IFN Working Paper No. 1327, 2020

General Revealed Preference Tests of Weak Separability and Utility Maximization with Incomplete Adjustment

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March 2020

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JEL Codes: C14; C60; D01; D10; E40

Key Words: Aggregation; Incomplete Adjustment; Revealed Preference; Weak Separability

ABSTRACT

We propose more general non-parametric revealed preference tests for weak separability and utility maximization with incomplete adjustment. Hence, these procedures account for a decision maker's inability to adjust his optimal allocation of the demanded goods and assets. Incomplete adjustment is especially important when modelling preferences of durable goods and assets. The procedures are based on a computationally attractive integer programming approach. Two empirical applications show that it is important to account for incomplete adjustment in consumer demand models of durable consumption goods and monetary assets.

* Hjertstrand thank Jan Wallander och Tom Hedelius stiftelse and Marianne och Marcus Wallenberg stiftelse for financial support.

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1. INTRODUCTION

That decision makers cannot instantaneously *adjust* to their optimal long-term equilibrium demand of goods and assets has been recognized in economics since at least Böhm-Bawerk (1888) and Keynes (1936). Kydland and Prescott (1982) argue that incomplete adjustment is important when modelling the aggregate economy, while others have argued and shown that various microeconomic decisions including those on capital, monetary and financial assets, and consumer durables are characterized by incomplete adjustment (See e.g., the recent work by Swofford and Whitney, 1988; Fleissig and Swofford, 1996; Jones et al., 2005; Elger et al., 2008; Jha and Longjam, 2006).[1] Incomplete adjustment may arise because of habit persistence, adjustment costs, the formation of expectations, or a combination of reasons. Koyck (1954) and Almon (1965) developed distributed lag models to account for incomplete adjustment in regression analysis.[2] Swofford and Whitney (1994) allow for incomplete adjustment in revealed preference tests.

In this paper, we propose more general and computationally efficient non-parametric revealed preference tests for weak separability and utility maximization that allow for incomplete adjustment. Utility maximization is a core concept in economics and the main underlying assumption for all rational choice theory. Weak separability is another key concept in economics: It establishes the fundamental link between aggregation of goods and the maximization principles, provides means of dividing the economy into sectors, and produces powerful parameter restrictions that enhance estimation of large scale demand systems.[3] This led Barnett

[1] See Hillinger (1996) for a historical overview of the theory of dynamic disequilibrium in economics.
[2] The theory of cointegration can be seen as a statistical expression of a more general notion of (dis)equilibrium in economics (Engle and Granger, 1987).
[3] A group of goods is weakly separable if the demanded quantities of these goods depend solely on the prices of the goods in the group (and not on the price of any outside good other than through total expenditure). This allows for a separation of the separable goods into a sub-utility or aggregator function.

and Choi (1989) to argue that weak separability is fundamental to all empirical research in order to avoid the structure of the economy becoming prohibitively difficult to model empirically.

Weak separability and utility maximization are often tested using revealed preference methods. These procedures are non-parametric in the sense that they are free from any assumption of the functional form of the utility function, and consequently, avoids any problems associated with model misspecification. Varian (1983) and Fleissig and Whitney (2003) proposed different revealed preference procedures to test for weak separability. More recently, Cherchye, Demuynck, De Rock and Hjertstrand (2015) introduced a computationally efficient test-procedure based on solving a mixed integer linear programming problem. However, none of these test-procedures is able to account for incomplete adjustment in the data.

Swofford and Whitney's (1994) revealed preference test for weak separability accounts for incomplete adjustment, but only allow the goods in the weakly separable block to be chosen with incomplete adjustment. In contrast, the revealed preference test proposed in this paper is very general since it allow goods in both the weakly separable and non-weakly separable blocks to be chosen with incomplete adjustment.[4]

Our procedure is computationally efficient and shares many of the computational advantages of Cherchye et al.'s (2015) procedure. Specifically, like theirs, our procedure is based on solving a problem with linear restrictions. In practice, our procedure is computationally only marginally more involved since it is based on minimizing a quadratic objective function while the objective in Cherchye et al.'s (2015) problem is linear. Thus, compared to Swofford and Whitney's (1994)

[4] Crawford (2010) give necessary and sufficient revealed preference conditions for the habits model, i.e., when lags of the demanded quantities enter as arguments in the utility function. Our framework is conceptually different from the habits model since it is based on a notion of disequilibrium.

procedure, which is based on finding a solution to a non-linear optimization problem with non-linear restrictions, our new method is considerably easier to implement.[5]

We apply our test-procedure for weak separability with incomplete adjustment to two data sets. The first application is to aggregated consumption and monetary U.S. quarterly data, partly used in Hjertstrand, Swofford and Whitney (2016). We test whether nine monetary and real-sector/consumption aggregates are weakly separable from all other goods. We find that five of these aggregates are weakly separable with considerable amounts of incomplete adjustment. These results show that it is important to allow for incomplete adjustment when modelling weak separability of monetary assets and durable consumption goods using data sampled on a relatively short time frequency. We also aggregate our data to obtain yearly observations. In this case, we find that eight (of the nine) aggregates are weakly separable with essentially zero amounts of incomplete adjustment. Thus, we find that it is much more important to account for incomplete adjustment in quarterly than in yearly data.

In the second application, we apply our test-procedure to disaggregated survey (micro) data over Spanish household expenditures for 25 durable and non-durable consumption goods and services for 1,585 households. It is common in empirical analyses of demand and consumption patterns over the life-cycle to assume that durable goods are weakly separable from non-durable goods and services.[6] We test whether this is a valid assumption and find that the data for almost 85 % of all households satisfy weak separability. 33 % of the households satisfy weak

[5] At the time of writing their article, Swofford and Whitney (1994) had to use a CRAY super-computer to implement their procedure, which included dividing the sample of 62 observations into two overlapping sub-samples of 40 observations each.

[6] See, for example, Browning and Collado (2001), Demuynck and Verriest (2013) and Luengo-Prado and Sevilla (2012).

separability with incomplete adjustment, which, in our view, shows that it is important to account for incomplete adjustment in demand modelling of durable goods in micro survey data.

In the next few sections, we present our general notion of weak separability with incomplete adjustment, discuss some special cases, and show by example how important incomplete adjustment can be. Then we show how we account for incomplete adjustment in a standard utility maximization model (i.e., without assuming weakly separable preferences).

2. INCOMPLETE ADJUSTMENT

2.1 Weak Separability with Incomplete Adjustment

Suppose that there are k goods and assets observed in the market.[7] Suppose that these k goods are split into two mutually exclusive blocks. Let $\boldsymbol{x}$ and $\boldsymbol{m}$ denote column vectors of the quantity data of the first and second blocks, respectively. Let $\boldsymbol{p}$ denote a row vector of the prices for the $\boldsymbol{x}$-goods and $\boldsymbol{r}$ denote a row vector of the prices for the $\boldsymbol{m}$-goods.

We assume that each block may contain durable and non-durable goods, and that the durable goods may be subject to incomplete adjustment. That is, the decision maker (DM) may fail to adjust optimal consumption allocations for the durable goods. In contrast, we assume that the DM is able to fully adjust optimal consumption for the non-durable goods. Let $\boldsymbol{x_D}$ and $\boldsymbol{x_{ND}}$ denote the durable and non-durable goods in $\boldsymbol{x}$, respectively. Let $\boldsymbol{p_D}$ and $\boldsymbol{p_{ND}}$ denote the prices of $\boldsymbol{x_D}$ and $\boldsymbol{x_{ND}}$. Analogously, let $\boldsymbol{m_D}$ and $\boldsymbol{m_{ND}}$ denote the durable and non-durable goods in $\boldsymbol{m}$, and let $\boldsymbol{r_D}$ and $\boldsymbol{r_{ND}}$ denote the prices of $\boldsymbol{m_D}$ and $\boldsymbol{m_{ND}}$. Suppose there are n observations on the prices and quantities and let the i^{th} observation of the prices be denoted $\boldsymbol{p^i} = (\boldsymbol{p^i_D}, \boldsymbol{p^i_{ND}})$ and $\boldsymbol{r^i} = (\boldsymbol{r^i_D}, \boldsymbol{r^i_{ND}})$, while the i^{th} observation of the quantities is denoted $\boldsymbol{x^i} = (\boldsymbol{x^i_D}, \boldsymbol{x^i_{ND}})$ and $\boldsymbol{m^i} = (\boldsymbol{m^i_D}, \boldsymbol{m^i_{ND}})$. We write:

[7] For compactness, we will refer to all goods and assets simply as "goods".

$$\mathbb{D} = \{(\boldsymbol{p_D^i}, \boldsymbol{p_{ND}^i}), (\boldsymbol{r_D^i}, \boldsymbol{r_{ND}^i}); (\boldsymbol{x_D^i}, \boldsymbol{x_{ND}^i}), (\boldsymbol{m_D^i}, \boldsymbol{m_{ND}^i})\}_{i=1,\ldots,n},$$

to signify all price-quantity observations and refer to $\mathbb{D}$ as "the data".

The DM's utility function is $u(\boldsymbol{x}, \boldsymbol{m}) = u(\boldsymbol{x_D}, \boldsymbol{x_{ND}}, \boldsymbol{m_D}, \boldsymbol{m_{ND}})$. The utility function is weakly separable in the $\boldsymbol{m}$ block of goods if there exists a macro function U and a sub-utility function V such that u can be written as $u(\boldsymbol{x_D}, \boldsymbol{x_{ND}}, \boldsymbol{m_D}, \boldsymbol{m_{ND}}) = U(\boldsymbol{x_D}, \boldsymbol{x_{ND}}, V(\boldsymbol{m_D}, \boldsymbol{m_{ND}}))$. Note that this model allows for durable and non-durable goods in the weakly separable block, $\boldsymbol{m}$, as well as in the non-separable block of goods, $\boldsymbol{x}$.

Our general notion of weak separability with incomplete adjustment is based on the idea that the DM, at each observation $i = 1, \ldots, n$, solves an overall utility maximization problem and a sub-utility maximization problem involving a sub-set of the goods. Specifically, the DM solves the overall utility maximization problem:

$$\max_{\{\boldsymbol{x_D}, \boldsymbol{x_{ND}}, \boldsymbol{m_D}, \boldsymbol{m_{ND}}\}} U(\boldsymbol{x_D}, \boldsymbol{x_{ND}}, V(\boldsymbol{m_D}, \boldsymbol{m_{ND}})), \quad (1)$$

subject to the three different budget constraints in (2)-(4). The standard overall budget constraint puts a restriction on the total outlay for all goods:

$$\boldsymbol{p}_D^i \boldsymbol{x_D} + \boldsymbol{p}_{ND}^i \boldsymbol{x_{ND}} + \boldsymbol{r}_D^i \boldsymbol{m_D} + \boldsymbol{r}_{ND}^i \boldsymbol{m_{ND}} \le Y^i, \quad (2)$$

where $Y^i$is total expenditure on all nondurable and durable goods. The second restriction imposes a constraint on expenditure for the durable goods in the non-separable block $\boldsymbol{x}$:

$$\boldsymbol{p}_D^i \boldsymbol{x_D} \le Y_D^i, \quad (3)$$

where Y_D^i denotes total expenditure on the durable goods $\boldsymbol{x_D}$. Finally, the third restriction imposes a constraint on expenditure of the durable goods in the separable block $\boldsymbol{m}$:

$$\boldsymbol{r}_D^i \boldsymbol{m_D} \le E_D^i, \quad (4)$$

where E_D^i is total expenditure on the durables $\boldsymbol{m_D}$.

The sub-utility maximization problem solved by the DM is given by:

$$\max_{\{\boldsymbol{m_D}, \boldsymbol{m_{ND}}\}} V(\boldsymbol{m_D}, \boldsymbol{m_{ND}}), \tag{5}$$

subject to the two budget constraints (4) and (6). The constraint (6) puts a restriction on total outlay for all separable goods ($\boldsymbol{m}$):

$$\boldsymbol{r}_D^i \boldsymbol{m_D} + \boldsymbol{r}_{ND}^i \boldsymbol{m_{ND}} \leq E^i, \tag{6}$$

where $E^i = \boldsymbol{r}_D^i \boldsymbol{m_D^i} + \boldsymbol{r}_{ND}^i \boldsymbol{m_{ND}^i}$ is total expenditure on the separable goods. The constraint (4) is equivalent to equation (4) above and puts a restriction on expenditure of the durable goods in the sub-group.

The Lagrangian to the overall utility maximization problem in (1)-(4) is:

$$L_U = U\big(\boldsymbol{x_D}, \boldsymbol{x_{ND}}, V(\boldsymbol{m_D}, \boldsymbol{m_{ND}})\big) + \tau^i\big[Y^i - \boldsymbol{p}_D^i \boldsymbol{x_D} - \boldsymbol{p}_{ND}^i \boldsymbol{x_{ND}} - \boldsymbol{r}_D^i \boldsymbol{m_D} - \boldsymbol{r}_{ND}^i \boldsymbol{m_{ND}}\big]$$

$$+\Omega^i\big[Y_D^i - \boldsymbol{p}_D^i \boldsymbol{x_D}\big] + \Theta_U^i\big[E_D^i - \boldsymbol{r}_D^i \boldsymbol{m_D}\big]$$

$$= U\big(\boldsymbol{x_D}, \boldsymbol{x_{ND}}, V(\boldsymbol{m_D}, \boldsymbol{m_{ND}})\big) + \tau^i\big[Y^i - \boldsymbol{p}_D^i \boldsymbol{x_D} - \boldsymbol{p}_{ND}^i \boldsymbol{x_{ND}} - \boldsymbol{r}_D^i \boldsymbol{m_D} - \boldsymbol{r}_{ND}^i \boldsymbol{m_{ND}}\big]$$

$$+\tau^i\left(\frac{\Omega^i}{\tau^i}\right)\big[Y_D^i - \boldsymbol{p}_D^i \boldsymbol{x_D}\big] + \tau^i\left(\frac{\Theta_U^i}{\tau^i}\right)\big[E_D^i - \boldsymbol{r}_D^i \boldsymbol{m_D}\big]. \tag{7}$$

τ^i is the Lagrange multiplier corresponding to the overall budget constraint $Y^i = \boldsymbol{p}_D^i \boldsymbol{x_D^i} + \boldsymbol{p}_{ND}^i \boldsymbol{x_{ND}^i} + \boldsymbol{r}_D^i \boldsymbol{m_D^i} + \boldsymbol{r}_{ND}^i \boldsymbol{m_{ND}^i}$, and can be interpreted as the marginal utility of total expenditure (i.e., $\tau^i = \partial U\big(\boldsymbol{x_D}, \boldsymbol{x_{ND}}, V(\boldsymbol{m_D}, \boldsymbol{m_{ND}})\big)/\partial Y^i$). Ω^i is the Lagrange multiplier for the budget constraint on the durable goods in $\boldsymbol{x}$, $Y_D^i = \boldsymbol{p}_D^i \boldsymbol{x_D^i}$, and thus corresponds to the marginal utility of expenditure on these durable goods (i.e., $\Omega^i = \partial U\big(\boldsymbol{x_D}, \boldsymbol{x_{ND}}, V(\boldsymbol{m_D}, \boldsymbol{m_{ND}})\big)/\partial Y_D^i$). This number is a function of prices $\boldsymbol{p}_D^i$ and expenditure Y_D^i and equals zero when expenditure on $\boldsymbol{x_D}$ is optimally adjusted. If Ω^i is negative then expenditure on these goods is greater than desired, and if Ω^i is positive then expenditure is less than desired.

Analogously, Θ_U^i is the Lagrange multiplier for the budget constraint on the durable goods in $\boldsymbol{m}$, $E_D^i = \boldsymbol{r}^i{}_D \boldsymbol{m}_{\boldsymbol{D}}^{\boldsymbol{i}}$, and corresponds to the marginal utility of expenditure on $\boldsymbol{m_D}$ (i.e., $\Theta_U^i = \partial U\big(\boldsymbol{x_D}, \boldsymbol{x_{ND}}, V(\boldsymbol{m_D}, \boldsymbol{m_{ND}})\big)/\partial E_D^i$). As such, it is a function of prices $\boldsymbol{r}_D^i$ and expenditure E_D^i, and negative (positive) when expenditure is greater (less) than desired. When expenditure on $\boldsymbol{m_D}$ is optimally adjusted, then Θ_U^i is effectively zero.

The multipliers Ω^i and Θ_U^i may be seen as measures of deviations from the optimal levels of expenditure for the durable goods $\boldsymbol{x_D}$ and $\boldsymbol{m_D}$, respectively. However, a problem with using Ω^i and Θ_U^i as measures of the degree of incomplete adjustment is that they are not invariant to monotonic transformations of utility and may therefore be difficult to interpret. Below, these numbers appear in the definition of a set of "virtual prices" as Ω^i/τ^i and Θ_U^i/τ^i. Specifically, the ratios, Ω^i/τ^i and Θ_U^i/τ^i represents the increments of overall utility from spending an additional dollar on the durable goods $\boldsymbol{x_D}$ and $\boldsymbol{m_D}$ relative to the marginal utility of total expenditure for all goods, respectively. Since these measures are ratios of marginal utilities, they are, by construction ordinal, and consequently, invariant to any monotonic transformations of utility. Hence, we interpret Ω^i/τ^i and Θ_U/τ^i as the overall amounts of incomplete adjustment, and define:

$$\mathrm{IA}_\Omega^i = \frac{\partial U\big(\boldsymbol{x_D}, \boldsymbol{x_{ND}}, V(\boldsymbol{m_D}, \boldsymbol{m_{ND}})\big)/\partial Y_D^i}{\partial U\big(\boldsymbol{x_D}, \boldsymbol{x_{ND}}, V(\boldsymbol{m_D}, \boldsymbol{m_{ND}})\big)/\partial Y^i} = \frac{\Omega^i}{\tau^i}, \tag{8}$$

and,

$$\mathrm{IA}_\Theta^i = \frac{\partial U\big(\boldsymbol{x_D}, \boldsymbol{x_{ND}}, V(\boldsymbol{m_D}, \boldsymbol{m_{ND}})\big)/\partial E_D^i}{\partial U\big(\boldsymbol{x_D}, \boldsymbol{x_{ND}}, V(\boldsymbol{m_D}, \boldsymbol{m_{ND}})\big)/\partial Y^i} = \frac{\Theta_U^i}{\tau^i}, \tag{9}$$

for all observations $i = 1, \dots, n$.

Using these definitions, we can write the first-order conditions of the Lagrangian, L_U, in (7) as:

$$\frac{\partial U(\boldsymbol{x_D}, \boldsymbol{x_{ND}}, V(\boldsymbol{m_D}, \boldsymbol{m_{ND}}))}{\partial \boldsymbol{x^i_{ND}}} = \tau^i \boldsymbol{p}^i_{ND}, \tag{10}$$

$$\frac{\partial U(\boldsymbol{x_D}, \boldsymbol{x_{ND}}, V(\boldsymbol{m_D}, \boldsymbol{m_{ND}}))}{\partial \boldsymbol{x^i_D}} = \tau^i \left(1 + \frac{\Omega^i}{\tau^i}\right) \boldsymbol{p}^i_D = \tau^i (1 + \mathrm{IA}^i_\Omega) \boldsymbol{p}^i_D, \tag{11}$$

$$\frac{\partial U(\boldsymbol{x_D}, \boldsymbol{x_{ND}}, V(\boldsymbol{m_D}, \boldsymbol{m_{ND}}))}{\partial \boldsymbol{m^i_{ND}}} = \frac{\partial U(\boldsymbol{x_D}, \boldsymbol{x_{ND}}, V(\boldsymbol{m_D}, \boldsymbol{m_{ND}}))}{\partial V(\boldsymbol{m^i_D}, \boldsymbol{m^i_{ND}})} \frac{\partial V(\boldsymbol{m_D}, \boldsymbol{m_{ND}})}{\partial \boldsymbol{m^i_{ND}}}$$

$$= \tau^i \boldsymbol{r}^i_{ND}, \tag{12}$$

$$\frac{\partial U(\boldsymbol{x_D}, \boldsymbol{x_{ND}}, V(\boldsymbol{m_D}, \boldsymbol{m_{ND}}))}{\partial \boldsymbol{m^i_D}} = \frac{\partial U(\boldsymbol{x_D}, \boldsymbol{x_{ND}}, V(\boldsymbol{m_D}, \boldsymbol{m_{ND}}))}{\partial V(\boldsymbol{m^i_D}, \boldsymbol{m^i_{ND}})} \frac{\partial V(\boldsymbol{m_D}, \boldsymbol{m_{ND}})}{\partial \boldsymbol{m^i_D}}$$

$$= \tau^i \left(1 + \frac{\Theta^i_U}{\tau^i}\right) \boldsymbol{r}^i_D = \tau^i (1 + \mathrm{IA}^i_\Theta) \boldsymbol{r}^i_D. \tag{13}$$

The Lagrangian of the sub-utility maximization problem in (4)-(6) is:

$$L_V = V(\boldsymbol{m_D}, \boldsymbol{m_{ND}}) + \mu^i [E^i - \boldsymbol{r}^i_D \boldsymbol{m_D} - \boldsymbol{r}^i_{ND} \boldsymbol{m_{ND}}] + \Theta^i_V [E^i_D - \boldsymbol{r}^i_D \boldsymbol{m_D}]$$

$$= V(\boldsymbol{m_D}, \boldsymbol{m_{ND}}) + \mu^i [E^i - \boldsymbol{r}^i_D \boldsymbol{m_D} - \boldsymbol{r}^i_{ND} \boldsymbol{m_{ND}}] + \mu^i \left(\frac{\Theta^i_V}{\mu^i}\right) [E^i_D - \boldsymbol{r}^i_D \boldsymbol{m_D}]. \tag{14}$$

The Lagrange multiplier μ^i can be interpreted as the marginal utility of sub-group expenditure (i.e., $\mu^i = \partial V(\boldsymbol{m_D}, \boldsymbol{m_{ND}})/\partial E^i$). The number Θ^i_V corresponds to the marginal utility of expenditure on the durable goods in the sub-group (i.e., $\Theta^i_V = \partial V(\boldsymbol{m_D}, \boldsymbol{m_{ND}})/\partial E^i_D$). Thus, Θ_V/μ represents the increment of sub-group utility from spending an additional dollar on the durable goods in the sub-group relative to the marginal utility of total expenditure of the goods in the sub-group. Hence, we interpret Θ_V/μ as the *a*mount of incomplete adjustment for the sub-group, and define:

$$\mathrm{IA}_V^i = \frac{\partial V(\boldsymbol{m_D}, \boldsymbol{m_{ND}})/\partial E_D^i}{\partial V(\boldsymbol{m_D}, \boldsymbol{m_{ND}})/\partial E^i} = \frac{\Theta_V^i}{\mu^i}, \tag{15}$$

for all observations $i = 1, \dots, n$.

The first-order conditions of the Lagrangian, L_V, in (14) are:

$$\frac{\partial V(\boldsymbol{m_D}, \boldsymbol{m_{ND}})}{\partial \boldsymbol{m_{ND}^i}} = \mu^i \boldsymbol{r}_{ND}^i, \tag{16}$$

$$\frac{\partial V(\boldsymbol{m_D}, \boldsymbol{m_{ND}})}{\partial \boldsymbol{m_D^i}} = \mu^i \left(1 + \frac{\Theta_V^i}{\mu^i}\right) \boldsymbol{r}_D^i = \mu^i \left(1 + \mathrm{IA}_V^i\right) \boldsymbol{r}_D^i. \tag{17}$$

The first-order condition (13) of L_U gives the prices of the durable goods in the separable block, $\boldsymbol{r_D}$, at which the optimal quantities, $\boldsymbol{m_D}$ are demanded. Analogously, (17) gives a set of equivalent conditions. These prices, denoted by $\left(1 + \mathrm{IA}_\Theta^i\right)\boldsymbol{r}_D^i$ in (13) and $\left(1 + \mathrm{IA}_V^i\right)\boldsymbol{r}_D^i$ in (17) are so called "virtual prices" of the constrained goods $\boldsymbol{m_D}$, and gives the prices at which the optimal bundle $\boldsymbol{m_D}$ is demanded in long-run equilibrium. Clearly, these virtual prices must be the same, in which case we must have $\left(1 + \mathrm{IA}_\Theta^i\right)\boldsymbol{r}_D^i = \left(1 + \mathrm{IA}_V^i\right)\boldsymbol{r}_D^i$. Thus, this implies $\mathrm{IA}_\Theta^i = \mathrm{IA}_V^i$, and we define the virtual prices as $\tilde{\boldsymbol{r}}_D^i = \left(1 + \mathrm{IA}^i\right)\boldsymbol{r}_D^i = \left(1 + \mathrm{IA}_\Theta^i\right)\boldsymbol{r}_D^i = \left(1 + \mathrm{IA}_V^i\right)\boldsymbol{r}_D^i$, where $\mathrm{IA}^i = \mathrm{IA}_\Theta^i = \mathrm{IA}_V^i$.

Analogously, the virtual prices for the durable goods in the non-separable block, $\left(1 + \mathrm{IA}_\Omega^i\right)\boldsymbol{p}_D^i$, in the first-order condition (11) of L_U, gives the prices at which the optimal bundle $\boldsymbol{x_D}$ is demanded in long-run equilibrium. We assume that the degree of incomplete adjustment, IA_Ω^i, is the same as IA_Θ^i and IA_V^i, and we define the virtual prices as $\widetilde{\boldsymbol{p}}_D^i = \left(1 + \mathrm{IA}^i\right)\boldsymbol{p}_D^i = \left(1 + \mathrm{IA}_\Omega^i\right)\boldsymbol{p}_D^i$, where $\mathrm{IA}^i = \mathrm{IA}_\Omega^i$.[8]

[8] This assumption is more restrictive than allowing the incomplete adjustment to vary between the separable and non-separable blocks. Thus, in a sense, our test is sufficient but not necessary as there could be solutions where the incomplete adjustment is different for the two categories of durable goods.

Given the virtual prices $\tilde{\boldsymbol{p}}_D^i$ and $\tilde{\boldsymbol{r}}_D^i$, the first-order conditions (10)-(13) and (16)-(17) has three important implications. First, substituting (16) into (12), and (17) into (13) gives for any good $l = 1, \dots, k$:

$$\frac{\partial U\big(\boldsymbol{x_D}, \boldsymbol{x_{ND}}, V(\boldsymbol{m_D}, \boldsymbol{m_{ND}})\big)}{\partial V\big(\boldsymbol{m_D^i}, \boldsymbol{m_{ND}^i}\big)} = \frac{\tau^i}{\mu^i}. \tag{18}$$

Second, our model of incomplete adjustment defined by the overall utility and sub-utility maximization problems in (1)-(4) and (4)-(6) is identical to a model that solves the following overall utility maximization problem:

$$\max_{\{\boldsymbol{x_D}, \boldsymbol{x_{ND}}, \boldsymbol{m_D}, \boldsymbol{m_{ND}}\}} U\big(\boldsymbol{x_D}, \boldsymbol{x_{ND}}, V(\boldsymbol{m_D}, \boldsymbol{m_{ND}})\big) \quad subject\ to$$

$$\tilde{\boldsymbol{p}}_D^i \boldsymbol{x_D} + \boldsymbol{p}_{ND}^i \boldsymbol{x_{ND}} + \tilde{\boldsymbol{r}}_D^i \boldsymbol{m_D} + \boldsymbol{r}_{ND}^i \boldsymbol{m_{ND}} \le \tilde{Y}^i, \tag{19}$$

where $\tilde{Y}^i = \tilde{\boldsymbol{p}}_D^i \boldsymbol{x_D^i} + \boldsymbol{p}_{ND}^i \boldsymbol{x_{ND}^i} + \tilde{\boldsymbol{r}}_D^i \boldsymbol{m_D^i} + \boldsymbol{r}_{ND}^i \boldsymbol{m_{ND}^i}$ is the "virtual expenditure" on all goods and also solves the following sub-utility maximization problem:

$$\max_{\{\boldsymbol{m_D}, \boldsymbol{m_{ND}}\}} V(\boldsymbol{m_D}, \boldsymbol{m_{ND}}) \quad subject\ to$$

$$\boldsymbol{r}_{ND}^i \boldsymbol{m_{ND}} + \tilde{\boldsymbol{r}}_D^i \boldsymbol{m_D} \le \tilde{E}^i, \tag{20}$$

where $\tilde{E}^i = \boldsymbol{r}_{ND}^i \boldsymbol{m}_{\boldsymbol{ND}}^i + \tilde{\boldsymbol{r}}_D^i \boldsymbol{m}_{\boldsymbol{D}}^i$ is the "virtual expenditure" on the goods in the sub-group.

Third, the first-order conditions allow us to define and classify weak separability with complete or incomplete adjustment: We say that the $\boldsymbol{m}$-goods are weakly separable with *complete adjustment* if the first-order conditions (10)-(13) and (16)-(17) hold with $\text{IA} = 0$. In this case, they reduce to the first-order conditions from the standard weakly separable utility maximization model.[9] The $\boldsymbol{m}$-goods are said to be weakly separable with *incomplete adjustment*

[9] Thus, for the standard weakly separable utility maximization model (i.e., with complete adjustment), the first-order conditions (17), (11) and (13) becomes: $\partial V(\boldsymbol{m_D}, \boldsymbol{m_{ND}})/\partial \boldsymbol{m_D^i} = \mu^i \boldsymbol{r}_D^i$, $\partial U\big(\boldsymbol{x_D}, \boldsymbol{x_{ND}}, V(\boldsymbol{m_D}, \boldsymbol{m_{ND}})\big)/\partial \boldsymbol{x_D^i} = \tau^i \boldsymbol{p}_D^i$ and $\partial U\big(\boldsymbol{x_D}, \boldsymbol{x_{ND}}, V(\boldsymbol{m_D}, \boldsymbol{m_{ND}})\big)/\partial \boldsymbol{m_D^i} = \tau^i \boldsymbol{r}_D^i$ (Varian, 1983, p.105).

if the first-order conditions (10)-(13) and (16)-(17) hold with $\text{IA} \neq 0$ (and $\text{IA} > -1$). Finally, the $\boldsymbol{m}$-goods are *not* weakly separable if the first-order conditions fail to hold for any value $\text{IA} > -1$.

In order to calculate the amount of incomplete adjustment in empirical applications, it is convenient to define the number:

$$\Psi^i = \Theta_V^i + \mu^i, \tag{21}$$

and express the amount of incomplete adjustment in terms of Ψ as:

$$\text{IA}^i = \frac{\Theta_V^i}{\mu^i} = \frac{\Psi^i - \mu^i}{\mu^i} = \frac{\Psi^i}{\mu^i} - 1. \tag{22}$$

Hence, by equation (22) we can interpret incomplete adjustment as a distance, since, in such case, it is defined as the difference between Ψ and μ normalized by μ. Clearly, if $\Psi^i = \mu^i$, or equivalently $\text{IA}^i = 0$, for all $i = 1, \dots, n$ then the $\boldsymbol{m}$-goods are weakly separable with *complete adjustment* but if $\Psi^i \neq \mu^i$, or equivalently $\text{IA}^i \neq 0$, for at least some i, then weak separability holds with *incomplete adjustment*.

In the example and applications of incomplete adjustment presented below we report the percentage incomplete adjustment (% IA^i) calculated as:

$$\%\ \text{IA}^i = 100 \times \left(\frac{\Psi^i}{\mu^i} - 1\right). \tag{23}$$

Thus, % IA is the percentage difference between μ and Ψ. Since incomplete adjustment concerns the DM's inability to fully adjust his optimal consumption allocation of durable goods within the observed time period, we interpret % IA as a unit-free measure of the DM's habit persistence, formation of expectations and adjustment costs.

2.2 Special cases

This section briefly discusses some special cases of our general model of incomplete adjustment. The first is when there are neither any durable goods in $\boldsymbol{x}$ nor any non-durable goods in $\boldsymbol{m}$, i.e., $\boldsymbol{x_D} = \boldsymbol{m_{ND}} = \emptyset$. This special case corresponds to Swofford and Whitney's (1994) model of incomplete adjustment.

The second model is when there are neither any non-durable goods in $\boldsymbol{x}$ nor any durable goods in $\boldsymbol{m}$, i.e., $\boldsymbol{x_{ND}} = \boldsymbol{m_D} = \emptyset$. If one interchanges the prices and quantities of the durable goods with the prices and quantities of the non-durable goods, the testable condition becomes that of Swofford and Whitney (1994). Thus, this special case corresponds to a model which only accounts for incomplete adjustment in the non-separable block.

The third model is when there are only non-durable goods in $\boldsymbol{x}$ and $\boldsymbol{m}$, i.e., $\boldsymbol{x_D} = \boldsymbol{m_D} = \emptyset$. This corresponds, of course, to the standard model of weakly separable utility maximization (with complete adjustment; See footnote 9).

Finally, the fourth special case is when there are only durable goods in $\boldsymbol{x}$ and $\boldsymbol{m}$, i.e. $\boldsymbol{x_{ND}} = \boldsymbol{m_{ND}} = \emptyset$. In this case, the first-order conditions become:

$$\frac{\partial U\big(\boldsymbol{x_D}, \boldsymbol{x_{ND}}, V(\boldsymbol{m_D}, \boldsymbol{m_{ND}})\big)}{\partial \boldsymbol{x_D^i}} = \tau^i \left(1 + \frac{\Omega^i}{\tau^i}\right) \boldsymbol{p}_D^i = \tau^i\big(1 + \mathrm{IA}_\Omega^i\big) = \tau^i\big(1 + \mathrm{IA}^i\big)\boldsymbol{p}_D^i,$$

$$\frac{\partial U\big(\boldsymbol{x_D}, \boldsymbol{x_{ND}}, V(\boldsymbol{m_D}, \boldsymbol{m_{ND}})\big)}{\partial \boldsymbol{m_D^i}} = \frac{\partial U\big(\boldsymbol{x_D}, \boldsymbol{x_{ND}}, V(\boldsymbol{m_D}, \boldsymbol{m_{ND}})\big)}{\partial V\big(\boldsymbol{m_D^i}, \boldsymbol{m_{ND}^i}\big)} \frac{\partial V(\boldsymbol{m_D}, \boldsymbol{m_{ND}})}{\partial \boldsymbol{m_D^i}}$$

$$= \tau^i \left(1 + \frac{\Theta_U^i}{\tau^i}\right) \boldsymbol{r}_D^i = \tau^i\big(1 + \mathrm{IA}_\Theta^i\big)\boldsymbol{r}_D^i = \tau^i\big(1 + \mathrm{IA}^i\big)\boldsymbol{r}_D^i,$$

$$\frac{\partial V(\boldsymbol{m_D}, \boldsymbol{m_{ND}})}{\partial \boldsymbol{m_D^i}} = \mu^i \left(1 + \frac{\Theta_V^i}{\mu^i}\right) \boldsymbol{r}_D^i = \mu^i\big(1 + \mathrm{IA}_V^i\big)\boldsymbol{r}_D^i = \mu^i\big(1 + \mathrm{IA}^i\big)\boldsymbol{r}_D^i.$$

By defining $\tilde{\tau}^i = \tau^i\big(1 + \mathrm{IA}^i\big)$ and $\tilde{\mu}^i = \mu^i\big(1 + \mathrm{IA}^i\big)$, we see that it is not possible to separately identify the degree of incomplete adjustment, IA, from the Lagrange multipliers.

Hence, with only durable goods, the weak separability model with incomplete adjustment is observationally equivalent to a model with complete adjustment. Consequently, in this case, incomplete adjustment does not have any testable implications, in which case, the model also corresponds to the standard model of weakly separable utility maximization (with complete adjustment).

2.3 A Parametric Example

It is illustrative to consider the effects of incomplete adjustment by means of a parametric example. Suppose that there are two non-durable goods $\boldsymbol{x} = \boldsymbol{x_{ND}} = (x_1, x_2)$ and two durable goods $\boldsymbol{m} = \boldsymbol{m_D} = (m_1, m_2)$, i.e., there are in total four goods ($k = 4$). Thus, this corresponds to a model of incomplete adjustment where there are only non-durable goods in the $\boldsymbol{x}$- block, and only durable goods in the $\boldsymbol{m}$- block.[10] The prices for the $\boldsymbol{x}$-goods are denoted $\boldsymbol{p} = \boldsymbol{p_{ND}} = (p_1, p_2)$ and the prices for the $\boldsymbol{m}$-goods are denoted $\boldsymbol{r} = \boldsymbol{r_D} = (r_1, r_2)$. Suppose that the utility function is:

$$u(\boldsymbol{x}, \boldsymbol{m}) = u(\boldsymbol{x_{ND}}, \boldsymbol{m_D}) = x_1 m_1 m_2 + \sqrt{x_2 m_1 m_2}.$$

This utility function is weakly separable in the (durable) $\boldsymbol{m}$-goods since it can be written as:

$$u(\boldsymbol{x}, \boldsymbol{m}) = u(\boldsymbol{x_{ND}}, \boldsymbol{m_D}) = U\big(\boldsymbol{x_{ND}}, V(\boldsymbol{m_D})\big) = x_1 V(\boldsymbol{m_D}) + \sqrt{x_2 V(\boldsymbol{m_D})},$$

where $V(\boldsymbol{m_D}) = V(m_1, m_2) = m_1 m_2$.

Solving the sub-utility maximization problem in (4)-(6) gives the conditional demand functions:

$$\tilde{m}_1 = \frac{E}{2r_1} \quad and \quad \tilde{m}_2 = \frac{E}{2r_2}.$$

The reduced form (overall) utility function is obtained by plugging in these solutions as:

[10] Thus, this setup corresponds to Swofford and Whitney's (1994) model of incomplete adjustment.

$$U\left(\boldsymbol{x_{ND}}, \tilde{V}(\boldsymbol{r_D}, E)\right) = x_1\tilde{V}(\boldsymbol{r_D}, E) + \sqrt{x_2\tilde{V}(\boldsymbol{r_D}, E)} = x_1\frac{(E)^2}{4r_1r_2} + \sqrt{x_2\frac{(E)^2}{4r_1r_2}},$$

where $\tilde{V}(\boldsymbol{r_D}, E) = \widetilde{m}_1\widetilde{m}_2 = (E)^2/4r_1r_2$ is the indirect sub-utility function corresponding to the sub-utility function $V(\boldsymbol{m_D}) = m_1m_2$. Solving the reduced form problem gives the optimal (unconditional) demand functions $(\bar{x}_1, \bar{x}_2)$ and the optimal allocation of sub-expenditure $\bar{E}$:

$$\bar{x}_1 = \frac{4p_2(Y)^3(3+\text{IA}) - 8p_2(Y)^3 - (p_1)^2r_1r_2(3+\text{IA})^2}{4p_1p_2(Y)^2(3+\text{IA})},$$

$$\bar{x}_2 = (3+\text{IA})^2\frac{(p_1)^2r_1r_2}{4(p_2)^2(Y)^2},$$

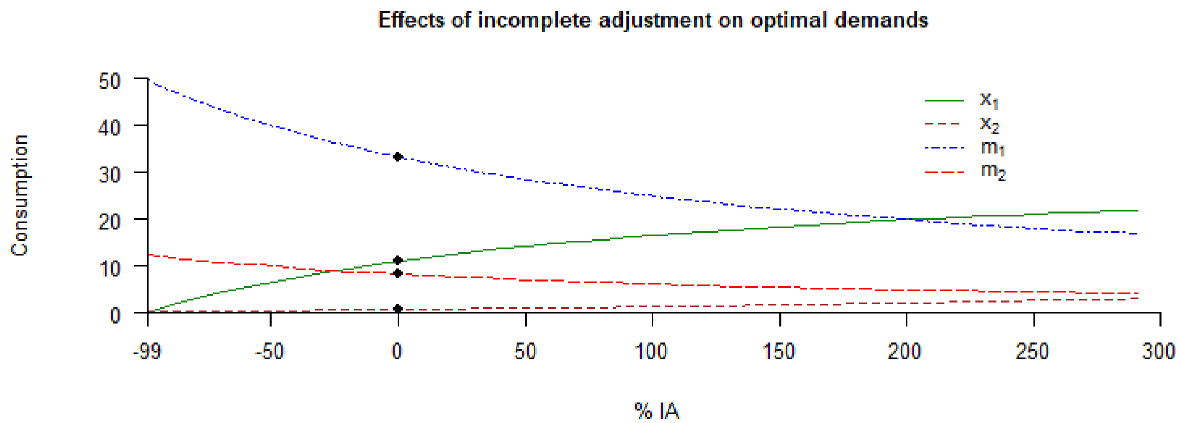
$$\bar{E} = \frac{2Y}{(3+\text{IA})}.$$

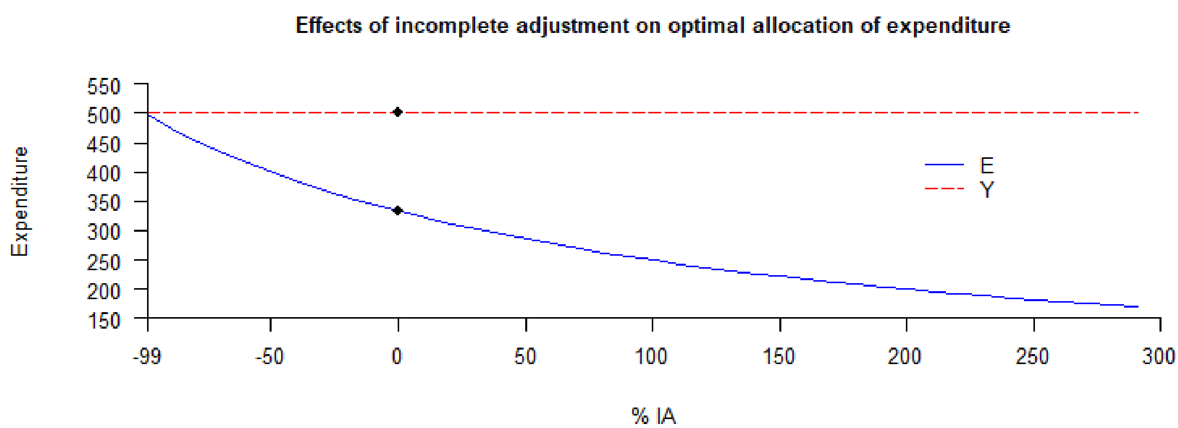
Plugging in the optimal allocation of sub-expenditure into the conditional demand functions $(\widetilde{m}_1, \widetilde{m}_2)$ gives the optimal (unconditional) sub-utility demand functions:

$$\bar{m}_1 = \frac{Y}{(3+\text{IA})r_1} \quad and \quad \bar{m}_2 = \frac{Y}{(3+\text{IA})r_2}.$$

The upper graph in Figure 1 plots the optimal demands $(\bar{x}_1, \bar{x}_2, \bar{m}_1, \bar{m}_2)$ (y-axis) for different values of percentage incomplete adjustment (i.e., % IA in (23)) (x-axis). We see that the demands vary quite considerably even for small amounts of incomplete adjustment (filled points correspond to optimal demands with complete adjustment, i.e., with % IA $= 0$). The lower plot in Figure 1 shows total expenditure (Y) and sub-expenditure for the separable $\boldsymbol{m}$-goods (E) for different values of % IA. Consistent with our theoretical results, when the amount of incomplete adjustment is negative, expenditure on the durable goods (E) is greater than desired (and increases for lower values of IA). In contrast, when the amount of incomplete adjustment is positive, expenditure on the durable goods is less than desired (and will eventually approach

Figure 1: Optimal demands and expenditures as functions of incomplete adjustment in the parametric example. Prices are: $p_1 = 15$, $p_2 = 0.5$, $r_1 = 5$ and $r_2 = 20$. Total expenditure is $Y = 500$. The percentage amount of incomplete adjustment (x-axis) vary between 99 and 300. Filled points are optimal demands and expenditures without incomplete adjustment.





zero in the limit as IA → ∞). As in the upper plot, there can be quite large differences in the optimal allocation of sub-expenditure even for values of incomplete adjustment close to zero.

These findings raise the question how much incomplete adjustment we should expect are present in observed data. In Section 4, we provide some empirical evidence to answer this question using both aggregated consumption and monetary (macro) data and disaggregated survey (micro) data. In the next section, we illustrate incomplete adjustment for utility maximization, that is, when the utility function is not weakly separable in the $\boldsymbol{m}$-goods.

2.4 Utility maximization with incomplete adjustment

Suppose that the $\boldsymbol{m}$-goods are chosen with incomplete adjustment according to the standard utility maximization problem:

$$\max_{\{\boldsymbol{x_D},\boldsymbol{x_{ND}},\boldsymbol{m_D},\boldsymbol{m_{ND}}\}} u(\boldsymbol{x_D},\boldsymbol{x_{ND}},\boldsymbol{m_D},\boldsymbol{m_{ND}}) \quad subject\ to$$

$$\boldsymbol{p}_D^i\boldsymbol{x_D} + \boldsymbol{p}_{ND}^i\boldsymbol{x_{ND}} + \boldsymbol{r}_D^i\boldsymbol{m_D} + \boldsymbol{r}_{ND}^i\boldsymbol{m_{ND}} \leq Y^i \quad and \quad \boldsymbol{p}_D^i\boldsymbol{x_D} + \boldsymbol{r}_D^i\boldsymbol{m_D} \leq Y_D^i, \tag{24}$$

τ is the marginal utility of total expenditure (i.e., $\tau = \partial u(\boldsymbol{x_D},\boldsymbol{x_{ND}},\boldsymbol{m_D},\boldsymbol{m_{ND}})/\partial Y$) and Θ is the marginal utility of expenditure on the durable goods (i.e., $\Theta = \partial u(\boldsymbol{x_D},\boldsymbol{x_{ND}},\boldsymbol{m_D},\boldsymbol{m_{ND}})/\partial Y_D$). Analogous to the weak separability case, we define the amount of incomplete adjustment, IA_u^i as the extra increment of utility of spending one dollar more on the durable goods relative to the marginal utility of total expenditure, i.e.,

$$\text{IA}_u^i = \frac{\partial u(\boldsymbol{x_D},\boldsymbol{x_{ND}},\boldsymbol{m_D},\boldsymbol{m_{ND}})/\partial Y_D^i}{\partial u(\boldsymbol{x_D},\boldsymbol{x_{ND}},\boldsymbol{m_D},\boldsymbol{m_{ND}})/\partial Y^i} = \frac{\Theta^i}{\tau^i},$$

for all $i = 1, \ldots, n$.

As in the weak separability model, the first-order conditions can be used to show that the model (24) is equivalent to a model that solves the following utility maximization problem:

$$\max_{\{\boldsymbol{x_D},\boldsymbol{x_{ND}},\boldsymbol{m_D},\boldsymbol{m_{ND}}\}} u(\boldsymbol{x_D},\boldsymbol{x_{ND}},\boldsymbol{m_D},\boldsymbol{m_{ND}}) \quad subject\ to$$

$$\tilde{\boldsymbol{p}}_D^i\boldsymbol{x_D} + \boldsymbol{p}_{ND}^i\boldsymbol{x_{ND}} + \tilde{\boldsymbol{r}}_D^i\boldsymbol{m_D} + \boldsymbol{r}_{ND}^i\boldsymbol{m_{ND}} \leq \tilde{Y}^i, \tag{25}$$

where $\tilde{Y}^i = \tilde{\boldsymbol{p}}_D^i\boldsymbol{x_D^i} + \boldsymbol{p}_{ND}^i\boldsymbol{x_{ND}^i} + \tilde{\boldsymbol{r}}_D^i\boldsymbol{m_D^i} + \boldsymbol{r}_{ND}^i\boldsymbol{m_{ND}^i}$ with $\tilde{\boldsymbol{p}}_D^i = (1 + \text{IA}_u^i)\boldsymbol{p}_D^i$ and $\tilde{\boldsymbol{r}}_D^i = (1 + \text{IA}_u^i)\boldsymbol{r}_D^i$. As above, for empirical purposes, it is convenient to define the number $\Gamma^i = \Theta^i + \tau^i$, and express the amount of incomplete adjustment as:

$$\text{IA}_u^i = \frac{\Theta^i}{\tau^i} = \frac{\Gamma^i - \tau^i}{\tau^i} = \frac{\Gamma^i}{\tau^i} - 1.$$

The percentage incomplete adjustment % IA_u^i is given by:

$$\%\,\mathrm{IA}_u^i = 100 \times \left(\frac{\Gamma^i}{\tau^i} - 1 \right), \tag{26}$$

for all $i = 1, \dots, n$.

3. TEST-PROCEDURES

In this section, we propose efficient non-parametric revealed preference procedures to implement the models proposed in the previous section.

3.1 Testing for weak separability with incomplete adjustment

We begin by defining the concept of rationalization with incomplete adjustment.

Definition 1. *The data* $\mathbb{D} = \{(\boldsymbol{p}_D^i, \boldsymbol{p}_{ND}^i), (\boldsymbol{r}_D^i, \boldsymbol{r}_{ND}^i); (\boldsymbol{x}_D^i, \boldsymbol{x}_{ND}^i), (\boldsymbol{m}_D^i, \boldsymbol{m}_{ND}^i)\}_{i=1,\dots,n}$ *can be rationalized with incomplete adjustment if there exists a well-behaved (i.e., continuous, strictly increasing and concave) macro function* U *and a well-behaved sub-utility function* V*, such that* $\{(\boldsymbol{x}_D^i, \boldsymbol{x}_{ND}^i), (\boldsymbol{m}_D^i, \boldsymbol{m}_{ND}^i)\}_{i=1,\dots,n}$ *solves the overall utility maximization problem in equations (1)-(4) and the sub-utility maximization problem in equations (4)-(6), or equivalently, solves the overall utility maximization problem in equation (19) and the sub-utility maximization problem in equation (20).*

Our non-parametric test-procedure is based on a revealed preference characterization for when the data $\mathbb{D}$ can be rationalized with incomplete adjustment. This characterization consists of a set of conditions that can be implemented by checking whether there exist numbers satisfying some inequalities. A simple intuition for one of the testable conditions follows from that both the sub-utility function and the overall utility function must satisfy concavity (by the definition of rationalizability), in which case it holds for all $i, j = 1, \dots, n$:

$$V(\boldsymbol{m}_{ND}^i, \boldsymbol{m}_D^i) \le V(\boldsymbol{m}_{ND}^j, \boldsymbol{m}_D^j) + \frac{\partial V}{\partial \boldsymbol{m}_{ND}^j}(\boldsymbol{m}_{ND}^i - \boldsymbol{m}_{ND}^j) + \frac{\partial V}{\partial \boldsymbol{m}_D^j}(\boldsymbol{m}_D^i - \boldsymbol{m}_D^j), \tag{27}$$

and,

$$U\left(\boldsymbol{x_{ND}^i}, \boldsymbol{x_D^i}, V\left(\boldsymbol{m_{ND}^i}, \boldsymbol{m_D^i}\right)\right) \leq U\left(\boldsymbol{x_{ND}^j}, \boldsymbol{x_D^j}, V\left(\boldsymbol{m_{ND}^j}, \boldsymbol{m_D^j}\right)\right) + \frac{\partial U}{\partial \boldsymbol{x_{ND}^j}}\left(\boldsymbol{x_{ND}^i} - \boldsymbol{x_{ND}^j}\right)$$

$$+ \frac{\partial U}{\partial \boldsymbol{x_D^j}}\left(\boldsymbol{x_D^i} - \boldsymbol{x_D^j}\right) + \frac{\partial U}{\partial V(\boldsymbol{m^j})}\left(V(\boldsymbol{m^i}) - V(\boldsymbol{m^j})\right). \tag{28}$$

Substituting the first-order conditions (10)-(13) and (16)-(17) gives:

$$V^i \leq V^j + \mu^j \boldsymbol{r}_{ND}^j\left(\boldsymbol{m_{ND}^i} - \boldsymbol{m_{ND}^j}\right) + \mu^j \tilde{\boldsymbol{r}}_D^i\left(\boldsymbol{m_D^i} - \boldsymbol{m_D^j}\right), \tag{29}$$

$$U^i \leq U^j + \tau^j \boldsymbol{p}_{ND}^j\left(\boldsymbol{x_{ND}^i} - \boldsymbol{x_{ND}^j}\right) + \tau^j \widetilde{\boldsymbol{p}}_D^i\left(\boldsymbol{x_D^i} - \boldsymbol{x_D^j}\right) + \frac{\tau^j}{\mu^j}\left(V^i - V^j\right), \tag{30}$$

where $V^i = V\left(\boldsymbol{m_{ND}^i}, \boldsymbol{m_D^i}\right)$ and $U^i = U\left(\boldsymbol{x_{ND}^i}, \boldsymbol{x_D^i}, V\left(\boldsymbol{m_{ND}^i}, \boldsymbol{m_D^i}\right)\right)$ are defined as utility indices.

Theorem 1 states our characterization (the proof is given in the appendix).

Theorem 1. *Consider the data set* $\mathbb{D} = \left\{\left(\boldsymbol{p_D^i}, \boldsymbol{p_{ND}^i}\right), \left(\boldsymbol{r_D^i}, \boldsymbol{r_{ND}^i}\right); \left(\boldsymbol{x_D^i}, \boldsymbol{x_{ND}^i}\right), \left(\boldsymbol{m_D^i}, \boldsymbol{m_{ND}^i}\right)\right\}_{i=1,\dots,n}$.

Conditions (a)-(c) are equivalent:

(a) The data $\mathbb{D}$ *can be rationalized with incomplete adjustment.*

(b) There exist numbers $V^i, U^i, \mu^i > 0, \tau^i > 0$ *and* $\Psi^i > 0$ *such that (for all* $i, j = 1, \dots, n$*):*

$$V^i \leq V^j + \mu^j \boldsymbol{r}_{ND}^j\left(\boldsymbol{m_{ND}^i} - \boldsymbol{m_{ND}^j}\right) + \Psi^j \boldsymbol{r}_D^j\left(\boldsymbol{m_D^i} - \boldsymbol{m_D^j}\right), \tag{31}$$

$$U^i \leq U^j + \tau^j \boldsymbol{p}_{ND}^j\left(\boldsymbol{x_{ND}^i} - \boldsymbol{x_{ND}^j}\right) + \frac{\tau^j}{\mu^j}\Psi^j \boldsymbol{p}_D^j\left(\boldsymbol{x_D^i} - \boldsymbol{x_D^j}\right) + \frac{\tau^j}{\mu^j}\left(V^i - V^j\right). \tag{32}$$

(c) There exist numbers $V^i, W^i, \mu^i > 0$ *and* $\Psi^i > 0$ *such that (for all* $i, j = 1, \dots, n$*):*

$$V^i \leq V^j + \mu^j \boldsymbol{r}_{ND}^j\left(\boldsymbol{m_{ND}^i} - \boldsymbol{m_{ND}^j}\right) + \Psi^j \boldsymbol{r}_D^j\left(\boldsymbol{m_D^i} - \boldsymbol{m_D^j}\right), \tag{33}$$

$$if\ \mu^i \boldsymbol{p}_{ND}^i\left(\boldsymbol{x_{ND}^i} - \boldsymbol{x_{ND}^j}\right) + \Psi^i \boldsymbol{p}_D^i\left(\boldsymbol{x_D^i} - \boldsymbol{x_D^j}\right) + \left(V^i - V^j\right) \geq 0 \quad then \quad W^i - W^j \geq 0, \tag{34}$$

$$if\ \mu^i \boldsymbol{p}_{ND}^i\left(\boldsymbol{x_{ND}^i} - \boldsymbol{x_{ND}^j}\right) + \Psi^i \boldsymbol{p}_D^i\left(\boldsymbol{x_D^i} - \boldsymbol{x_D^j}\right) + \left(V^i - V^j\right) > 0 \quad then \quad W^i - W^j > 0. \tag{35}$$

The inequalities (31) and (32) in condition *(b)* in Theorem 1 are equivalent to (29) and (30) from the identity in equation (22). However, (31) and (32) are not very attractive from a computational perspective since parts of them are highly non-linear; for example, the inequalities (32) contains the non-linear term $\tau^j\left(V^i - V^j\right)/\mu^j$. Thus, any procedure to implement condition *(b)* is prone to solve a complex non-linear optimization problem with non-linear constraints (which grows quadratically in the number of observations). Consequently, the optimand in any such procedure may be badly behaved with saddle points and local optima.

Instead, we base our procedure on condition *(c)* in Theorem 1. In contrast to the ones in condition *(b)*, the inequalities in condition *(c)* are linear and therefore overcome the disadvantages with the non-linear procedures. However, the inequalities (34) and (35) are not practically operational in their current form since we need to capture the logical relation between the right- and left-hand sides of them. We follow a recent idea by Cherchye et al. (2015) and use binary variables to link the two sides. Specifically, it is straightforward to show that (34) and (35) are equivalent to that there exist binary numbers X^{ij} for all $i, j = 1, \ldots, n$ such that the following (linear) inequalities hold: [11]

$$V^i \leq V^j + \mu^j \boldsymbol{r}_{ND}^j\left(\boldsymbol{m}_{\boldsymbol{ND}}^{\boldsymbol{i}} - \boldsymbol{m}_{\boldsymbol{ND}}^{\boldsymbol{j}}\right) + \Psi^j \boldsymbol{r}_D^j\left(\boldsymbol{m}_{\boldsymbol{D}}^{\boldsymbol{i}} - \boldsymbol{m}_{\boldsymbol{D}}^{\boldsymbol{j}}\right), \qquad (\text{c}.1)$$

$$W^i - W^j - X^{ij} \leq -\varepsilon, \qquad (\text{c}.2)$$

$$\left(X^{ij} - 1\right) \leq W^i - W^j, \qquad (\text{c}.3)$$

$$\mu^i \boldsymbol{p}_{ND}^i\left(\boldsymbol{x}_{\boldsymbol{ND}}^{\boldsymbol{i}} - \boldsymbol{x}_{\boldsymbol{ND}}^{\boldsymbol{j}}\right) + \Psi^i \boldsymbol{p}_D^i\left(\boldsymbol{x}_{\boldsymbol{D}}^{\boldsymbol{i}} - \boldsymbol{x}_{\boldsymbol{D}}^{\boldsymbol{j}}\right) + \left(V^i - V^j\right) - X^{ij} A^i \leq -\varepsilon, \qquad (\text{c}.4)$$

$$\left(X^{ij} - 1\right) A^j \leq \mu^j \boldsymbol{p}_{ND}^j\left(\boldsymbol{x}_{\boldsymbol{ND}}^{\boldsymbol{i}} - \boldsymbol{x}_{\boldsymbol{ND}}^{\boldsymbol{j}}\right) + \Psi^j \boldsymbol{p}_D^j\left(\boldsymbol{x}_{\boldsymbol{D}}^{\boldsymbol{i}} - \boldsymbol{x}_{\boldsymbol{D}}^{\boldsymbol{j}}\right) + \left(V^i - V^j\right), \qquad (\text{c}.5)$$

$$0 \leq V^i \leq 1, \qquad (\text{c}.6)$$

$$0 \leq W^i \leq 1 - \varepsilon, \qquad (\text{c}.7)$$

$$\varepsilon \leq \mu^i \leq 1, \qquad (\text{c}.8)$$

[11] Cherchye et al. (2015, Theorem 4) used this approach to formulate a new test for standard weakly separable utility maximization (with complete adjustment).

$$\varepsilon \leq \Psi^i \leq 1, \qquad (c.9)$$

$$X^{ij} \in \{0,1\}. \qquad (c.10)$$

Conditions (c.2) and (c.3) reproduce the right-hand side of the inequalities (34) and (35), while conditions (c.4) and (c.5) reproduce the left-hand sides of (34) and (35). X^{ij} are binary (0-1) variables that captures the logical relation in (34) and (35), and equals one if and only if $W^i - W^j \geq 0$. Moreover, ε is a small positive number and A^i is a fixed number larger than $\boldsymbol{p}_{ND}^i \boldsymbol{x}_{ND}^i + \boldsymbol{p}_D^i \boldsymbol{x}_D^i + 1$.[12]

We propose calculating the minimal amount of incomplete adjustment by minimizing the sum of squared deviations between Ψ^i and μ^i as suggested by the definition of incomplete adjustment in equation (22) and the discussion in the paragraph following (22). More formally, we calculate the minimal amount of incomplete adjustment such that the $\boldsymbol{m}$-goods are weakly separable from all other goods by solving the following mixed integer (binary) quadratic programming (MIQP) problem:[13]

$$\min_{\{V^i, W^i, \mu^i, \Psi^i, X^{ij}\}_{i,j=1,\dots,n}} \sum_{i=1}^{n} \left(\Psi^i - \mu^i\right)^2 \quad \mathit{subject\ to}\ (c.1) - (c.10). \qquad (36)$$

Let $\widehat{\Psi}^i$ and $\hat{\mu}^i$ for all observations $i = 1, \dots, n$ be the optimal solutions from this MIQP problem. We use $\widehat{\Psi}^i$ and $\hat{\mu}^i$ to calculate the percentage amount of incomplete adjustment required to rationalize the data with incomplete adjustment from equation (23).

3.2 Utility Maximization with Incomplete Adjustment

[12] We set $\varepsilon = 10^{-6}$ and $A^i = \boldsymbol{p}_{ND}^i \boldsymbol{x}_{ND}^i + \boldsymbol{p}_D^i \boldsymbol{x}_D^i + 2$ in our empirical applications in Section 4.

[13] A MIQP problem is very similar to a standard quadratic programming (QP) problem in the sense that it minimizes a quadratic objective function subject to a set of linear constraints, but in contrast to a QP problem some variables in a MIQP problem may also take binary (integer) values.

In this section, we present an analogous test-procedure for utility maximization with incomplete adjustment. We say that the data $\mathbb{D} = \{(\boldsymbol{p}_{\boldsymbol{D}}^{\boldsymbol{i}}, \boldsymbol{p}_{\boldsymbol{ND}}^{\boldsymbol{i}}), (\boldsymbol{r}_{\boldsymbol{D}}^{\boldsymbol{i}}, \boldsymbol{r}_{\boldsymbol{ND}}^{\boldsymbol{i}}); (\boldsymbol{x}_{\boldsymbol{D}}^{\boldsymbol{i}}, \boldsymbol{x}_{\boldsymbol{ND}}^{\boldsymbol{i}}), (\boldsymbol{m}_{\boldsymbol{D}}^{\boldsymbol{i}}, \boldsymbol{m}_{\boldsymbol{ND}}^{\boldsymbol{i}})\}_{i=1,\dots,n}$ can be *utility-rationalized with incomplete adjustment* if there exists a well-behaved utility function u such that $\{(\boldsymbol{x}_{\boldsymbol{D}}^{\boldsymbol{i}}, \boldsymbol{x}_{\boldsymbol{ND}}^{\boldsymbol{i}}), (\boldsymbol{m}_{\boldsymbol{D}}^{\boldsymbol{i}}, \boldsymbol{m}_{\boldsymbol{ND}}^{\boldsymbol{i}})\}_{i=1,\dots,n}$ solves the utility maximization problem in equation (24), or equivalently, the utility maximization problem in equation (25).

Theorem 2 states the non-parametric revealed preference characterization for utility maximization model with incomplete adjustment.[14]

Theorem 2. *Consider the data set* $\mathbb{D} = \{(\boldsymbol{p}_{\boldsymbol{D}}^{\boldsymbol{i}}, \boldsymbol{p}_{\boldsymbol{ND}}^{\boldsymbol{i}}), (\boldsymbol{r}_{\boldsymbol{D}}^{\boldsymbol{i}}, \boldsymbol{r}_{\boldsymbol{ND}}^{\boldsymbol{i}}); (\boldsymbol{x}_{\boldsymbol{D}}^{\boldsymbol{i}}, \boldsymbol{x}_{\boldsymbol{ND}}^{\boldsymbol{i}}), (\boldsymbol{m}_{\boldsymbol{D}}^{\boldsymbol{i}}, \boldsymbol{m}_{\boldsymbol{ND}}^{\boldsymbol{i}})\}_{i=1,\dots,n}$. *The data* $\mathbb{D}$ *can be utility-rationalized with incomplete adjustment if and only if there exist numbers* $U^i, \tau^i > 0$ *and* $\Gamma^i > 0$ *such that (for all* $i, j = 1, \dots, n$*):*

$$U^i \leq U^j + \tau^j \boldsymbol{p}_{ND}^j(\boldsymbol{x}_{\boldsymbol{ND}}^{\boldsymbol{i}} - \boldsymbol{x}_{\boldsymbol{ND}}^{\boldsymbol{j}}) + \Gamma^j \boldsymbol{p}_D^j(\boldsymbol{x}_{\boldsymbol{D}}^{\boldsymbol{i}} - \boldsymbol{x}_{\boldsymbol{D}}^{\boldsymbol{j}}) + \tau^j \boldsymbol{r}_{ND}^j(\boldsymbol{m}_{\boldsymbol{ND}}^{\boldsymbol{i}} - \boldsymbol{m}_{\boldsymbol{ND}}^{\boldsymbol{j}})$$
$$+\Gamma^j \boldsymbol{r}_D^j(\boldsymbol{m}_{\boldsymbol{D}}^{\boldsymbol{i}} - \boldsymbol{m}_{\boldsymbol{D}}^{\boldsymbol{j}}). \tag{37}$$

Note that the inequalities (37) reduce to the well-known Afriat inequalities (See Varian, 1983) whenever $\tau^j = \Gamma^j$ holds for all $j = 1, \dots, n$.[15] The inequalities (37) are linear and therefore suitable for empirical implementations. The minimal amount of incomplete adjustment that rationalizes the data in the quadratic norm can be calculated by solving the following quadratic programming (QP) problem:

$$\min_{\{U^i, \tau^i > 0, \Gamma^i > 0\}_{i=1,\dots,n}} \sum_{i=1}^{n} (\tau^i - \Gamma^i)^2 \quad \textit{subject to } (37). \tag{38}$$

[14] The testable inequalities in Theorem 2 can be derived similarly to the testable inequalities (29) and (30) in the weak separability case.

[15] We show by example in the Supplementary material accompanying the paper that this model is refutable. Since the weak separability model is nested within the utility maximization model it follows that both models can be refuted with observed data.

Let $\hat{\tau}^i$ and $\hat{\Gamma}^i$ for all observations $i = 1, \ldots, n$ be the optimal solutions from this QP problem. We use $\hat{\tau}^i$ and $\hat{\Gamma}^i$ to calculate the percentage amount of incomplete adjustment required to rationalize the data with incomplete adjustment from equation (26).

4. APPLICATIONS

In this section, we illustrate the importance of incomplete adjustment by applying our test-procedures to two data sets. The first application uses aggregated U.S. data over durable and non-durable consumption goods, services, leisure and monetary assets. The second application uses rich micro survey data of Spanish household expenditures for durable and non-durable goods and services. Our results show strong evidence that it is important to account for incomplete adjustment in consumer demand modelling of durable goods and monetary assets.

In presenting our results, we use four summary statistics. Let $\%\,\widehat{\mathrm{IA}}^i$ for $i = 1, \ldots, n$ be the calculated percentage amount of incomplete adjustment for the n observations. We define the mean percentage incomplete adjustment as:

$$\text{Mean } \%\text{ IA} = \frac{1}{n}\sum_{i=1}^{n} \%\,\widehat{\mathrm{IA}}^i, \tag{39}$$

the maximum absolute percentage incomplete adjustment as:

$$\text{Max absolute } \%\text{ IA} = max_{i=1,\ldots,n}\left\{\left|\%\,\widehat{\mathrm{IA}}^i\right|\right\}, \tag{40}$$

the percentage root mean squared incomplete adjustment as:

$$\%\text{ root mean squared IA} = \sqrt{\sum_{i=1}^{n}\left(\%\,\widehat{\mathrm{IA}}^i\right)^2}, \tag{41}$$

and finally, the percentage number of adjusted time periods as:

$$\%\text{ of adjusted periods} = 100 \times \frac{1}{n}\sum_{i=1}^{n} I_{\%\,\widehat{\mathrm{IA}}^i \neq 0}, \tag{42}$$

where $I_{a \neq 0}$ denotes the indicator function: $I = 1$ if a $\neq 0$, and zero otherwise.

4.1 Applications to Aggregate Consumption and Monetary Data

Hjertstrand et al. (2016) tested weak separability of durable and non-durable consumption goods, services, leisure and monetary assets in order to identify appropriate economic aggregates. Barnett (1980) showed that weak separability of monetary goods is a necessary condition to construct monetary aggregates that are broader than currency and consistent with the economic theory of aggregation over goods. This is important because theoretically valid monetary aggregates restrict the effect of money on real economic activity. Moreover, weak separability of major categories of consumption expenditure from leisure and monetary goods must hold in order for consumption to have a stable relationship with income. We add to this literature by identifying what monetary and consumption aggregates that are being used by the U.S. public by applying our new test-procedures with incomplete adjustment. In particular, since "money" is usually treated as a durable good with an infinite life-span and assumed to retain some utility beyond the holding period (Serletis, 2007), applying tests that account for incomplete adjustment in the money demand literature is important in order to draw accurate conclusions.

We use U.S. quarterly data covering the period 2000Q1 – 2011Q3, which gives in total $n = 47$ observations. Since a quarterly frequency may be considered as a relatively short time span for durable and monetary goods, we also aggregate our data into yearly observations ranging from 2000-2010. This gives in total $n = 11$ observations and allows us to check whether there is more incomplete adjustment in quarterly than in yearly data.

The data on durable and non-durable consumption goods and other non-monetary goods are the same as in Hjertstrand et al. (2016), and are described in detail as goods (a)-(d) in Table 1.

Table 1: Goods and services in aggregated macro data

		Consumption goods
(a)	NDUR:	Real expenditures on nondurables
(b)	DUR:	Real expenditures on durables
		Other non-monetary goods
(c)	SER:	Real expenditures on services
(d)	LEIS:	Hours of leisure
		Monetary assets
(e)	CUR+DD:	Currency plus demand deposits
(f)	TC:	Traveler's checks
(g)	OCDCB and OCDTH:	Other checkable deposits at commercial banks and thrifts
(h)	SD-CB and SD-TH:	Savings deposits at commercial banks and thrifts
(i)	STDCB and STDTH:	Small time deposits at commercial banks and thrifts
(j)	MMFR and MMFI:	Retail and institutional money market mutual funds
(k)	TB:	Treasury bills
(l)	CP:	Commerical paper
(m)	LTD:	Large time deposits
(n)	RP:	Repurchase agreements

The prices of services and nondurables are the respective implicit price deflators. The price of durables is a user cost. An annualized ten percent depreciation rate was applied each quarter to annualize expenditures on durables to make it compatible with annualized expenditures on services and nondurables. Leisure is calculated as 98 hours minus average hours worked per week during the quarter.[16]

We use the same monetary assets as in Hjertstrand et al. (2016), which are described in Table 1 as assets (e)-(m). Data on these assets were obtained from the Center for Financial Stability (CFS).[17] In addition, we include repurchase agreements (RP) in our analysis, which is described

[16] See Hjertstrand et al. (2016) for the original sources of these data.
[17] www.centerforfinancialstability.org.

as asset (n) in Table 1.[18] Hjertstrand et al. (2016) excluded RP from their analysis and thus assumed that the goods and assets (a)-(m) in Table 1 were weakly separable from RP. By including RP in our analysis, we avoid making any such assumption.

Demand deposits, other checkable deposits and savings deposits were adjusted for retail sweeps.[19] All monetary assets were deflated by the implicit price deflator to obtain real per capita balances. The user costs for the monetary assets were multiplied by the implicit price deflator to yield nominal prices of a dollar of real balances.

We assume a representative agent with preferences over the 18 goods and assets in Table 1. Using the test-procedure specified in equation (38), we begin our analysis by testing whether the data can be rationalized by the utility maximization model with incomplete adjustment. The top left graph in Figure 2 plots the minimal amount of incomplete adjustment necessary to rationalize the data.[20]

The filled blue points give the amount of incomplete adjustment at every yearly observation, and shows that the yearly data pass utility maximization with complete adjustment (i.e., with zero amount of incomplete adjustment). The solid (red) line gives the amount of incomplete adjustment for the quarterly data. As seen from the graph, this amount is essentially zero up until the beginning or slightly after the crises began in 2008, after which it increases slightly in the latter part of the sample. This is confirmed by the summary statistics of the amount of incomplete adjustment over all observations given in Table 2.

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[18] Repurchase agreements are short terms loans of securities to financial institutions with the understanding that the securities will be repurchased at a higher price at a later (overnight or other) time.

[19] Jones, Dutkowsky and Elger (2005) found sweep adjustment to be important in identifying appropriate monetary aggregates. At one time the data could be adjusted for both retail and commercial sweeps. However, the CFS sweep adjusts on its own, using an econometric model.

[20] The CPU time to solve the QP problem (38) using the quarterly data (with 47 observations) was 6 seconds.

Figure 2: Minimal amount of incomplete adjustment for utility maximization and the weakly separable structures (i)-(ii) and (vii)-(ix) from Table 3.

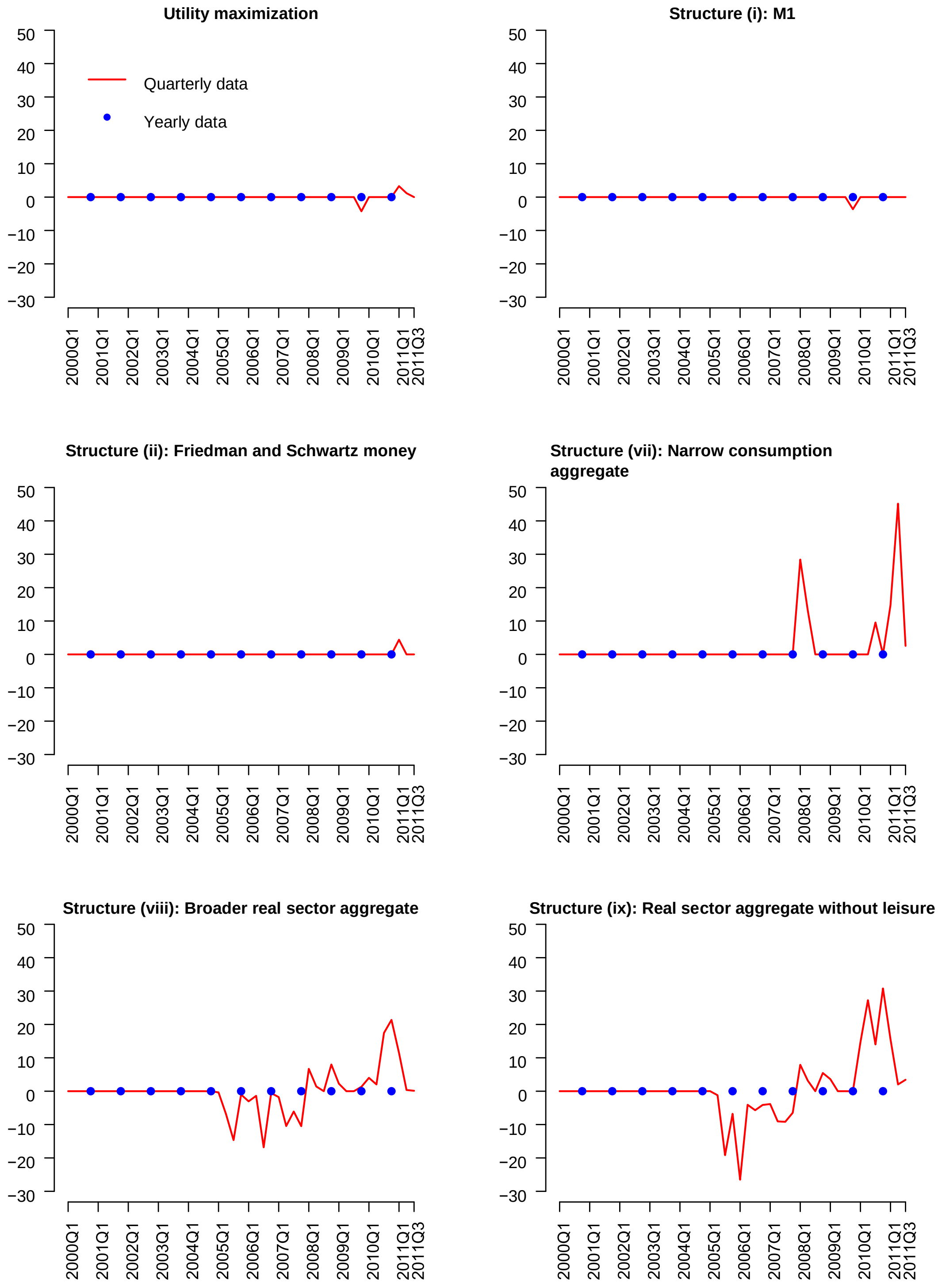

Table 2: Summary statistics of the amount of incomplete adjustment for the utility maximization model using aggregated consumption and monetary data.

	Mean % IA	Max absolute % IA	% root mean squared IA	% of adjusted periods
Quarterly data	0.0042	4.2596	0.8026	97.8723
Yearly data	0	0	0	0

Next, we continue our analysis by testing whether the preferences of the representative agent are weakly separable in the groups of monetary assets and durable and non-durable consumption goods that are defined in Table 3. Structures (i)-(vi) are different monetary aggregates. In particular, Structure (i) is the FED aggregate M1 and Structure (ii) is the modern analog of what Friedman and Schwartz (1963) called money in the U.S. Structure (iii) is equivalent to the FED old liquidity aggregate L. Structure (iv) is the FED M2 aggregate and Structure (v) is M3 as formerly designed by the FED. Structure (vi) is what is commonly known as MZero or an aggregate including all financial assets with zero capital risk. Structures (vii) and (viii) are narrow and broader real-sector consumption aggregates, respectively. Structure (ix) is a real-sector consumption aggregate including services, durable and non-durable goods, but not leisure.

Using the test-procedure for weak separability with incomplete adjustment in equation (36), we test if the nine utility structures in Table 3 are weakly separable using both the quarterly and yearly data. Let us first consider the results for the quarterly data. We find that the monetary aggregates in Structures (iii)-(vi) are not weakly separable from all other goods. In contrast, we find that the monetary aggregates in Structures (i) and (ii), M1 and “Friedman and Schwartz” money, and the real sector aggregates in Structures (vii)-(ix) are weakly separable withincomplete adjustment.[21] Figure 2 plots the minimal amount of incomplete adjustment

[21] The average CPU time to solve the MIQP problem (37) over all Structures (i)-(ix) was 108.97 seconds for the quarterly data (47 observations) and 0.25 seconds for the yearly data (11 observations).

necessary to rationalize the data for these five structures. As seen from the plots, the amount of incomplete

Table 3: Utility structures tested for weak separability

Monetary aggregates

(i) $u(\mathrm{DUR}, \mathrm{NDUR}, \mathrm{SER}, \mathrm{LEIS}, \mathbf{V}(\mathbf{CUR} + \mathbf{DD}, \mathbf{TC}, \mathbf{OCD} - \mathbf{CB}, \mathbf{OCD} - \mathbf{TH}), \mathrm{SD} - \mathrm{CB}, \mathrm{SD} - \mathrm{TH}, \mathrm{MMMF} - \mathrm{R}, \mathrm{STD} - \mathrm{CB}, \mathrm{STD} - \mathrm{TH}, \mathrm{MMMF} - \mathrm{I}, \mathrm{T} - \mathrm{BILLS}, \mathrm{CP}, \mathrm{LTD}, \mathrm{RP})$

(ii) $u(\mathrm{DUR}, \mathrm{NDUR}, \mathrm{SER}, \mathrm{LEIS}, \mathbf{V}(\mathbf{CUR} + \mathbf{DD}, \mathbf{TC}, \mathbf{OCD} - \mathbf{CB}, \mathbf{OCD} - \mathbf{TH}, \mathbf{SD} - \mathbf{CB}, \mathbf{SD} - \mathbf{TH}), \mathrm{MMMF} - \mathrm{R}, \mathrm{STD} - \mathrm{CB}, \mathrm{STD} - \mathrm{TH}, \mathrm{MMMF} - \mathrm{I}, \mathrm{T} - \mathrm{BILLS}, \mathrm{CP}, \mathrm{LTD}, \mathrm{RP})$

(iii) $u\big(\mathrm{DUR}, \mathrm{NDUR}, \mathrm{SER}, \mathrm{LEIS}, \mathbf{V}(\mathbf{CUR} + \mathbf{DD}, \mathbf{TC}, \mathbf{OCD} - \mathbf{CB}, \mathbf{OCD} - \mathbf{TH}, \mathbf{SD} - \mathbf{CB}, \mathbf{SD} - \mathbf{TH}, \mathbf{MMMF} - \mathbf{R}, \mathbf{STD} - \mathbf{CB}, \mathbf{STD} - \mathbf{TH}, \mathbf{MMMF} - \mathbf{I}, \mathbf{T} - \mathbf{BILLS}, \mathbf{CP}, \mathbf{LTD}, \mathbf{RP})\big)$

(iv) $u(\mathrm{DUR}, \mathrm{NDUR}, \mathrm{SER}, \mathrm{LEIS}, \mathbf{V}(\mathbf{CUR} + \mathbf{DD}, \mathbf{TC}, \mathbf{OCD} - \mathbf{CB}, \mathbf{OCD} - \mathbf{TH}, \mathbf{SD} - \mathbf{CB}, \mathbf{SD} - \mathbf{TH}, \mathbf{MMMF} - \mathbf{R}, \mathbf{STD} - \mathbf{CB}, \mathbf{STD} - \mathbf{TH}), \mathrm{MMMF} - \mathrm{I}, \mathrm{T} - \mathrm{BILLS}, \mathrm{CP}, \mathrm{LTD}, \mathrm{RP})$

(v) $u(\mathrm{DUR}, \mathrm{NDUR}, \mathrm{SER}, \mathrm{LEIS}, \mathbf{V}(\mathbf{CUR} + \mathbf{DD}, \mathbf{TC}, \mathbf{OCD} - \mathbf{CB}, \mathbf{OCD} - \mathbf{TH}, \mathbf{SD} - \mathbf{CB}, \mathbf{SD} - \mathbf{TH}, \mathbf{MMMF} - \mathbf{R}, \mathbf{STD} - \mathbf{CB}, \mathbf{STD} - \mathbf{TH}, \mathbf{MMMF} - \mathbf{I}, \mathbf{LTD}), \mathrm{T} - \mathrm{BILLS}, \mathrm{CP}, \mathrm{RP})$

(vi) $(u(\mathrm{DUR}, \mathrm{NDUR}, \mathrm{SER}, \mathrm{LEIS}, \mathbf{V}(\mathbf{CUR} + \mathbf{DD}, \mathbf{TC}, \mathbf{OCD} - \mathbf{CB}, \mathbf{OCD} - \mathbf{TH}, \mathbf{SD} - \mathbf{CB}, \mathbf{SD} - \mathbf{TH}, \mathbf{MMMF} - \mathbf{R}, \mathbf{MMMF} - \mathbf{I}), \mathrm{STD} - \mathrm{CB}, \mathrm{STD} - \mathrm{TH}, \mathrm{T} - \mathrm{BILLS}, \mathrm{CP}, \mathrm{LTD}, \mathrm{RP})$

Real-sector and consumption aggregates

(vii) $u(\mathbf{V}(\mathbf{NDUR}, \mathbf{SER}), \mathrm{DUR}, \mathrm{LEIS}, \mathrm{CUR} + \mathrm{DD}, \mathrm{TC}, \mathrm{OCD} - \mathrm{CB}, \mathrm{OCD} - \mathrm{TH}, \mathrm{SD} - \mathrm{CB}, \mathrm{SD} - \mathrm{TH}, \mathrm{MMMF} - \mathrm{R}, \mathrm{STD} - \mathrm{CB}, \mathrm{STD} - \mathrm{TH}, \mathrm{MMMF} - \mathrm{I}, \mathrm{T} - \mathrm{BILLS}, \mathrm{CP}, \mathrm{LTD}, \mathrm{RP})$

(viii) $u(\mathbf{V}(\mathbf{DUR}, \mathbf{NDUR}, \mathbf{SER}, \mathbf{LEIS}), \mathrm{CUR} + \mathrm{DD}, \mathrm{TC}, \mathrm{OCD} - \mathrm{CB}, \mathrm{OCD} - \mathrm{TH}, \mathrm{SD} - \mathrm{CB}, \mathrm{SD} - \mathrm{TH}, \mathrm{MMMF} - \mathrm{R}, \mathrm{STD} - \mathrm{CB}, \mathrm{STD} - \mathrm{TH}, \mathrm{MMMF} - \mathrm{I}, \mathrm{T} - \mathrm{BILLS}, \mathrm{CP}, \mathrm{LTD}, \mathrm{RP})$

(ix) $u(\mathbf{V}(\mathbf{DUR}, \mathbf{NDUR}, \mathbf{SER}), \mathrm{LEIS}, \mathrm{CUR} + \mathrm{DD}, \mathrm{TC}, \mathrm{OCD} - \mathrm{CB}, \mathrm{OCD} - \mathrm{TH}, \mathrm{SD} - \mathrm{CB}, \mathrm{SD} - \mathrm{TH}, \mathrm{MMMF} - \mathrm{R}, \mathrm{STD} - \mathrm{CB}, \mathrm{STD} - \mathrm{TH}, \mathrm{MMMF} - \mathrm{I}, \mathrm{T} - \mathrm{BILLS}, \mathrm{CP}, \mathrm{LTD}, \mathrm{RP})$

adjustment is higher for the quarterly data (solid line) than for the yearly data (filled points). This shows that while a year is enough for the representative agent to adjust his costs and form his expectations, a quarter of a year is too short.

Looking closer at the quarterly data in Figure 2, we see that the monetary aggregates can be rationalized with a very small amount of incomplete adjustment. This is confirmed by Table 4 which presents the summary statistics (39)-(42) of the amount of incomplete adjustment over all observations.

Table 4: Summary statistics of the amount of incomplete adjustment for the weak separability model using quarterly aggregated consumption and monetary data.

	Mean % IA	Max absolute % IA	% root mean squared IA	% of adjusted periods
Structure (i)	-0.0772	3.6270	0.5291	23.4043
Structure (ii)	0.0936	4.3971	0.6414	2.1277
Structure (vii)	2.4171	45.1655	8.4248	12.7660
Structure (viii)	0.0614	21.3563	6.2686	55.3191
Structure (ix)	0.6742	30.7887	9.0883	57.4468

Note: Structure (i) is M1 with 3 components and Structure (ii) is Friedman and Schwartz money with 6 components. Structures (vii), (viii) and (ix) are real-sector aggregates with 2, 4 and 3 components, respectively. Structures (iii)-(vi) were found not to be weakly separable.

In contrast, as seen from Figure 2 and Table 4, the real sector aggregates require considerably higher amounts of incomplete adjustment to satisfy weak separability.

For both the monetary and real sector aggregates, the amount of incomplete adjustment is usually higher in the latter part of the sample, and especially at the time of or slightly after the financial crises in 2008. This may not be surprising given the liquidity crises and the resulting loss of confidence in loan institutes which may have altered the expectations of creditors.

Let us compare the results to Hjertstrand et al. (2016), which used the same quarterly data but excluded the monetary asset RP in their analysis. They found that the four aggregates in Structures (i)-(ii) and (vii)-(viii) in Table 3 were weakly separable. In contrast, they also found that the broader monetary aggregate in Structure (iii) (Fed L) was weakly separable but that the real sector aggregate in Structure (ix) was not weakly separable. Thus, our results are overall consistent with and confirm the findings in Hjertstrand et al. (2016).

Finally, consider the summary statistics for the yearly data in Table 5.

Table 5: Summary statistics of the amount of incomplete adjustment for the weak separability model using yearly aggregated consumption and monetary data.

	Mean % IA	Max absolute % IA	% root mean squared IA	% of adjusted periods
Structure (i)	< 1.0e-10	< 1.0e-10	< 1.0e-10	36.3636
Structure (ii)	< 1.0e-10	< 1.0e-10	< 1.0e-10	18.1818
Structure (iii)	< 1.0e-10	< 1.0e-10	< 1.0e-10	9.0909
Structure (iv)	< 1.0e-10	< 1.0e-10	< 1.0e-10	18.1818
Structure (v)	< 1.0e-10	< 1.0e-10	< 1.0e-10	9.0909
Structure (vii)	0	0	0	0
Structure (viii)	< 1.0e-10	< 1.0e-10	< 1.0e-10	9.0909
Structure (ix)	< 1.0e-10	< 1.0e-10	< 1.0e-10	36.3636

Note: Structure (i) is M1 with 3 components; Structure (ii) is Friedman and Schwartz money with 6 components and Structure (iii) is broad money with 13 components. Structures (iv) and (v) are FED M2 and old FED M3 with 9 and 11 components, respectively. Structures (vii)-(ix) are real-sector aggregates with 2, 4 and 3 components, respectively. Structure (vi) was found not to be weakly separable.

We see that all aggregates besides Structure (vi), MZero, are weakly separable. Structure (vii) is weakly separable without incomplete adjustment while the other structures are weakly separable with essentially zero amounts of incomplete adjustment. Thus, this provides further evidence that the amount of incomplete adjustment is higher in quarterly than in yearly data.[22]

[22] Given the convergence properties of numerical optimization procedures, it seems reasonable to classify an amount of incomplete adjustment less than 1.0e-10 as zero.

4.2 Application to Household Survey Data

Our second application is to panel survey data over disaggregated Spanish household expenditures on durable and non-durable consumption goods and services (Encuesta Continua de Presupuestos Familiares, abbreviated ECPF). The data was obtained from Crawford (2010) and is a quarterly budget survey ranging from 1985-1997 that interviews households for up to a maximum of eight consecutive quarters on their consumption expenditures.[23] The data consists of 25 durable and non-durable goods and services and are specified in more detail in Table 6.[24] A common assumption in empirical models of demand and consumption patterns over the life-cycle is that durable goods are weakly separable from non-durable goods and services. Our purpose is to test this assumption. We do so by testing whether the durable goods (a)-(g) and the service equivalents to durable goods, education (x) and medical services (y), are weakly separable from the other goods and services. We include education and medical expenditures in the separable group because they increase the stock of human capital and can therefore be "consumed" over a long time span.[25] Since we test weak separability on time series data sets for each individual household, we avoid making any assumption of a representative agent. Moreover, by testing weak separability for each household, we avoid making any preference homogeneity assumption between households. We exclude households with less than 8 observations which give us in total data on 1,585 households.

[23] These data has been used in a wide variety of applications in revealed preference. For example, Cherchye et al. (2015) used it to test whether households' utility functions are weakly separable in food categories.

[24] We have classified all 25 goods and services into durables, non-durables and semi-durables and services using UN's "classication of individual consumption according to purpose" (COICOP, https://unstats.un.org/unsd/cr/registry/regcst.asp?Cl=5.).

[25] This reason for why these services should be regarded as durables is elaborated by the ILO, which in their consumer price index manual writes: "For some analytical purposes, it may be appropriate to treat certain kind of services such as education and health, as the service equivalent to durable goods. Expenditures on such services can be viewed as investments that augment the stock of human capital. Another characteristic that education and health services share with durable goods is that they are often so expensive that their purchase has to be financed by borrowing or by running down other assets" (ILO consumer price index manual, 2004, chapter 3.25).

Table 6: Classification of ECPF data according to COICOP.

	Durable goods
(a)	Durables at home (e.g. furniture and appliances)
(b)	Small durables at home
(c)	Durable medicines (e.g. spectacles, crutches and wheelchairs)
(d)	Cars
(e)	Durables at home (e.g. tv and music)
(f)	Small durables (e.g. books, toys and CDs)
(g)	Personal small durables (e.g. hair-dryer, shavors, coams, lighters and suitcases)
	Non-durable and semi-durable goods
(h)	Food and non-alcoholic drinks at home
(i)	Alcohol
(j)	Restaurants and bars
(k)	Tobacco
(l)	Non-durables at home (e.g. cleaning products)
(m)	Non-durable medicines
(n)	Petrol
(o)	Personal non-durables (e.g. toothpaste and soap)
(p)	Clothing and footwear
(q)	Energy at home (e.g. heating by electricity)
	Services
(r)	Services at home (e.g. heating not electricity, water and furniture repair)
(s)	Personal services
(t)	House rent (includes imputed rent)
(u)	Transportation
(v)	Travelling
(w)	Leisure (e.g. cinema, theatre and clubs for sports)
(x)	Education
(y)	Medical services

Table 7 presents results of the fraction of households satisfying weak separability.[26]

[26] We also ran the utility maximization problem in equation (38). We found 1444 households that satisfied the utility maximization model with complete adjustment and 140 households that satisfied the model with incomplete adjustment. Thus, only one household failed to satisfy the utility maximization model (with or without incomplete adjustment). The average amount of incomplete adjustment over all time periods and households were 3.9214% and the average % root mean squared IA was 14.379%. Detailed results are available in the Supplementary material accompanying the paper.

Table 7: Summary statistics for pass rates of weak separability in the ECPF data

Total number of households:	1585
Number of households satisfying weak separability with complete adjustment:	904 (57.03 %)
Number of households satisfying weak separability with incomplete adjustment:	440 (32.74 %)

We find 904 households $(\sim 57\ \%)$ whose preferences can be rationalized by weak separability with complete adjustment, and a further 440 households $(\sim 33\ \%)$ whose preferences can be rationalized by weak separability with incomplete adjustment.

Figure 3 contains histograms of the various summary statistics (39)-(42) over the 440 households satisfying weak separability with incomplete adjustment. We find that most households have an average incomplete adjustment close to zero, but that there are a few households with very large amounts of incomplete adjustment. This is more visible from the histograms of the maximum absolute values (upper right histogram) and the % root mean squared incomplete adjustment (lower left histogram). The lower right histogram shows that there are relatively few time periods containing incomplete adjustment for most households. Finally, Table 8 presents the mean, standard deviation, minimum, median, first and third quartiles and maximum values of the summary statistics over all 440 households. Looking, for example, at the third quartile for the max absolute % IA, we see that the maximal absolute value of IA for 75 % of the households is equal to 9.169 or less. The results in this table confirms the findings from Figure 3, and shows that most households have reasonable values of incomplete adjustment but that there are a few households with one of few time periods with very large values of incomplete adjustment.

Figure 3: Histograms of the summary statistics (39)-(42) over the 440 households satisfying weak separability with incomplete adjustment in the ECPF data.

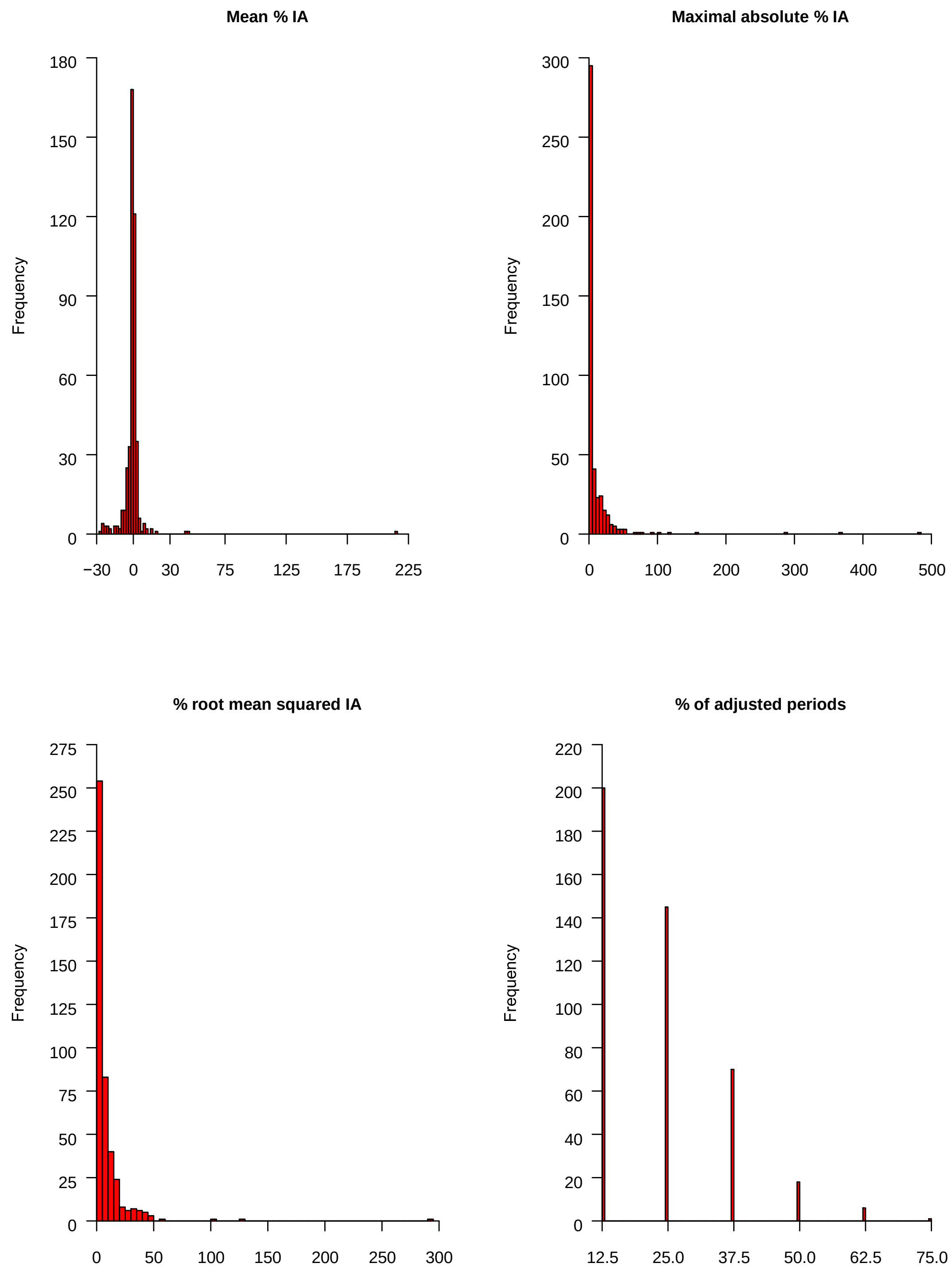

Table 8: Summary statistics over the 440 households satisfying weak separability with incomplete adjustment in the ECPF data

	Mean % IA	Max absolute % IA	% root mean squared IA	% of adjusted periods
Mean	-0.5747	10.1101	8.4753	22.9545
Min	-26.7958	0	0	12.5000
First quartile	-1.6695	0	1.3706	12.5000
Median	-0.2971	0	3.8244	25.0000
Third quartile	0.7209	9.1690	9.3719	25.0000
Max	215.3214	480.6327	291.6445	75.0000
Std. dev.	11.9190	35.1413	18.0763	11.9351

We believe that our two applications illustrate that incomplete adjustment is important when testing for weak separability of durable goods and monetary assets, and that modelling preferences for such goods should account for incomplete adjustment.

5. CONCLUSIONS

We propose a more general revealed preference test-procedures for weakly separable utility maximization and utility maximization with incomplete adjustment. The procedures are based on a computationally attractive integer programming approach.

Two empirical applications show that it is important to allow for incomplete adjustment when modelling preferences with both micro-panel data and aggregate data. With each type of data incomplete adjustment is important when testing for weakly separable utility maximization involving durable goods and assets.

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APPENDIX: Proof of Theorem 1

The proof that (i) implies (ii) essentially follows from the discussion in Section 3.1. By replacing derivatives with sub-gradients, it holds that (i) implies (ii) in the non-differentiable case (See Varian, 1983, p.104-105, for an analogous argument). The equivalence between conditions (ii) and (iii) follows from Cherchye et al. (2015, Theorem 2). Hence, it suffices to show that (ii) implies (i).

Suppose that there exist numbers $V^i, U^i, \mu^i > 0, \tau^i > 0$ and $\Psi^i > 0$ satisfying the inequalities (31) and (32) in condition *(b)*. For all $(\boldsymbol{x_{ND}}, \boldsymbol{x_D}, \boldsymbol{m_{ND}}, \boldsymbol{m_D}) \in \mathbb{R}_+^{\boldsymbol{k}}$**,** we define the functions:

$$V(\boldsymbol{m_{ND}}, \boldsymbol{m_D}) = min_{j=1,\dots,n}\{V^j + \mu^j \boldsymbol{r_{ND}^j}(\boldsymbol{m_{ND}} - \boldsymbol{m_{ND}^j}) + \Psi^j \boldsymbol{r_D^j}(\boldsymbol{m_D} - \boldsymbol{m_D^j})\},$$

$$U(\boldsymbol{x_{ND}}, \boldsymbol{x_D}, V(\boldsymbol{m_{ND}}, \boldsymbol{m_D})) =$$

$$min_{j=1,\dots,n}\left\{U^j + \tau^j \boldsymbol{p_{ND}^j}(\boldsymbol{x_{ND}} - \boldsymbol{x_{ND}^j}) + \frac{\tau^j}{\mu^j}\Psi^j \boldsymbol{p_D^j}(\boldsymbol{x_D} - \boldsymbol{x_D^j}) + \frac{\tau^j}{\mu^j}(V(\boldsymbol{m}) - V^j)\right\}.$$

It follows from Varian (1983) (and Afriat's theorem) that the function V is continuous, strictly increasing and concave. Moreover, we have $V(\boldsymbol{m_{ND}^i}, \boldsymbol{m_D^i}) \leq V^i$, since

$$V(\boldsymbol{m_{ND}^i}, \boldsymbol{m_D^i}) \leq V^i + \mu^i \boldsymbol{r_{ND}^i}(\boldsymbol{m_{ND}^i} - \boldsymbol{m_{ND}^i}) + \Psi^i \boldsymbol{r_D^i}(\boldsymbol{m_D^i} - \boldsymbol{m_D^i}) = V^i.$$

Setting the weak inequality strict yields:

$$V^i > V(\boldsymbol{m_{ND}^i}, \boldsymbol{m_D^i}) = V^a + \mu^a \boldsymbol{r_{ND}^a}(\boldsymbol{m_{ND}^i} - \boldsymbol{m_{ND}^a}) + \Psi^a \boldsymbol{r_D^a}(\boldsymbol{m_D^i} - \boldsymbol{m_D^a}),$$

for some $a = 1, \dots, n$, which violates the inequalities (31). Thus, $V(\boldsymbol{m_{ND}^i}, \boldsymbol{m_D^i}) = V^i$. Next, we show that $V(\boldsymbol{m_{ND}}, \boldsymbol{m_D})$ rationalizes the data $\{(\boldsymbol{r_D^i}, \boldsymbol{r_{ND}^i}); (\boldsymbol{m_D^i}, \boldsymbol{m_{ND}^i})\}_{i=1,\dots,n}$. Suppose $\boldsymbol{r_{ND}^i}\boldsymbol{m_{ND}^i} + \tilde{r}_D^i \boldsymbol{m_D^i} \geq \boldsymbol{r_{ND}^i}\boldsymbol{m_{ND}} + \tilde{r}_D^i \boldsymbol{m_D}$. For all $(\boldsymbol{m_{ND}}, \boldsymbol{m_D})$, we have

$$V(\boldsymbol{m_{ND}}, \boldsymbol{m_D}) \leq V^i + \mu^i \boldsymbol{r_{ND}^i}(\boldsymbol{m_{ND}} - \boldsymbol{m_{ND}^i}) + \Psi^i \boldsymbol{r_D^i}(\boldsymbol{m_D} - \boldsymbol{m_D^i})$$

$$= V^i + \mu^i \boldsymbol{r_{ND}^i}(\boldsymbol{m_{ND}} - \boldsymbol{m_{ND}^i}) + \mu^i(1 + \mathrm{IA}^i)\boldsymbol{r_D^i}(\boldsymbol{m_D} - \boldsymbol{m_D^i})$$

$$= V^i + \mu^i \boldsymbol{r_{ND}^i}(\boldsymbol{m_{ND}} - \boldsymbol{m_{ND}^i}) + \mu^i \tilde{r}_D^i(\boldsymbol{m_D} - \boldsymbol{m_D^i})$$

$$= V^i + \mu^i\left(\boldsymbol{r_{ND}^i}(\boldsymbol{m_{ND}} - \boldsymbol{m_{ND}^i}) + \tilde{r}_D^i(\boldsymbol{m_D} - \boldsymbol{m_D^i})\right)$$

$\leq V^i$

$= V\left(\boldsymbol{m_{ND}^i}, \boldsymbol{m_D^i}\right),$

where the first equality follows from equation (22) and where we have defined $\tilde{\boldsymbol{r}}_D^i = \left(1 + \mathrm{IA}^i\right)\boldsymbol{r}_D^i$.

Consider next the function $U\left(\boldsymbol{x_{ND}}, \boldsymbol{x_D}, V(\boldsymbol{m_{ND}}, \boldsymbol{m_D})\right)$. Following Varian (1983), it can be verified that the function $U\left(\boldsymbol{x_{ND}}, \boldsymbol{x_D}, V(\boldsymbol{m_{ND}}, \boldsymbol{m_D})\right)$ is continuous, strictly increasing, and concave in $(\boldsymbol{x_{ND}}, \boldsymbol{x_D})$ and V. Also we have $U\left(\boldsymbol{x_{ND}^i}, \boldsymbol{x_D^i}, V\left(\boldsymbol{m_{ND}^i}, \boldsymbol{m_D^i}\right)\right) = U^i$ by using the same argument as above. Finally, we show that $U\left(\boldsymbol{x_{ND}}, \boldsymbol{x_D}, V(\boldsymbol{m_{ND}}, \boldsymbol{m_D})\right)$ rationalizes the data $\mathbb{D} = \left\{\left(\boldsymbol{p_D^i}, \boldsymbol{p_{ND}^i}\right), \left(\boldsymbol{r_D^i}, \boldsymbol{r_{ND}^i}\right); \left(\boldsymbol{x_D^i}, \boldsymbol{x_{ND}^i}\right), \left(\boldsymbol{m_D^i}, \boldsymbol{m_{ND}^i}\right)\right\}_{i=1,\dots,n}$. Suppose $\boldsymbol{p_{ND}^i}\boldsymbol{x_{ND}^i} + \tilde{\boldsymbol{p}}_D^i\boldsymbol{x_D^i} + \boldsymbol{r_{ND}^i}\boldsymbol{m_{ND}^i} + \tilde{\boldsymbol{r}}_D^i\boldsymbol{m_D^i} \geq \boldsymbol{p_{ND}^i}\boldsymbol{x_{ND}} + \tilde{\boldsymbol{p}}_D^i\boldsymbol{x_D} + \boldsymbol{r_{ND}^i}\boldsymbol{m_{ND}} + \tilde{\boldsymbol{r}}_D^i\boldsymbol{m_D}$. For all $(\boldsymbol{x_{ND}}, \boldsymbol{x_D}, \boldsymbol{m_{ND}}, \boldsymbol{m_D})$**,** we have:

$U\left(\boldsymbol{x_{ND}}, \boldsymbol{x_D}, V(\boldsymbol{m_{ND}}, \boldsymbol{m_D})\right)$

$$\leq U^i + \tau^i\boldsymbol{p_{ND}^i}\left(\boldsymbol{x_{ND}} - \boldsymbol{x_{ND}^i}\right) + \frac{\tau^i}{\mu^i}\Psi^i\boldsymbol{p_D^i}\left(\boldsymbol{x_D} - \boldsymbol{x_D^i}\right) + \frac{\tau^i}{\mu^i}\left(V(\boldsymbol{m_{ND}}, \boldsymbol{m_D}) - V^i\right)$$

$$\leq U^i + \tau^i\boldsymbol{p_{ND}^i}\left(\boldsymbol{x_{ND}} - \boldsymbol{x_{ND}^i}\right) + \frac{\tau^i}{\mu^i}\Psi^i\boldsymbol{p_D^i}\left(\boldsymbol{x_D} - \boldsymbol{x_D^i}\right)$$

$$+\frac{\tau^i}{\mu^i}\left(\left(V^i + \mu^i\boldsymbol{r_{ND}^i}\left(\boldsymbol{m_{ND}} - \boldsymbol{m_{ND}^i}\right) + \Psi^i\boldsymbol{r_D^i}\left(\boldsymbol{m_D} - \boldsymbol{m_D^i}\right)\right) - V^i\right)$$

$$= U^i + \tau^i\boldsymbol{p_{ND}^i}\left(\boldsymbol{x_{ND}} - \boldsymbol{x_{ND}^i}\right) + \frac{\tau^i}{\mu^i}\Psi^i\boldsymbol{p_D^i}\left(\boldsymbol{x_D} - \boldsymbol{x_D^i}\right)$$

$$+\frac{\tau^i}{\mu^i}\left(\mu^i\boldsymbol{r_{ND}^i}\left(\boldsymbol{m_{ND}} - \boldsymbol{m_{ND}^i}\right) + \Psi^i\boldsymbol{r_D^i}\left(\boldsymbol{m_D} - \boldsymbol{m_D^i}\right)\right)$$

$$= U^i + \tau^i\boldsymbol{p_{ND}^i}\left(\boldsymbol{x_{ND}} - \boldsymbol{x_{ND}^i}\right) + \frac{\tau^i}{\mu^i}\Psi^i\boldsymbol{p_D^i}\left(\boldsymbol{x_D} - \boldsymbol{x_D^i}\right) + \tau^i\boldsymbol{r_{ND}^i}\left(\boldsymbol{m_{ND}} - \boldsymbol{m_{ND}^i}\right)$$

$$+\frac{\tau^i}{\mu^i}\Psi^i\boldsymbol{r_D^i}\left(\boldsymbol{m_D} - \boldsymbol{m_D^i}\right)$$

$$= U^i + \tau^i(\boldsymbol{p}^{\boldsymbol{i}}_{\boldsymbol{ND}}(\boldsymbol{x}_{\boldsymbol{ND}} - \boldsymbol{x}^{\boldsymbol{i}}_{\boldsymbol{ND}}) + \frac{\Psi^i}{\mu^i}\boldsymbol{p}^{\boldsymbol{i}}_{\boldsymbol{D}}(\boldsymbol{x}_{\boldsymbol{D}} - \boldsymbol{x}^{\boldsymbol{i}}_{\boldsymbol{D}}) + \boldsymbol{r}^{\boldsymbol{i}}_{\boldsymbol{ND}}(\boldsymbol{m}_{\boldsymbol{ND}} - \boldsymbol{m}^{\boldsymbol{i}}_{\boldsymbol{ND}})$$

$$+ \frac{\Psi^i}{\mu^i}\boldsymbol{r}^{\boldsymbol{i}}_{\boldsymbol{D}}(\boldsymbol{m}_{\boldsymbol{D}} - \boldsymbol{m}^{\boldsymbol{i}}_{\boldsymbol{D}}))$$

$$= U^i + \tau^i(\boldsymbol{p}^{\boldsymbol{i}}_{\boldsymbol{ND}}(\boldsymbol{x}_{\boldsymbol{ND}} - \boldsymbol{x}^{\boldsymbol{i}}_{\boldsymbol{ND}}) + (1 + \mathrm{IA}^i)\boldsymbol{p}^{\boldsymbol{i}}_{\boldsymbol{D}}(\boldsymbol{x}_{\boldsymbol{D}} - \boldsymbol{x}^{\boldsymbol{i}}_{\boldsymbol{D}}) + \boldsymbol{r}^{\boldsymbol{i}}_{\boldsymbol{ND}}(\boldsymbol{m}_{\boldsymbol{ND}} - \boldsymbol{m}^{\boldsymbol{i}}_{\boldsymbol{ND}})$$

$$+(1 + \mathrm{IA}^i)\boldsymbol{r}^{\boldsymbol{i}}_{\boldsymbol{D}}(\boldsymbol{m}_{\boldsymbol{D}} - \boldsymbol{m}^{\boldsymbol{i}}_{\boldsymbol{D}}))$$

$$= U^i + \tau^i\left(\boldsymbol{p}^{\boldsymbol{i}}_{\boldsymbol{ND}}(\boldsymbol{x}_{\boldsymbol{ND}} - \boldsymbol{x}^{\boldsymbol{i}}_{\boldsymbol{ND}}) + \widetilde{\boldsymbol{p}}^i_D(\boldsymbol{x}_{\boldsymbol{D}} - \boldsymbol{x}^{\boldsymbol{i}}_{\boldsymbol{D}}) + \boldsymbol{r}^{\boldsymbol{i}}_{\boldsymbol{ND}}(\boldsymbol{m}_{\boldsymbol{ND}} - \boldsymbol{m}^{\boldsymbol{i}}_{\boldsymbol{ND}}) + \tilde{\boldsymbol{r}}^i_D(\boldsymbol{m}_{\boldsymbol{D}} - \boldsymbol{m}^{\boldsymbol{i}}_{\boldsymbol{D}})\right)$$

$$\leq U^i$$

$$= U\left(\boldsymbol{x}^{\boldsymbol{i}}_{\boldsymbol{ND}}, \boldsymbol{x}^{\boldsymbol{i}}_{\boldsymbol{D}}, V(\boldsymbol{m}^{\boldsymbol{i}}_{\boldsymbol{ND}}, \boldsymbol{m}^{\boldsymbol{i}}_{\boldsymbol{D}})\right),$$

where $\widetilde{\boldsymbol{p}}^i = (1 + \mathrm{IA}^i)\boldsymbol{p}^i$. This completes the proof.