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PRODUCT DIFFERENTIATION AND THE SUSTAINABILITY OF COLLUSION

(preliminary version, January 22, 1992)

by

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ABSTRACT

This paper examines the incentives to differentiate products horizontally in a repeated game framework. The main findings are the following; If firms are patient with respect to future profits (i.e the discount factor is high) they would want to choose an intermediate degree of differentiation. The lower the discount factor, the more are firms forced to increase differentiation in order to sustain collusion. In the special case where differentiation is totally exogenous to the firms, it is shown that monopoly pricing is easier to sustain on markets where products are relatively differentiated. In case monopoly pricing is not sustainable, lowering the collusive price will always enable firms to cooperate successfully. Moreover, these constrained monopoly prices will be lower the greater the substitutability.

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I INTRODUCTION

This paper examines the incentives to differentiate products in a collusive duopoly. The idea is that rational firms might want to to use differentiation to increase profits, but also to make collusion easier to sustain. However, these interests may conflict so no conclusions can be made without giving the problem some structure.

To begin with, let us abstract from the question of differentiation. During the last ten years an extensive literature on collusive behavior has evolved. The term collusion covers everything from explicit cartels to agreements formed without explicit exchange of information. Since collusion is considered illegal in most countries, no firm taking part in a collusive arrangement would of course admit this, nor would it supply accurate data concerning the price–cost margin. However, although the existence of collusion is difficult to prove, most economists would probably agree on the phenomenon being more than a theoretical artifact.

To sustain collusive behavior, a certain amount of coordination is needed between firms with the implication that collusion is most likely to be found on fairly concentrated markets. Whenever it seems possible to coordinate pricing decisions, it is of course tempting for firms to collectively raise prices above the non–cooperative level and thereby increase profits. However, such a situation is far from stable. As long as the collusive prices are above the non–cooperative level there are incentives to cheat on the other members of the collusive club. By lowering his price unilaterally by a small amount (or by increasing output by a large amount), a cheating firm may capture a large fraction of the market and thus make a substantial short–term gain. Hence, for collusion to be sustainable, there must be some punishment penalizing a cheater.

The so called "Folk theorem" captures the essence of what is said above. It says that collusion is sustainable if i) there is an infinite time horizon, ii) firms' strategies are to go back to the non–cooperative price forever after if anyone cheats, and iii) the discount factor is high enough. If "tomorrow" is important enough, i.e the discount factor is high enough, the short–run gains from cheating will be outweighed by the reduction in future profits streams and collusion will be sustainable. The higher the punishment payoffs and the cheating payoffs, the higher the discount factor has to be for collusion not to break down. Conversely, the higher the collusive payoffs, the lower may the discount factor be.

Now, assume there is a possibility also to choose certain product characteristics in each period. The hypothesis of this paper is that the optimal degree of product differentiation will then be a function of the discount factor. First, as implied by the "Folk theorem", almost any collusive agreement is sustainable if the discount factor is high. Then, rational firms would want to maximize profits both with respect to price and with respect to differentiation. Second, if the discount factor is low, what happens tomorrow will not be very important. Intuitively, one might suspect that firms would then be forced to make the deviation gains as small as possible by making products remote substitutes. On the other hand, a better strategy might be to maximize the collusive payoffs, by choosing differentiation appropriately.

Stating the goal of the article a bit more carefully, we want to generate some behavioral implication on how collusive firms would choose prices and product design optimally at various discount factors. Since many (maybe most) markets are characterized by competition between only a few firms, making collusive agreements possible, insights of this kind would be of more than minor interest. The framework chosen will be a repeated game version of a model similar to the 1929 Hotelling model.

These are the main findings; In the special case where differentiation is totally exogenous to the firms, it is shown that monopoly pricing is easier to sustain on markets where products are relatively differentiated. In case monopoly pricing is not sustainable, lowering the collusive price will always enable firms to cooperate successfully. In the more interesting case where product design is endogenized the results are the following; If the discount factor is high, firms would want to commit to an intermediate degree of differentiation since then, joint profits are the highest possible to attain. If the discount factor is lowered, firms are forced to to **increase** differentiation in order to sustain collusion. Unless the discount factor is extremely low, prices will be the unconstrained monopoly prices conditional on locations.

The article is organized as follows; Section II contains a brief survey of the literature on product differentiation. We argue for the choice of a modified version of the 1929 Hotelling model as a suitable framework but the limitations of such a framework are also discussed. In section III, the theory of repeated games is discussed. Furthermore, the timing and the strategies of the players are described in detail. We end up with a general expression for the minimal discount factor at which collusion can be sustained. Basically, the problem discussed in this paper is how rational firms would choose prices and product design when constrained by this discount factor restriction. In section IV, the basic model is presented. In section V, the firms' pricing decisions are discussed under the assumption of exogenous product design. We ask what prices will maximize joint profits, what prices a deviator would choose given the collusive prices and what prices would constitute an equilibrium when firms are not colluding. Using this input, an explicit expression for the minimal discount factor needed to sustain monopoly pricing is derived in section VI. This is done under two assumptions. First, product design is thought to be totally exogenous from the firms' point of view. Second, firms are free to pick any design once price cooperation has broken down. In section VII, product design is endogenized. First, we derive the highest collusive price consistent with sustainability, given a certain design and discount factor.

Then, firm profits are maximized with respect to design, yielding the optimal design as a function of the discount factor. Finally, some concluding remarks are presented in section VIII.

II A BRIEF SURVEY OF THE LITERATURE

Following Eaton and Lipsey's survey, the literature on product differentiation can be divided into two branches. The so called address branch is characterized by consumers' preferences being spread over some continuous parameter space, describing the products. Thus, consumers' tastes differ. The most notable example is the 1929 Hotelling model, later modified by d'Aspremont, Gabszewicz and Thisse [1979] and by Neven [1985]. The model is a two–period duopoly game where firms make irreversible choices of product design in period 1 and compete in prices in period 2. Product differentiation is one–dimensional and there is no consensus among consumers whether one brand is better than another when equally priced. On the contrary, each consumer has a favorite position and the less similar the existing brands are to this favorite, the lower is utility.

In addition to the Hotelling model, there are other variations on the same theme. One example is the 1979 Salop model, where consumers' preferences are described by a circle rather than a straight line. This model has mainly been used to analyze free entry equilibria. Not quite a member of the address branch, but still very similar to the Hotelling model, is the Shaked and Sutton 1982 model of a market for vertically differentiated products. Here, different qualities can be **objectively** ranked, but due to income differences, some people will prefer highly priced high–quality goods to inexpensive low–quality goods and vice versa.

There has been some work done on the sustainability of supergame equilibria within the address branch. Independently of the author, Chang [1991] has analyzed the connection between the discount factor restriction and the degree of differentiation for exogenous locations in the d'Aspremont, Gabszewicz and Thisse version of Hotelling's model. His results are basically identical with the results of section VI(i) in this paper. Within the same framework, Chang has endogenized product design in an unpublished paper [1990] in a way that is quite similar to the analysis of section VII of this paper. His work is even slightly more general in that he allows a fixed cost for redesigning products.

The other main branch is the so called non–address branch. Here, the number and nature of the varieties possible to produce is exogenously given, and preferences are generally represented by a single consumer. These models mostly deal with free entry equilibria. Seminal papers in the area are Spence [1976a, 1976b] and Dixit and Stiglitz [1977]

Within the non–address branch, Deneckere [1983] has shown that price setting supports more tacit collusion than quantity setting when goods are complements or very good substitutes while the opposite is true for moderate or poor substitutes. However, his analysis assumes exogenous product design and a duopoly market. The latter assumption has been relaxed by Majerus [1988] who shows that price setting is superior to quantity setting, from a social point of view, if the number of firms are greater or equal to three. It has come to our knowledge that there has been some unpublished work done by Martin [1989] and Ross [1990] in addition to these references, also within the non–address branch.

This paper analyzes the properties of a modified version of the 1929 Hotelling model, due to d'Aspremont, Gabszewicz and Thisse. There are several reasons for this. First, in the address branch approach, differentiation is a continuous variable which simplifies computations. Moreover, allowing for heterogeneous preferences is also a very appealing characteristic of this branch. Secondly, since the framework is a collusive situation we are not primarily interested in free–entry equilibria. Therefore, the 1979 modification by Salop does not add anything essential. Thirdly, unlike the original Hotelling model, the d'Aspremont, Gabszewicz and Thisse modification ensures the existence of price equilibria at all locations. Finally, it should be mentioned that the similar work conducted by Chang was not known to the author until fairly recently when mentioned in referee report concerning an earlier draft of this paper

Choosing the Hotelling model certainly involves making some rather simplistic assumptions. Differentiation is one–dimensional and quality is not high or low in any objective sense. Clearly, there is a limited number of goods that will fit this description. Furthermore, the number of players are restricted to two and there is no question of entry. Finally, each firm is allowed to produce only one specific variety. Hopefully, what is gained in terms of tractability is not totally lost in terms of realism.

III THE REPEATED GAME FRAMEWORK

A collusive agreement can be seen as a contract between firms which is not enforceable by the legal system. Therefore, such a contract also has to be a subgame perfect Nash equilibrium (SPE) to be sustainable. Collusion is typically dealt with in infinitely repeated game settings where there is always an underlying one–period base game with one, or more, Nash equilibria (NE).

To begin with, let us abstract from the possibility to differentiate products. One SPE of the repeated game is to play the "competitive" one–shot NE every day from now to eternity but, as mentioned above, cooperation can also be sustained as an equilibrium if the discount factor is high enough. This is possible if the one–shot NE is being used as a punishment as suggested by Friedman [1971]. Then, the punishment strategies themselves form a SPE of the entire game so a deviator cannot avoid being punished. No one will take advantage of the fact that the cooperative solution is not a one–shot NE if the one–shot gain by deviating is smaller then the losses in terms of reduced future profit streams. Thus, making the discount factor ,δ, arbitrarily large will also make the discounted stream of profit reductions arbitrarily large and no deviation will take place.

More formally; Let Π^c be the cooperative per period payoff for a colluding firm. Π^d is the one–shot gain from deviating by undercutting the rival, while Π^p is the NE punishment payoff following a deviation from the period after the deviation and henceforth. Then, for collusion to be sustainable;

$$\frac{\Pi^c}{1-\delta} \geq \Pi^d + \frac{\delta\Pi^p}{1-\delta} \qquad \text{or} \qquad \delta \geq \frac{\Pi^d - \Pi^c}{\Pi^d - \Pi^p} \equiv \gamma$$

Clearly, γ puts a restriction on the discount factor. Cooperation is sustainable if and only if $\gamma \leq \delta$. γ is increasing in Π^p and Π^d but decreasing in Π^c.

Let us now introduce the possibility of differentiation. Clearly, Π^p , Π^d and Π^c are likely to be affected by the degree of differentiation, and it should therefore be possible to derive a function γ(a), where a denotes the degree of differentiation. The problem of this paper is to describe how rational collusive firms would choose prices and product design subject to;

$$\delta \geq \frac{\Pi^d(a) - \Pi^c(a)}{\Pi^d(a) - \Pi^p(a)} \equiv \gamma(a) \tag{3.1}$$

The game played is the following; The time horizon is infinite. In period τ, firms decide on two variables, namely the price of period τ and the product characteristics of period τ+1, i.e "next years design". There is no cost associated with these decisions[1]. There is a collusive agreement specifying the collusive prices and designs. If a firm deviates with

[1]This assumption is made in order to high–light the importance of product differentiation. Of course, for high enough costs associated with changing product design, it would no longer be a relevant variable.

respect to price in period τ, the strategies are to play the one–shot NE prices and designs in periods $\tau+1$ to eternity. If a firm deviates in period τ with respect to next periods design, this will be detected immediately and the strategies are then to play the one–shot NE prices and designs in periods $\tau+1$ to eternity.

Firms having to decide on next years design one year in advance reflects that it takes some time developing a new design. Surely, this is an important feature in many industries. It is also relatively easy to gather information on the competitors future product design. This is reflected in all price games being competitive once one of the firms has deviated in product design.

It should be noted that a simplified version of the game is considered in section VI(i). There, the discount factor restriction is derived assuming that product design is exogenous to the firm. This gives some conditions for when collusion is likely to be easily sustained in case changing design is impossible or very costly.

IV THE MODEL

There are two firms denoted 1 and 2. Consumers are assumed to be uniformly distributed by taste along a line of unit length and the two firms are located at points a_1 and $(1-a_2)$ in this one–dimensional product space. Each period, consumers buy at most one unit of a good which is homogeneous in all other respects than the distance between consumer preference and product design. There is a disutility cost associated with not being able to buy the favorite good in the product space. The utility of a consumer with taste $\theta \in [0, 1]$ is:

$$(4.1) \qquad U(\theta) = \begin{cases} s - t(\theta - a_1)^2 - P_1 & \text{if buying from firm 1} \\ s - t(1 - a_2 - \theta)^2 - P_2 & \text{if buying from firm 2} \\ 0 & \text{otherwise} \end{cases}$$

where s is the reservation price before disutility costs are deducted, t times the squared distance gives the total disutility cost and P_1 and P_2 are the prices charged by the firms. This is basically the d'Aspremont, Gabszewicz and Thisse 1979 modification of Hotelling's original 1929 formulation, guaranteeing price equilibria for all kinds of locational choices. The consumers' utility levels, given firm locations at a_1 and a_2 and prices P_1 and P_2, are shown graphically in figure 4.1. Of course, consumers make their purchase from the firm whose product characteristic and price gives them the highest utility, (if positive) or refuse to buy at all if prices are too high. Both firms have constant and identical marginal costs which are normalized to zero.

One very weak assumption is made concerning the relative size of the reservation price, s. The first part is equivalent to saying that the equilibrium payoffs in absence of cooperation are always lower than the collusive payoffs[2]. The second part prevents the collusive prices from being unbounded.

Assumption 1: $\frac{5}{4} t \leq s < \infty$

[2]This is demonstrated in the proof of lemma 6 which can be found in appendix.

V PRICING STRATEGIES

In this section, the design specified by the collusive agreement will be exogenous. We calculate the prices maximizing joint profits, the one–shot NE prices and the optimal deviation prices. These prices are used as input in the following section where we derive the minimal discount factor at which monopoly pricing is sustainable in a repeated game. For simplicity, it is assumed that the collusive agreement specifies symmetric locations.

Assumption 2: The collusive agreement specifies symmetric locations so that $a_1=a_2=a$.

V(i) The joint profit maximizing price

A priori, it is not clear whether full market coverage is optimal under monopoly pricing. Intuitively, the higher the reservation price, the more profitable to cover the entire market. As it turns out;

Lemma 1: If assumption 1 holds, monopoly pricing will imply full market coverage.

Proof: In appendix

By lemma 1, if $a \leq 1/4$ profits will be maximized by raising prices until the consumer located at $\theta=1/2$ is indifferent between buying and not buying[3]. Similarly, if $a \geq 1/4$, the consumers located at the endpoints will have zero utility at the profit maximizing price. Let P^c and Π^c denote the monopoly price and the corresponding per firm profit. Then;

[3]Choosing a lower price would create no additional demand.

(5.1) $$\Pi^c(a) = \frac{P^c(a)}{2} = \frac{1}{2}\left[s - t(1/2 - a)^2\right] \qquad a \leq 1/4$$

(5.2) $$\Pi^c(a) = \frac{P^c(a)}{2} = \frac{1}{2}\left[s - ta^2\right] \qquad a \geq 1/4$$

There is a strictly positive relationship between Π^c and a when $a<1/4$, and a negative relationship when $a>1/4$. Consequently, monopoly profits are highest at $a=1/4$.

V(ii) The punishment price

The punishment prices are simply the prices of the unique NE of the one–shot base game. Figure 4.1 showed the utility of consumers when firms are located at a_1 and $(1-a_2)$, charging prices P_1 and P_2. θ^* denotes the consumer who is indifferent between the two firms. Algebraically, θ^* is given by:

$$s - t(\theta^* - a_1)^2 - P_1 = s - t(1 - a_2 - \theta^*)^2 - P_2$$

Solving for θ^* and noting that the demand functions, D_i, are given by θ^* and $1-\theta^*$ respectively, we arrive at:

(5.3) $$D_i = \frac{1 - a_j + a_i}{2} + \frac{P_j - P_i}{2t(1 - a_i - a_j)}$$

where $i,j \in \{1, 2\}$ and $i \neq j$. Each firm maximizes its profit, taking product design and the competitor's price as given. For firm i, profits are;

(5.4) $$\Pi_i = P_i\left[\frac{1 - a_j + a_i}{2} + \frac{P_j - P_i}{2t(1 - a_i - a_j)}\right]$$

Π_i being concave in P_i, straightforward differentiation yields the following reaction function for firm i:

$$P_i = \frac{1}{2}\left[\, t(1 - 2a_j + a_j^2 - a_i^2) + P_j \,\right]$$

These reaction functions are upward sloping, implying that prices are strategic complements. Solving for the equilibrium prices, with p denoting punishment, we have:

$$P_i^p = \frac{t}{3}\,(3 - 4a_j - 2a_i + a_j^2 - a_i^2)$$

Substituting the equilibrium prices into (5.4) and rearranging, we end up with the following punishment payoff for firm i:

$$\Pi_i^p = \frac{t(1 - a_i - a_j)(3 + a_i - a_j)^2}{18} \tag{5.5}$$

or, in the symmetric case;

$$\Pi_i^p = \frac{t(1-2a)}{2} \tag{5.6}$$

Finally we can state;

Lemma 2: If assumptions 1 and 2 hold there will be full market coverage in the punishment phase.

Proof: In appendix

V(iii) The deviation price

Since monopoly prices are not one–shot equilibrium prices, it might pay to deviate from the collusive agreement. There are two possible deviation strategies. In both cases one firm lowers its' price to make a short run gain by stealing the competitors customers. However, for some locations it might be optimal to steal only some part of the competitor's customers while for other locations, capturing the entire market might be more profitable.

When stealing the entire market, a deviating firm will have to lower its price to the extent that the most distant consumer is indifferent between the firms' products. Let subscripts w and h denote a "whole" theft and a "half" theft respectively and let superscript d denote deviation. Then, the more aggressive strategy will give the following profits to the deviator;

(5.7) $$P_w^d(a) = \Pi_w^d(a) = P^c(a) - t(1 - 2a)$$

From the definition of $P^c(a)$ in section V(i), it follows that;

(5.8) $$\Pi_w^d(a) = \frac{1}{4}(4s - 5t + 12at - 4a^2t) \qquad a \leq 1/4$$

(5.9) $$\Pi_w^d(a) = s - t + 2at - a^2t \qquad a \geq 1/4$$

In the less aggressive case, a deviating firm faces the following profit function;

(5.10) $$\Pi_i = P_i\left[\frac{1}{2} + \frac{P^c(a) - P_i}{2t(1 - 2a)}\right]$$

which is simply expression (5.4), firms being located symmetrically and the competitor charging $P^c(a)$. By profit maximization, the optimal deviation price then equals;

$$P_h^d(a) = \frac{1}{2}(P^c(a) + t(1-2a)) \tag{5.11}$$

Substituting this into (5.10) we have;

$$\Pi_h^d(a) = \frac{(2at - P^c(a) - t)^2}{8t(1-2a)} \tag{5.12}$$

Inserting the collusive prices from section V(i), we end up with the following deviation payoffs;

$$\Pi_h^d(a) = \frac{[4s + 3t - 4at - 4a^2t]^2}{128t(1-2a)} \qquad a \leq 1/4 \tag{5.13}$$

$$\Pi_h^d(a) = \frac{[s + t - 2at - a^2t]^2}{8t(1-2a)} \qquad a \geq 1/4 \tag{5.14}$$

Now it only remains to derive conditions for the relative profitability of the two deviation strategies. By lemma 3, it follows, quite intuitively, that the aggressive deviation strategy dominates for a larger set of locations the higher the reservation price, s. Moreover, if products are very close substitutes, the aggressive strategy will always dominate.

Lemma 3: Define $k \equiv s/t$ and $\hat{a}_1 \equiv 3-\sqrt{k+6}$ and $\hat{a}_2 \equiv (7-2\sqrt{k+9})/2$. Then;

i) if $5/4 \leq k \leq 25/16$, $\Pi_h^d(a) \geq \Pi_w^d(a)$ for $a \epsilon [0, \hat{a}_1]$ where $\hat{a}_1 \epsilon [1/4, 1/2]$. Consequently, $\Pi_h^d(a) \leq \Pi_w^d(a)$ for $a \epsilon [\hat{a}_1, 1/2]$.

ii) if $25/16 \leq k \leq 52/16$, $\Pi_h^d(a) \geq \Pi_w^d(a)$ for $a \epsilon [0, \hat{a}_2]$ where $\hat{a}_2 \epsilon [0, 1/4]$. Consequently, $\Pi_h^d(a) \leq \Pi_w^d(a)$ for $a \epsilon [\hat{a}_2, 1/2]$.

iii) if $k \geq 52/16$, $\Pi_h^d(a) \leq \Pi_w^d(a)$, $a \epsilon [0, 1/2]$.

Proof: In appendix

VI THE DISCOUNT FACTOR RESTRICTION

Having derived the collusive payoffs, the punishment payoffs and the deviation payoffs, we are now in a position to also derive expressions for the minimal discount factor needed to sustain monopoly pricing, conditional on locations. First, we do this keeping the assumption of totally fixed locations. However, even if the collusive design is exogenous, firms that have an opportunity to change design are likely to do so once cooperation breaks down. The implications of this for the punishment payoffs as well as for the discount factor restriction is also discussed.

In order to summarize the payoffs derived so far (assuming fixed locations), let us define a_h as the set of locations for which the less aggressive deviation strategy is more profitable. In analogy, a_w denotes the set of locations for which capturing the entire market is more profitable. Thus,

Definition 1: $a_h \equiv \{a \,|\, \Pi_h^d(a) \geq \Pi_w^d(a)\}$

$$a_w \equiv \{a \,|\, \Pi_h^d(a) \leq \Pi_w^d(a)\}$$

Using this notation and defining $k \equiv s/t$ we have;

$$\text{(5.1)} \qquad \Pi^c(a) = \frac{t}{2}\left[k - (1/2 - a)^2\right] \qquad a \leq 1/4$$

$$\text{(5.2)} \qquad \Pi^c(a) = \frac{t}{2}\left[k - a^2\right] \qquad a \geq 1/4$$

$$\text{(5.6)} \qquad \Pi^p = \frac{t(1-2a)}{2} \qquad \forall a$$

$$\text{(5.8)} \qquad \Pi^d(a) = \frac{t}{4}(4k - 5 + 12a - 4a^2) \qquad a \leq 1/4 \text{ and } a \in a_w$$

$$\text{(5.9)} \qquad \Pi^d(a) = t(k - 1 + 2a - a^2) \qquad a \geq 1/4 \text{ and } a \in a_w$$

$$\text{(5.13)} \qquad \Pi^d(a) = \frac{t[4k + 3 - 4a - 4a^2]^2}{128(1 - 2a)} \qquad a \leq 1/4 \text{ and } a \in a_h$$

$$\text{(5.14)} \qquad \Pi^d(a) = \frac{t[k + 1 - 2a - a^2]^2}{8(1 - 2a)} \qquad a \geq 1/4 \text{ and } a \in a_h$$

VI(i) The case of fixed design in the punishment phase

First, assume $20/16 \leq k \leq 25/16$. Then, by lemma 3, we can insert (5.1), (5.2), (5.6), (5.9), (5.13) and (5.14) into expression (3.1) yielding;

$$\gamma(a) = \begin{cases} \dfrac{4k-4a^2+12a-5}{4k-4a^2-20a+11} & 0\leq a\leq 1/4 \\ \dfrac{k-a^2+2a-1}{k-a^2-6a+3} & 1/4\leq a\leq \hat{a}_1 \\ \dfrac{k-a^2+4a-2}{2k-2a^2+6a-3} & \hat{a}_1\leq a\leq 1/2 \end{cases}$$

In analogy, when $25/16\leq k\leq 52/16$ we insert (5.1), (5.2), (5.6), (5.8), (5.9) and (5.13) into expression (3.1) arriving at;

$$\gamma(a) = \begin{cases} \dfrac{4k-4a^2+12a-5}{4k-4a^2-20a+11} & 0\leq a\leq \hat{a}_2 \\ \dfrac{4k-4a^2+20a-9}{2(4k-4a^2+16a-7)} & \hat{a}_2\leq a\leq 1/4 \\ \dfrac{k-a^2+4a-2}{2k-2a^2+6a-3} & 1/4\leq a\leq 1/2 \end{cases}$$

Finally, if $k\geq 52/16$ we insert (5.1), (5.2), (5.6), (5.8) and (5.9) into expression (3.1). Then;

$$\gamma(a) = \begin{cases} \dfrac{4k-4a^2+20a-9}{2(4k-4a^2+16a-7)} & 0 \leq a \leq 1/4 \\ \dfrac{k-a^2+4a-2}{2k-2a^2+6a-3} & 1/4 \leq a \leq 1/2 \end{cases}$$

Lemma 4: For all $k\geq 20/16$, $\gamma(a)$ is continuous and increasing in a. Moreover, $0<\gamma(a)\leq 1/2$.

Proof: Straightforward

Lemma 5: If monopoly pricing is not sustainable, the best firms can do is to lower the collusive price until some price P^* at which the discount factor restriction binds. There will exist a P^* such that $P^p \leq P^* \leq P^c$ for every design and discount factor. Moreover, P^* will be lower the greater the substitutability.

Proof: In appendix.

VI(ii) The case of variable design in the punishment phase

If firms are able to change locations, there is no reason to believe that they would stick to the design specified by the collusive agreement once cooperation has broken down. Differentiating (5.5) with respect to a_i, we have;

$$\frac{\partial \Pi_i^p}{\partial a_i} = \frac{-t(a_i - a_j + 3)(3a_i + a_j + 1)}{18} < 0$$

so both firms gain by increasing differentiation in the punishment phase and, of course, they have nothing to lose by doing so once cooperation has broken down. That is, $a_i=a_j=0$, which means $P^p=t$ and;

(6.1) $$\Pi_i^p = \Pi^p = \frac{t}{2}$$

Introducing the ability to change design also offers the possibility to deviate in product design rather than prices. However;

Lemma 6: Deviations in product design will never occur if assumption 1 holds.

Proof: In appendix

Since the collusive design is kept exogenous, the only difference from section VI(i) is that now the punishment payoff is given by (6.1) instead of (5.6). This minor change result in the following discount factor restriction[4].

First, assume $20/16 \leq k \leq 25/16$. Then, by lemma 3, we can insert (5.1), (5.2), (5.9), (5.13), (5.14) and (6.1) into expression (3.1) yielding;

$$\gamma(a) = \begin{cases} \dfrac{16a^4-96a^3-8a^2(4k-23)+24a(4k-5)+(4k-5)^2}{16a^4+32a^3-8a^2(4k+1)-8a(4k-13)+16k^2+24k-55} & 0 \leq a \leq 1/4 \\[2ex] \dfrac{a^4-4a^3-2a^2(k-3)+4a(k-1)+(k-1)^2}{a^4+4a^3-2a^2(k-1)-4a(k-1)+(k-1)(k+3)} & 1/4 \leq a \leq \hat{a}_1 \\[2ex] \dfrac{k-a^2+4a-2}{2k-2a^2+4a-3} & \hat{a}_1 \leq a \leq 1/2 \end{cases}$$

In analogy, when $25/16 \leq k \leq 52/16$ we insert (5.1), (5.2), (5.8), (5.9), (5.13) and (6.1) into expression (3.1) arriving at;

$$\gamma(a) = \begin{cases} \dfrac{16a^4-96a^3-8a^2(4k-23)+24a(4k-5)+(4k-5)^2}{16a^4+32a^3-8a^2(4k+1)-8a(4k-13)+16k^2+24k-55} & 0 \leq a \leq \hat{a}_2 \\[2ex] \dfrac{4k-4a^2+20a-9}{8k-8a^2+24a-14} & \hat{a}_2 \leq a \leq 1/4 \\[2ex] \dfrac{k-a^2+4a-2}{2k-2a^2+4a-3} & 1/4 \leq a \leq 1/2 \end{cases}$$

Finally, if $k \geq 52/16$ we insert (5.1), (5.2), (5.8), (5.9) and (6.1) into expression (3.1). Then;

[4]Of course, the argument of $\gamma(a)$ now refers to the design specified by the collusive agreement and not to the location chosen in case cooperation breaks down.

$$\gamma(a) = \begin{cases} \dfrac{4k-4a^2+20a-9}{8k-8a^2+24a-14} & 0 \le a \le 1/4 \\[2ex] \dfrac{k-a^2+4a-2}{2k-2a^2+4a-3} & 1/4 \le a \le 1/2 \end{cases}$$

Lemma 7: For all $k \geq 20/16$, $\gamma(a)$ is continuous and increasing in a. Moreover, $0<\gamma(a)\leq 1$.

Proof: Straightforward

To conclude; Given assumption 1 so that $k>20/16$, monopoly pricing will be less demanding to sustain, in terms of the discount factor, the more differentiated the products are. Graphically, a typical $\gamma(a)$ is shown in figure 6.1. We can also see that, regardless of locations, $\gamma(a)$ approaches 1/2 as k approaches infinity. Hence, ceteris paribus, the trade–off between sustainability and product similarity becomes less important the higher the consumers' reservation price. In other words, given that k is large, the degree of differentiation plays a very small role when a firm decides whether to deviate or not. These conclusions are valid both when locations are totally fixed and when firms may change locations once cooperation breaks down.

Since $\gamma(a)$ is increasing in Π^P, allowing firms to change product design in the punishment phase has the obvious implication of shifting the $\gamma(a)$–function upwards making monopoly pricing more difficult to sustain. This shift is in fact quite substantial. For example, if products are close substitutes, and k is small (close to 5/4) the discount factor has to be close to one in the latter case. As a comparison, monopoly pricing is always sustainable for discount factors higher than 1/2 when locations are fixed.

VII THE OPTIMAL DEGREE OF DIFFERENTIATION

Until now, the design specified by the collusive agreement has been regarded exogenous. In this part we let firms choose design in a rational way.

The analysis is carried out in three steps. First, the best collusive price, $P^*(a)$, will be derived as a function of the discount factor, still treating product design as fixed. This price is defined as the price maximizing joint profits subject to the discount factor restriction of section VI(ii). Second, the best collusive profit $\Pi^*(a;\delta)$ will be defined for all $a \in [0, 1/2]$ as a function of the discount factor. Third, the optimal collusive design, $a^*(\delta)$, defined as;

$$a^*(\delta) = \max_{a} \Pi^*(a;\delta)$$

will be characterized.

When $\delta \geq \gamma(a)$, monopoly pricing is sustainable by definition so;

Lemma 8: If $\delta \geq \gamma(a)$, the best collusive price, $P^*(a)$, is the monopoly price $P^c(a)$.

Proof: Trivial

If, however, the monopoly price cannot be sustained given a certain location and discount factor, maybe choosing some other price will allow firms to collude successfully. Choosing a collusive price higher than the monopoly price may only have a negative effect on sustainability. The temptation to deviate increases while collusive profits are lowered. Choosing a collusive price lower than the monopoly price lowers both deviation profits and collusive profits and the net effect may very well be a mitigation of the discount factor restriction. Hence, if $\delta < \gamma(a)$ we may define the best collusive price as;

$$P^*(a) = \max P \quad \text{s.t} \quad \delta \geq g(a, P) = \frac{\Pi^d(P, a) - \Pi(P)}{\Pi^d(P, a) - \Pi^p}$$

where $\delta \geq g(a, P)$ is the general discount factor restriction for $P \leq P^c$ and where $\Pi(P)=P/2$ by symmetry.

Inserting (6.1) for the punishment payoff and expressions (5.7) and (5.12) for the deviation payoff (replacing P^c by P), the discount factor restriction equals;

$$\delta \geq g(a, P) = \frac{4at + P - 2t}{4at + 2P - 3t} \qquad P \epsilon P_w \tag{7.1}$$

$$\delta \geq g(a, P) = \frac{4a^2t^2 + 4at(P-t) + (P-t)^2}{4a^2t^2 - 4at(P-t) + (P-t)(P+3t)} \qquad P \epsilon P_h \tag{7.2}$$

where P_w denotes the set of collusive prices for which the aggressive deviation strategy is most profitable. P_h is defined analogously for the less aggressive deviation strategy. Now, it can be shown that if $a>1/4$, the discount factor restriction cannot be mitigated by choosing some price $P<P^c$. That is, if $a>1/4$ collusion cannot be sustained unless $\delta \geq \gamma(a)$

Lemma 9: When $a>1/4$, collusion cannot be sustained unless $\delta \geq \gamma(a)$. When $a \leq 1/4$ collusion may be sustained even when $\delta < \gamma(a)$

Proof: We have already argued that choosing $P>P^c$ may only make the discount factor restriction more demanding since it decreases collusive profits while deviation profits are increased and punishment profits are unchanged. For $P<P^c$ to mitigate the discount factor restriction, we must have $\partial g(a, P)/\partial P>0$ for some price range. We will show that this is impossible for $a>1/4$ while it always possible for $a \leq 1/4$.

First, assume $P \epsilon P_w$. Differentiating (7.1), we have

$$\frac{\partial g(a,\ P)}{\partial P} = \frac{t(1-4a)}{(4at + 2P - 3t)^2}$$

which is negative for a>1/4 and positive for a<1/4.

Second, assuming $P \epsilon P_h$, $\Pi_h^d(a) \geq \Pi_w^d(a)$ by definition. For an arbitrary collusive price, these functions are defined by expressions (5.7) and (5.12) (inserting P instead of P^c). Solving for P from this inequality, it follows that $P \epsilon P_h$ is equivalent to $P<3t(1-2a)$. Now, differentiating (7.2) and denoting the denominator by B,

$$\frac{\partial g(a,\ P)}{\partial P} = \frac{4t(1-2a)}{B^2}[P^2 - 4a^2t^2 - t(2P-t)]$$

which is positive if $P>t(2a+1)$ and vice versa. Thus, for the less aggressive strategy to dominate at the same time as $\partial g(a, P)/\partial P>0$, we must have $t(2a+1) \leq P \leq 3t(1-2a)$. This, however, implies $a \leq 1/4$.

To summarize; If $a \leq 1/4$ $\partial g(a, P)/\partial P>0$ for $P>t(2a+1)$. Moreover, for $P>3t(1-2a) \geq t(2a+1)$ the aggressive deviation strategy will be most profitable while the opposite is true for $t(2a+1)<P<3t(1-2a)$. If $a>1/4$, $\partial g(a, P)/\partial P<0$ for all prices and, consequently, the discount factor restriction cannot be satisfied unless $\delta \geq \gamma(a)$.

QED

Now assume $a \leq 1/4$ and $\delta < \gamma(a)$. From the proof of lemma 9 we know that $P \epsilon P_w$ is equivalent to $P>3t(1-2a)$ and that $P \epsilon P_h$ is equivalent to $P<3t(1-2a)$. Moreover, we know that

$\partial g(a, P)/\partial P>0$ for $P>t(2a+1)$. Finally, we know that $t(2a+1)\leq 3t(1-2a)$. Consequently, if there exist a P^* it is found by simply lowering prices until $g(a, P)=\delta$[5]. Thus, if a P^* exist it will be given either by;

$$(7.3) \qquad P^*(a) = \frac{t[4a(1-\delta) + 3\delta - 2]}{2\delta-1} \qquad P^*\geq 3t(1-2a)$$

or by;

$$(7.4) \qquad P^*(a) = \frac{t[(\delta+1)(1-2a) + 2\sqrt{\delta(\delta-2a)(1-2a)}]}{1-\delta} \qquad P^*\leq 3t(1-2a)$$

where $P^*(a)$ is solved from (7.1) and (7.2)

Lemma 10: Assume $a\leq 1/4$ and $\delta<\gamma(a)$. $P^*(a)$ is then decreasing in a. Moreover, it exists if and only if $a\leq\delta/2$. Hence, a collusive price will exist even for very low discount factors given that differentiation is large enough. Finally, (7.3) and (7.4) both equal $3t(1-2a)$ at $a=(3\delta-1)/(8\delta-2)$ so $P^*(a)$ is continuous.

Proof: In appendix

Lemma 8, 9 and 10 now allows us to characterize the best collusive per firm profit, $\Pi^*(a;\delta)$, since by symmetry, $\Pi^*(a;\delta)=P^*(a)/2$.

[5]There is no discountinuity in $g(a, P)$ at $P=3t(1-2a)$.

$$
\Pi^*(a;\delta) = \begin{cases}
[s - t(1/2-a)^2]/2 & a \leq 1/4 \quad \delta \geq \gamma(a) \\
[s - ta^2]/2 & a \geq 1/4 \quad \delta \geq \gamma(a) \\
\dfrac{t[4a(1-\delta) + 3\delta - 2]}{4\delta-2} & a \leq \dfrac{3\delta-1}{8\delta-2} \leq \dfrac{\delta}{2} \quad \delta < \gamma(a) \\
\dfrac{t[(\delta+1)(1-2a) + 2\sqrt{\delta(\delta-2a)(1-2a)}]}{2-2\delta} & \dfrac{3\delta-1}{8\delta-2} \leq a \leq \dfrac{\delta}{2} \quad \delta < \gamma(a) \\
\text{nonexistent} & a > \min\{1/4,\ \delta/2\} \quad \delta < \gamma(a)
\end{cases}
$$

Let us define $\hat{a} \equiv \{a \mid \delta = \gamma(a)\}$. That is, $\hat{a}$ is the minimal degree of differentiation needed to sustain monopoly pricing when the discount factor is δ. Using this definition, $a^*(\delta)$ can now be characterized.

Theorem: If $\delta \geq \gamma(1/4)$, then $a^*(\delta) = 1/4$ and if $\delta < \gamma(1/4)$, then $a^*(\delta) = \max\{0, \hat{a}\}$

Proof: If $\delta \geq \gamma(1/4)$, monopoly pricing will be sustainable even for $a=1/4$ at which monopoly profits are the highest possible. Now assume $\delta < \gamma(1/4)$. $\gamma(a)$ being increasing in a, it is clear that $\hat{a} < 1/4$. From expression (5.1) it is also obvious that $a^* \geq \hat{a}$ since for $a < \hat{a}$ monopoly pricing is sustainable by definition and monopoly profits are increasing in a for all $a < 1/4$. However, $a^* > \hat{a}$ is not possible since by lemma 10, Π^* is decreasing in a when $\delta < \gamma(a)$. Hence, it is profitable to lower a until monopoly pricing is sustainable or, if that is not possible, until $a=0$. Hence $a^* = \max\{\hat{a}, 0\}$. QED

Graphically, $a^*(\delta)$ is shown in figure 7.1.

VIII CONCLUDING REMARKS

When extended into a repeated game, the Hotelling model has the following implications; If product design is exogenous, monopoly pricing will be easier to sustain on markets where products are more differentiated. In case monopoly pricing is not sustainable, lowering the collusive price will always enable firms to cooperate successfully. Moreover, these constrained monopoly prices will be lower the greater the substitutability.

In case firms may choose product design, monopoly pricing is also easier to sustain the more differentiated the products are. Monopoly prices are maximal at an intermediate degree of differentiation. Consequently, if the discount factor is high, firms would choose this amount of differentiation. If the discount factor is low, rational firms will increase differentiation, still charging the unconstrained monopoly price. The lower the discount factor, the more differentiated will products be.

APPENDIX

PROOF OF LEMMA 1

It is obvious that the larger the reservation price, s, the more profitable it will be to cover the entire market. We therefore want to derive a condition under which s is so large that full market coverage is always optimal.

Assume that the price maximizing joint profits, $\hat{P}$, is so high that there is not full market coverage. Moreover, assume **a≤1/4**. Then the demand facing firm i will be given by $\hat{\theta}$ such that;

$$U(\hat{\theta}) = s - t(\hat{\theta} - a)^2 - \hat{P} = 0$$

or

$$D_i = \hat{\theta} = a + \frac{\sqrt{s - \hat{P}}}{\sqrt{t}}$$

Maximizing profits given this demand function we have;

$$\hat{P} = \frac{2}{9}\left[3s - a^2 t + a\sqrt{t(a^2 t + 3s)}\right]$$

If, on the other hand, full market coverage is optimal, the collusive price will be;

$$P^c = s - t(1/2 - a)^2$$

from expression (5.1). Since $\hat{P}=P^c$ is a permissible choice, partial market coverage will be optimal, from the firms' point of view, if and only if $\hat{P} > P^c$. This condition is equivalent to;

$$s < \frac{t(4a^2 - 8a + 3)}{4}$$

Since the right–hand–side is decreasing in a, the inequality will never hold if $s>3t/4$. This will always be the case, however, due to assumption 1. Consequently, partial market coverage is not consistent with profit maximization.

Now, if **$a \geq 1/4$**, and profit maximization implies partial market coverage, the consumers closest to the endpoints will choose not to buy. The indifferent consumers are located at $\hat{\theta}$ and $1-\hat{\theta}$ such that;

$$U(\hat{\theta}) = U(1-\hat{\theta}) = s - t(a - \hat{\theta})^2 - \hat{P} = 0$$

Firm i will then face the demand function;

$$D_i = 1/2 - \hat{\theta} = 1/2 - a + \frac{\sqrt{s - \hat{P}}}{\sqrt{t}}$$

Maximizing profits given this demand function we have;

$$\hat{P} = \frac{(1 - 2a)\sqrt{t(4a^2t - 4at + 12s + t)}}{18} - \frac{4a^2t - 4at - 12s + t}{18}$$

On the other hand, if profit maximization implies full market coverage, the collusive price is;

$$P^c = s - ta^2$$

from expression (5.2). Again, since $\hat{P}=P^c$ is a permissible choice, partial market coverage will be most profitable if and only if $\hat{P} > P^c$ which is equivalent to;

$$s < at(a + 1)$$

Since the right–hand–side is increasing in a, the inequality will never hold if s>3t/4. This is always the case, however, due to assumption 1. Consequently, partial market coverage is not consistent with profit maximization in this case either. QED

PROOF OF LEMMA 2

When **a≤1/4** there is full market coverage whenever the consumer located at $\theta=1/2$ enjoys a weakly positive utility level. This condition amounts to

$$U(1/2) = s - t(1/2 - a)^2 - P^p \geq 0$$

The punishment price from section V(ii), $P^p=t(1-2a)$, maximally equals t. Inserting this, we have;

$$U(1/2) = s - t(5/4 - a + a^2)$$

which is positive by assumption 1.

When **a≥1/4**, the analogous condition will be to ensure the endpoint consumers a weakly positive utility level. That is;

$$U(0) = U(1) = s - ta^2 - P^P \geq 0$$

Inserting P^P=t, we have;

$$U(0) = U(1) = s - t(1 + a^2)$$

which is positive by assumption 1. QED

PROOF OF LEMMA 3

Consider a certain location and a corresponding monopoly price, $P^c(a)$. When choosing $P^d_h(a)$ rationally, $P^d_h(a)=P^d_w(a)$ is a permissible choice. Therefore, $P^d_h(a)>P^d_w(a)$ must imply $\Pi^d_h(a)>\Pi^d_w(a)$. Using expressions (5.7) and (5.11), we may define;

$$P^d_h(a)-P^d_w(a) = \frac{3t(1-2a) \; - \; P^c(a)}{2}$$

In case **a≤1/4**, we know from expression (5.1) that $P^c(a) = s - t(1/2-a)^2$. Inserting this, letting $k \equiv s/t$, we have,

$$P^d_h(a)-P^d_w(a) = \frac{t}{8}\left[4a^2 - 28a - 4k + 13\right]$$

which is decreasing in a and equals zero for $a=\hat{a}_1=(7-2\sqrt{k+9})/2$.

i) When $20/16 \leq k \leq 25/16$ $\hat{a}_1 \geq 1/4$ so $P^d_h(a) \geq P^d_w(a)$ for all $a \epsilon [0, 1/4]$ making the less aggressive strategy most profitable.

ii) When $25/16 \leq k \leq 52/16$, $0 \geq \hat{a}_1 \geq 1/4$ so $P_h^d(a) \geq P_w^d(a)$ $a \epsilon [0, \hat{a}_1]$, making the less aggressive strategy most profitable, while the aggressive strategy is of course most profitable for $a \epsilon [\hat{a}_1, 1/4]$

iii) When $k \geq 52/16$, $\hat{a}_1 \leq 0$ so the aggressive strategy will be most profitable for all $a \epsilon [0, 1/4]$

In case **a≥1/4** we know from expression (5.2) that $P^c(a) = s - ta^2$. Then;

$$P_h^d(a) - P_w^d(a) = \frac{t}{2}\left[a^2 - 6a - k + 3\right]$$

which is decreasing in a and equals zero for $a = \hat{a}_2 = 3 - \sqrt{k+6}$.

i) When $20/16 \leq k \leq 25/16$, $1/4 \leq \hat{a}_2 \leq 0.307$, so $P_h^d(a) \geq P_w^d(a)$ $a \epsilon [1/4, \hat{a}_2]$, making the less aggressive strategy most profitable, while the aggressive strategy is of course most profitable for $a \epsilon [\hat{a}_2, 1/2]$

ii) When $k \geq 25/16$, $\hat{a}_2 \leq 1/4$ making the aggressive strategy most profitable for all $a \epsilon [1/4, 1/2]$

QED

PROOF OF LEMMA 5

In the general case, where $P \leq P^c$, the collusive per firm profits are simply $P/2$. The optimal deviation profits are given by (5.7) and (5.12) (replacing P^c by P) while the punishment payoffs are the same as before, namely (5.6). It may easily be shown that the aggressive deviation strategy dominates for $P \geq 3t(1-2a)$ while the less aggressive strategy dominates for $P \leq 3t(1-2a)$. Thus, in the general case, the discount factor restriction is;

$$\delta \geq g(a, P) = \frac{P - 2t(1-2a)}{2P - 3t(1-2a)} \qquad P \geq 3t(1-2a)$$

$$\delta \geq g(a, P) = \frac{P - t(1-2a)}{P + 3t(1-2a)} \qquad P \leq 3t(1-2a)$$

These expressions both equal 1/3 at $P=3t(1-2a)$ so $g(a, P)$ is continuous. In addition, $g(a, P)$ is increasing in P. Defining P^* as the maximal collusive price possible to charge without violating $\delta \geq g(a, P)$, it follows directly that P^* is the price that makes $\delta = g(a, P)$. Noting that $P=P^P=t(1-2a)<3t(1-2a)$ implies $g=0$ it follows that there will exist a $P^P \leq P^* \leq P^c$ for any $0<\delta<\gamma(a)$.

Solving for P^* and noting that $\delta<1/2$ is necessary for the discount factor to be a restriction, we have;

$$P^* = \frac{t(2-3\delta)(1-2a)}{1-2\delta} \qquad 1/3 \leq \delta \leq 1/2$$

$$P^* = \frac{t(3\delta+1)(1-2a)}{1-\delta} \qquad 0 \leq \delta \leq 1/3$$

which is decreasing in a implying lower profits on markets where products are close substitutes in case monopoly pricing is not sustainable..

QED

PROOF OF LEMMA 6

Assume that a firm deviates in product design. All subsequent payoffs will then be punishment payoffs according to the equilibrium strategies. Consequently, this kind of deviation will not occur if $\Pi^c > \Pi^p$, Π^p being conditional on locations after the deviation.

From differentiating (5.5) with respect to a_i we know that if firm i deviates in location, he will choose $a_i=0$. Moreover, since collusion has broken down anyway, firm j will have no incentive to choose a location other than $a_j=0$. Therefore, $\Pi_i^p=\Pi_j^p=t/2$ will be the payoffs forever after the deviation. Then, if **a≤1/4**, the per period gain from cooperating is;

$$\Pi^c(a) - \Pi_i^p(0) = \frac{1}{2}\left[s - \frac{5}{4}t + at(1-a)\right]$$

where $\Pi^c(a)$ is defined as in expression (5.1). Evidently, requiring this to be positive is an extremely weak assumption. All it says is that cooperative payoffs should be larger than the payoffs of the noncooperative game. $\Pi^c(a)-\Pi_i^p(0)$ is increasing in a and will therefore be minimized at a=0 which means $\Pi^c(0)-\Pi_i^p(0)=(4s-5t)/8$. This expression is positive due to assumption 1 and no deviation in product design will therefore occur.

If **a≥1/4**, the per period gain from cooperating is;

$$\Pi^c(a) - \Pi_i^p(0) = \frac{1}{2}\left[s - t(1 + a^2)\right]$$

where $\Pi^c(a)$ is defined by expression (5.2). $\Pi^c(a)-\Pi_i^p(0)$ is minimized at a=1/2. $\Pi^c(1/2)-\Pi_i^p(0)=(4s-5t)/8$ which is positive due to assumption 1. Thus, no deviation in product design will occur in this case either. QED

PROOF OF LEMMA 10

First, it will be shown that (7.3) and (7.4) are both decreasing in a. Let us begin with (7.3). Differentiation yields;

$$\frac{\partial P^*}{\partial a} = \frac{4t(1-\delta)}{2\delta-1}$$

which is negative if $\delta<1/2$. Moreover, we know that if the aggressive deviation strategy is optimal for $P=P^*$, it will also be optimal for $P=P^c>P^*$ since then $P>P^*>3t(1-2a)$. But for $P=P^c$, the aggressive deviation strategy may dominate only if $k\geq 25/16$ which implies $\gamma(a)\leq 1/2$ for $a\leq 1/4$. Consequently, for the discount factor to be a restriction when $P^*\geq 3t(1-2a)$ it must be the case that $\delta<1/2$.

Now, consider (7.4). We know that $\partial g(a, P)/\partial P\geq 0$ for $t(2a+1)\leq P\leq P^c$. Hence, $P=t(2a+1)$ minimizes the right–hand side of (7.2) yielding $g(a, t(2a+1))=2a$. Consequently, $\delta\geq 2a$, is a necessary and sufficient condition for the existence of P^*. Differentiating (6.4), we have;

$$\frac{\partial P^*}{\partial a} = \frac{2t[(\delta+1)\sqrt{(1-2a)(\delta-2a)} - \sqrt{\delta}(1-4a+\delta)]}{\sqrt{(1-2a)(\delta-2a)}(1-\delta)}$$

The denominator is obviously positive and well defined since $\delta>2a$. Denoting the numerator by N, we have;

$$\frac{\partial N}{\partial \delta} = \frac{2t(4a-1-3\delta)}{2\sqrt{\delta}}\left[1 + \frac{\sqrt{\delta(1-2a)}}{\sqrt{\delta-2a}}\right] < 0 \qquad a\leq 1/4$$

so N is maximal for $\delta=2a$ at which $N=2t\sqrt{2a}(2a-1)<0$. Hence, $\partial P^*/\partial a<0$ also for $P^*\leq 3t(1-2a)$. Finally, continuity follows from (7.3) and (7.4) being equal to $3t(1-2a)$ at;

$$a = \frac{3\delta - 1}{8\delta - 2}$$

Hence, if $\delta>1/2$, $P^{*}>3t(1-2a)$ for all $a\leq 1/4$. If $1/3\leq\delta\leq 1/2$, $P^{*}>3t(1-2a)$ for small a's and $P^{*}<3t(1-2a)$ for a's close to $\delta/2\leq 1/4$ while if $\delta<1/3$ $P^{*}<3t(1-2a)$ for all a's smaller than $\delta/2<1/4$.

QED

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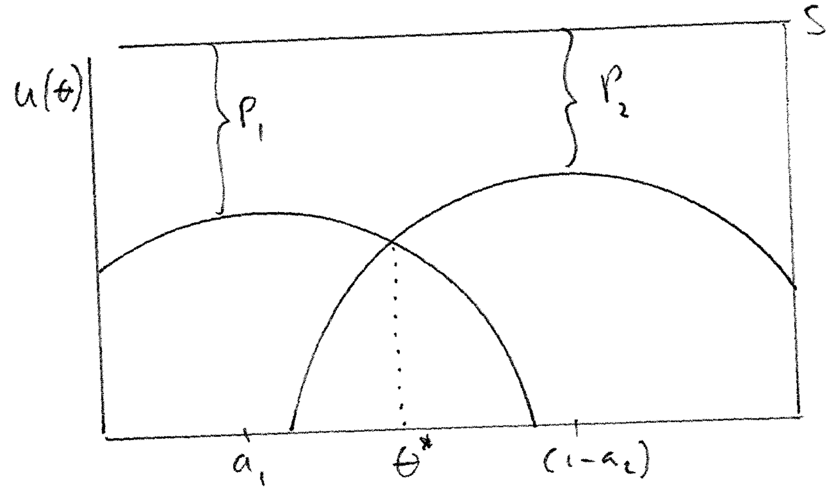
FIGURES

(4.1)



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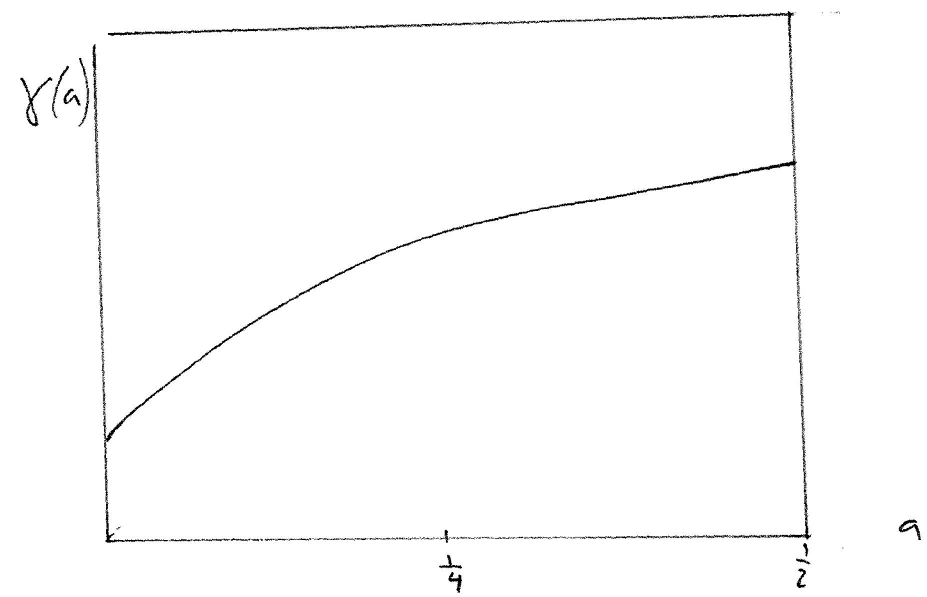

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