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Supply Function Equilibria: Step Functions and Continuous Representations

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Supply Function Equilibria: Step functions and continuous representations[1]

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Abstract

In most wholesale electricity markets generators must submit step-function offers of supply to a uniform price auction, and the market is cleared at the price of the most expensive offer needed to meet realised demand. Such markets can most elegantly be modelled as the pure-strategy, Nash Equilibrium of continuous supply functions, in which each supplier has a unique profit maximising choice of supply function given the choices of other suppliers. Critics argue that the discreteness and discontinuity of the required steps can rule out pure-strategy equilibria and may result in price instability. This paper argues that if prices must be selected from a finite set the resulting step function converges to the continuous supply function as the number of steps increases, reconciling the apparently very disparate approaches to modelling electricity markets.

Key words Auctions, supply function equilibria, convergence of step-functions, electricity markets

JEL codes: C62, D43, D44, L11, L13, L94

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INTRODUCTION

This paper fills an increasingly embarrassing gap between theory and reality in multi-bid auction markets such as electricity wholesale markets. The leading equilibrium theory underpinning market analysis and the econometric estimation of strategic bidding behaviour in electricity auctions assumes that generating companies offer a piecewise differentiable supply function, specifying the amounts they are willing to supply at each price. The market operator aggregates these supplies and clears the market at the lowest price at which supply is equal to demand – the Market Clearing Price (MCP). Generators on this theory choose their offers by optimising against the smooth residual demand, which gives well-defined first-order conditions. In reality, wholesale markets require offers to take the form of a step function, and the resulting residual demand facing any generator is also a step function, whose derivative is zero almost everywhere.

Faced with this, economists have chosen either to model the market as a discrete unit auction, which typically leads to complex mixed strategy equilibria, or have argued that with enough steps, the residual demand can be smoothed and then treated as differentiable. The difference between these approaches appears dramatic, and it is the purpose of this paper to demonstrate that in a well-defined sense it can be legitimate to approximate step-functions by smooth differentiable functions, and hence to draw on the well-developed theory associated with continuous supply functions.

To prove this result, we develop a new discrete model that has a pure-strategy equilibrium, which converges to the equilibrium of the limit game with continuous supply functions. Similar to Dahlquist/Lax-Richtmyer's equivalence theorem (LeVeque, 2007), convergence requires that the discrete system is consistent with the continuous system – the first-order conditions of the two systems converge - and that the discrete solution is stable, i.e. the difference between the two solutions does not grow at each step. Moreover, solutions should exist and globally maximize profits of the agents in both the discrete and continuous system.

To our knowledge we are the first to prove convergence of equilibria in multi-unit auctions to equilibria in divisible good auctions in this rigorous manner. The new discrete model can be useful for other purposes. For example, it has the potential to enhance the accuracy in econometric studies of bidding in auctions, as our discrete model sidesteps the problem of how to smooth stepped residual demand curves, which has been a somewhat arbitrary and therefore disputed process in previous empirical studies of electricity auctions.

1.1 Modelling electricity markets

Electricity liberalisation typically creates a number of wholesale electricity markets. The balancing market is needed to secure real-time balancing services, to ensure that supply and demand can be instantaneously matched. The day-ahead or spot market provides hourly or half-hourly prices for adjusting contract positions, which themselves are traded in over-the-counter (OTC) or futures markets. If traders are competitive and the markets liquid, there should be a close relationship between the contract, spot and balancing prices, otherwise profitable arbitrage would be possible. In such cases one can talk about a single wholesale spot price.

The two key markets that we wish to model are the day-ahead market and the balancing market (in the English Electricity Pool they were combined). In most such markets there is a separate auction for each delivery period, which is typically a half-hour or hour. Normally, the post-2001 British balancing mechanism being an exception, the markets are organised as uniform price auctions. Thus all accepted bids and offers pay or are paid the market clearing price (MCP) and all purchase bids with a price limit higher than the MCP and all sales offers with a price limit lower than the MCP are executed. Rationing of excess supply at the clearing price may be necessary and so market designs must specify how rationing will take place, normally by pro-rata on-the-margin rationing (Kremer and Nyborg, 2004a).

Producers submit non-decreasing step function offers to the auction (and in some markets agents, normally retailers, may submit non-increasing demands). With its offer the producer states how much power it is willing to generate at each price. The Amsterdam Power Exchange (APX) provides a good example and the bid and offer ladders that determine the MCP can be readily downloaded.[5] The successive offers specify a quantity that would be available at a fixed per unit price. The smallest step in the ladder is given by the number of allowed decimals in the offer. Thus all prices and quantities in an offer have to be a multiple of the price tick size and quantity multiple, respectively. Table 1 summarizes these and other offer constraints for some of the

[5] At http://www.apxgroup.com/marketdata/powernl/public/aggregated_curves/curves.html.

electricity markets in U.S. and Europe. In particular it is worth noting that most electricity markets have significantly more possible quantity levels compared with possible price levels. In that sense, the quantity multiple is small relative to the price tick size. Most markets also have a constraint on the maximum number of allowed steps per bidder. Typically the number of units or gensets is very large in electricity markets, (over 200 in Britain) so even if only 3-5 steps per unit is allowed, there can still be many steps in the market.

Table 1: Constraints on the supply functions in various electricity markets.

Market	*Installed capacity*	*Max steps*	*Price range*	*Price tick size*	*Quantity multiple*	*No. quantities/ No. prices*
Nord Pool spot	90,000 *MW*	64 *per bidder*	0-5,000 *NOK/MWh*	0.1 *NOK/MWh*	0.1 *MWh*	18
ERCOT balancing	70,000 *MW*	40 *per bidder*	-$1,000/*MWh*- $1,000/*MWh*	*$0.01/MWh*	0.01 *MWh*	35
PJM	160,000 *MW*	10 *per genset*	*0-$1,000/MWh*	$0.01/*MWh*	0.01 *MWh*	160
UK (NETA)	80,000 *MW*	5 *per genset*	-£9,999/*MWh*- £9,999/*MWh*	£0.01/*MWh*	0.001 *MWh*	4
Spain Intra-day market	46,000 *MW*	5 *per genset*	*Yearly cap on revenues*	€0.01/*MWh*	0.1 *MWh*	—

Offers are submitted ahead of time (typically the day before) and may have to be valid for an extended period (e.g. 48 half-hour periods in the English Pool) during which demand can vary significantly. Plant may fail suddenly, requiring replacement at short notice, so the residual demand (i.e. the total demand *less* the supply accepted at each price from other generators) may shift suddenly with an individual failure, again increasing the range over which offers are required.

Green and Newbery (1992) argued that the natural way to model such a market was to adapt Klemperer and Meyer's (1989) supply function equilibrium (SFE) formulation, in which firms make offers before the realization of demand is revealed. Units of electricity are assumed to be divisible, so firms offer continuous supply functions (SFs) to the auction. Accordingly, residual demand is piece-wise differentiable and firms have a well-defined piece-wise continuous marginal revenue, which offers the prospect of a well-defined best response function at each point. An equilibrium is such that each firm ensures that given the supplies offered by all other firms, it is maximising its profits for each realization of demand.

With a uniform price auction and a continuous SF the effect of lowering the price to capture the marginal unit lowers the price for the large quantity of inframarginal units (the 'price' effect) while only capturing an infinitesimal sale (the 'quantity' effect). As a result very collusive supply function equilibria can be supported.

The first order conditions for the Nash equilibrium for each demand realization satisfy a set of linked differential equations, which under various simplifying assumptions can be solved analytically, although for realistic specifications of costs numerical integration is normally required (Anderson and Hu, 2008; Baldick and Hogan, 2002; Holmberg, 2008). This approach opened the way for a large number of papers deriving solutions under various assumptions. Analytical solutions can be found for the case of equal and constant marginal costs and linear marginal costs.[6] Closed form solutions are also available for symmetric firms and perfectly inelastic demand (Rudkevich et al, 1998; Anderson and Philpott, 2002). The literature on numerical algorithms for finding SFE of markets with asymmetric firms and general cost functions (Holmberg, 2008; Anderson and Hu, 2008) is particularly relevant to our investigation. For example, numerical instabilities often arise in computation especially when mark-ups are small (Baldick and Hogan, 2002; Holmberg, 2008). Our analysis amplifies this observation, namely, the relationship we establish between the discrete and continuous cases relies on mark-ups that are positive and bounded away from zero. Finally, the SFE model has also been extended to account for transmission constraints (Wilson, 2008).

Green and Newbery (1992) argued that the large number of possible steps meant that, given the uncertainty about, and variability of, demand, such steps could reasonably be approximated by continuous and piecewise differentiable functions. von der Fehr and Harbord (1993), however, argued that the ladders were step functions that were not continuously differentiable, and it would be inappropriate to

[6] In general there is a continuum of equilibria bounded above and below, although these collapse to a unique equilibrium under certain conditions, such as free entry or limited capacity (Newbery, 1998, Holmberg, 2007). For the case of linear marginal costs, there is a unique linear SF equilibrium, (Klemperer and Meyer, 1989; Green, 1996; Baldick et al., 2004) although the general analytic solution can still be characterised as a closed form solution and

assume that they were. Instead, their paper models the electricity market as a multiple-unit auction. Costs were assumed to be common knowledge. Each genset could submit a single bid for its entire capacity (and so quantities submitted are chosen from a discrete set). The bid is selected from a continuum of prices (although in all existing electricity markets the set of prices is finite). Demand was perfectly inelastic up to a price cap and drawn from a probability distribution with finite support, and the market price was set at the bid of the marginal unit called to meet demand, as in a uniform-price auction.

The authors specifically contrasted this with the Green and Newbery supply function approach. The contrast was sharp - a step function (or ladder) of bids combined with inelastic demand gives rise to a residual demand schedule facing any bidder that is also a step function, and whose marginal revenue is either at the residual demand price or is discontinuous at the steps. Competition is therefore almost everywhere in prices, with winner takes all over the whole step. Thus the 'price' effect, which can be made infinitesimally small in their model, of stealing some market is no longer larger than the now significant 'quantity' effect. Not surprisingly such Bertrand competition often destroys any pure strategy, and if demand uncertainty is sufficiently large the only equilibrium has mixed strategies in which the firms randomise over a distribution of possible prices. As these equilibria are hard to solve, the examples typically only have one step, so the step lengths are large, as are the supports of the price distributions. Solving for the mixed strategy equilibrium with a more realistic number of steps proved extremely difficult, so the result was destructive, in the sense that existing supply function models were claimed to be flawed but suitable auction models were intractable.

In a similar spirit, Supatgiat, Zhang and Birge (2001) build a step-function model motivated by the special Californian PX market design, a multi-round non-sealed bid auction. Generators submit a single price offer chosen from a set of possible price levels for their entire output, so that the number of rounds is not to increase unreasonably before convergence. Generators are assumed to be non-pivotal, so the solution is typically close to a Bertrand equilibrium at each step, although they cannot rule out multiple equilibria nor mixed strategies when demand is stochastic.

In a subsequent paper, Fabra, von de Fehr and Harbord (2006) extended their

solved numerically (Newbery, 2002; 2008).

analysis in various important directions, although (for the most part) under an extremely strong restriction on the timing of demand realization. Whereas the 1993 model was, plausibly, one in which the bids were submitted *before* the realization of demand, in this later paper the bids are made *after* the realization of demand. With two firms with variable costs $c_1 \leq c_2 = c$ facing a price cap P, and each submitting a single bid for the whole of their capacity, the pure strategy equilibrium is readily found. If demand is low enough for either firm to supply the entire market, the equilibrium is Bertrand (price $p = c$). If both firms are required to meet demand, one of them offers its supply at the price cap, $p = P$, while the other supplier submits an offer price sufficiently low so as to make undercutting unprofitable. This simple model is extended to allow multiple bids (b_{in}, k_{in}), where b_{in} is the n-th bid of generator i for an amount k_{in}. This allows a step function bid for each generator that might be expected to more closely match a smooth supply function. The authors also extend the model to allow long-lived but single bids with varying demand. Not surprisingly, this has an effect only when both low and high demand realizations occur with positive probability (i.e. cover the range where either the capacity of only one or both firms are required to meet demand). In such cases demand variability or uncertainty destroys all pure strategy equilibria, leaving a unique mixed-strategy equilibrium in which both suppliers submit bids that strictly exceed c. It is possible (but difficult) to compute the mixed strategy equilibrium when both suppliers have the same capacity (but possibly different costs).

Choosing a mixed strategy in prices means that prices will be inherently volatile or unstable, even if exactly the same demand is realised each day at the same time with the same generating sets available for dispatch and the same level of contract cover. It is clearly the case that spot prices are indeed very volatile, even at the same level of realised demand as can be seen by plotting prices against generation output, that can be downloaded from various power exchange websites. It is not unusual for prices to vary by a factor of 10 for the same level of output. Nevertheless, there are many explanations for such volatility apart from suppliers randomising over price offers. Most power exchanges such as the APX are effectively residual markets in which contract portfolio positions are adjusted to expected supply and demand. As contract positions, demand, imports and exports, as well as plant availability, vary over short periods of time, so will the necessity of buying and selling in the APX and hence so would the position of a smooth SF (if such were allowed). One can see this visually by looking at successive days' bid and offer ladders from e.g. the APX web site.

Despite the theoretical problem pointed out by von der Fehr and colleagues, three empirical studies of the balancing market in Texas (ERCOT) suggest that the continuous representation is approximately correct in describing the behaviour of the largest producers in this market (Niu et al., 2005; Hortacsu and Puller, 2008; Sioshansi and Oren, 2007). Sweeting (2007) similarly estimates best responses to realizations of a smoothed residual demand schedule in the English Electricity Pool and is able to convincingly characterise the various phases of market evolution and the exercise of market power. Wolak (2001) has also used observed bidding behaviour to back out the unobserved underlying cost and contract positions of generators bidding into the Australian market. He notes that continuity of the SF allows each price quantity pair to be a best response and hence does not depend on the distribution of shocks, whereas the choice of an optimal step function will depend on the distribution of the shocks, and can only be an approximation to the continuous representation. Nevertheless Wolak is content to smooth the ex post observed stepped residual demand schedule to compute its derivative and hence find the best response supply, which is then compared with the actual supply (chosen before the residual demand was realised).

These empirical papers all start with the observed outcome to test whether generators are maximising their profits and, explicitly or implicitly accept the key assumptions underlying the SFE model, because residual demand is smoothened and it is assumed that producers use pure-strategies, so that producers know their competitors' offer functions. They can reach no conclusions on whether the market is in equilibrium, or whether the recovered supply functions would give rise to an equilibrium, especially, as there is significant arbitrariness in how to smooth the observed residual demand of a producer.

1.2 Reconciling step and continuous supply functions

The central question raised by these criticisms and empirical applications is whether smoothing and/or increasing the number of steps in the ladder, combined with the need to bid before demand is realised, can reconcile the discrete and continuous approaches to modelling electricity markets. Do markets with uncertain or variable demand and sufficiently finely graduated bidding ladders converge to supply function equilibria, or do they remain resolutely and significantly different? The central claim of this paper is that under well-defined conditions, convergence can be assured, providing an intellectually solid basis for accepting the SFE approach. As such it marks a major step forward in the theory of supply function equilibria. We also conjecture that there may be a wider class of cases in which convergence can be established, but leave that for further investigation.

Fabra et al (2006) argue that the difference between the two approaches derives from the finite benefit of infinitesimal price undercutting in the ladder model. But this argument assumes that prices can be infinitely finely varied. In practice, the price tick size cannot be less than the smallest unit of account (e.g. 1 US cent, 1 pence, normally per MWh), and might be further restricted, as in the multi-round California PX auction. In this case, the undercutting strategy is not necessarily profitable, because the price reduction cannot be made arbitrarily small. Whereas von der Fehr and Harbord (1993) considered the extreme case when the set of quantities is finite and the set of prices is infinite, this paper considers the other extreme when the set of quantities is infinite and the set of prices is finite. Our assumption concurs with the observation in Section 1.1 that the quantity multiple is often small relative to the price tick size. Restrictions on the number of allowed steps per bidder/production unit might be important for determining the equilibrium as well, but this issue is left for future research. We show that, with sufficiently many allowed steps in the bid curves, the step function and the market-clearing price (MCP) generally converge to the supply functions and price predicted by the SFE model. As in Dahlquist/Lax-Richtmyer's equivalence theorem (LeVeque, 2007), convergence requires that the discrete system is consistent with the continuous system – the first-order conditions of the two systems converge - and that the discrete solution is stable, i.e. the difference between the two solutions does not grow at each step. Moreover, solutions should exist and globally maximize profits of the agents in both the discrete and continuous system. The use of the Dahlquist/Lax-Richtmyer's equivalence theorem is a standard procedure when analyzing convergence of numerical methods, but it seems that we are the first to apply this theorem to the convergence of Nash equilibria.

Our existence and convergence result suggest that with a negligible quantity multiple and sufficiently many steps, discrete supply functions are deterministic (and hence so is the price for each realization, *cet. par.*) and a continuous supply function equilibrium is a valid approximation of bidding in such electricity auctions.

Our model has parallels in the theoretical work by Anderson and Xu (2004). They analyse a duopoly model that reflects two important features of the Australian electricity market, in which prices and quantities are specified separately. They assume demand is random but inelastic, with an elastic outside supply at some price, *P*, which effectively sets a price ceiling. At the day-ahead stage, each of two generators simultaneously chooses ten prices, which are then published. Subsequently (nearer to the time of dispatch) each generator decides how much to offer at each of its chosen prices. Demand is then realised and both generators are paid the MCP. Anderson and Xu are able to show that, under certain conditions, the second stage has a pure strategy equilibrium in quantities, although the first stage only has mixed strategies in the choice of prices. The second stage of their game has similarities with our model, because prices are discrete in both models. On the other hand, generators' chosen price vectors generally differ as the declared prices are chosen by randomising over a continuous range of prices. In our paper, however, the available price levels are given by the market design and accordingly are the same for all firms. Moreover, Anderson and Xu (2004) do not compare their discrete equilibrium with a continuous SFE.

Wolak (2004) develops a similar model of the Australian market to that of Anderson and Xu, but Wolak derives a best response rather than an equilibrium, and each producer is assumed to know both competitors' selected price grid and their offers when making its own offer. The model by Wolak (2004) is quasi-discrete in the sense that residual demand of the analysed producer is smoothed by an algorithm that involves arbitrary parameters, before the best response is calculated. This model is applied empirically to recover the cost function of a producer from observed bids. The same model is used by Gans and Wolak (2007) to assess the impact of vertical integration between a large electricity retailer and a large electricity generator in the Australian market. A problem with the arbitrary smoothing is that it introduces several degrees of freedom in the empirical model, and it has even been claimed that the model becomes so general that the first-order condition of a producer cannot be rejected.

Anderson and Hu (2008) study an auction in which supply functions submitted and the residual demand function are continuous. To numerically calculate approximate equilibria of the continuous system, they approximate continuous supply functions with piece-wise linear supply functions and discretise the demand distribution. They show that equilibria of this approximation converge to equilibria in the original continuous model. The piece-wise linear bid functions are carefully chosen to avoid the influence of kinks in the residual demand curves. These approximate bid curves are drawn so that all producers have locally well-defined derivatives in their residual demand curves for all possible discrete demand realizations. Anderson and Hu's discrete model is motivated by its computational properties. By contrast we deal with the worst kinks possible, i.e. steps, and we do so explicitly. Because we want to prove equilibrium convergence for a more problematic case, which is relevant for real electricity markets where convergence has been disputed both empirically and theoretically.

Kastl (2008) analyzes divisible-good auctions with certain demand and private values, i.e. bidders have incomplete information. This set-up, which was introduced by Wilson (1979), is mainly used to analyze treasury auctions. Kastl considers both uniform-price and discriminatory auctions. He assumes that both quantities and prices are chosen from continuous sets, but the maximum number of steps is restricted. He verifies consistency, i.e. that the first-order condition (the Euler condition) of the stepped bid curve converges to the first-order condition of a continuous bid-curve when the number of steps becomes unbounded. But he does not verify stability, nor that solutions exists and globally maximize agents' profits in the discrete and continuous systems, which all are necessary conditions for the convergence of Nash equilibria in the discrete system to Nash equilibria in the continuous systems.

More generally, the convergence problem under study is related to the seminal paper by Dasgupta and Maskin (1986) on games with discontinuous profits. They show that if payoffs are discontinuous, then Nash equilibria in games with finite approximations of the strategy space of a limit game may not necessarily converge to Nash equilibria of the limit game. Later Simon (1987) showed that convergence may depend on how the strategy space is approximated. This intuitively explains why NE in the model by von der Fehr and Harbord (1993), in which payoffs are discontinuous, do not necessarily converge to continuous SFE, and also why it is not surprising that NE in our discrete model, in which payoffs are continuous, converge to continuous SFE. However, Dasgupta and Maskin (1986) and Simon (1987) derive their convergence results for a limit game in which the strategy space has a finite dimension. Thus they do not consider the stability property, which is often important when the limit game has infinitely many dimensions (a continuous supply function has infinitely many price/quantity pairs).

2 THE MODEL AND ANALYSIS

Consider a uniform price auction and assume that excess supply is rationed pro-rata on-the-margin. We calculate a pure strategy Nash equilibrium of a one-shot game, in which each risk-neutral electricity producer, *i,* chooses a step supply function to maximise its expected profit, $\mathrm{E}(\pi_i)$. There are M price levels, p_j, $j=1,2,\ldots M$, with the price tick $\Delta p_j = p_j - p_{j-1}$. In most of our analysis price levels are assumed to be equidistant and then we let Δp denote the price tick-size. The minimum quantity increment is zero - quantities can be continuously varied.

Generator i ($i=1,..,N$) submits a supply vector $\mathbf{s}_i$ consisting of maximum quantities $\{s_i^1,\ldots,s_i^M\}$ it is willing to produce at each price level $\{p_1,\ldots,p_M\}$. The step length $\Delta s_i^j = s_i^j - s_i^{j-1} \geq 0$: offers must be non-decreasing in price and bounded above by the capacity $\bar{s}_i$ of Generator i. Let $\mathbf{s} = \{\mathbf{s_1},\ldots,\mathbf{s_N}\}$ and denote competitors' collective quantity offers at price p_j as s_{-i}^j and the total market offer as s^j. In the continuous model the set of individual supply functions is $\{s_i(p)\}_{i=1}^N$. The cost function of firm i, $C_i(s_i)$, is a smooth, increasing and convex function up to the capacity constraint $\bar{s}_i$. Costs are common knowledge. Electricity consumers are non-strategic. Their demand is stepped and the minimum demand at each price is $d^j + \varepsilon$, where ε is an additive demand shock. Decremental demand is $\Delta d^j = d^j - d^{j-1} \leq 0$, with $\Delta d^j \geq \Delta d^{j+1}$, corresponding to a continuously differentiable concave deterministic demand curve, $d(p)$, in the continuous case. The latter is such that $\lim_{\Delta p_j \to 0} \frac{\Delta d^j}{\Delta p_j} = d'(p_j)$ and $\lim_{\Delta p_j \to 0} d^j = d(p_j)$. Note that Δp_j is a local tick-size and that other tick-sizes Δp_k are fixed when these limits are calculated, so that p_j is fixed and $p_j \to p_{j+1}$. The additive demand shock has a continuous non-zero probability density, $g(\varepsilon)$, with support on $\left[\underline{\varepsilon}, \overline{\varepsilon}\right]$.

Let $\tau^j = s^j - d^j$ be the total deterministic net supply (excluding the stochastic shock) at price p_j, and define the increase in net supply from a positive increment in price as $\Delta\tau^j = \tau^j - \tau^{j-1}$. Similarly, the residual deterministic net supply is $\tau_{-i}^j = s_{-i}^j - d^j$ and its increase is $\Delta\tau_{-i}^j = \tau_{-i}^j - \tau_{-i}^{j-1}$.

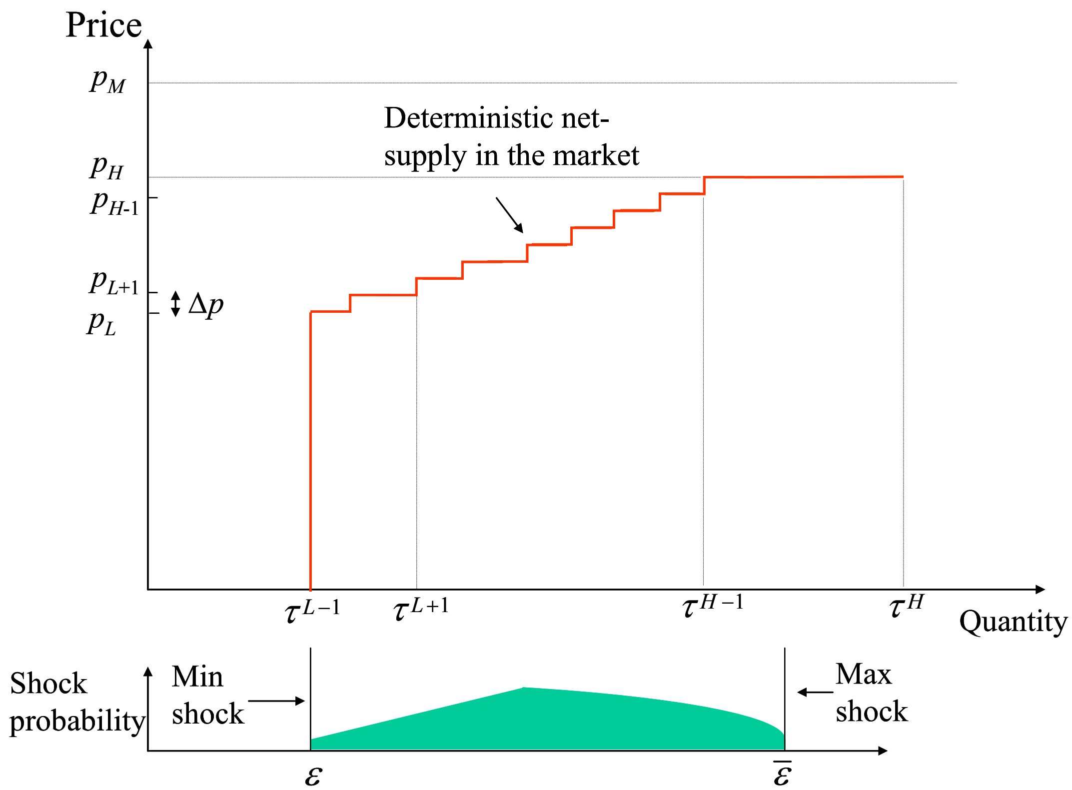

Figure 1. Stepped supply, demand shocks and key price levels.

The Market Clearing Price (MCP) is the lowest price at which the deterministic net-supply equals the stochastic demand shock. Thus the equilibrium price as a function of the demand shock is left continuous, and the MCP equals p_j if $\varepsilon \in \left(\tau^{j-1}, \tau^j\right]$. Given chosen step functions, the market clearing price can be calculated for each demand shock in the interval $\left[\underline{\varepsilon}, \overline{\varepsilon}\right]$. The lowest and highest prices that are realized are denoted by p_L and p_H, respectively, where $1 \le L < H \le M$. Both depend on the available number of price levels, M, as well as the initial (or boundary) conditions, and these various price levels and the demand shocks are shown in Figure 1. The lowest and highest realized prices in the corresponding continuous system are a and b respectively.

2.1 First-order conditions

With pro-rata on-the-margin rationing, all supply offers below the MCP, p_j, are accepted, while offers at p_j are rationed pro-rata. Thus for $\varepsilon \in \left(\tau^{j-1}, \tau^j\right]$, $\varepsilon - \tau^{j-1}$ is excess demand at $p_{j\text{-}1}$, so the accepted supply of a generator i is given by:

$$s_i(\varepsilon) = s_i^{j-1} + \frac{\Delta s_i^j \left(\varepsilon - \tau^{j-1}\right)}{\Delta \tau^j} = \varepsilon - \tau_{-i}^{j-1} - \frac{\Delta \tau_{-i}^j \left(\varepsilon - \tau^{j-1}\right)}{\Delta \tau^j}, \tag{1}$$

(making use of the fact that $\tau^j = \tau_{-i}^j + s_i^j$ and $\Delta\tau^j = \Delta\tau_{-i}^j + \Delta s_i^j$). Hence, the contribution to the expected profit of generator i from realizations $\varepsilon \in \left(\tau^{j-1}, \tau^j\right]$ is:

$$\begin{gathered} E_i^j = \int_{\tau^{j-1}}^{\tau^j} \left[p_j s_i - C_i(s_i)\right] g(\varepsilon) d\varepsilon = \\ \int_{s_i^{j-1}+\tau_{-i}^{j-1}}^{s_i^j+\tau_{-i}^j} \left[p_j \left(\varepsilon - \tau_{-i}^{j-1} - \frac{\Delta \tau_{-i}^j \left(\varepsilon - \tau^{j-1}\right)}{\Delta \tau^j} \right) - C_i \left(\varepsilon - \tau_{-i}^{j-1} - \frac{\Delta \tau_{-i}^j \left(\varepsilon - \tau^{j-1}\right)}{\Delta \tau^j} \right) \right] g(\varepsilon) d\varepsilon, \end{gathered} \tag{2}$$

where again $\tau^j = \tau_{-i}^j + s_i^j$. Generator i's total expected profit is

$$\mathrm{E}\left(\pi_i(\mathbf{s})\right) = \sum_{j=1}^{M} E_i^j \left(s_i^j, s_i^{j-1}\right). \tag{3}$$

The Nash equilibrium is found by deriving the best response of each firm given its competitors' chosen stepped supply functions. The first order conditions are found by differentiating the expected profit in (3). Proposition 1 characterises these first order conditions over the range of possible intersections of aggregate supply with demand (i.e. over the range on which it has positive probability). All proofs are given in the appendix.

Proposition 1. With discrete supply function offers, $\Gamma_i^j(\mathbf{s}) = \partial\, \mathrm{E}\left(\pi_i(\mathbf{s})\right) / \partial s_i^j$ is always well-defined, and the first-order condition for the supply of firm i at a price level j, such that $\underline{\varepsilon} \le \tau^j \le \overline{\varepsilon}$, is given by:

$$0=\Gamma_i^j(\mathbf{s})=\frac{\partial\,\mathrm{E}(\pi_i(\mathbf{s}))}{\partial s_i^j}=-\Delta p s_i^j g(\tau^j)+\int_{\tau^{j-1}}^{\tau^j}\left[p_j-C_i'(s_i(\varepsilon))\right]\left(\frac{\Delta\tau_{-i}^j(\varepsilon-\tau^{j-1})}{(\Delta\tau^j)^2}\right)g(\varepsilon)d\varepsilon+$$

$$+\int_{\tau^j}^{\tau^{j+1}}\left[p_{j+1}-C_i'(s_i(\varepsilon))\right]\Delta\tau_{-i}^{j+1}\left(\frac{\Delta\tau^{j+1}-(\varepsilon-\tau^j)}{(\Delta\tau^{j+1})^2}\right)g(\varepsilon)d\varepsilon, \tag{4}$$

where $s_i(\varepsilon)$ is given by (1) if $\varepsilon\in\left[\tau^{j-1},\tau^j\right]$.

The first point to note, *pace* Dasgupta and Maskin's (1986) result for games with discontinuous profits, is that expected profits $\mathrm{E}(\pi_i(\mathbf{s}))$ are differentiable. Thus expected profit is continuous in the strategy variables, and convergence should be less problematic. The first-order condition can be intuitively interpreted as follows. When calculating $\Gamma_i^j(\mathbf{s})=\partial\,\mathrm{E}(\pi_i(\mathbf{s}))/\partial s_i^j$, supply is increased at p_j, while holding the supply at all other price levels constant. This implies that the offer price of one (infinitesimally small) unit of power is decreased from p_{j+1} to p_j. This decreases the MCP for the event when the unit is price-setting, i.e. when $\varepsilon=\tau^j$. This event brings a negative contribution to the expected profit, which corresponds to the first term in the first-order condition. On the other hand, because of the rationing mechanism, decreasing the price of one unit (weakly) increases the supply for demand outcomes $\varepsilon\in\left(\tau^{j-1},\tau^{j+1}\right]$. This brings a positive contribution to the expected profit, which corresponds to the two integrals in the first-order condition. The first integral represents $\varepsilon\in\left(\tau^{j-1},\tau^j\right]$ when the MCP is p_j, and the other integral represents $\varepsilon\in\left(\tau^j,\tau^{j+1}\right]$ when the MCP is p_{j+1}.

The first-order condition in Proposition 1 is not directly applicable to parts of the offer curve that are always or never accepted in equilibrium. The appendix shows that, because of pro-rata rationing, a producer's profit is maximized if offers that are never accepted are offered with a perfectly elastic supply (until the capacity constraint binds) at p_H, so that $s_i^H=\bar{s}_i$, and offers that are always accepted are offered below p_L. In particular, we assume that

$$s_i^j=s_i^{L-1}\ \text{if}\ j<L, \tag{5}$$

because this offer curve discourages NE deviations that undercut the price level p_L.

In summary, equilibrium supply is constant for $p<p_L$, satisfies (4) for $p \in [p_L, p_H)$ and jumps to $\bar{s}_i$ at p_H. Definition 1 gives the notation for a set of solutions, meaning a list of simultaneous solutions, one for each player i and price level p_j.

Note that the difference equation in (4) is of the second-order. Thus solutions, should they exist, would be indexed by two boundary conditions that could appear in a variety of forms, e.g., initial and final (boundary) values or, as we shall do, two boundary values at the upper end of the interval. As argued above, one of the boundary conditions is pinned down by the capacity constraint $s_i^H = \bar{s}_i$. This leaves each firm with one remaining free parameter, s_i^{H-1}, that will be tied down with a second boundary condition, $s_i^{H-1} = \hat{k}_i$, for some constant $\hat{k}_i$. This latter condition corresponds to the single boundary condition needed for the continuous case, presented shortly.

Definition 1. *By* $\left\{\left\{\hat{s}_i^j\right\}_{j=L}^{j=H}\right\}_{i=1}^N$ *or* $\left\{\hat{s}_i^j\right\}_{L,1}^{H,N}$ *we denote a set of discrete solutions to the system of difference equations (4) given two boundary conditions* $\hat{s}_i^H = \bar{s}_i$ *and* $\hat{s}_i^{H-1} = \hat{k}_i$ for some constant $\hat{k}_i$. *We say this set is a segment of a discrete SFE if the set of strategies* $\left\{s_i^j\right\}_{j,i=1}^{H,N}$ *formed by taking* $s_i^j = \hat{s}_i^L$ *if j<L and* $s_i^j = \hat{s}_i^j$ *if* $L \le j \le H$ *is an SFE for the discrete game.*

Section 3 studies convergence of equilibria of the discrete system to equilibria of the continuous system. The system of first-order conditions in the continuous case is given by Klemperer and Meyer (1989):

$$-s_i(p) + \left[p - C_i'(s_i(p))\right]\left(s_{-i}'(p) - d'(p)\right) = 0 . \qquad (6)$$

This system has one degree of freedom, and hence an infinite number of potential solutions. As shown by Baldick and Hogan (2001), the system of differential equations can be written in the standard form of an ordinary differential equation (ODE):

$$s_i'(p) = \frac{d'(p)}{N-1} - \frac{s_i(p)}{p - C_i'(s_i(p))} + \frac{1}{N-1}\sum_k \frac{s_k(p)}{p - C_k'(s_k(p))} . \qquad (7)$$

We can therefore index the continuum of continuous SFE by a boundary condition

$s_i(b) = k_i$. In Section 3, we will link the discrete and continuous boundary conditions by requiring $k_i = \lim_{\Delta p \to 0} \hat{k}_i$, where we note that $\hat{k}_i$ depends on Δp or, equivalently, on M.

The assumed shape of the offer curves in the never-price-setting region is the same as for the discrete system; bids that are always accepted are perfectly inelastic and bids that are never accepted are perfectly elastic. This shape also discourages competitors from deviating from a potential NE, and is accordingly most supportive of an NE:

$$s_i(p) = s_i(a) \text{ if } p < a \text{ and } s_i(p) = \bar{s}_i \text{ if } p > b. \tag{8}$$

The next definition provides the notation for solutions to the continuous system.

Definition 2. *By* $\{\hat{s}_i(p)\}_{i=1}^N$ *we denote a set of continuous solutions to the system of the differential equations* (7) *on the interval* $[a,b]$. *We say* $\{\hat{s}_i(p)\}_{i=1}^N$ *is a segment of a continuous SFE if the set of strategies* $\{s_i(p)\}_{i=1}^N$ *formed by taking* $s_i(p) = \hat{s}_i(a)$ *if* $p < a$, $s_i(p) = \hat{s}_i(p)$ *if* $p \in [a,b]$, *and* $s_i(b+) = \bar{s}_i$ *is an SFE.*

2.2 Sufficient conditions

Here we show that a *non-decreasing* solution of either the discrete or continuous stationary conditions, presented above, must be an SFE if assumptions 1a and 2 (discrete case) or 1b (continuous case) below are satisfied. That is, the non-decreasing condition acts rather like a second-order condition in ensuring sufficiency. These results are of independent interest. For example, Proposition **3**, on the sufficiency in the continuous case, extends the symmetric case presented in claim 7 and the text following in Klemperer and Meyer (1989).

Assumption 1a. A binding price cap, i.e. $H=M$, or sufficiently large production capacities, ensures that there is no producer in the discrete system that can increase its profit by decreasing its supply at p_H. If there are unilateral deviations such that the price is higher than p_H with a positive probability, then the profit of the deviating firm decreases by an amount bounded away from zero.

The assumption is always satisfied for non-pivotal firms. In this case, no producer can unilaterally deviate and push the price above p_H, as competitors offer all of their capacity at the price p_H. Pivotal producers would be able to deviate and push the price above p_H if $H<M$. Still Assumption 1a is satisfied if such deviations are strictly non-profitable, i.e. p_H is sufficiently high or the firm is not sufficiently pivotal. If $H=M$, i.e. the price cap binds, then there is no limit on how pivotal the firms are allowed to be, assumption 1a is satisfied anyway. See Genc and Reynolds (2004) for a more detailed analysis of pivotal producers' impact on the range of supply function equilibria. For technical reasons assumption 1a rules out borderline cases where there are withholding deviations that do not change pivotal producer's profits. Given that the continuous and discrete solutions converge, we can use this technical condition to ensure that assumption 1a is satisfied if and only if assumption 1b (below) is satisfied, which is useful when we, in Section 3, verify convergence of the discrete and continuous equilibria, i.e. that global second-order conditions in the two systems have the same signs.

Assumption 1b. A binding price cap or sufficiently large production capacities ensures that there is no producer in the continuous system that can increase its profit by decreasing its supply at b. If there are unilateral deviations such that the price is higher than b with a positive probability, then the profit of the deviating firm decreases by an amount bounded away from zero.

Generally $\tau^H > \bar{\varepsilon}$, so the first step of the stepped supply curve – as we move "backwards" from $\bar{\varepsilon}$ toward $\underline{\varepsilon}$ – is special. Typically the solution of the discrete system of equations would converge to a set of curves with significantly different slopes at p^{H-1} and p^{H-2}. To avoid this potential problem we make Assumption 2, which ensures that the discrete first-order condition of the highest-price step is consistent with the first-order conditions of the other steps, i.e. the set of first-order equations at step H-1 converges to the set of first-order equation at step H-2 as $\Delta p \to 0$. Details of this assumption appear in Lemma **2** in the Appendix.

Assumption 2. Given $\left\{\hat{s}_i^H\right\}_{i=1}^N = \left\{\bar{s}_i\right\}_{i=1}^N$, $\left\{\hat{s}_i^{H-1}\right\}_{i=1}^N = \left\{\hat{k}_i\right\}_{i=1}^N$ and $\left\{k_i\right\}_{i=1}^N$, the discrete boundary values $\left\{\hat{k}_i\right\}_{i=1}^N$ converges to their limit $\left\{k_i\right\}_{i=1}^N$ in such a way that $\left\{\Gamma_i^{H-1}(\mathbf{s})\right\}_{i=1}^N \to \left\{\Gamma_i^{H-2}(\mathbf{s})\right\}_{i=1}^N$ as $\Delta p \to 0$.

The set of limits $\left\{k_i\right\}_{i=1}^N$ may also serve as boundary conditions for a set of continuous solutions, as assumed in Section 3, but this is not necessary. Appendix

Lemma **2** shows that there is always at least one set of $\left\{\hat{s}_i^{H-1}\right\}_{i=1}^{N}$ for which Assumption 2 is satisfied.

Proposition **2** says that solutions of the first-order difference equations that are non-decreasing everywhere in the region of possible demand realizations are essentially discrete SFE. This result relies on the assumption that $\Delta d^j \geq \Delta d^{j+1}$, i.e. concave demand.

Proposition 2 *Consider a set* $\left\{\hat{s}_i^j\right\}_{L,1}^{H,N}$ *of solutions to the discrete first-order conditions* (4) *under the usual boundary conditions* $\hat{s}_i^H = \bar{s}_i$ *and* $s_i^{H-1} = \hat{k}_i$ *. Suppose Assumption 1a and Assumption 2 hold. Suppose further that* $0 \leq s_i^j - s_i^{j-1} \leq W\Delta p$ *(where W is some positive constant) for j* = *L,…,H*-1, *and each i*=1, …, *N, independent of M. If the discrete strategy* $\left\{\hat{s}_i^j\right\}_L^H$ *is non-decreasing for each generator i then, for sufficiently large M,* $\left\{\hat{s}_i^j\right\}_{L,1}^{H,N}$ *is a segment of a discrete SFE.*

The analogous sufficiency result for continuous SFE with concave demand is given by

Proposition 3. *Let Assumption 1b hold. If each* $\hat{s}_i(p)$ *is non-decreasing on* [*a*,*b*] then $\left\{\hat{s}_i(p)\right\}_{i=1}^{N}$ *is a segment of a continuous SFE.*

3. CONVERGENCE OF DISCRETE AND CONTINUOUS SFE

This section states (and the appendix proves) the central result of the paper: that for a market for which a continuous SFE exists, a discrete SFE also exists and converges to the continuous SFE as $\Delta p \to 0$. The steps in the convergence proof are related to the steps in the proof of Dahlquist's equivalence theorem[7] for discrete approximations of ODEs (LeVeque, 2007). Up to this point, the convergence proof is about first-order optimality, or stationary, conditions posed as ODEs. We then depart from the theory of ODEs in order to prove convergence of the equilibria themselves. Fortunately this turns out to follow relatively easily from convergence of the first-order solutions: if assumption 1b is satisfied and if demand is concave, then it can be shown that monotonically increasing

[7] The more general Lax-Richtmyer equivalence theorem applies to partial differential equations.

solutions to the continuous first-order conditions yield monotonicity of the discrete first-order solutions, giving discrete SFE as described in Proposition **2**.

In order to avoid singularities in (5) when we later apply approximation theory for ODEs, we make:

Assumption 3. *Initial values* $\{\widehat{s}_i(b)\}_{i=1}^{N}$ *and the support of the demand shocks* $[\underline{\varepsilon}, \overline{\varepsilon}]$ *are such that the set of solutions* $\{\widehat{s}_i(p)\}_{i=1}^{N}$ *of* (4) *exists, are bounded, increasing and differentiable on the interval* [*a*,*b*]. *We also assume that the mark-up* $p - C_i'(\widehat{s}_i(p))$ *is positive for each i and each p* $\in$ [*a*,*b*].[8]

We present our main result and then lay out the proof strategy; technicalities are relegated to the appendix. Our task is to relate continuous solutions to solutions of the discrete system (4). Recall also that p_L and p_H are the lowest and highest realized prices, and that the indices *L* and *H* vary with *M* (and the boundary conditions).

Theorem 1. *Let Assumptions* 1*b and* 3 *hold, then:*

a) $\{\widehat{s}_i(p)\}_{i=1}^{N}$ *is a segment of a continuous SFE.*

b) In addition, suppose as $M \to \infty$ *that* $p_L \to a$, $p_H \to b$, *and Assumption 2 holds. Then there exists a set of solutions* $\{\widehat{s}_i^{\,j}\}_{L,1}^{H,N}$ *of the difference equations* (4), *under the usual boundary conditions* $\widehat{s}_i^H = \overline{s}_i$ *and* $s_i^{H-1} = \hat{k}_i$, *that is a segment of a discrete SFE and converges to* $\{\widehat{s}_i(p)\}_{i=1}^{N}$ *in the interval* [*a*,*b*] *as* $M \to \infty$.

The meaning of convergence in this result is that if *j* is chosen to depend on *M* such that $p_j \to p \in [a,b]$ as $M \to \infty$, then $\widehat{s}_i^{\,j} \to \widehat{s}_i(p)$ as $M \to \infty$ for each *i*.

One implication of Theorem 1 is that with a sufficient number of steps, existence of discrete SFE is ensured if a corresponding continuous SFE exists. As an example, Klemperer and Meyer (1989) establish the existence of continuous SF equilibria if firms are symmetric, ε has strictly positive density everywhere on its

[8] This is a non-restrictive constraint, because profit-maximizing producers with a non-negative output would never bid below their marginal cost, and solutions with prices below

support $\left[\underline{\varepsilon}, \overline{\varepsilon}\right]$, the cost function is C_2 and convex, and the demand function $D(p, \varepsilon)$ is C_2, concave and with a negative first derivative.

Before outlining the proof of Theorem 1, we mention two departures of this result from the literature. The first is a technical point, namely it is not standard to approximate ODEs by systems that are both non-linear and implicit (since solving an approximating system then requires an iterative procedure at each step of the integration). Nevertheless our convergence proof has to deal with systems of difference equations in Proposition 1 that are implicit and non-linear; we extend the framework of Leveque (2007) for this purpose. Second, and more important, in the study of SFEs there is little if any work that relates *ex ante* discrete games to their continuous counterparts by convergence analysis. Recall how Anderson and Hu (2008) discretise a continuous SFE system in order to get a numerically convenient discrete system with straightforward convergence to the continuous solution. This is a (valuable) numerical scheme for approximating continuous SFE. By contrast, we start with a class of self-contained discrete games and demonstrate both existence and convergence of SFE for the discrete system to those of the continuous system. This is a hitherto missing bridge from continuous SFE theory to discrete SFE practice.

The first step in proving Theorem 1 is to verify that the discrete system of stationary conditions in Proposition 1 is consistent with the stationary conditions for continuous SFE written as the ODE (5). Lemma 4 of the appendix shows this to be the case by using the positive mark-up assumption to avoid a singularity in the equations at the point where mark-ups are zero.

That the discrete system is a consistent approximation of the continuous one implies the former set of equations converges to the latter as the number of price steps M goes to infinity. Thus as $M\to\infty$, the second-order difference equation in (4) converges to a differential equation of the first-order, which corresponds to the Klemperer and Meyer equation. However, this does not ensure that a discrete solution will exist or, if it does, that it will converge to the continuous solution, because if the error increases at each step, it could explode when the number of steps becomes large – this describes what is called the unstable case. Hence the second step in the convergence analysis is to establish

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the marginal cost would never constitute Nash equilibria.

existence and stability. Proposition 4 in the Appendix states that the discrete solution exists and is stable, and shows that the solution of the discrete first-order system does indeed converge to the solution of the continuous first-order system as $M\to\infty$. As an illustration of the discrepancy between consistency and convergence, the following can be noted: to prove consistency in our model it would have been enough to assume that $\hat{s}_i(p)/(p - C_i'(\hat{s}_i(p)))$ is bounded, which would allow for zero mark-ups when supply is zero. However, the error grows at an infinite rate when the mark-up is zero at zero supply, so the continuous and discrete solutions do not necessarily converge at this point. This is related to the instability near zero supply that has been observed when continuous SFE are calculated by means of standard numerical integration methods (Baldick and Hogan, 2002; Holmberg, 2008).

Up to this point, the proof has shown existence and convergence of solutions of the discrete stationary conditions to those of the continuous stationary conditions. The final step of the convergence proof uses the observation that a stationary solution of either the discrete or continuous system is actually a Nash equilibrium strategy if it is increasing in price: see Propositions **2** and **3** in Section 2. It follows from the consistency property that if there is a continuously differentiable SFE with each player's strategy having positive gradient $\hat{s}_i'(p) > 0$ for all p of interest (in a closed interval), then the discrete solution, for which $\frac{s_i^{j+1} - s_i^j}{\Delta p_j} \to \hat{s}_i'(p)$, must also be increasing, and the proof of Theorem 1 is complete.

Note that the convergence result is valid for general cost functions, asymmetric producers and general probability distributions of the demand shock. From Proposition 1 we know that the latter influences the first-order condition for a finite number of steps, but apparently this dependence disappears in the limit, as it does in the continuous case.

Appendix Proposition 5 reverses the implication of Theorem 1 to show that if a solution of discrete first-order conditions is non-decreasing and converges to a set of smooth functions (one per player) with positive mark-ups, then the limiting set of functions is a continuous SFE. That is, the family of increasing continuous SFE with positive mark-ups is asymptotically in one-to-one correspondence with the family of corresponding discrete SFE. This is in itself a useful contribution to existence results for continuous SFEs.

3.1 Example

Consider a market with two symmetric firms that have infinite production capacity. Each producer has linear increasing marginal costs $C'_i = s_i$. Demand at each price level is by assumption given by $D(p_j, \varepsilon) = \varepsilon - 0.5 p_j$. The demand shock, ε, is assumed to be uniformly distributed on the interval [1.5,3.5], i.e. $g(\varepsilon)$ = 0.5 in this range.

In the continuous case, there is a continuum of symmetric solutions to the first-order condition in (6). The chosen solution depends on the end-condition. Klemperer and Meyer (1989) and Green and Newbery (1992) show that in the continuous case, the symmetric solution slopes upwards between the marginal cost curve and the Cournot schedule, while it slopes downwards (or backwards) outside this wedge. The Cournot schedule is the set of Cournot solutions that would result for all possible realizations of the demand shock, and the continuous SFE is vertical at this line (with price on the *y*-axis). In the other extreme, when price equals marginal cost the solution becomes horizontal. Thus a continuous symmetric solution constitutes an SFE if and only if the solution is within the wedge for all realized prices. Fig. 2 plots the most and least competitive continuous SFE. All solutions of the differential equations (4) or (5) in-between the most and least competitive continuous cases are also continuous SFE.[9]

For the marginal cost and demand curves assumed in this example, the discrete first-order condition in Proposition 1 can be simplified to:

$$-\Delta p s_i^j + \frac{1}{2}\left(p_j - \frac{c}{3}\left(s_i^{j-1} + 2s_i^j\right)\right)\Delta\tau_{-i}^j + \frac{1}{2}\left(p_{j+1} - \frac{c}{3}\left(2s_i^j + s_i^{j+1}\right)\right)\Delta\tau_{-i}^{j+1} = 0. \quad (9)$$

In a symmetric duopoly equilibrium with $\Delta d = -0.5\Delta p$, $\Delta\tau_{-i}^j = s_i^j - s_i^{j-1} + 0.5\Delta p$. Thus the first-order condition can be written:

$$-\Delta p s_i^j + \frac{1}{2}\left(p_j - \frac{c}{3}\left(s_i^{j-1} + 2s_i^j\right)\right)\left(s_i^j - s_i^{j-1} + 0.5\Delta p\right) +$$
$$\frac{1}{2}\left(p_{j+1} - \frac{c}{3}\left(2s_i^j + s_i^{j+1}\right)\right)\left(s_i^{j+1} - s_i^j + 0.5\Delta p\right) = 0.$$

[9] The dotted continuous SFs are very close to the stepped SF and for the most competitive case are essentially indistinguishable.

In Fig. 2 the discrete solutions with the same end-conditions as the most and least competitive SFE respectively are plotted. The offers at the price level *H*-1 have been calculated using boundary conditions given in Appendix Lemma 2, so that Assumption 2 is satisfied. Thus with a sufficiently small Δp these solutions will be discrete SFE according to Theorem 1, and so will all discrete solutions in-between them. Our experience is that we need a much smaller tick-size in the most competitive case compared to the least competitive case in order to get a monotonic solution. We believe that it is related to that convergence is poorer when mark-ups are small due to the singularity at zero mark-ups.

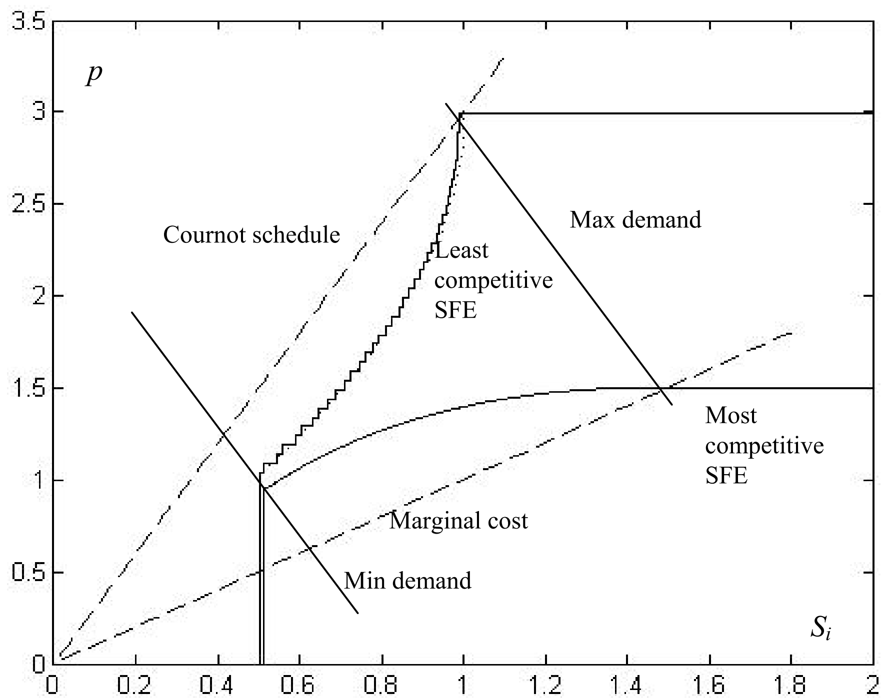

Figure 2. The most and least competitive continuous SFE (dotted) and their discrete approximations (solid). The discrete approximations have a tick-size of Δp=0.05 (non-competive case) and Δp=0.001 (competitive case).

3.3 Conjectured convergence in actual electricity markets

Anderson and Xu (2008) only solve for a very simple example with two firms each choosing one price in the first stage of the Australian market, noting that to solve for the mixed strategy for multiple steps would be challenging. For similar reasons von der Fehr and Harbord (1993) only consider mixed equilibria in which each firm chooses one price. An interesting conjecture is that if firms can choose a large but finite number of prices from a larger set of possible prices, then the range over which each price is sampled may shrink as the number of possible price choices increases, particularly if the prices themselves must be discrete. It may then be possible to demonstrate convergence of step SFEs to the continuous SFEs even when the possible price steps are smaller than the quantity steps. If so, the price instability at any level of demand would be small, and errors in using continuous representations also small.

It follows from classical existence results that NE in finite approximations of a limit game converge to the NE in the limit game if the strategy space is finite-dimensional, convex, compact and payoffs are continuous and quasi-concave (Dasgupta and Maskin, 1986; Simon, 1987). These results are not directly applicable to our case, as our limit game has an infinite-dimensional strategy space. But the general results can anyway be used to make very reasonable conjectures.. As the strategy space in the von der Fehr and Harbord model is convex and compact, it is our belief that their equilibrium fails to converge to a continuous SFE because of the payoff discontinuity; payoffs in their model can be significantly increased by slightly undercutting competitors' offers. Thus we argue that the risk of price instability would be mitigated if payoffs could be made continuous. For example, if costs are private information to some extent as in Parisio and Bosco (2003), then uncertainty about competitors' offers would make expected profits continuous. In spite of this additional uncertainty, we believe that pure-strategy equilibria in such a market can be approximated by a continuous SFE if demand uncertainty dominates uncertainty about competitor's production costs.

Further, it would be helpful if the market design did not require stepped offers. For example, Nord Pool (in the Nordic countries) and Powernext (in France) make a linear interpolation of volumes between each adjacent pair of submitted price steps. Anderson and Hu (2008) show that equilibria in such auctions converge to continuous SFE provided that the piece-wise linear offer curves are constructed to avoid the influence of kinks in residual demand. But we believe that their result is true for more general circumstances, as payoffs are continuous in such a market design, unless producers choose to make stepped offers. Continuous payoffs, because of piece-wise linear offers or uncertainty about competitors' production costs, are helpful but only guarantee convergence to SFE in the limit. To ensure price stability in a discrete system, an SFE must exist in the limit game, the quantity multiple needs to be sufficiently small, and the allowed number of steps sufficiently large.

4 CONCLUSIONS

Green and Newbery (1992), and Newbery (1998) assume that the allowed number of steps in the supply function bids of electricity auctions is so large that equilibrium bids can be approximated by continuous SFE. This is a very attractive assumption, because it implies that a pure-strategy equilibrium can be calculated analytically for simple cases and numerically for general cost functions and asymmetric producers. The pure-strategy equilibrium that has inherently stable prices also justifies empirical approaches that enable observers to deduce contract positions, marginal costs and the price-cost mark-up from observed bids, as in Wolak (2001).

von der Fehr and Harbord (1993), however, argue that as long as the number of steps is finite, then continuous SFE are not a valid representation of bidding in electricity auctions. Under the extreme assumption that prices can be chosen from a continuous distribution so that the price tick size is negligible, von der Fehr and Harbord (1993) show that uniform price electricity auctions have an inherent price instability. If demand variation is sufficiently large, so that no producer is pivotal at minimum demand and at least one firm is pivotal at maximum demand, then there are no pure strategy Nash equilibria, only mixed strategy Nash equilibria. The intuition behind the non-existence of pure strategy Nash equilibria is that producers slightly undercut each other's step bids until mark-ups are zero. Whenever producers are pivotal they have profitable deviations from such an outcome.

We claim that the von der Fehr and Harbord result is not driven by the stepped form of the supply functions, but rather by their discreteness assumption. We consider the other extreme in which the price tick size is significant and the quantity multiple is negligible. We show that in this case step equilibria converge to continuous supply function equilibria. The intuition for the existence of pure strategy equilibria is that with a significant price tick size, it is not necessarily profitable to undercut perfectly elastic segments in competitors' bids.

Our results imply that the concern that electricity auctions have an inherent price instability and that they cannot be modelled by continuous SFE is not necessarily correct. We also claim that this potential problem can be avoided if tick sizes are such that the number of price levels is small compared to the number of quantity levels, which is the case in many electricity markets. To avoid price instability, we also recommend that restrictions in the number of steps should be as lax as possible, even if some restrictions are probably administratively necessary. Restricting the number of steps increases each producer's incremental supply offered at each step, encouraging price randomisation.

Our recommendation to have small quantity multiples contrasts with that of Kremer and Nyborg (2004b) who recommend a large minimum quantity increment relative to the price tick size to encourage competitive bidding. A problem with their analysis is that they only consider first-order conditions; they do not verify that pure-strategy equilibria exist by checking second-order conditions. We believe that their recommendation is correct for markets in which bidders are non-pivotal for all demand realizations, because in such markets pure strategy equilibria with very low mark-ups are possible. For example, von der Fehr and Harbord's (1993) model has a Bertrand equilibrium in this case. However, when one or several producers are pivotal for some demand realization, encouraging producers to undercut competitors' bids can lead to non-existence of pure strategy Nash equilibria and not necessarily lower average mark-ups (von der Fehr and Harbord, 1993).

Even if mark-ups would be lower also in this case, the market participants would bear the cost of uncertainty caused by the inherent price instability. As undercutting incentives are only problematic when producers are pivotal, it is possible that an optimal market design would have a price tick-size that increases with the price. This could be achieved by limiting the number of non-zero digits rather than the number of decimals in the bids, or by requiring a minimum percentage increment in successive prices, as in some multi-round auctions. If this is an attractive option, it should be noted that the first-order condition in Proposition 1 is valid even if the tick size varies with the price.

Because of a singularity at zero mark-up, equilibrium bid-curves tend to be numerically unstable and easily non-monotonic near such points (Baldick and Hogan, 2002; Holmberg, 2008). We have the same experience with our stepped offer curves. The policy implication is that smaller tick-sizes, and even smaller quantity multiples, are needed in competitive markets with small mark-ups in order to get stable prices

General convergence results for finite-dimensional games by Dasgupta and Maskin (1986) and Simon (1987) are not necessarily applicable to our problem, which is infinitely-dimensional in the limit. But their results suggest that the risk of non-convergence and price instability in electricity auctions would be lower if payoffs were continuous, for example by allowing piece-wise linear offers as in Nord Pool. Existence of continuous and discrete pure-strategy SFE is problematic if the demand curve is sufficiently convex or if production costs are sufficiently non-convex.

If an electricity market would fail to have a pure-strategy NE due to large quantity increments, then problems caused by instability might not be too severe for levels of demand when no generator is pivotal and the MCP were close to system marginal cost. We also conjecture that if mixed strategy equilibria occur, then the price instability at any level of demand would be small if there are many available price and quantity levels.

Recently, it has been empirically verified that large producers in the balancing market of Texas (ERCOT) approximately bid in accordance with the first-order condition for continuous supply functions (Niu et al., 2005; Hortascu and Puller, 2007; Sioshansi and Oren, 2007). It is possible that the new discrete model could improve the accuracy of such empirical studies, because the new first-order condition considers the influence by the demand uncertainty on stepped offers. This effect has previously been considered by Wolak (2004) in an empirical analysis of the Australian market, but this market is quite different from most other markets, as producers choose their own price grid in Australia. Moreover, our discrete model side-steps the problem of how to smooth the residual demand curve. The smoothing process has been a somewhat arbitrary and therefore disputed part of previous empirical studies, which rely on continuous or quasi-discrete models (Wolak, 2004) of bidding in the electricity market. In case discrete NE are useful as a method of numerically calculating approximate SFE, it should be noticed that the assumed price tick size does not necessarily have to correspond to the tick size of the studied auction. In a numerically efficient solver, it might be of interest to vary the tick size with the price.

We show that never-accepted out-of-equilibrium bids of rational producers are perfectly elastic in uniform-price procurement auctions with stepped supply functions and pro-rata on-the-margin rationing. This theoretical prediction can be used to empirically test whether producers in electricity auctions believe that some of their offers are accepted with zero-probability, which is assumed in many theoretical models of electricity auctions. A by-product of our analysis is the result that any set of, not necessarily symmetric, solutions to Klemperer and Meyer's system of differential equations constitute a continuous SFE if supply functions are increasing for all realized prices, demand is concave, and if there are no profitable deviations at the highest realized price, because of a price cap or because competitors' have sufficiently large excess capacity.

Finally, we would not claim that the apparent tension between tractable but unrealistic continuous SFEs and realistic but intractable step SFEs is the only, or even the main, problem in modelling electricity markets. First, there are multiple SFE if some offers are always accepted or never accepted. Then under reasonable conditions, there is a continuum of continuous SFE bounded by (in the short run) a least and most profitable SFE. Second, the position of the SFEs depends on the contract position of all the generators, and determining the choice of contracts and their impact on the spot market is a hard and important problem. The greater the extent of contract cover, the less will be the incentive for spot market manipulation (Newbery, 1995), and as electricity demand is very inelastic and markets typically concentrated, this is an important determinant of market performance. Newbery (1998) argued that these can be related, in that incumbents can choose contract positions to keep both the contract and average spot price at the entry-deterring level, thus simultaneously solving for prices, contract positions, and embedding the short-run SFE within a longer run investment and entry equilibrium. A full long-run model of the electricity market should also be able to investigate whether some market power is required for (or inimical to) adequate investment in reserve capacity to maintain adequate security of supply. With such a model one could also make a proper assessment of how many competing generators are needed to deliver a workably competitive but secure electricity market.

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APPENDIX - PROOFS OF PROPOSITIONS

Proof of Proposition 1: To find an equilibrium we need to determine the best response of firm *i* given its competitors' bids. The best response necessarily satisfies a first-order condition for each price level, found by differentiating (2) with respect to s_i^j and s_i^{j-1}, noting that the limits are functions of s_i^j and s_i^{j-1}, as $\tau^j = \tau_{-i}^j + s_i^j$:

$$\frac{\partial E_i^j}{\partial s_i^j} = \int_{\tau^{j-1}}^{\tau^j} \left(p_j - C_i'(\cdot) \right) \frac{\Delta \tau_{-i}^j \left(\varepsilon - \tau^{j-1} \right)}{\left(\Delta \tau^j \right)^2} g(\varepsilon) d\varepsilon + \left[p_j s_i^j - C_i\left(s_i^j\right) \right] g\left(\tau^j\right) \tag{10}$$

and

$$\frac{\partial E_i^j}{\partial s_i^{j-1}} = \int_{\tau^{j-1}}^{\tau^j} \left(p_j - C_i'(\cdot) \right) \Delta \tau_{-i}^j \frac{\left(\tau^j - \varepsilon \right)}{\left(\Delta \tau^j \right)^2} g(\varepsilon) d\varepsilon - \left[p_j s_i^{j-1} - C_i\left(s_i^{j-1}\right) \right] g\left(\tau^{j-1}\right).$$

From the last expression it follows that:

$$\frac{\partial E_i^{j+1}}{\partial s_i^j} = \int_{\tau^j}^{\tau^{j+1}} \left[p_{j+1} - C_i'(\cdot) \right] \Delta \tau_{-i}^{j+1} \frac{\left(\tau^{j+1} - \varepsilon \right)}{\left(\Delta \tau^{j+1} \right)^2} g(\varepsilon) d\varepsilon - \left[p_{j+1} s_i^j - C_i\left(s_i^j\right) \right] g\left(\tau^j\right). \tag{11}$$

Combining (10) and (11) gives the first-order condition for step supply functions:

$$\begin{aligned} \frac{\partial \operatorname{E}(\pi_i(\mathbf{s}))}{\partial s_i^j} &= \frac{\partial E_i^j}{\partial s_i^j} + \frac{\partial E_i^{j+1}}{\partial s_i^j} = -\Delta p s_i^j g\left(\tau^j\right) + \int_{\tau^{j-1}}^{\tau^j} \left[p_j - C_i'\left(s_i(\varepsilon)\right) \right] \frac{\Delta \tau_{-i}^j \left(\varepsilon - \tau^{j-1} \right)}{\left(\Delta \tau^j \right)^2} g(\varepsilon) d\varepsilon + \\ &+ \int_{\tau^j}^{\tau^{j+1}} \left[p_{j+1} - C_i'\left(s_i(\varepsilon)\right) \right] \Delta \tau_{-i}^{j+1} \frac{\left(\tau^{j+1} - \varepsilon \right)}{\left(\Delta \tau^{j+1} \right)^2} g(\varepsilon) d\varepsilon = 0, \end{aligned} \tag{12}$$

where $s_i(\varepsilon)$ is given by (1) if $\varepsilon \in \left[\tau^{j-1}, \tau^j\right]$.

$\partial \operatorname{E}(\pi_i(\mathbf{s})) / \partial s_i^j$ is always well-defined, as from our definitions and assumed restrictions on the bids it follows that $\Delta \tau^j \geq \Delta \tau_{-i}^j \geq 0$ and $\Delta \tau^j \geq \varepsilon - \tau^{j-1} \geq 0$ if $\varepsilon \in \left[\tau^{j-1}, \tau^j\right]$. □

The first-order condition in Proposition 1 is not directly applicable to parts of the offer curve that are never accepted in equilibrium, i.e. for price levels p_j such that $\tau^j > \bar{\varepsilon}$. Let p_H be the highest price level that is realized with a positive probability. By differentiating the expected profit in (3), one can show that

$$0 < \frac{\partial \mathrm{E}(\pi_i(\mathbf{s}))}{\partial s_i^H} = \int_{\tau^{H-1}}^{\bar{\varepsilon}} \left[p_H - C_i'(s_i(\varepsilon)) \right] \left(\frac{\Delta \tau_{-i}^H (\varepsilon - \tau^{H-1})}{(\Delta \tau^H)^2} \right) g(\varepsilon) d\varepsilon \,,$$

because $g(\varepsilon) = 0$ for $\varepsilon > \bar{\varepsilon}$. Thus to maximize its expected profit a firm should offer all of its remaining capacity at p_H. The intuition for this result is as follows: due to pro-rata on-the-margin rationing, maximizing the supply at p_H maximizes the firm's share of the accepted supply at p_H, and, because of the bounded range of demand shocks, there is no risk that an increased supply at p_H will lead to a lower price for any realized event. Hence $s_i^H = \bar{s}_i$. Our discreteness and uncertainty assumptions should not be critical for this result. Intuitively, we expect never-accepted offers to be perfectly elastic in any uniform price auction with stepped supply functions and pro-rata on the margin rationing.

Now, consider offers that are always infra-marginal. Let p_L be the lowest price that is realized with positive probability. Differentiate expected profit in (3):

$$0 < \frac{\partial \mathrm{E}(\pi_i(\mathbf{s}))}{\partial s_i^{L-1}} = \int_{\tau^{n-1}}^{\tau^n} \left[p_L - C_i'(s_i(\varepsilon)) \right] \Delta \tau_{-i}^L \left(\frac{\Delta \tau^L - (\varepsilon - \tau^{L-1})}{(\Delta \tau^L)^2} \right) g(\varepsilon) d\varepsilon \quad \text{if} \quad \tau^{L-1} < \underline{\varepsilon} \,,$$

because $g(\varepsilon) = 0$ for $\varepsilon < \underline{\varepsilon}$. Hence $\tau^{L-1} = \underline{\varepsilon}$. This result makes sense intuitively. To increase the accepted supply with pro-rata on-the-margin rationing at the price level p_L, infra-marginal offers that are never price-setting should be offered below p_L rather than at p_L. Again, we intuitively believe that always-accepted offers are generally offered below p_L in any uniform price auction with stepped supply functions and a pro-rata on the margin rationing mechanism.

Lemma 1 below derives a Taylor expansion of the discrete first-order condition - very useful when we show that discrete SFE converge to continuous SFE.

Lemma 1. If the differences $s_i^{j+1} - s_i^j$ are of the order Δp_j, then the discrete first-order condition in (12) can approximated by the following Taylor series expansion in Δp_j:

$$-\Delta p_j s_i^j g(\tau^j) + \left[p_j - C_i'(s_i^j) \right] \Delta \tau_{-i}^j g(\tau^j) + O(\Delta p_j^2) = 0 \text{ if } L \le j < H\text{-}1 \text{ and}$$

$$- \Delta p_j s_i^{H-1} g\left(\tau^{H-1}\right) + \frac{\left[p_{H-1} - C_i'\left(s_i^{H-1}\right)\right] g\left(\tau^{H-1}\right)\Delta\tau_{-i}^{H-1}}{2}$$
$$+ \left[p_{H-1} - C_i'\left(s_i^{H-1}\right)\right]\Delta\tau_{-i}^{H} g\left(\tau^{H-1}\right)\frac{\left(\bar{\varepsilon} - \tau^{H-1}\right)}{\left(\Delta\tau^{H}\right)} + O\left((\Delta p)^2\right) = 0.$$

Proof: Let $\Delta p_j = \Delta p_{j+1}$, then the first-order condition in (12) can be written:

$$\Gamma_i^j(\mathbf{s}) = \frac{\partial \mathrm{E}(\pi_i(\mathbf{s}))}{\partial s_i^j} = -\Delta p_j s_i^j g\left(\tau^j\right) + \int_{\tau^j}^{\tau^{j+1}} \left[p_{j+1} - C_i'(\cdot)\right]\frac{\Delta\tau_{-i}^{j+1}}{\Delta\tau^{j+1}} g(\varepsilon)d\varepsilon +$$
$$\int_{\tau^{j-1}}^{\tau^j} \left[p_j - C_i'(\cdot)\right]\frac{\Delta\tau_{-i}^{j}\left(\varepsilon - \tau^{j-1}\right)}{\left(\Delta\tau^j\right)^2} g(\varepsilon)d\varepsilon - \int_{\tau^j}^{\tau^{j+1}} \left[p_{j+1} - C_i'(\cdot)\right]\frac{\Delta\tau_{-i}^{j+1}\left(\varepsilon - \tau^{j}\right)}{\left(\Delta\tau^{j+1}\right)^2} g(\varepsilon)d\varepsilon = 0. \tag{13}$$

First assume that $L \le j < H$-1. Straightforward differentiation yields:

$$\lim_{\Delta p\to 0}\frac{\partial\Gamma_i^j(\mathbf{s})}{\partial\Delta\tau^{j+1}} = \lim_{\Delta p\to 0}\frac{\partial\Gamma_i^j(\mathbf{s})}{\partial\Delta\tau^{j}} = 0, \text{ and}$$

$$\lim_{\Delta p\to 0}\frac{\partial\Gamma_i^j(\mathbf{s})}{\partial\Delta\tau_{-i}^{j+1}} = \frac{1}{2}\left[p_j - C_i'\left(s_i^j\right)\right] g\left(\tau^j\right) = \lim_{\Delta p\to 0}\frac{\partial\Gamma_i^j(\mathbf{s})}{\partial\Delta\tau_{-i}^{j}}$$

where the data at step *j*, e.g. s_i^j, are deemed fixed while $\Delta p \to 0$ implies data at step *j*+1converge to their respective values at step *j*. The difference between $\Delta\tau_{-i}^j$ and $\Delta\tau_{-i}^{j+1}$ is of the second-order. Thus considering the derivatives above, we get:

$$\Gamma_i^j(\mathbf{s}) = -\Delta p s_i^j g\left(\tau^j\right) + \left[p_j - C_i'\left(s_i^j\right)\right] g\left(\tau^j\right)\Delta\tau_{-i}^j + O\left((\Delta p)^2\right). \tag{14}$$

Next, perform the corresponding derivation for j=H-1 where

$$\frac{\partial\Gamma_i^{H-1}(\mathbf{s})}{\partial\Delta\tau^H} = \left[p_{H-1} - C_i'\left(s_i^{H-1}\right)\right]\Delta\tau_{-i}^H g\left(\tau^{H-1}\right)\frac{\left[\left(\Delta\tau^H\right)\left(\bar{\varepsilon} - \tau^{H-1}\right) + \left(\tau^H - \bar{\varepsilon}\right)^2 - \left(\Delta\tau^H\right)^2\right]}{\left(\Delta\tau^H\right)^3}$$
$$+ O(\Delta p^2),$$
$$\frac{\partial\Gamma_i^{H-1}(\mathbf{s})}{\partial\Delta\tau_{-i}^H} = -\left[p_{H-1} - C_i'\left(s_i^{H-1}\right)\right]\frac{\left(\tau^H - \bar{\varepsilon}\right)^2 - \left(\Delta\tau^H\right)^2}{2\left(\Delta\tau^H\right)^2} g\left(\tau^{H-1}\right) + O(\Delta p^2),$$
$$\frac{\partial\Gamma_i^{H-1}(\mathbf{s})}{\partial\Delta\tau_{-i}^{H-1}} = \frac{1}{2}\left[p_{H-1} - C_i'\left(s_i^{H-1}\right)\right] g\left(\tau^{H-1}\right) + O(\Delta p^2).$$

Also $\lim_{\Delta p\to 0}\frac{\partial\Gamma_i^{H-1}(\mathbf{s})}{\partial\Delta\tau^{H-1}} = 0$. Considering the above derivatives, it can be shown that:

$$\Gamma_i^{H-1}(\mathbf{s}) = -\Delta p s_i^{H-1} g\left(\tau^{H-1}\right) + \frac{\left[p_{H-1} - C_i'\left(s_i^{H-1}\right)\right] g\left(\tau^{H-1}\right)\Delta\tau_{-i}^{H-1}}{2}$$
$$+\left[p_{H-1} - C_i'\left(s_i^{H-1}\right)\right]\Delta\tau_{-i}^{H} g\left(\tau^{H-1}\right)\frac{\left(\bar{\varepsilon} - \tau^{H-1}\right)}{\left(\Delta\tau^{H}\right)} + O\left((\Delta p)^2\right).$$

□

The result in Lemma 1 can be used to prove the following:

Lemma 2. Assume that the differences $s_i^{j+1} - s_i^j$ are of the order Δp_j The discrete first-order equation for the highest-price step is consistent with the first-order equations for lower steps if and only if $\lim_{\Delta p \to 0} \frac{\left(\bar{\varepsilon} - \tau^{H-1}\right)\Delta\tau_{-i}^{H}}{\Delta\tau_{-i}^{H-1}\Delta\tau^{H}} = \frac{1}{2}$. In particular, this condition is satisfied if $\tau^{H-1} = \bar{\varepsilon} - \sum_{i=1}^{N} \frac{\Delta p k_i}{2(N-1)\left[p_H - C_i'(k_i)\right]}$ and

$$s_i^{H-1} = s_i^{H} - \Delta\tau^{H} + \frac{k_i \Delta p_j \Delta\tau^{H}}{2\left[p_{H-1} - C_i'(k_i)\right]\left(\bar{\varepsilon} - \tau^{H-1}\right)}, \text{ where } k_i = \lim_{\Delta p \to 0} s_i^{H-1}.$$

Proof: It follows from Lemma 1 that consistency is equivalent to the condition that $\lim_{\Delta p \to 0} \frac{\left(\bar{\varepsilon} - \tau^{H-1}\right)\Delta\tau_{-i}^{H}}{\Delta\tau_{-i}^{H-1}\Delta\tau^{H}} = \frac{1}{2}$, because under this condition, the first-order equation for the highest-price step converges to the first-order equation for the price level p^{H-2}. Given this condition, we also have from Lemma 1 that

$$\lim_{\Delta p \to 0} \frac{s_i^{H-1} g\left(\tau^{H}\right)}{2} = \lim_{\Delta p \to 0}\left[p_{H-1} - C_i'\left(s_i^{H-1}\right)\right]\frac{\Delta\tau_{-i}^{H}}{\Delta p_j} g\left(\tau^{H-1}\right)\frac{\left(\bar{\varepsilon} - \tau^{H-1}\right)}{\left(\Delta\tau^{H}\right)}.$$

In particular, this limiting condition is satisfied if

$$\Delta\tau_{-i}^{H} = \frac{k_i \Delta p_j \Delta\tau^{H}}{2\left[p_{H-1} - C_i'(k_i)\right]\left(\bar{\varepsilon} - \tau^{H-1}\right)}. \tag{15}$$

Summing over all firms yields:

$$(N-1)\Delta\tau^{H} = \frac{\Delta\tau^{H}}{\left(\bar{\varepsilon} - \tau^{H-1}\right)}\sum_{i=1}^{N}\frac{k_i \Delta p_j}{2\left[p_{H-1} - C_i'(k_i)\right]},$$

which can be simplified to:

$$\tau^{H-1} = \bar{\varepsilon} - \frac{1}{(N-1)} \sum_{i=1}^{N} \frac{k_i \Delta p_j}{2\left[p_{H-1} - C_i'(k_i)\right]}.$$

Given this result we can now calculate s_i^{H-1} from (15):

$$s_i^{H-1} = s_i^H - \Delta\tau^H + \frac{k_i \Delta p_j \Delta\tau^H}{2\left[p_{H-1} - C_i'(k_i)\right]\left(\bar{\varepsilon} - \tau^{H-1}\right)}. \qquad \square$$

Proof of sufficient conditions

In both the discrete and continuous case, only non-decreasing solutions of the first-order system can constitute valid SFE, because electricity auctions do not accept decreasing offers. Thus a necessary condition for an SFE is that solutions are non-decreasing. Proposition **2** shows that being non-decreasing is also a sufficient condition for a discrete SFE (so the non-decreasing condition acts rather like a second order condition in ensuring sufficiency). Note that the result relies on Assumption 1a and the assumptions that $\Delta d^j \geq \Delta d^{j+1}$, i.e. concave demand.

Proof of Proposition 2 : Consider a set of non-decreasing solutions, for some price range [p_L ,p_H] to a system of discrete first-order conditions as in Proposition 1. The shock distribution is such that p_L and p_H are the lowest and highest realized prices. Denote the solution by $\breve{\mathbf{s}} = \{\breve{\mathbf{s}}_\mathbf{1}, \ldots, \breve{\mathbf{s}}_\mathbf{N}\}$. In what follows it will be shown that an arbitrary chosen firm i has no incentive to unilaterally deviate from the supply schedule $\breve{\mathbf{s}}_i = \{\breve{s}_i^1, \ldots, \breve{s}_i^M\}$ to any $\mathbf{s}_i = \{s_i^1, \ldots, s_i^M\}$ given that Δp is sufficiently small and that competitors stick to $\breve{\mathbf{s}}_{-i} = \{\breve{s}_{-i}^1, \ldots, \breve{s}_{-i}^M\}$. Thus $\breve{\mathbf{s}} = \{\breve{\mathbf{s}}_\mathbf{1}, \ldots, \breve{\mathbf{s}}_\mathbf{N}\}$ constitutes a Nash equilibrium. Now, assume that competitors stick to $\breve{\mathbf{s}}_{-i} = \{\breve{s}_{-i}^1, \ldots, \breve{s}_{-i}^M\}$ and calculate the total differential of the expected profit of firm i for some $\mathbf{s}_i$:

$$d\,\mathrm{E}(\pi_i(\mathbf{s}_i)) = \sum_{j=1}^{M} \frac{\partial\,\mathrm{E}(\pi_i(\mathbf{s}_i))}{\partial s_i^j} ds_i^j \ .$$

In the limit when the number of price levels approaches infinity, we get from Lemma **1** (applicable as difference $s_i^{j+1} - s_i^j$ are of the order Δp_j)

$$\lim_{M\to\infty} d\,\mathrm{E}(\pi_i(\mathbf{s}_i)) = \sum_{j=1}^{M} \left\{-\Delta p s_i^j + \left[p_{j+1} - C_i'\left(s_i^j\right)\right]\Delta\breve{\tau}_{-i}^{j+1}\right\} g\left(s_i^j + \breve{\tau}_{-i}^j\right) ds_i^j \ . \qquad (16)$$

Note that, under assumption 2, this limit result is valid also for j=H-1. For the solution $\breve{\mathbf{s}}_i = \{\breve{s}_i^1, \ldots, \breve{s}_i^M\}$ we know from Lemma **1** that

$$-\Delta p \breve{s}_i^j + \left[p_{j+1} - C_i'\left(\breve{s}_i^j\right)\right] \Delta \breve{\tau}_{-i}^{j+1} = O\left(\Delta p^2\right), \forall j \in L, \ldots, M. \tag{17}$$

Using the expression above, we can deduce that:

$$\begin{aligned} & -\Delta p s_i^j + \left[p_{j+1} - C_i'\left(s_i^j\right)\right] \Delta \breve{\tau}_{-i}^{j+1} = \\ & \Delta p\left(\breve{s}_i^j - s_i^j\right) + \left[C_i'\left(\breve{s}_i^j\right) - C_i'\left(s_i^j\right)\right] \Delta \breve{\tau}_{-i}^{j+1} + O\left(\Delta p^2\right) \forall j \in L \ldots H. \end{aligned} \tag{18}$$

The cost function is increasing and convex by assumption. It now follows from (16) that for any supply schedule $\mathbf{s}_i$ that differs from $\breve{\mathbf{s}}_i$ at some price $j \in L \ldots H-1$, the expected profit can be increased by the following adjustment of the supply schedule (if the tick size is sufficiently small):

1) Marginally increase supply at each price level $j \in L, \ldots, H-1$, for which $\breve{s}_i^j > s_i^j$, because for this case (18) implies that $-\Delta p s_i^j + \left[p_{j+1} - C_i'\left(s_i^j\right)\right] \Delta \breve{\tau}_{-i}^{j+1} \geq 0$ if Δp is sufficiently small.

2) Marginally decrease supply at each price level $j \in L, \ldots, H-1$, for which $\breve{s}_i^j < s_i^j$, because for this case (18) implies that $-\Delta p s_i^j + \left[p_{j+1} - C_i'\left(s_i^j\right)\right] \Delta \breve{\tau}_{-i}^{j+1} \leq 0$ if Δp is sufficiently small. This analysis applies to any firm and it implies that there are no profitable unilateral deviations at the price levels $j \in L, \ldots, H-1$ from the equilibrium candidate $\breve{\mathbf{s}} = \{\breve{\mathbf{s}}_\mathbf{1}, \ldots, \breve{\mathbf{s}}_\mathbf{N}\}$.

The next step is to prove that there are no profitable unilateral deviations for the other price levels either. According to Assumption 1a there are no profitable deviations in which the supply at p_H (and higher price levels) is less than maximum capacity. It is possible to push the market price below p_L, the lowest realized price in the potential equilibrium. However, as shown below such deviations would not be profitable. Equation (5) and the assumption that $\Delta d^j \geq \Delta d^{j+1}$ together imply that $\Delta \breve{\tau}_{-i}^{j+1} \geq \Delta \breve{\tau}_{-i}^j$ for all $j \leq L$. Thus it follows from (17) that

$$\lim_{M \to \infty} -\Delta p \breve{s}_i^j + \left[p_{j+1} - C_i'\left(s_i^j\right)\right] \Delta \breve{\tau}_{-i}^{j+1} < O\left(\Delta p^2\right), \text{for all } j < L.$$

The probability density $g\left(s_i^j + \breve{\tau}_{-i}^j\right)$ is positive for some $j < L$ only if $\breve{s}_i^j < s_i^j$. Hence,

$$\lim_{M \to \infty} - \Delta p s_i^j + \left[p_{j+1} - C_i'\left(s_i^j\right)\right] \Delta \breve{\tau}_{-i}^{j+1} < O\left(\Delta p^2\right), \text{for all } j < L, \text{such that } g\left(s_i^j + \breve{\tau}_{-i}^j\right) > 0. \quad (19)$$

It now follows from (16) and (19) that for any supply schedule $\mathbf{s}_i$ that differs from $\breve{\mathbf{s}}_i$ at some price $j < L$, such that $g\left(s_i^j + \breve{\tau}_{-i}^j\right) > 0$ the expected profit can be increased by marginally decreasing s_i^j if Δp is sufficiently small. This analysis applies to any firm and it accordingly implies that there are no profitable unilateral deviations at price levels $j < L$ from the equilibrium candidate $\breve{\mathbf{s}} = \{\breve{\mathbf{s}}_1, \ldots, \breve{\mathbf{s}}_N\}$. We have now shown this result for every price level. Accordingly, we can conclude that $\breve{\mathbf{s}}_i = \left\{\breve{s}_i^1, \ldots, \breve{s}_i^M\right\}$ globally maximizes the expected profit of firm *i* and that $\breve{\mathbf{s}} = \{\breve{\mathbf{s}}_1, \ldots, \breve{\mathbf{s}}_N\}$ constitutes a Nash equilibrium. □

Proposition **3** states that a set of increasing solutions to the continuous first-order conditions is a sufficient condition for supply function equilibrium if assumption 1b is satisfied and the demand curve is concave.

Proof of Proposition 3: Let *X* be a potential equilibrium, i.e. all SFs are non-decreasing and satisfy the continuous f.o.c. in (6) in the whole price-setting region, (a,b). Consider an arbitrary firm *i*. Assume that its competitors follow the potential equilibrium strategy. The question is whether it will be a best response of firm *i* to do the same. The profit of producer *i* for the outcome *ε* is given by

$$\pi_i(p,\varepsilon) = \left(\varepsilon + D(p) - S_{-i}^X(p)\right)p - C_i\left(\varepsilon + D(p) - S_{-i}^X(p)\right).$$

Hence

$$\frac{\partial \pi_i(p,\varepsilon)}{\partial p} = \left[D'(p) - S_{-i}'^X(p)\right]\left[p - C_i'\left(\varepsilon + D(p) - S_{-i}^X(p)\right)\right] + \varepsilon + D(p) - S_{-i}^X(p). \quad (20)$$

From the first-order condition in (6) it is known that

$$\left[D'(p) - S_{-i}'^X(p)\right]\left[p - C_i'\left(S_i^X(p)\right)\right] + S_i^X(p) = 0 \quad \forall p \in [a,b].$$

Subtracting this expression from (20) yields:

$$\frac{\partial \pi_i(\varepsilon,p)}{\partial p} = \left[S_{-i}'^X(p) - D'(p)\right]\left[C_i'\left(\underbrace{\varepsilon + D(p) - S_{-i}^X(p)}_{S_i}\right) - C_i'\left(S_i^X(p)\right)\right] + \left(\underbrace{\varepsilon + D(p) - S_{-i}^X(p)}_{S_i} - S_i^X(p)\right) \quad \forall p \in \left[p^X(\underline{\varepsilon}), p^X(\overline{\varepsilon})\right]. \tag{21}$$

Due to monotonicity of the supply functions we know that ${S_{-i}^X}'(p) - D'(p) \geq 0$ and that

$$p(S_i) \leq p\left(S_i^X\right) \Leftrightarrow S_i \geq S_i^X \Leftrightarrow C_i'(S_i) \geq C_i'\left(S_i^X\right).$$

Thus for every $p(S_i) \in \left[p^X(\underline{\varepsilon}), p^X(\overline{\varepsilon})\right] = [a,b]$ we can conclude from (21) that

$$\frac{\partial \pi_i(\varepsilon,S_i)}{\partial p} \geq 0 \text{ if } p(S_i) \leq p\left(S_i^X\right) \text{ and}$$

$$\frac{\partial \pi_i(\varepsilon,S_i)}{\partial p} \leq 0 \text{ if } p(S_i) \geq p\left(S_i^X\right).$$

Hence, given $S_{-i}^X(p)$ and ε, the profit of firm *i* is pseudo-concave in the price range $[a,b)$ and the profit maximum is given by the first-order condition if prices are restricted to this range. The next step in the proof is to rule out profitable deviations outside this price range. According to Assumption 1b there are no profitable deviations in which the supply at *b* (and higher prices) is less than maximum capacity. It is possible to push market prices below *a,* but as will be shown such deviations will be unprofitable. The assumptions in (8) imply that all supply functions of the potential equilibrium are perfectly inelastic below *a*. This assumption and concavity of the demand curve implies that $\frac{d\left\{D'(p) - S_{-i}'^X(p)\right\}}{dp} \leq 0$ if $p \leq a$. Thus

$$\frac{\partial^2 \pi_i(p,\varepsilon)}{\partial p^2} = \underbrace{\left[D''(p) - S_{-i}''^X(p)\right]}_{\leq 0}\underbrace{\left[p - C_i'\left(\varepsilon + D(p) - S_{-i}^X(p)\right)\right]}_{\geq 0} + \underbrace{\left[D'(p) - S_{-i}'^X(p)\right]}_{\leq 0}\left[\underbrace{1 - C_i''\underbrace{\left(\varepsilon + D(p) - S_{-i}^X(p)\right)}_{\geq 0}\underbrace{\left[D'(p) - S_{-i}'^X(p)\right]}_{\leq 0}}_{\geq 0}\right] + \underbrace{\left[D'(p) - S_{-i}'^X(p)\right]}_{\leq 0} \leq 0 \;\forall p \in \left[C_i'(S_i),a\right].$$

Hence, given that competitors stick to their potential equilibrium strategies $S_{-i}^{X}(p)$, the profit function is concave in the range $[C_i'(S_i), a]$. Offering supply below marginal cost can never be profit maximizing. Thus we can conclude that $S_i^X(p)$ must be a best response to $S_{-i}^{X}(p)$. This is true for any firm and we can conclude that X is an equilibrium. □

Proof of convergence

Lemma **3** below states that the system of first-order conditions implied by Proposition 1 has a unique solution for the price level p_{j-1} if Δp_j is sufficiently small and if supplies for the two previous steps, p_j and p_{j+1}, are known and satisfy certain properties and if producers never bid below their marginal cost. We will later use Lemma **3** iteratively to ensures that we will be able to find unique solutions to the discrete first-order condition for multiple price levels under some specified circumstances.

Lemma 3. *For a sufficiently small local tick-size $\Delta p_j = \Delta p_{j+1}$, assume that the known supplies at price levels p_{j+1} and p_j are given by a pair of differentiable vector functions $\{s_i^{j+1}(\Delta p_j)\}_{i=1}^{N}$ and $\{s_i^{j}(\Delta p_j)\}_{i=1}^{N}$. It is assumed that there exists $\delta > 0$, s.t.* $p_{j+1} - C_i'(s_i^{j+1}) \geq \delta > 0 \; \forall i = 1 \ldots N$ and $p_j - C_i'(s_i^{j}) \geq \delta > 0 \; \forall i = 1 \ldots N$. *We also assume a positive constant K can be found such that* $0 \leq s_i^{j+1}(\Delta p_j) - s_i^{j}(\Delta p_j) \leq K \Delta p_j$. *Under these circumstances, there exists a unique differentiable vector function $\{s_i^{j-1}(\Delta p_j)\}_{i=1}^{N}$ that together with $\{s_i^{j}(\Delta p_j)\}_{i=1}^{N}$ and $\{s_i^{j+1}(\Delta p_j)\}_{i=1}^{N}$ satisfy the first-order condition in Proposition 1 for the price level j.*

Proof: We want to determine $\{s_i^{j-1}(\Delta p_j)\}_{i=1}^{N}$, i.e. a set of solutions for price level j-1 as a function of the local tick-size Δp_j. The implicit function Γ is defined by the first-order condition in Proposition 1:

$$\Gamma_i^j\left(\mathbf{s}^{j-1},\Delta p_j\right)=-\Delta p_j\left(\tau^j-\tau_{-i}^j\right)g\left(\tau^j\right)+\int_{\tau^{j-1}}^{\tau^j}\left[p_j-C_i'\left(s_i(\varepsilon)\right)\right]\frac{\Delta\tau_{-i}^j\left(\varepsilon-\tau^{j-1}\right)}{\left(\Delta\tau^j\right)^2}g(\varepsilon)d\varepsilon$$
$$+\int_{\tau^j}^{\tau^{j+1}}\left[p_{j+1}-C_i'\left(s_i(\varepsilon)\right)\right]\Delta\tau_{-i}^{j+1}\frac{\left(\tau^{j+1}-\varepsilon\right)}{\left(\Delta\tau^{j+1}\right)^2}g(\varepsilon)d\varepsilon=0. \tag{22}$$

Note that τ^{j+1}, τ^j, τ_{-i}^{j+1} and τ_{-i}^j indirectly depend on Δp_j and that this dependence is given by $\left\{s_i^j\left(\Delta p_j\right)\right\}_{i=1}^N$ and $\left\{s_i^{j+1}\left(\Delta p_j\right)\right\}_{i=1}^N$, whereas the functions $\tau^{j-1}\left(\Delta p_j\right)$ and $\tau_{-i}^{j-1}\left(\Delta p_j\right)$ are unknown.

The first step in the application of the implicit function theorem is to fix a point for which (22) is satisfied for all firms. This is straightforward, because it is easy to show that $s_i^{j-1}=s_i^j$ is a solution to (22) when $\Delta p_j=0$. The next step is to prove that the Jacobian $\left(\frac{\partial\Gamma_i^j}{\partial s_i^{j-1}}\right)$ is invertible at this fixed point. By differentiating (22), it is straightforward to show that:

$$\frac{\partial\Gamma_i^j}{\partial\tau_{-i}^{j-1}}=\int_{\tau^{j-1}}^{\tau^j}C_i''\left(s_i(\varepsilon)\right)\frac{\Delta\tau_{-i}^j\left(\varepsilon-\tau^{j-1}\right)\left(\tau^j-\varepsilon\right)}{\left(\Delta\tau^j\right)^3}g(\varepsilon)d\varepsilon-\int_{\tau^{j-1}}^{\tau^j}\left[p_j-C_i'\left(s_i(\varepsilon)\right)\right]\frac{\left(\varepsilon-\tau^{j-1}\right)}{\left(\Delta\tau^j\right)^2}g(\varepsilon)d\varepsilon$$

Evaluating the integrals in the limit as we get closer to the fixed point yields

$$\lim_{s_i^{j-1}\to s_i^j}\frac{\partial\Gamma_i^j}{\partial\tau_{-i}^{j-1}}=-\frac{\left[p_j-C_i'\left(s_i^j\right)\right]g\left(\tau_i^j\right)}{2}<0. \tag{23}$$

Similarly,

$$\frac{\partial\Gamma_i^j}{\partial\tau^{j-1}}=\int_{\tau^{j-1}}^{\tau^j}\left[C_i''\left(s_i(\varepsilon)\right)\right]\frac{\left(\Delta\tau_{-i}^j\right)^2\left(\varepsilon-\tau^{j-1}\right)\left(\varepsilon-\tau^j\right)}{\left(\Delta\tau^j\right)^4}g(\varepsilon)d\varepsilon$$
$$+\int_{\tau^{j-1}}^{\tau^j}\left[p_j-C_i'\left(s_i(\varepsilon)\right)\right]\Delta\tau_{-i}^j\frac{2\varepsilon-\tau^{j-1}-\tau^j}{\left(\Delta\tau^j\right)^3}g(\varepsilon)d\varepsilon$$

and

$$\lim_{s_i^{j-1}\to s_i^j}\frac{\partial\Gamma_i^j}{\partial\tau^{j-1}}=0. \tag{24}$$

For convenience let $\alpha_i = \frac{\partial \Gamma_i^j}{\partial \tau^{j-1}}$ and $\beta_i = \frac{\partial \Gamma_i^j}{\partial \tau^{j-1}} + \frac{\partial \Gamma_i^j}{\partial \tau_{-i}^{j-1}}$. We know that $\frac{\partial \Gamma_j^j}{\partial s_i^{j-1}} = \frac{\partial \Gamma_i^j}{\partial \tau^{j-1}} = \alpha_{\mathrm{i}}$ and that $\frac{\partial \Gamma_i^j}{\partial s_k^{j-1}} = \frac{\partial \Gamma_i^j}{\partial \tau^{j-1}} + \frac{\partial \Gamma_i^j}{\partial \tau_{-i}^{j-1}} = \beta_{\mathrm{i}}\ \forall k \neq i$. Accordingly, it follows from (23) and (24) that at the fixed point when $\Delta p_j = 0$ we have that

$$\beta_i < \alpha_i \text{ and } \beta_i < 0. \tag{25}$$

The Jacobian matrix of the functions $\Gamma_1^j \ldots \Gamma_N^j$ is:

$$J_1 = \begin{bmatrix} \alpha_1 & \beta_1 & \ldots & \beta_1 \\ \beta_2 & \alpha_2 & \ldots & \beta_2 \\ \vdots & \vdots & & \vdots \\ \beta_N & \beta_N & \ldots & \alpha_N \end{bmatrix}.$$

To verify that the matrix is invertible, we want to prove that its determinant is non-zero. The non-zero property of the determinant is unaltered if we divide each row *i* by the factor $\beta_i < 0$.

$$J_2 = \begin{bmatrix} \alpha_1 / \beta_1 & 1 & \ldots & 1 \\ 1 & \alpha_2 / \beta_2 & \ldots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \ldots & \alpha_N / \beta_N \end{bmatrix}.$$

The determinant cannot acquire a (but may lose its) non-zero property if one row is replaced by a linear combination of the rows. In the next step, each row (except for the last row) is subtracted by the row below.

$$J_3 = \begin{bmatrix} \alpha_1 / \beta_1 - 1 & 1 - \alpha_2 / \beta_2 & 0 & \ldots & 0 \\ 0 & \alpha_2 / \beta_2 - 1 & 1 - \alpha_3 / \beta_3 & \ldots & 0 \\ 0 & 0 & \alpha_3 / \beta_3 - 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & \ldots & \alpha_N / \beta_N \end{bmatrix}$$

It can now be shown that $|J_3| = \sum_{j=1}^{N} \prod_{k \neq j} (\alpha_k / \beta_k - 1)$. By means of (25) we are now ready to conclude that $|J_3| \neq 0$. Consequently, we can also conclude that $|J_2| \neq 0$ and that

$|J_1| \neq 0$. Thus the Jacobian matrix of the functions $\Gamma_1^j \ldots \Gamma_N^j$ with respect to s_k^{j-1} is invertible at the fixed point. Moreover, as $\{s_i^j(\Delta p_j)\}_{i=1}^N$ and $\{s_i^{j+1}(\Delta p_j)\}_{i=1}^N$ are assumed to be differentiable, it is straightforward to verify that the functions $\Gamma_1^j \ldots \Gamma_N^j$ are continuously differentiable in Δp_j. Thus we can conclude from the Implicit Function Theorem that for sufficiently small Δp_j, there is a unique and differentiable solution to the discrete equation in Proposition 1 around the fixed point given by $s_i^{j-1} = s_i^j$ and $\Delta p_j = 0$.

In the final step, solutions not in the neighbourhood of the fixed point are ruled out for sufficiently small Δp_j. The property that $0 \leq s_i^{j+1} - s_i^j \leq K\Delta p_j$, for some finite constant K implies that the integral $\int_{\tau^{j-1}}^{\tau^j} \left[p_j - C_i'(s_i(\varepsilon)) \right] \frac{\Delta \tau_{-i}^j (\varepsilon - \tau^{j-1})}{(\Delta \tau^j)^2} g(\varepsilon) d\varepsilon$ must be of the order Δp_j if a vector $\{s_i^{j-1}\}_{i=1}^N$ is to satisfy the first-order condition in Proposition 1. Together with the assumption that marginal costs are non-decreasing and the constraint that $p_j - C_i'(s_i^j) \geq \delta > 0 \; \forall i = 1 \ldots N$, it follows that any solution vector $\{s_i^{j-1}\}_{i=1}^N$ must have the property that differences $s_i^j - s_i^{j-1}$ are of the order Δp_j, otherwise the first-order condition cannot be satisfied for each firm. Thus solutions not in the neighbourhood of the fixed point can be ruled out, ensuring a unique solution to the discrete equation in Proposition 1 for sufficiently small Δp_j. □

Lemma 4. *Under Assumption 2, the difference equation in Proposition* 1 *is consistent with the continuous equation in (7) if* $\{\widehat{s}_i(p)\}_{i=1}^N$ is bounded and

$p - C_i'(\widehat{s}_i(p)) \geq \delta > 0 \; \forall i = 1 \ldots N$.

Proof: A discrete approximation of an ordinary differential equation is consistent if the local truncation error is infinitesimally small when the step length is infinitesimally small (LeVeque, 2007). The local truncation error is the discrepancy between the continuous slope and its discrete estimate when discrete values s_i^j are replaced with samples of the continuous solution $\widehat{s}_i(p_j)$. The continuous first-order condition in (7) and the constraint $p - C_i'(\hat{s}_i(p)) \geq \delta > 0$, imply that $\left\{\hat{s}_i'(p)\right\}_{i=1}^{N}$ are bounded. Thus differences $\hat{s}_i(p_{j+1}) - \hat{s}_i(p_j)$ will be of the order Δp_j. Hence, we can use the Taylor approximation from Lemma **1** to approximate the first-order condition in (12):

$$-\Delta p_j s_i^j g(\tau^j) + \left[p_j - C_i'(s_i^j)\right] \Delta \tau_{-i}^j g(s_i^j + \tau_{-i}^j) + O(\Delta p_j^2) = 0.$$

Note that this approximation is valid for $j=H$-1 as well if assumption 2 is satisfied. We have assumed that g is bounded away from zero. Thus

$$-\Delta p_j s_i^j + \left[p_j - C_i'(s_i^j)\right](\Delta s_{-i}^j - \Delta d^j) + O(\Delta p_j^2) = 0. \tag{26}$$

This lemma considers prices for which mark-ups are bounded away from zero. Hence, (26) can be rewritten as:

$$\frac{-\Delta p_j s_i^j + O(\Delta p_j^2)}{p_j - C_i'(s_i^j)} + \Delta s_{-i}^j - \Delta d^j = 0. \tag{27}$$

Summing the corresponding expressions of all firms and then dividing by N-1 yields:

$$\Delta s^j - \frac{N}{N-1}\Delta d^j - \frac{1}{N-1}\sum_k \frac{\Delta p_j s_k^j + O(\Delta p_j^2)}{p_j - C_k'(s_k^j)} = 0. \tag{28}$$

By subtracting (27) from (28) followed by some rearrangements we obtain:

$$\frac{s_i^j - s_i^{j-1}}{\Delta p_j} = \frac{\Delta d^j}{\Delta p_j (N-1)} - \frac{s_i^j + O(\Delta p_j)}{p_j - C_i'(s_i^j)} + \frac{1}{N-1}\sum_k \frac{s_k^j + O(\Delta p_j)}{p_j - C_k'(s_k^j)}. \tag{29}$$

We know from the definition of the demand in the continuous system that $d'(p_j) = \lim_{\Delta p_j \to 0} \frac{\Delta d^j}{\Delta p_j}$. Hence,

$$\lim_{\Delta p_j \to 0} \frac{s_i^j - s_i^{j-1}}{\Delta p_j} = \frac{d'(p_j)}{N-1} - \frac{s_i^j}{p_j - C_i'(s_i^j)} + \frac{1}{N-1}\sum_k \frac{s_k^j}{p_j - C_k'(s_k^j)}. \tag{30}$$

It remains to show that if s_i^j and s_k^j in the right hand side of (30) are replaced by samples of the continuous solution $\hat{s}_i(p_j)$ and $\hat{s}_k(p_j)$ then the right hand side converges to $\hat{s}_i'(p_j)$. But this follows from (7). Thus the local truncation error is zero and we can conclude that the discrete system is a consistent approximation of the continuous system. □

We use this consistency property when proving convergence below. Recall that L and H are the lowest and highest price indices, j, such that price p_j occurs with positive probability, and varies with M (and the initial or boundary conditions).

Proposition 4 *Let* $\{\hat{s}_i(p)\}$ *be a solution on the interval* $[a, b]$ *that satisfies Assumption* 3. *Consider the discrete first-order system (in Proposition* 1) *with initial conditions* $\hat{s}_i^{H-1}$ *and* $\hat{s}_i^{H} = \bar{s}_i$ *for each i. If as* $M \to \infty$ *we have* $p_H \to b$ *and that* $\hat{s}_i^{H-1}$ *converges to* $k_i = \hat{s}_i(b)$ *in a way consistent with Assumption 2, then for sufficiently large M there exists a unique discrete solution* $\{\hat{s}_i^j\}_{i=1}^N$. *As the number of steps grows (M → ∞),* $\{\hat{s}_i^j\}_{i=1}^N$ *converges to* $\{\hat{s}_i(p)\}$ *in the interval* $[a, b]$.

Proof: Lemma **4** states that the discrete equation is a consistent approximation of the continuous equation. To show that the discrete solution converges to the continuous solution, we need to prove that the discrete solution exists and is stable, i.e. the error does not explode as the number of steps increases without limit. The proof is inspired by LeVeque's **(**2007) convergence proof for general one-step methods.

Define the vector of global errors at the price p_j, $\mathbf{E^j} = \mathbf{s^j} - \hat{\mathbf{s}}(\mathbf{p_j})$ and the corresponding vector for the local truncation error:

$$\upsilon_i^j = \frac{\hat{s}_i(p_{j+1}) - \hat{s}_i(p_j)}{\Delta p_j} - \hat{s}'_i(p_j).$$

It is useful to introduce a Lipschitz constant λ (LeVeque, 2007). Let it be some constant that satisfies the inequality[10]

$$\left\| \frac{p - C_i'(\hat{s}_i(p)) + \hat{s}_i(p) C_i''(\hat{s}_i(p))}{\left[p - C_i'(\hat{s}_i(p))\right]^2} \right\|_\infty + \frac{1}{N-1} \sum_k \left\| \frac{p - C_k'(\hat{s}_k(p)) + \hat{s}_k(p) C_k''(\hat{s}_k(p))}{\left[p - C_k'(\hat{s}_k(p))\right]^2} \right\|_\infty < \lambda, \forall p \in (a, b). \tag{31}$$

Such a Lipschitz constant exists since we have assumed the mark-up, which appears in the denominator of each fraction in (31), is bounded away from zero, the cost function is twice continuously differentiable, and the prices and corresponding strategy values are bounded. For sufficiently small Δp, λ puts a bound on the sensitivity of the vector s^{j-1} to small changes in the solution of the previous step. It is also useful to introduce another constant κ, such that

$$\frac{d'(p)}{N-1}+\left\|\frac{\widehat{s}_i(p)}{p-C_i'(\widehat{s}_i(p))}\right\|+\frac{1}{N-1}\sum_k\left\|\frac{\widehat{s}_k(p)}{p-C_k'(\widehat{s}_k(p))}\right\|<\kappa\ \ \forall p\in(a,b). \qquad (32)$$

The constant κ will bound the difference between the vectors s^j and s^{j-1}. Again we know that such a constant will exist, because the continuous solutions are bounded and mark-ups are bounded away from zero on the interval.

One problem with the highest price step is that differences $s_i^H - s_i^{H-1}$ are finite also for infinitesimally small Δp, which makes it problematic to use Lemma 3. But there is a way around this problem. Assumption 2 and Lemma 2 imply that $\bar{\varepsilon}-\hat{\tau}^{H-1}$ and $\int_{\tau^{H-1}}^{\bar{\varepsilon}}\left[p_H - C_i'(s_i(\varepsilon))\right]\Delta\tau_{-i}^H\left(\frac{\Delta\tau^H-(\varepsilon-\tau^{H-1})}{(\Delta\tau^H)^2}\right)g(\varepsilon)d\varepsilon$ are both of the order Δp. It is straightforward to verify that this condition can replace the condition that differences $s_i^H - s_i^{H-1}$ are of the order Δp in Lemma 3. Thus it follows that $\{s_i^{H-2}\}_{i=1}^N$ can be uniquely determined if Δp is sufficiently small. For sufficiently small Δp, it now follows from (29) and (31) that the global error satisfies the following inequality:

$$\begin{aligned}\left\|\mathbf{E^{H-2}}\right\|_\infty &= \left\|\mathbf{s^{H-2}}-\widehat{\mathbf{s}}(\mathbf{p_{H-2}})\right\|_\infty \le \left\|\mathbf{E^{H-1}}\right\|_\infty + \lambda\Delta p\left\|\mathbf{s^{H-1}}-\widehat{\mathbf{s}}(\mathbf{p_{H-1}})\right\|_\infty + \Delta p\left\|\boldsymbol{\upsilon}^{\mathbf{H-1}}\right\|_\infty = \\ &= (1+\lambda\Delta p)\left\|\mathbf{E^{H-1}}\right\|_\infty + \Delta p\left\|\boldsymbol{\upsilon}^{\mathbf{H-1}}\right\|_\infty .\end{aligned}$$

Thus if Δp is sufficiently small, so that the initial error $\left\|\mathbf{E^{H-1}}\right\|_\infty$ and the local truncation error $\left\|\boldsymbol{\upsilon}^{\mathbf{H-1}}\right\|$ are small enough, then $\left\|\mathbf{E^{H-2}}\right\|_\infty$ is sufficiently small. It now follows from the assumed properties of the continuous solution that $s_i^{H-1}-s_i^{H-2}\ge 0$ and that $p_{H-2}-C_i'(s_i^{H-2})\ge\delta>0\ \forall i=1\ldots N$. We know from (29) and (32) that

[10] Note that $\|\cdot\|_\infty$ is the max-norm, i.e. $\left\|E^j\right\|_\infty = \max_{1\le i\le N}\left|E_i^j\right|$ (LeVeque, 2007).

$s_i^{H-1} - s_i^{H-2} \le \kappa \Delta p$. Thus if Δp is sufficiently small, then the argument for the vector $\mathbf{s^{H-2}}$ can be repeated iteratively to prove that the vector $\mathbf{s^k}$ $\forall k = L,...,H-3$ can be uniquely determined and that

$$\left\|\mathbf{E^k}\right\|_\infty = \left\|\mathbf{s^k} - \hat{\mathbf{s}}(\mathbf{p_k})\right\|_\infty < (1+\lambda\Delta p)\left\|\mathbf{E^{k+1}}\right\|_\infty + \Delta p\left\|\boldsymbol{\upsilon}^{\mathbf{k+1}}\right\|_\infty . \quad (33)$$

Let $\upsilon_{\max}^k = \max\left\{\left\|\boldsymbol{\upsilon}^{\mathbf{n}}\right\|_\infty\right\}_{n=k}^{H-1}$. From the inequality in (33), we can show by induction that

$$\begin{aligned} &\left\|\mathbf{E^k}\right\|_\infty \le (1+\lambda\Delta p)^{H-1-k}\left\|\mathbf{E^{H-1}}\right\|_\infty + \Delta p \sum_{m=k+1}^{H-1}\left\|\boldsymbol{\upsilon}^{\mathbf{m}}\right\|_\infty (1+\lambda\Delta p)^{H-1-m} < \\ &(1+\lambda\Delta p)^{H-1-k}\left(\left\|\mathbf{E^{H-1}}\right\|_\infty + (H-k-1)\Delta p\,\upsilon_{\max}^{k+1}\right) \le (1+\lambda\Delta p)^{H-1-k}\left(\left\|\mathbf{E^{H-1}}\right\|_\infty + (H-L)\Delta p\,\upsilon_{\max}^{L}\right). \end{aligned} \quad (34)$$

In the limit as $\Delta p \to 0$ then $\left\|\mathbf{E^{H-1}}\right\|_\infty \to 0$, $\upsilon_{\max}^{L+1} \to 0$ (because of Lemma **4**), $(H-L)\Delta p \to b-a$, and $(1+\lambda\Delta p)^{H-L} \to e^{\lambda(b-a)}$. Thus from (34) $\left\|\mathbf{E^k}\right\|_\infty \to 0$ when $\Delta p \to 0$, proving that the discrete solution converges to the continuous one. □

Given the results of Propositions 2-4 we are now ready to prove Theorem 1:

Proof of Theorem 1: Part (a) is a restatement of Proposition **3**. To show part (b), given Propositions **4** and **2**, it is sufficient to verify that Assumption 1a holds and to show that each player's discrete strategy – that exists and is convergent to the continuous strategy – satisfies $0 \le s_i^{j+1} - s_i^j \le W\Delta p$. Convergence implies that $\frac{s_{j+1}^i - s_j^i}{\Delta p}$ converges to $\hat{s}_i'(p)$ *if* $p_j \to p$, uniformly in [*a,b*]. We know that $\hat{s}_i'(p)$ is bounded (due to the positive mark-up assumption) and strictly positive, uniformly in [*a,b*]. Thus $0 \le s_i^{j+1} - s_i^j \le W\Delta p$. Convergence of competitors' supply curves implies that the difference between a producer's profits in the discrete and continuous system will converge to zero, and this is also true for all possible deviations of the producer. Hence, in the limit, assumption 1a is satisfied if assumption 1b is satisfied. □

The result below ensures that whenever a discrete equilibrium exists in the limit, when the number of steps becomes arbitrarily large, then there always exists a corresponding continuous equilibrium. This reverses the implication of Theorem 1, and thereby establishes that the family of discrete NE is, asymptotically, in one-to-one correspondence with the family of continuous equilibria.

Proposition 5. *Assume for a sufficiently large number of steps M that there exists a discrete solution* $\{\widehat{s}_i^{\,j}\}_{i=1}^N$ *that is a segment of a discrete SFE. The solution satisfies Assumption* 1*a, Assumption* 2 *and the inequality* $0 \le s_i^{j+1} - s_i^j \le W\Delta p$ *(where W is some positive constant). Moreover it is stable, so that it converges to a set of continuous functions* $\{\widetilde{s}_i(p)\}_{i=1}^N$ *on* [*a, b*], where $a = \lim_{\Delta p \to 0} p_L$ and $b = \lim_{\Delta p \to 0} p_H$. *Then* $\{\widetilde{s}_i(p)\}_{i=1}^N$ *is a segment of a continuous SFE, if* $\{\widetilde{s}_i(p)\}_{i=1}^N$ *is increasing in the interval* [*a, b*] *and if it satisfies the property* $p - C_i'(\widetilde{s}_i(p)) \ge \delta > 0 \; \forall i = 1 \ldots N$ *in this interval.*

Proof: Given the assumed properties of $\{\widetilde{s}_i(p)\}_{i=1}^N$ and that $\{\widetilde{s}_i(p)\}_{i=1}^N$ satisfies the discrete first-order condition in Proposition 1 when $\Delta p \to 0$, it follows from Assumption 2 and Lemma **4** that $\{\widetilde{s}_i(p)\}_{i=1}^N$ will satisfy the continuous first-order condition in (6). Convergence of competitors' supply curves implies that the difference between a producer's profits in the discrete and continuous system will converge to zero, and this is also true for all possible deviations of the producer. Hence, in the limit, assumption 1b is satisfied if assumption 1a is satisfied. As $\{\widetilde{s}_i(p)\}_{i=1}^N$ is a set of increasing functions in the interval [*a, b*], it now follows from Proposition **3** that $\{\widetilde{s}_i(p)\}_{i=1}^N$ is a segment of a continuous SFE. □