residual demand, γ_i^{res} , and applying the inverse elasticity rule (Tirole, 2003).

Supply functions are monotonically increasing and continuous in the price range $p_- \leq p \leq p_+$. Thus the equilibrium price $p(\varepsilon)$ is continuous and monotonically increasing in the demand range $\varepsilon_- \leq \varepsilon \leq \varepsilon_+$. Accordingly, if (13) is fulfilled for all $\varepsilon \in (\varepsilon_-, \varepsilon_+)$, then it must also be fulfilled for all $p \in (p_-, p_+)$. Moreover, the considered equilibrium candidates are symmetric, so $S_i(p) \equiv S_j(p)$, and (13) can be rewritten as

$$\widehat{S}_i(p) - (N-1)\widehat{S}_i'(p)[p - C''(\widehat{S}_i(p))] = 0, \quad \forall p \in (p_-, p_+). \quad (15)$$

In the subsequent two subsections, non-smooth symmetric SFE will be ruled out. Thus all symmetric SFE are given by (15).

Lemma 1 below rules out smooth symmetric transitions to a perfectly inelastic supply and isolated points in the (p, S) space where $S'_i(p) = 0$, whenever $S > 0$. This ensures that the control variable u is bounded and that optimal control theory can be relied upon when deriving the first-order condition for any smooth segment.¹⁰ Note that Lemma 1 does not rule out smooth symmetric transitions to a perfectly inelastic supply at $S=0$. Indeed, in Section 4.3, it is shown that in the case $N=2$ and $C''(0) > 0$, one gets $S'_i(p) = 0$ at $S=0$. It is easy to verify, however, that (15) is still satisfied when $S'_i(p) = 0$ at $S=0$, as p is bounded by the price cap. Anyway, the point $S=0$ will be considered separately in parts of the subsequent analysis, as $S'_i(p) = 0$ cannot be ruled at this point.

Lemma 1: No symmetric equilibria exist that, for a finite positive supply bounded away from zero, have smooth symmetric transitions to a perfectly inelastic supply.

Proof: See Appendix.

3.2 Symmetric SFE with perfectly elastic segments do not exist

Now consider symmetric equilibrium candidates $\check{S}_i(p)$ in which all producers have segments with perfectly elastic supply at some price p_0 , i.e. $\Delta\check{S}_i(p_0) \equiv \check{S}_i(p_0+) - \check{S}_i(p_0) > 0$. In such a case, supply rationing is needed for some demand outcomes. In what follows, I show that any producer will find it profitable to unilaterally deviate from the equilibrium candidate. He increases his expected profit by undercutting p_0 with units that, for the equilibrium candidate, are offered at p_0 (see Figure 3). The intuition is the same as for Bertrand competition, where

¹⁰ It follows from (5) that $\widehat{S}_i'(p) = 0$ would violate $u < \infty$, which is required in optimal control theory (Chiang, 2001; Seierstad and Sydsaeter 1987).

producers undercut one another's horizontal bids down to marginal cost. As marginal costs are monotonically increasing, Bertrand equilibria can be ruled out in all price intervals. However, as indicated in Figure 3, undercutting competitors' perfectly elastic segment at p_0 necessarily implies that the equilibrium price is lowered just below the demand level $\varepsilon' = \tilde{S}(p_0)$, and this may contribute negatively to expected profits. To make sure that there really exist profitable deviations, a formal proof using optimal control theory is presented in Proposition 2. Optimal control theory is not applicable when total supply is perfectly inelastic just below p_0 , as the control variable $u = p'$ must be finite. This case is analysed separately in Proposition 3. Note that one implication of Propositions 2 and 3 is that symmetric SFE with perfectly elastic segments at the price cap can be excluded. Negative mark-ups are ruled out in Proposition 1. This obvious result is useful when proving Propositions 2 and 3.

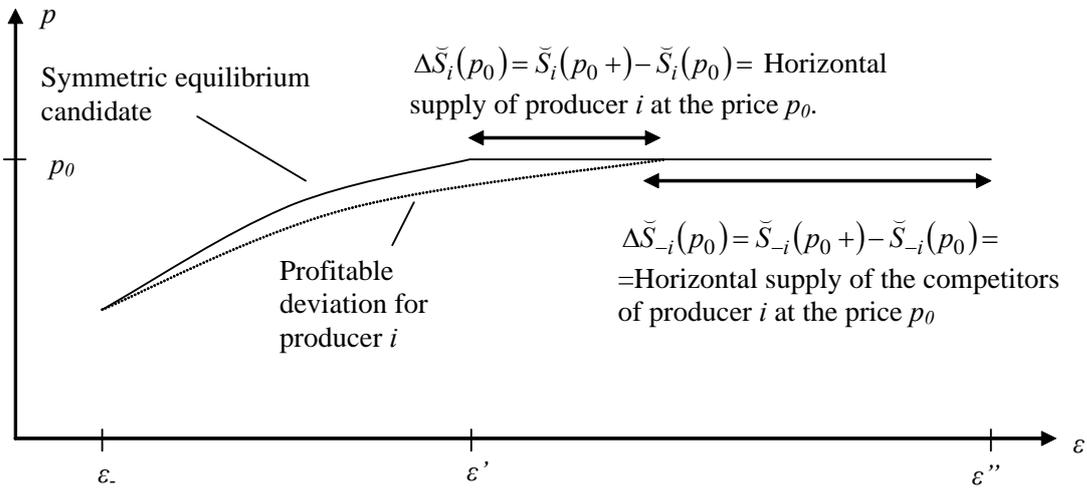


Figure 3. Symmetric equilibria with perfectly elastic segments can be ruled out. Any producer will find it profitable to slightly undercut competitors' horizontal supply.

Proposition 1: In equilibrium no production units are offered for sale below their marginal cost.

Proof: See Appendix.

Proposition 2: For positive supply, there are no symmetric supply function equilibria with perfectly elastic segments at p_0 when market supply is elastic just below p_0 .

Proof: See Appendix.

Proposition 3: For positive supply, there are no symmetric supply function equilibria with perfectly elastic segments at p_0 when market supply is perfectly inelastic just below p_0 .

Proof: See Appendix.

Proposition 1 rules out perfectly elastic segments starting at zero supply if the price is equal to or lower than $C'(0)$. Proposition 3 can be used to rule out such segments at price levels $p_0 > C'(0)$, as this implies that the supply is perfectly inelastic between $C'(0)$ and p_0 .

3.3. The equilibrium price is not discontinuous

Assume that there is a discontinuity in the price at $\varepsilon_L \leq \bar{\varepsilon}$, at which the price jumps from p_L to p_U . This means that all producers have a perfectly inelastic supply in the interval (p_L, p_U) , i.e. the slope constraint $0 \leq S_i'(p)$ binds in this price interval. As a result, any producer that bids just below p_L can increase expected profit by deviating. He can significantly increase the price for some units offered at and slightly below p_L as in Figure 4. This significantly increases the price for demand outcomes just below ε_L , while the reduction in sales is small. Thus the deviation should increase the expected profit. However, it is not completely certain while, as is indicated in Figure 4, the deviation necessarily implies that the deviating firm's sales are reduced just below the demand level ε' as well, and in this case there is no significant price increase. Thus the deviation may result in lower profits from demand outcomes just below ε' . To make sure that there really exist profitable deviations, a formal proof using optimal control theory is presented in Proposition 4. This proposition also rules out discontinuities in the equilibrium price at the demand outcome for which the offered market capacity starts to bind. Thus in a symmetric equilibrium all supply functions must be elastic up to the price cap.

Proposition 4: For symmetric equilibria there are no discontinuities in the equilibrium price if demand is positive.

Proof: See Appendix.

Supply is perfectly inelastic below the lowest bid. Thus if minimum demand equals zero, there is always a discontinuity in the equilibrium price at zero supply. The case when minimum demand is negative, as in a balancing market, is considered in Section 5.

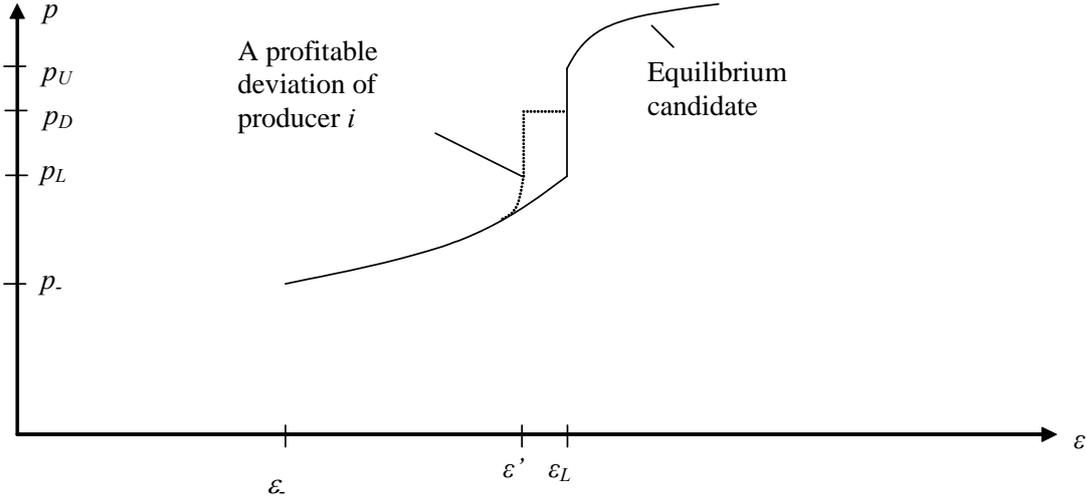


Figure 4. *Discontinuities in the equilibrium price do not exist. Any producer will find it profitable to deviate.*

3.4 No capacity is withheld in equilibrium

If producers are not required by law to offer all of their available capacity to the procurement auction, will firms withhold capacity in the equilibrium of the one shot game? Proposition 5 ensures that they do not. Instead of withholding some units, it is always better to offer these units at the price cap. Thus producers' bids will be exhausted exactly when the total capacity constraint binds, i.e. at $\varepsilon = \bar{\varepsilon}$.

Proposition 5: If $\bar{p} > C'\left(\frac{\bar{\varepsilon}}{N}\right)$ no capacity is withheld from supply in equilibrium.

Proof: Consider a unit that is withheld from supply by producer i in a potential equilibrium. Then there is a profitable deviation for producer i in which he offers the unit at a price equal to the price cap. This deviation strategy will not negatively affect the sales of other units or the equilibrium price. Furthermore, because $\bar{p} > C'\left(\frac{\bar{\varepsilon}}{N}\right)$ and that there is a positive probability that demand exceeds or equals the total capacity of all producers, expected profit from the previously withheld unit will be positive. Accordingly, the deviation increases the expected profit of producer i . Thus there are no equilibria for which units are withheld from supply.

3.5 There is a unique equilibrium candidate fulfilling the necessary conditions

For symmetric equilibrium candidates, Section 3.2 rules out perfectly elastic segments in supply functions. In Section 3.3, it is proven that all supply functions must be elastic from the lowest bid to the price cap. Thus one realises that supply functions $S_i(p)$ must be continuous and consist of piece-wise smooth and monotonically increasing segments from the lowest bid to the price cap. Every smooth segment fulfils the first-order condition in (15). As supply functions are continuous also at junctions between smooth segments and as the cost function is twice continuously differentiable, it follows from (15) that also the slope of supply functions are continuous at junctions, i.e. there are no kinks. Proposition 5 ensures that no capacity is withheld from the procurement auction in equilibrium. Thus symmetric equilibria must fulfil the first-order condition in (15) for $\varepsilon \in (0, \bar{\varepsilon}]$, and the equilibrium price must reach the price cap at $\varepsilon = \bar{\varepsilon}$.

The differential equation in (15) is solved by Rudkevich et al. (1998) and Anderson and Philpott (2002). There is exactly one solution that fulfils the terminal condition $p(\bar{\varepsilon}) = \bar{p}$:

$$p(\varepsilon) = \frac{\bar{p}\varepsilon^{N-1}}{\varepsilon^{-N-1}} + (N-1)\varepsilon^{N-1} \int_{\varepsilon}^{\bar{\varepsilon}} \frac{C'(x/N)dx}{x^N} \quad \forall \varepsilon \in [0, \bar{\varepsilon}]. \quad (16)$$

3.6. The unique candidate is a SFE

In Section 3.5 it was shown that there is a unique symmetric SFE candidate, given by (16), which fulfils the necessary first-order condition and end-condition. This unique candidate is denoted by $S_j^X(p)$. In this section it will be verified that the unique candidate also fulfils a second-order condition, i.e. given the residual demand $\varepsilon - S_{-i}^X(p)$, $S_i^X(p)$ is a best response for firm i . It is sufficient to show that this response globally maximises firm i 's profit for every demand outcome.

It is obvious that no firm can improve its profit in the range $\varepsilon > \bar{\varepsilon}$, as, in this range, all producers sell all of their capacity at the maximum price. For $\varepsilon \in \left[\frac{\bar{\varepsilon}}{N}, \bar{\varepsilon}\right]$, there is some price $\tilde{p}(\varepsilon)$, such that the capacity constraint of producer i binds if his last production unit is offered at or below $\tilde{p}(\varepsilon)$. It can never be profitable for producer i to push the price below $\tilde{p}(\varepsilon)$, as

firm i 's supply cannot be increased beyond the capacity constraint. For $\varepsilon \in \left[0, \frac{\bar{\varepsilon}}{N}\right)$, let us set

$\tilde{p}(\varepsilon) = C'(0)$ in this interval, as it is never profitable to offer units for sale below marginal

cost (see Proposition 1). Thus firm i 's best price must be in the range $p \in [\tilde{p}(\varepsilon), \bar{p}]$ if

$\varepsilon \in [0, \bar{\varepsilon}]$. Given $S_{-i}^X(p)$, neither capacity constraints nor the price cap bind in this price

interval, except at the boundaries. Thus the profit of producer i for the outcome ε is given by

$$\pi_X(\varepsilon, p) = [\varepsilon - S_{-i}^X(p)]p - C(\varepsilon - S_{-i}^X(p)) \quad \forall p \in [\tilde{p}(\varepsilon), \bar{p}]$$

and

$$\frac{\partial \pi_X(\varepsilon, p)}{\partial p} = -S'_{-i}^X(p)[p - C'(\varepsilon - S_{-i}^X(p))] + \varepsilon - S_{-i}^X(p) \quad \forall p \in [\tilde{p}(\varepsilon), \bar{p}]. \quad (17)$$

From the first-order condition in (15) it is known that

$$-S'_{-i}^X(p)[p - C'(S_i^X(p))] + S_i^X(p) = 0 \quad \forall p \in [C'(0), \bar{p}].$$

Subtracting this expression from (17) yields:

$$\frac{\partial \pi_X(\varepsilon, p)}{\partial p} = S'_{-i}^X(p) \left[C' \left(\underbrace{\varepsilon - S_{-i}^X(p)}_{S_i} \right) - C'(S_i^X(p)) \right] + \left(\underbrace{\varepsilon - S_{-i}^X(p)}_{S_i} - S_i^X(p) \right) \quad \forall p \in [\tilde{p}(\varepsilon), \bar{p}].$$

As $\varepsilon - S_{-i}^X(p) = S_i^X(p)$ if $p = p^X(\varepsilon)$, it is straightforward to conclude that $\frac{\partial \pi_X(\varepsilon, p)}{\partial p} > 0$

$\forall p \in [\tilde{p}(\varepsilon), p^X(\varepsilon))$ and $\frac{\partial \pi_X(\varepsilon, p)}{\partial p} < 0 \quad \forall p \in (p^X(\varepsilon), \bar{p}]$. Hence, given $S_{-i}^X(p)$, $p^X(\varepsilon)$ is

producer i 's globally optimal price for each ε . Thus, the equilibrium candidate is a SFE.¹¹

¹¹ Klemperer and Meyer (1989) present a corresponding proof for two symmetric firms without capacity constraints facing an elastic demand. In the last step of the proof it is not considered that $\varepsilon - S_{-i}^X(p)$ and $S_i^X(p)$ will generally differ for large deviations. With this simplification it can be shown that $\pi_X(\varepsilon, p)$ is locally concave in the price at $p^X(\varepsilon)$, but this is not sufficient to guarantee the existence of an equilibrium.

4. CHARACTERISING THE UNIQUE SYMMETRIC SFE

It has been shown that with a price cap and capacity constraints, a unique, symmetric SFE exists. This is good news for comparative statics. For symmetric equilibria, equation (16) continues to be valid even when the number of firms, marginal costs, the price cap or capacity constraints change.

4.1. Mark-ups

In a market with perfect competition, the equilibrium price is set by the marginal cost of the marginal production unit. The marginal costs of alternative cheaper or more expensive generators do not influence the price. What would happen under imperfect competition? Equation (16) shows that the equilibrium price of the unique symmetric SFE is given by a term related to the price cap and a term weighting the marginal costs of generators more expensive than the marginal unit. Thus as in the competitive case, generators cheaper than the marginal unit do not affect the equilibrium price. However, the price of the marginal unit of a producer is limited by the cost of the alternative, competitors' generators with a higher marginal cost. Thus the marginal costs of generators more expensive than the marginal unit influence the size of the mark-ups and accordingly the bid of the marginal unit. It is evident that for the term with weighted marginal costs, the weight decreases with increased demand. Furthermore, all weights are positive and integrate to less than or equal to one, as shown in the following calculation:

$$(N-1)\varepsilon^{N-1} \int_{\varepsilon}^{\bar{\varepsilon}} \frac{dx}{x^N} = \left[-\frac{\varepsilon^{N-1}}{x^{N-1}} \right]_{\varepsilon}^{\bar{\varepsilon}} = 1 - \frac{\varepsilon^{N-1}}{\bar{\varepsilon}^{N-1}} \leq 1.$$

According to Proposition 1, the equilibrium price never falls below the marginal cost of the marginal unit. As a result, producers will choose a positive mark-up for every positive demand. This can be shown by manipulating (16) as follows:

$$\begin{aligned} p(\varepsilon) &= \varepsilon^{N-1} \left[\frac{\bar{p}}{\varepsilon^{-N-1}} + (N-1) \int_{\varepsilon}^{\bar{\varepsilon}} \frac{C'(x/N) dx}{x^N} \right] \geq \varepsilon^{N-1} \left[\frac{\bar{p}}{\varepsilon^{-N-1}} + (N-1) C'(\varepsilon/N) \int_{\varepsilon}^{\bar{\varepsilon}} \frac{dx}{x^N} \right] = \\ &= \varepsilon^{N-1} \left\{ \frac{\bar{p}}{\varepsilon^{-N-1}} - C'(\varepsilon/N) \left[\frac{1}{x^{N-1}} \right]_{\varepsilon}^{\bar{\varepsilon}} \right\} = \varepsilon^{N-1} \left[\frac{\bar{p}}{\varepsilon^{-N-1}} - C'(\varepsilon/N) \left(\frac{1}{\varepsilon^{-N-1}} - \frac{1}{\bar{\varepsilon}^{N-1}} \right) \right] = \\ &= \frac{\varepsilon^{N-1}}{\varepsilon^{-N-1}} \underbrace{\left[\bar{p} - C'(\varepsilon/N) \right]}_{>0} + C'(\varepsilon/N) > C'(\varepsilon/N). \end{aligned}$$

4.2. Comparative statics

For any positive demand outcome, it is clear from (16) that the equilibrium price will increase if the price cap is increased. Equation (16) can also be used to investigate the effect of a symmetric change in the capacity constraints:

$$\frac{\partial p(\varepsilon)}{\partial \varepsilon} = \frac{-(N-1) \left[\bar{p} - C'(\varepsilon/N) \right] \varepsilon^{N-1}}{\varepsilon^{-N}} < 0, \text{ if } \varepsilon > 0.$$

That is, increased capacity decreases the price for all positive demand outcomes.

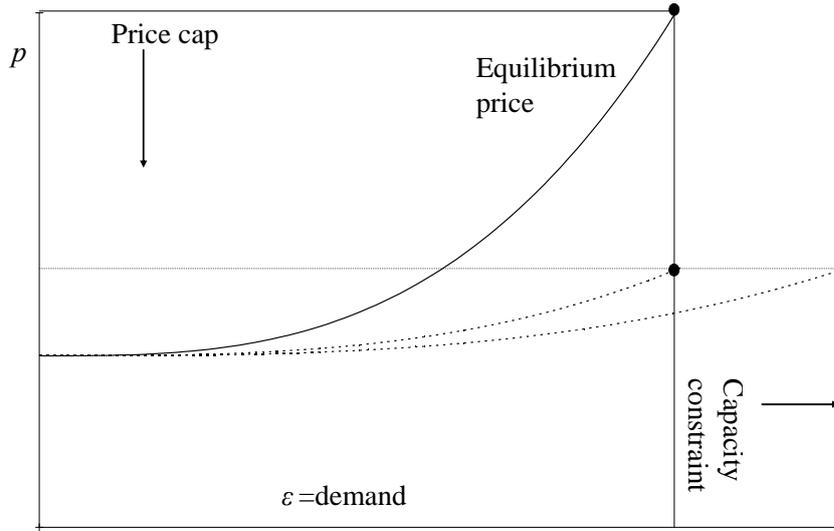


Figure 5. Reducing the price cap \bar{p} and/or increasing the total capacity constraint of the market ε push down the equilibrium price for every demand.

What happens if the number of producers increases? Let the total capacity and aggregated cost function be fixed, i.e. independent of the number of firms. Denote the total cost to meet demand by $C_{tot}(\varepsilon)$. In the unique symmetric SFE, this total cost is N times the cost of each symmetric producer. Hence,

$$C_{tot}(\varepsilon) = NC(S_i) = NC(\varepsilon/N).$$

Thus

$$C_{tot}'(\varepsilon) = C'(\varepsilon/N). \tag{18}$$

Combining (16) and (18), the equilibrium price of the unique symmetric SFE can be written as

$$p(\varepsilon) = \varepsilon^{N-1} \left[\frac{\bar{p}}{\varepsilon^{-N-1}} + (N-1) \int_{\varepsilon}^{\bar{\varepsilon}} \frac{C_{tot}'(x) dx}{x^N} \right] \text{ if } \varepsilon \geq 0.$$

The cost function is twice continuously differentiable and strictly convex. Thus $C_{tot}''(\varepsilon) > 0$.

Now, using integration by parts, the equilibrium price can be rewritten to yield

$$\begin{aligned}
p(\varepsilon) &= \bar{p} \left(\frac{\varepsilon}{\varepsilon} \right)^{N-1} - \left[\left(\frac{\varepsilon}{x} \right)^{N-1} C_{tot}'(x) \right]_{\varepsilon}^{\bar{\varepsilon}} + \int_{\varepsilon}^{\bar{\varepsilon}} \left(\frac{\varepsilon}{x} \right)^{N-1} C_{tot}''(x) dx = \\
&= \underbrace{\left[\bar{p} - C_{tot}' \left(\frac{\varepsilon}{\varepsilon} \right) \right]}_{>0} \underbrace{\left(\frac{\varepsilon}{\varepsilon} \right)^{N-1}}_{\leq 1} + C_{tot}'(\varepsilon) + \int_{\varepsilon}^{\bar{\varepsilon}} \underbrace{\left(\frac{\varepsilon}{x} \right)^{N-1}}_{\leq 1} \underbrace{C_{tot}''(x)}_{>0} dx.
\end{aligned} \tag{19}$$

It is evident that all terms are positive and that the first and last term decrease with N , unless $\varepsilon = \bar{\varepsilon}$. The middle term is not influenced by N at all. Hence, for every positive demand below $\bar{\varepsilon}$, the equilibrium price of the unique symmetric SFE decreases when the number of symmetric producers increases. From equation (19), it can also be noted that the equilibrium price approaches the marginal cost of the marginal unit as the number of symmetric producers approaches infinity.

What happens if entrants increase total capacity? This can be viewed as a combination of an increase in the number of producers and an increase in the total capacity and it has been established that both decrease the equilibrium price for every positive demand.

From equation (19) it is easy to verify that $p(0) = C_{tot}'(0) = C'(0)$, which is also proven by Klemperer and Meyer (1989). Hence, the equilibrium price equals the marginal cost of the marginal unit for zero demand. The intuition behind this is that firms' first units are not price-setting for any other units. Thus the first unit is sold under Bertrand competition. The first unit in private value models of uniform-price auctions is also offered at marginal cost (Krishna, 2002).

4.3. The slope of the equilibrium price

Using (16) and integration by parts it can be shown that

$$\begin{aligned}
\frac{\partial p(\varepsilon)}{\partial \varepsilon} &= \frac{(N-1)\bar{p}\varepsilon^{N-2}}{\varepsilon^{N-1}} + (N-1)^2 \varepsilon^{N-2} \int_{\varepsilon}^{\bar{\varepsilon}} \frac{C'(x/N) dx}{x^N} - (N-1)\varepsilon^{N-1} \frac{C'(\varepsilon/N)}{\varepsilon^N} = \\
&= \frac{(N-1)\varepsilon^{N-2} \left(\bar{p} - C'(\bar{\varepsilon}/N) \right)}{\varepsilon^{N-1}} + \frac{(N-1)\varepsilon^{N-2}}{N} \int_{\varepsilon}^{\bar{\varepsilon}} \frac{C''(x/N) dx}{x^{N-1}} \quad \forall \varepsilon \in (0, \bar{\varepsilon}).
\end{aligned} \tag{20}$$

Thus $0 < \frac{\partial p(\varepsilon)}{\partial \varepsilon} < \infty$ for $\varepsilon \in (0, \bar{\varepsilon})$. What happens if $\varepsilon \rightarrow 0+$? For $N=2$, $\lim_{\varepsilon \downarrow 0} \frac{\partial p(\varepsilon)}{\partial \varepsilon} = \infty$, as

$C''(\varepsilon) > 0$ and $\int_0^{\bar{\varepsilon}} \frac{dx}{x} \rightarrow \infty$. For $N > 2$, the limit is of the type $0 \cdot \infty$, but it can be written in the

form $\frac{\infty}{\infty}$. Hence, the limit can be calculated by means of l'Hospital's rule (Abramowitz and

Stegun, 1972):

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0+} \frac{\partial p(\varepsilon)}{\partial \varepsilon} &= \lim_{\varepsilon \rightarrow 0+} \frac{(N-1)\varepsilon^{N-2} \int_{\varepsilon}^{\bar{\varepsilon}} \frac{C''(x/N) dx}{x^{N-1}}}{N} = \lim_{\varepsilon \rightarrow 0+} \frac{(N-1) \int_{\varepsilon}^{\bar{\varepsilon}} \frac{C''(x/N) dx}{x^{N-1}}}{N\varepsilon^{2-N}} = \\ &= \lim_{\varepsilon \rightarrow 0+} \frac{(N-1) \frac{C''(\varepsilon/N)}{\varepsilon^{N-1}}}{(2-N)N\varepsilon^{1-N}} = \frac{(N-1)C''(0)}{(N-2)N}. \end{aligned} \quad (21)$$

Thus $0 < \frac{\partial p(\varepsilon)}{\partial \varepsilon} < \infty$ for $N > 2$, as the cost function is strictly convex and twice continuously differentiable by assumption.

5. BALANCING MARKETS

Relative to production costs, storage of electric energy is expensive. As a result, stored electric energy is negligible in most power systems and power consumption and production must be roughly in balance at all times. Because most electric power is sold on forward markets or with long-term agreements but neither consumption nor production is fully predictable, adjustments have to be made in real-time in order to maintain balance. The balancing market is an important component in this process. It is an auction in which the independent system operator (ISO) can buy additional power (increments) from producers or sell power back (decrements). The latter occurs if contracted production exceeds the realised total demand. A producer can offer decrements if his contracted production is larger than his current inflexible power production, which cannot be regulated in real-time.

That the ISO's demand can be both negative and positive in the balancing market ensures that the support of demand's probability density includes zero demand, which was assumed in Section 2. Moreover, the demand is particularly inelastic in real-time. Further, unexpected failures in power plants may result in real-time demand exceeding the supply, especially as only a fraction of the power production is sufficiently flexible to be regulated in real-time.

Thus the derived unique symmetric SFE should be of particular relevance to real-time and balancing markets.

Decrements can be analysed analogously to increments, but require additional notation and slightly different assumptions. In the case where a perfectly inelastic segment of total (negative) additional supply coincides with negative perfectly inelastic demand, the highest price will be chosen, i.e. the best price of the ISO. SFE are not influenced by whether the supply correspondence is represented by left or right continuous supply functions. But if one assumes left continuous supply functions for up-regulation, the analogous assumption for down-regulation is right continuous supply functions. The total decrement capacity is denoted by $\underline{\varepsilon}$. This reflects contracted flexible production which can be bought back and turned off. Producers will not buy back power if the price exceeds the marginal cost. Instead they will use their market power to lower price below marginal cost. As a result, a price floor, \underline{p} , is needed for decrements. It is assumed that $\underline{p} \leq C'\left(\frac{\underline{\varepsilon}}{N}\right)$. One can use arguments analogous to

the increment case to show that a unique, symmetric SFE also exists for decrements. Thus

$$p(\varepsilon) = \begin{cases} \frac{\overline{p\varepsilon}^{N-1}}{\varepsilon^{N-1}} + (N-1)\varepsilon^{N-1} \int_{\varepsilon}^{\overline{\varepsilon}} \frac{C'(x/N)dx}{x^N} & \text{if } \varepsilon \geq 0 \\ \frac{\underline{p\varepsilon}^{N-1}}{\underline{\varepsilon}^{N-1}} + (N-1)\varepsilon^{N-1} \int_{\underline{\varepsilon}}^{\varepsilon} \frac{C'(x/N)dx}{x^N} & \text{if } \varepsilon < 0 \end{cases} \quad (22)$$

In Section 4.2 it was verified that $p(0)=C'(0)$ for all symmetric SFE. Thus the equilibrium price is continuous at $\varepsilon=0$.

In some balancing markets, e.g. in Norway, up-regulation bids cannot be lower than the day-ahead price, p_d , and down-regulation bids cannot be higher than the day-ahead price. This introduces an extra constraint on real-time bids and may change the properties of the SFE for small imbalances. As an example, assume that $p_d > c_0$. In this case up-regulation bids will start with a horizontal segment at p_d as shown in Figure 6. The horizontal segment stops at the demand for which $p(\varepsilon)$ in (22) equals p_d . Above this demand level the equilibrium is the same as without the extra constraint. At zero demand there is a vertical segment down to c_0 , which has been econometrically analysed by Skytte (1999) for the Norwegian balancing market. It can be shown that given the extra constraint associated with the day-ahead price, there is no profitable deviation from the outlined equilibrium. When $p_d < c_0$, the extra

constraint has a similar influence on down-regulation bids.

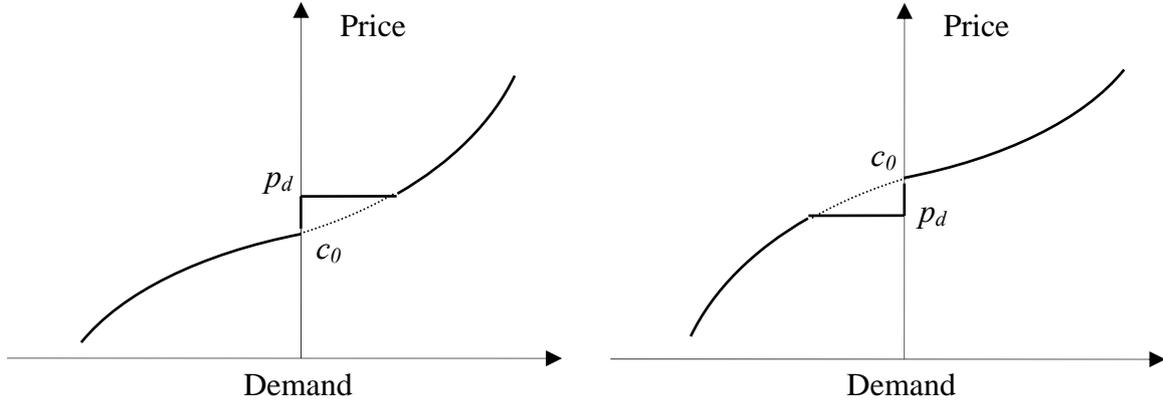


Figure 6. The left figure outlines the unique equilibrium if up-regulation bids cannot be lower than the day-ahead price and $p_d > c_0$. The right figure outlines the unique equilibrium if down-regulation bids cannot be higher than the day-ahead price and $p_d < c_0$.

6. A NUMERICAL ILLUSTRATION OF THE UNIQUE SFE

When the cost function is polynomial in form, it is straightforward to analytically calculate the equilibrium price as a function of the demand by means of (22). Here the equilibrium is illustrated with a simple example of a quadratic cost function, i.e. linear marginal costs:

$$C'_{tot}(x) = c_0 + kx.$$

The result for $N > 2$ is

$$p(\varepsilon) = \begin{cases} c_0 + (\bar{p} - c_0) \frac{\varepsilon^{N-1}}{\varepsilon^{-N-1}} + \frac{k(N-1)\varepsilon}{(N-2)} \left(1 - \frac{\varepsilon^{N-2}}{\varepsilon^{-N-2}} \right) & \text{if } \varepsilon \geq 0 \\ c_0 + (\underline{p} - c_0) \frac{\varepsilon^{N-1}}{\underline{\varepsilon}^{-N-1}} + \frac{k(N-1)\varepsilon}{(N-2)} \left(1 - \frac{\varepsilon^{N-2}}{\underline{\varepsilon}^{-N-2}} \right) & \text{if } \varepsilon < 0. \end{cases} \quad (23)$$

The auctioneer's demand is negative in sales auctions and positive in procurement auctions. Both are relevant for balancing markets. Equation (23) is used in Figure 7 to illustrate the effect of the number of producers on the equilibrium price. For positive (negative) demand, more producers implies reduced mark-ups (mark-downs). In Section 4.2, this was proven for all strictly convex and twice continuously differentiable cost functions. For $N=100$, the market price is very close to marginal cost, except near the capacity constraints. In general, residual demand is less elastic and mark-ups more extreme close to the capacity constraints. This is in agreement with the inverse elasticity rule in (14). For negative demand (decrements), the market price is below marginal cost. Oligopoly producers use their market

power to buy back power at a price below their marginal cost. Note also that in all cases, price equals marginal cost at zero supply/demand.

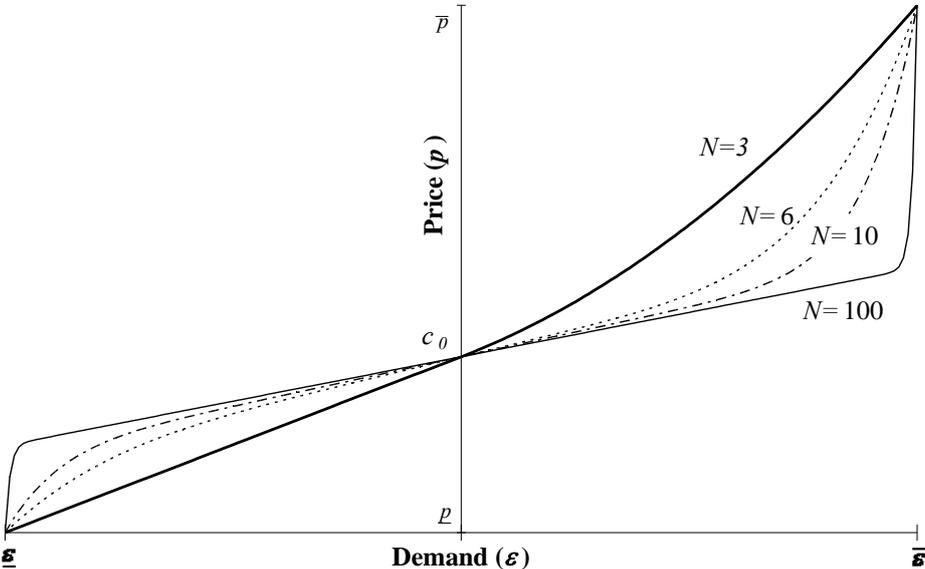


Figure 7. *The unique symmetric supply function equilibrium for linear marginal costs and N symmetric producers. Demand is negative in sales auctions and positive in procurement auctions.*

In Figure 7, demand is given by the additional power needed in real-time to maintain balance. The supply function equilibria can also be plotted as a function of total electricity demand, including power q that has been contracted in advance (see Figure 8). If one applies inelastic demand and the positive Loss-of-Load-Probability to the supply function equilibria of the spot market with forward contracts calculated by Anderson and Xu (2005), one gets a unique equilibrium reminiscent to the result in Figure 8. When comparing the two results, note that c_0 in Figure 8 denotes the marginal cost at the imbalance zero, i.e. it is the marginal cost when total demand equals q . Contracts influence on the supply function equilibrium have also been considered by Newbery (1998) and Green (1999), who both in addition determine forward contracts endogenously.

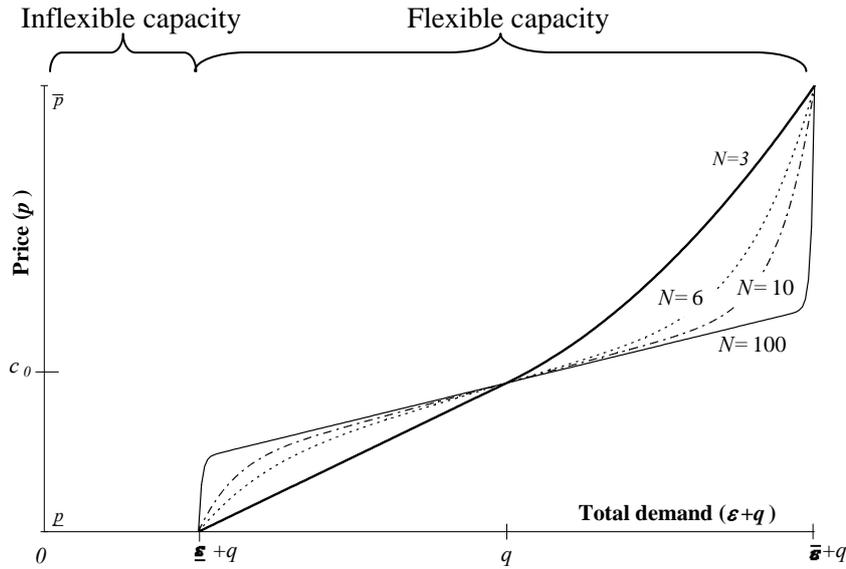


Figure 8. *The unique symmetric supply function equilibrium as a function of the total demand $\varepsilon+q$, where q is power contracted on forward markets.*

7. CONCLUSIONS

Multiplicity of equilibria is one basic criticism of the supply function equilibrium (SFE), an established model of strategic bidding in electricity markets. It is well known that capacity constraints reduce the set of SFE (Green and Newbery, 1992; Newbery, 1998; Baldick and Hogan, 2002). Genc and Reynolds (2004) have recently shown that the range of SFE can be reduced even further by considering pivotal suppliers, at least for perfectly inelastic demand, symmetric firms, constant marginal costs, a price cap, and a specific load function (which corresponds to a specific probability density of demand). This paper allows general cost functions, a general probability density of demand, and constrains the range of SFE even further. It is argued that there is always a positive Loss-of-Load-Probability, and it is shown that this leads to a unique symmetric SFE, i.e. there is only one SFE with symmetric supply functions. In real-time, demand may exceed the market capacity in any delivery period due to demand shocks or sufficiently many unexpected simultaneous failures in power plants. To avoid inconsistencies in the model, one may only consider outages for generators who cannot bid strategically in real-time, e.g. generators with must-run power plants or generators who export to the considered market. The uniqueness result is sensitive to the support of the probability distribution of the demand, but the equilibrium is otherwise insensitive to the probability density. An arbitrarily small risk of power shortage is enough to yield uniqueness. However, as some demand outcomes are very unlikely, this may result in a long learning period before the market finds the unique symmetric SFE.

Price caps are used in most electric power markets. The market price in the unique symmetric supply function equilibrium reaches the price cap exactly when the capacity constraints bind. It is shown that if the price cap is decreased or capacity constraints increased, the equilibrium price decreases for each positive demand outcome. That is, changing these constraints affects prices also for outcomes when the constraints are non-binding. Increasing the number of producers also decreases the equilibrium price for every level of positive demand. Mark-ups are zero at zero supply and positive for every positive supply.

Using unlikely events to refine a set of multiple Nash equilibria is an established method in game theory. One can for example compare with the trembling hand concept, in which players make mistakes with infinitely small probabilities (Mas-Colell et al., 1995). Further, hockey-stick bidding in some US electricity auctions (Hurlbut, 2004) lends empirical support to the end-condition of the equilibrium. This phenomenon means that some firms offer their last units of power at an extremely high price, such as the price cap.

In the European Union, abuse of market power is illegal and it is plausible that firms do not dare to bid with extreme mark-ups. However, the price cap of the model might be interpreted more generally, for example, as the highest bid acceptable without risking interference by the regulator. In this case, monitoring by the regulator is needed to ensure that no capacity is withheld from the market. Otherwise, firms might be tempted to withhold power, increasing the risk of power shortages and the probability that the market price equals the true price cap. It is possible that regulators intervene on the basis of high demand weighted average prices rather than high spike prices. If so, producers' total expected profits are capped at some level informally specified by the regulator. If one only considers symmetric SFE candidates that fulfil the first-order condition derived by Klemperer and Meyer, then there should still be a unique symmetric equilibrium. There is only one such candidate for which expected profits of producers are at the maximum level. For candidates where profits are lower, there is always a profitable deviation, within the total profit cap, e.g. by increasing the price near market capacity.

Perfectly inelastic demand is a realistic assumption for real-time markets and helps simplify the analysis. However, the uniqueness result is expected to hold also for elastic demand. Symmetry is not required to achieve a unique equilibrium, but as two recent papers demonstrate, asymmetry is likely to change the characteristics of the equilibrium (Holmberg, 2005a, 2005b). In particular, asymmetric equilibria will typically include supply functions with kinks and vertical and horizontal segments. This paper rules out such irregularities for

symmetric equilibria with smooth cost functions. One exception is when the day-ahead price sets a minimum level for up-regulation bids, as in the balancing market of Norway.

Technically this paper does not rule out asymmetric SFE in a market with symmetric firms. However, if minimum demand is non-positive, one can use a proof of Klemperer and Meyer (1989) to rule out smooth asymmetric SFE and techniques used in Holmberg (2005a) to rule out asymmetric SFE with vertical and horizontal segments.

Continuous SFE, analysed in this paper, has been criticized for not considering the quantity discreteness of real electricity auctions. To address this issue, von der Fehr and Harbord (1993) introduced an alternative model with stepped supply functions. Whilst their critique is partly justified, empirical studies are needed to determine which model is actually best suited to represent strategic bidding in electricity auctions. The conclusion might very well depend on the auction design. Two recent empirical studies of ERCOT (a balancing market in Texas) suggest that the bids of the two to three largest firms do indeed match the first-order condition of the continuous SFE (Hortascu and Puller, 2004; Sioshansi 2005).

As in the original model by Klemperer and Meyer (1989) the analysis is restricted to a one-shot game. Further, the assumptions leading to uniqueness are particularly realistic for and may only be applicable to balancing and real-time markets. To get a complete understanding of competition in electricity markets a full intertemporal equilibrium with entry, endogenous investments and endogenous forward contracting is needed; a work that has already been started by Newbery (1998) and Green (1999). Even if only a small part of all power is turned over on balancing markets, they represent the final stage in the multistage game of an electricity market, and this stage must be solved before equilibria in previous stages can be determined.

8. REFERENCES

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APPENDIX

Proof of Lemma 1:

The result follows from the symmetric first-order condition in (15). If $\varepsilon > A > 0$ — i.e. $\widehat{S}_i(p) > \frac{A}{N} > 0$ — then $0 < m \leq \widehat{S}_i'(p)$, where m is a number independent of ε . Thus $\frac{1}{\widehat{S}_i'(p)}$ is bounded for positive supply bounded away from zero.

If a transition to $\widehat{S}_i'(p_0) = 0$ (the left derivative) is smooth from the left, then $\widehat{S}_i'(p)$ is twice continuously differentiable and monotonically increasing in some interval below, but arbitrarily close to p_0 . From the argument above it follows that $\frac{1}{\widehat{S}_i'(p)}$ is bounded for p arbitrarily close to p_0 . Thus a smooth transition to $\widehat{S}_i'(p_0) = 0$ from the left can be ruled out. With similar reasoning it can be shown that there are no smooth transitions from the right to a perfectly inelastic supply if the positive supply is bounded away from zero.

Proof of Proposition 1:

Assume that there is an equilibrium $S_i^Z(p)$, in which producer i offers production for sale below marginal cost, i.e. there are some prices p , for which $C'[S_i^Z(p)] > p$. Denote this set of prices by \mathcal{P} . Then there exists a profitable deviation for producer i . Adjust the supply of producer i such that units previously offered below their marginal cost are now offered at their marginal cost. Formally, $C'[S_i(p)] = p$ for all $p \in \mathcal{P}$, where $S_i(p)$ is the adjusted supply function. The supply is unchanged for all other p , i.e. $S_i(p) = S_i^Z(p) \forall p \notin \mathcal{P}$. $S_i(p)$ is non-decreasing like $S_i^Z(p)$, as $C()$ is strictly convex and increasing. The contribution to expected

profits from units that are offered at or above their marginal costs are not negatively affected by the deviation. Their contribution might even increase as the equilibrium price increases for some imbalance outcomes. Now consider a unit that was previously offered below its marginal cost c_0 . Let ε_0 denote the imbalance for which the market price reaches c_0 in the assumed equilibrium. After the deviation, the price will reach c_0 at an imbalance $\varepsilon \leq \varepsilon_0$. Moreover, market prices will not decrease for any positive imbalances. Thus the positive contribution of the considered unit to the expected profit is either increased or unchanged. Furthermore, after the deviation, the unit is never sold below marginal cost. The same reasoning is true for all units offered below their marginal cost. Thus the deviation increases the expected profit of producer i and, in equilibrium, no production is offered below its marginal cost.

Proof of Proposition 2

Consider a symmetric SFE candidate with perfectly elastic segments at $p_0 \leq \bar{p}$. Denote supply functions following the equilibrium candidate by \check{S}_i . Thus $\Delta\check{S}_i(p_0) = \check{S}_i(p_0+) - \check{S}_i(p_0) > 0$ and $\Delta\check{S}_{-i}(p_0) > 0$. All considered supply functions are twice continuously differentiable in some price interval $[p_-, p_0]$, see Section 2. The market supply is elastic just below p_0 . Thus $0 < \check{S}'_{ii}(p_0) < \infty$ (the left derivative). Further, a sufficiently large p can be chosen such that $0 < \check{S}'_i(p) < \infty$ for all $p \in [p_-, p_0)$.

Now consider unilateral deviations $S_i(p)$ of player i , where all his bids above p_0 and below p_- are unchanged. Let $\varepsilon_- = S(p_-) = \check{S}(p_-)$, $\varepsilon' = S(p_0)$ and $\varepsilon'' = S(p_0+) = \check{S}(p_0+)$. For demand outcomes $\varepsilon \in (\varepsilon', \varepsilon'')$, the supply at p_0 has to be rationed somehow. The accepted ration of the perfectly elastic supply of producer i is given by $R(\varepsilon - \varepsilon', \Delta S_i(p_0), \Delta\check{S}_{-i}(p_0))$, where $\Delta S_i(p_0) = \varepsilon'' - \varepsilon' - \Delta\check{S}_{-i}(p_0)$.

To keep the equilibrium candidate, $\check{S}_i(p_0)$ must be the best response of the considered deviation strategies. The best response can be derived from

$$\begin{aligned}
& \text{Max}_{p(\varepsilon)} \int_{\varepsilon_-}^{\varepsilon'} \left\{ \left[\varepsilon - \tilde{S}_{-i}(p(\varepsilon)) \right] p(\varepsilon) - C(\varepsilon - \tilde{S}_{-i}(p(\varepsilon))) \right\} f(\varepsilon) d\varepsilon + F(\varepsilon') \\
& \text{s.t. } u = p'(\varepsilon) \quad 0 \leq u \leq \frac{1}{\tilde{S}'_{-i}(p(\varepsilon))} \\
& p(\varepsilon_-) = p_- \quad p(\varepsilon') = p_0.
\end{aligned} \tag{24}$$

The final value of the optimal control problem, $F()$, returns the contribution to the expected profit from the rationed supply at p_0 .

$$\begin{aligned}
F(\varepsilon') &= \int_{\varepsilon'}^{\varepsilon''} \left\{ \left[\varepsilon' - \tilde{S}_{-i}(p_0) + R(\varepsilon - \varepsilon', \varepsilon'' - \varepsilon' - \Delta\tilde{S}_{-i}, \Delta\tilde{S}_{-i}) \right] p_0 \right. \\
&\quad \left. - C \left[\varepsilon' - \tilde{S}_{-i}(p_0) + R(\varepsilon - \varepsilon', \varepsilon'' - \varepsilon' - \Delta\tilde{S}_{-i}, \Delta\tilde{S}_{-i}) \right] \right\} f(\varepsilon) d\varepsilon
\end{aligned} \tag{25}$$

The slope constraints $0 \leq u \leq \frac{1}{\tilde{S}'_{-i}(p(\varepsilon))}$ might bind for $\varepsilon \in [\varepsilon', \varepsilon_-]$ if there is a profitable

deviation from $\tilde{S}_i(p)$, i.e. $\tilde{S}_i(p)$ is not an equilibrium component. However, as in Section 3.1, the slope constraints can be disregarded when a necessary condition for $\tilde{S}_i(p)$ is derived under the assumption that $\tilde{S}_i(p)$ is a SFE.

The Hamiltonian, the Max H condition and the equations of motion are the same as for the optimal control problem in (7) (Léonard and van Long, 1992). In particular, $\lambda(\varepsilon) \equiv 0$ for $\varepsilon \in [\varepsilon_-, \varepsilon']$ as in (10). The transversality condition associated with the terminal constraint at the right end-point is (Léonard and van Long, 1992)

$$H(u, p, \lambda, \varepsilon') + \frac{\partial F(\varepsilon')}{\partial \varepsilon'} = 0. \tag{26}$$

The first term is the marginal value of increasing ε' . The second term, which is negative, represents the marginal loss in final value.

Now, it will be shown that the transversality condition in (26) cannot be fulfilled at ε' . It is known that $p(\varepsilon') = p_0$ and $R(0, \Delta S_i, \Delta \tilde{S}_{-i}) = 0$. These relations, combined with (8), (10) and (25), imply¹²

¹² Note that the first term in (8) cancels out one of the terms given by Leibniz' theorem (Abramowitz and Stegun, 1972) when differentiating the integral in (25).

$$H(u, p, \lambda, \varepsilon') + \frac{\partial F(\varepsilon')}{\partial \varepsilon'} = \int_{\varepsilon'}^{\varepsilon''} \left\{ (p_0 - C'(\cdot)) \left[1 + \frac{dR(\varepsilon - \varepsilon', \varepsilon'' - \varepsilon' - \Delta \tilde{S}_{-i}, \Delta \tilde{S}_{-i})}{d\varepsilon'} \right] \right\} f(\varepsilon) d\varepsilon = \quad (27)$$

$$\int_{\varepsilon'}^{\varepsilon''} \left\{ (p_0 - C'(\cdot)) [1 - R_1(\cdot) - R_2(\cdot)] \right\} f(\varepsilon) d\varepsilon.$$

Costs are strictly convex and Proposition 1 ensures that there are no equilibria with negative mark-ups. Thus $p_0 > C'$ for $\varepsilon \in [\varepsilon', \varepsilon'')$ and $p_0 \geq C'$ for $\varepsilon = \varepsilon''$. Thus the combination of (2) and (27) implies that

$$H(u, p, \lambda, \varepsilon') + \frac{\partial F(\varepsilon')}{\partial \varepsilon'} > 0. \quad (28)$$

For the equilibrium candidate, the marginal value of continuing, i.e. increasing ε' , is larger than the marginal loss in final value. The reason is that by slightly undercutting p_0 , as in Figure 3, producer i can sell significantly more. The relation in (28) is true as long as producer i has a perfectly elastic supply remaining at p_0 . Hence, equilibria of the type $\tilde{S}_i(p)$ can be excluded.

Proof of Proposition 3

Use the same notation as in Proposition 2, but now consider the case when aggregate supply is perfectly inelastic just below p_0 . Denote supply functions following the equilibrium candidate by the superscript H . Isolated perfectly inelastic points are ruled out by Lemma 1. Thus the equilibrium supply must be perfectly inelastic in a price interval below p_0 . Consider the following deviation: producer i can offer some units previously offered at p_0 at the price $p_0 - \eta$, where η is positive and infinitesimally small. As in Proposition 2, the perfectly elastic aggregate supply starts at ε' . Let $\varepsilon' = \varepsilon'_H$ in the potential equilibrium. The optimal $\varepsilon' \in [\varepsilon'_H, \varepsilon'']$ is then given by

$$\begin{aligned} \text{Max}_{\varepsilon'} \Omega &= \int_{\varepsilon'_H}^{\varepsilon'} \left\{ \varepsilon - S_{-i}^H(p_0) \right\} (p_0 - \eta) - C \left[\varepsilon - S_{-i}^H(p_0) \right] f(\varepsilon) d\varepsilon + \\ &+ \int_{\varepsilon'}^{\varepsilon''} \left\{ \varepsilon' - S_{-i}^H(p_0) + R(\varepsilon - \varepsilon', \varepsilon'' - \varepsilon' - \Delta S_{-i}^H, \Delta S_{-i}^H) \right\} p_0 + \\ &- C \left[\varepsilon' - S_{-i}^H(p_0) + R(\varepsilon - \varepsilon', \varepsilon'' - \varepsilon' - \Delta S_{-i}^H, \Delta S_{-i}^H) \right] f(\varepsilon) d\varepsilon. \end{aligned}$$

Thus¹³

¹³ Recall that $R(0, \Delta S_i(p_0), \Delta S_{-i}(p_0)) = 0$.

$$\begin{aligned} \frac{\partial \Omega}{\partial \varepsilon'} &= \left\{ \varepsilon' - S_{-i}^H(p_0) \right\} [p_0 - \eta] - C(\varepsilon' - S_{-i}^H(p_0)) \left\{ f(\varepsilon') + \right. \\ &- \left. \left\{ \varepsilon' - S_{-i}^H(p_0) \right\} p_0 - C[\varepsilon' - S_{-i}^H(p_0)] \right\} f(\varepsilon') + \\ &+ \int_{\varepsilon'}^{\varepsilon''} \left[1 + \frac{dR(\varepsilon - \varepsilon', \varepsilon'' - \varepsilon' - \Delta S_{-i}^H, \Delta S_{-i}^H)}{d\varepsilon'} \right] [p_0 - C'(\cdot)] f(\varepsilon) d\varepsilon. \end{aligned}$$

In order to keep the potential equilibrium with discontinuous supply functions at p_0 , $\varepsilon' = \varepsilon'_H$ must be optimal.

$$\frac{\partial \Omega}{\partial \varepsilon'} \Big|_{\varepsilon' = \varepsilon'_H} = - \left\{ \varepsilon'_H - S_{-i}^H(p_0) \right\} f(\varepsilon'_H) + \int_{\varepsilon'_H}^{\varepsilon''} \left\{ [1 - R_1(\cdot) - R_2(\cdot)] [p_0 - C'(\cdot)] f(\varepsilon) d\varepsilon \right\}. \quad (29)$$

The first term is negative but infinitesimally small, as η is infinitesimally small. It is known from the proof of Proposition 2 that the second term is positive and bounded away from zero.

Thus $\frac{\partial \Omega}{\partial \varepsilon'} \Big|_{\varepsilon' = \varepsilon'_H} > 0$. Hence, producer i will find it profitable to deviate by slightly reducing

the price of his perfectly elastic supply at p_0 , i.e. $\varepsilon' > \varepsilon'_H$ for the optimal ε' . Accordingly, symmetric supply function equilibria with perfectly elastic segments can be ruled out when supply functions are perfectly inelastic just below p_0 .

Proof of Proposition 4

Consider a symmetric equilibrium candidate with a discontinuity in the price at $\varepsilon_L > 0$.

Denote its upper price by p_U and its lower by p_L . Denote the equilibrium candidate by \tilde{S}_i . All considered supply functions are twice continuously differentiable in some price interval $[p_-, p_L]$, see Section 2. Equilibria with perfectly elastic segments are ruled out in Section 3.2. Furthermore, smooth transitions to a perfectly inelastic supply are ruled out in Lemma 1. Thus $0 < \tilde{S}'_{il}(p_L) < \infty$ (the left derivative). Further, a sufficiently large p_- can be chosen such that $0 < \tilde{S}'_i(p) < \infty$ for all $p \in [p_-, p_L]$, i.e. neither of the slope constraints bind just below p_L .

Now consider the following deviation strategy for producer i : leave the supply above p_U and below p_- unchanged, increase the bids for the production units offered at and just below p_L and offer them at a price $p_D \in (p_L, p_U)$ instead. If it is optimal to change the bids for a positive number of units, the deviation is more profitable than the equilibrium strategy and the equilibrium can be knocked out. Whether this occurs can be investigated using an optimal

control problem similar to (7) but with an added final value. The final value considers the contribution to expected profit from units sold at the price p_D .

$$\begin{aligned} \text{Max}_{p(\varepsilon)} \int_{\varepsilon_-}^{\varepsilon'} \left\{ \left[\varepsilon - \tilde{S}_{-i}(p(\varepsilon)) \right] p(\varepsilon) - C(\varepsilon - \tilde{S}_{-i}(p(\varepsilon))) \right\} f(\varepsilon) d\varepsilon + F(\varepsilon') \\ \text{s.t. } u = p'(\varepsilon) \quad 0 \leq u \leq \frac{1}{\tilde{S}_{-i}'(p(\varepsilon))} \\ p(\varepsilon_-) = p_- \quad p(\varepsilon') = p_L \end{aligned} \quad (30)$$

where

$$F(\varepsilon') = \int_{\varepsilon'}^{\varepsilon_L} \left\{ \left(\varepsilon - \tilde{S}_{-i}(p_L) \right) p_D - C(\varepsilon - \tilde{S}_{-i}(p_L)) \right\} f(\varepsilon) d\varepsilon. \quad (31)$$

The slope constraints $0 \leq u \leq \frac{1}{\tilde{S}_{-i}'(p(\varepsilon))}$ may bind for $\varepsilon \in [\varepsilon_-, \varepsilon']$ if there is a profitable

deviation from $\tilde{S}_i(p)$, i.e. $\tilde{S}_i(p)$ is not an equilibrium. However, as in Section 3.1, the slope constraints can be disregarded when a necessary condition for $\tilde{S}_i(p)$ is derived under the assumption that $\tilde{S}_i(p)$ is a SFE.

The Hamiltonian, the Max H condition and the equations of motion are the same as for the optimal control problem in (7) (Léonard and van Long, 1992). In particular $\lambda(\varepsilon) \equiv 0$ for

$\varepsilon \in [\varepsilon_-, \varepsilon']$. The transversality condition associated with the terminal constraint at the right end-point is (Léonard and van Long, 1992)

$$H(u, p, \lambda, \varepsilon') + \frac{\partial F(\varepsilon')}{\partial \varepsilon'} = 0.$$

From (8), (10), and (31) we get

$$\begin{aligned} H(u, p, \lambda, \varepsilon') + \frac{\partial F(\varepsilon')}{\partial \varepsilon'} = \left\{ \left[\varepsilon' - \tilde{S}_{-i}(p_L) \right] p_L - C(\varepsilon' - \tilde{S}_{-i}(p_L)) \right\} f(\varepsilon') + \\ - \left\{ \left(\varepsilon' - \tilde{S}_{-i}(p_L) \right) p_D - C(\varepsilon' - \tilde{S}_{-i}(p_L)) \right\} f(\varepsilon'). \end{aligned}$$

The relation must be true for $\varepsilon' = \varepsilon_L$, otherwise $\tilde{S}_i(p)$ cannot be part of an equilibrium:

$$H(u, p, \lambda, \varepsilon_L) + \frac{\partial F(\varepsilon')}{\partial \varepsilon'} \Big|_{\varepsilon'=\varepsilon_L} = \underbrace{\left[\varepsilon_L - \tilde{S}_{-i}(p_L) \right]}_{=\tilde{S}_i(p_L) > 0} \underbrace{(p_L - p_D)}_{-} f(\varepsilon_L) < 0.$$

Thus the transversality condition cannot be fulfilled for equilibrium candidates with a discontinuity in the price. The marginal value of continuing, i.e. increasing ε' , is less than the

marginal loss in final value. Thus, as in Figure 4, any producer will find it profitable to decrease ε' and raise the price for some production units offered just below p_L .