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THE WAGE POLICY ON A FIRM WHEN RECRUITMENT IS A WAGE DEPENDENT POISSON PROCESS AND WAGES ARE DOWNWARD RIGID

by

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ABSTRACT AND HEADNOTE

The paper contains an analysis of a firm's optimal wage and recruitment policy, when the dynamic labor supply to the firm is specified as a Poisson-distributed flow of job-applicants, whose reservation wages are distributed in a fairly general way.Provided that the firm is not allowed to lower its wage level over time, it has recently been shown by the author that the optimal policy is to establish a constant wage level over future employment states. The paper explores the further properties of the optimal wage and its response to changes in the parameters of the decision problem.

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Introduction

Since the publishing of the articles of Mortensen (1971) and Phelps (1971) there has been a growing number of studies, which analyze firm behavior in a labor market environment based on the assumption that job applicants are imperfectly informed about the exact location of job and wage offers. In such a situation job applicants are engaged in search for favorable offers and the outcome of the search is subject to uncertainty. This fact is recognized in the search literature which focuses on the behavior of the job searcher (often identified with the unemployed). Clearly, the search activity will also result in realizations of employment events, as experienced by a recruiting firm, that are random.

However, almost all studies that pay attention to the demand for labor in a job search framework are content just to use the implication that the supply of labor to the firm is dynamically wage dependent. They ignore the random character of the employment process and assume that the firm acts as if the flows of hires (and quits) are deterministically dependent on its offered wage level. Consequently they derive the optimal policy of the firm by using deterministic optimal control methods. Important contributions within this tradition are besides the seminal ones just mentioned Salop (1973), Pissarides (1976), Siven (1979), Virén (1979) and Leban (1982a,b). One exception to the rule is a joint article by Eaton and Watts (1977), in which the stochastic nature of the labor supply to the firm is explicitly considered. The structure of their model was too complicated to allow any analytical characterization of its solution, but simulation highlighted some interesting properties of its equilibrium state.

To my knowledge, there has as yet been no attempt to apply the established theory of stochastic processes and their control to derive analytical properties of a firm's wage and employment policy. Nevertheless it is a straightforward procedure to reformulate the labor flows to and from the firm in terms of such birth-and-death-processes that typically constitute a queueing system. So a relevant alternative to the deterministic approach is to formulate a model, where hires and quits of employees are treated as equivalent to arrivals and departures in an queueing model, the intensities of which are dependent on the firm's wage level.

This paper is a first attempt to formulate such a model and to analyze the properties of its optimal solution. In at least one respect the model is too simple: there exist only hires but no quits or other separations of employees. The remaining assumptions - to be presented shortly - have indeed a simplifying character, but not to an extent that seems to endanger the relevance of the model. One deficiency is admittedly the assumption of a completely static environment; it would be worth while to treat at least a part of it as time dependent.

One crucial feature of the model deserves to be mentioned at this introductory stage: we assume that the firm can never lower its (money) wage over time. The standard (deterministic) models do not include this restriction and as a consequence they produce a declining optimal wage along an employment expansion path, a not very satisfactory representation of observed wage behaviour as pointed out by Virén (1979) and Leban (1982a,b). The assumption of downward wage rigidity will have important implications for the structure of the optimal wage policy.

Although there is no precedence to the present analysis as applied to a firm regulating its recruitment flow, there have been some applications within the field of optimal control of queues, where prices are used as a control; Low (1974) is perhaps the best known example. In the present context it is worth drawing attention to a series of articles by Deshmukh and Chitke (1976), Deshmukh and Winston (1979) and Lippman (1980) which apply models with a formal resemblance to the present one, to the problem of an industry, organized as a price cartel, regulating the flow of entry of new competitors. The feasibility region of the price/wage control as well as the dependence of the arrival rate on the control are features that distinguish these models from the present one.

The paper is organized as follows: First we present the basic assumptions of the model. Then the structure of the optimal policy as well as the closed form solution of the functional equation of the decision problem are presented as they have been recently derived by the author (Schager (1986)). In the next section the analytical properties of the value function is studied in order to establish further characteristics of the optimal wage policy. A short discussion of the dynamics of the decision process follows and in the next section comparative statics effects on the optimal wage policy are analyzed. The paper ends with a summary and some comments about the prospects of the queue control approach in future search theory research.

Basic Assumptions

The firm in the model is a one-product production unit, where labor (measured in number of employees) is the only variable factor of production. The production function is linear up to a fixed capacity limit. The market for the firm's product is competitive, i.e., the firm is a price-taker on the product market (but the capacity limit may alternatively be interpreted as a sales constraint on the firm at the given price).

The stock of employees at the firm will never decrease but the firm can aim at increasing it by creating job vacancies with an offered wage that must be paid to all employees alike. Such job and wage offers are announced in a labor market in which there exists a stock of potential employees, who are qualified for a job at the firm and constantly searching for a favorable wage offer. The contact process between these job-applicants and the firm can be seen as a Poisson process if the number of applicants is large and if there is also a large number of other firms which attract their attention.

According to results from labor market search literature, each job-applicant should under fairly general conditions apply a search strategy, involving the calculation of a 'reservation' wage such that a job offer should be accepted if and only if the corresponding wage exceeds the reservation wage (see Zuckerman (1983) and references there). Let the distribution function of reservation wages over all job applicants be $F(\cdot)$ and denote the intensity of the contact process γ. Then the recruitment process will also be Poisson with the intensity $\gamma \cdot F(w)$, if the firm pays the wage w.

We will return to the possible properties of $F(\cdot)$ later on, as they are of importance when we want to establish more specific characteristics of the firm's optimal wage policy. For the result reported in the next section we only need to assume that the corresponding probability density function $f(\cdot)$ is continuous.

All these assumptions amount to establish the equivalence to a queuing model with exponentially distributed interarrival times with a mean equal to $[\gamma \cdot F(w)]^{-1}$ and a reward rate function equal to $r(i,w) = (pa-w)i$, p being the product price, a the constant productivity per employee, i the number of employees, $i \leqslant N$, N being the capacity limit.

The Decision Problem and the Structure of Its Solution

We assume that the decision criterion of the firm is total expected discounted profits, calculated over an infinite horizon (during which p, a, γ, etc. are expected to remain fixed). Which wage policy, non-decreasing over time, will lead to a maximization of this criterion?

By applying a dynamic programming technique to this decision problem which is in fact a Markov decision process in two states (the number of employees and the wage level) and one non-negative control (the wage increase), it is shown in Schager (1986) that the optimal wage policy is to choose one wage level at the decision instant and keep it fixed for all future. Furthermore, if at this instant the number of employees is n and the wage level w_n, the optimal wage level is the solution to

$$V^*(n,w_n) = \max_{w \geqslant w_n}\{V(n,w)\}$$

$$V(n,w) = \frac{(pa-w)}{\alpha}\left\{n + \frac{\lambda}{\alpha}\left[1 - \left(\frac{\lambda}{\lambda+\alpha}\right)^{N-n}\right]\right\}, \qquad (1)$$

$\lambda = \gamma \cdot F(w)$; α is the discount intensity.

As soon as the optimality of a constant wage level has been established, the derivation of V(n,w) is simple enough. V(i,w) must obey the functional equation

$$V(i,w) = \frac{(pa-w)i}{\lambda+\alpha} + \frac{\lambda}{\lambda+\alpha}V(i+1,w) \qquad i=n,\ldots,N-1$$

with the terminal condition

$$V(N,w) = \frac{(pa-w)N}{\alpha}$$

as the firm in employment state N will earn a profit rate (pa-w)N forever. The solution to the functional equation is easily found by standard methods of linear difference equations.

The Properties of the Value Function

The search for the optimal wage policy has been reduced to the problem of finding the wage level $w^* \geqslant w_n$ which maximizes V(n,w). The intuitive meaning of (1) is not transparent as it was very concisely derived from the functional equation, which in turn is based on theorems on continuous time Markov decision processes. Let us try to interpret the expression for V(n,w) and first observe that by the formula for a geometric series we have

$$\begin{aligned} V(n,w) &= \frac{pa-w}{\alpha}\left\{n + \frac{\lambda}{\alpha}\left[1 - \left(\frac{\lambda}{\lambda+\alpha}\right)^{N-n}\right]\right\} = \\ &= \frac{pa-w}{\alpha}\left[n + \sum_{i=1}^{N-n}\left(\frac{\lambda}{\lambda+\alpha}\right)^{i}\right] \end{aligned} \qquad (2)$$

$\frac{\lambda}{\lambda+\alpha}$ can be interpreted as the expected discount factor, when the elapsed time period is exponentially distributed with mean λ^{-1}. (To see this, note that the discount factor is $e^{-\alpha t}$, when the time period is t. As t has the pdf $\lambda \cdot e^{-\lambda t}$ on $[0,\infty]$, the interpretation follows immediately.) As now every employee will earn a total discounted profit of $\frac{pa-w}{\alpha}$, calculated from the point of time he is hired, the meaning of (2) is quite clear. The n employees, who are already in place, will contribu-

te $\frac{pa-w}{\alpha}$ each. The first new employee will be hired after a time period, which is exponentially distributed with mean λ^{-1} and then contribute $\frac{pa-w}{\alpha}$. The expected value of his contribution is then $\frac{\lambda}{\lambda+\alpha} \cdot \frac{pa-w}{\alpha}$. The second new employee will be hired after two such time periods and hence the expected value of his contribution is $(\frac{\lambda}{\lambda+\alpha})^2 \cdot \frac{pa-w}{\alpha}$ and so on up to the (N-n):th and last new employee with an expected contribution of $(\frac{\lambda}{\lambda+\alpha})^{N-n} \cdot \frac{pa-w}{\alpha}$.
V(n,w) is simply the sum of all these expected contributions to total discounted profits.

As the properties of the series $\sum_{i=1}^{m} (\frac{\lambda}{\lambda+\alpha})^i$ are of crucial importance for the behavior of V(n,w) we present the following relations concerning $G_m(\lambda) = \sum_{i=1}^{m} (\frac{\lambda}{\lambda+\alpha})^i$ (without proofs as they are easily verified).

$$G_m(\lambda) < m; \qquad \lim_{m\to\infty} G_m(\lambda) = \frac{\lambda}{\alpha}$$

$$0 < G_m(\lambda) - G_{m-1}(\lambda) = (\frac{\lambda}{\lambda+\alpha})^m < 1$$

$$-1 < G_{m+1}(\lambda) - G_m(\lambda) - [G_m(\lambda)-G_{m-1}(\lambda)] =$$

$$= -\frac{\alpha}{\lambda+\alpha} \cdot (\frac{\lambda}{\lambda+\alpha})^m < 0.$$

Thus $G_m(\lambda)$ is bounded, strictly increasing and strictly concave in m (for $\lambda > 0$). It also holds that $G_m(\lambda)$ is strictly increasing and strictly concave in λ. We have

$$G_m'(\lambda) = \frac{\alpha}{\lambda(\lambda+\alpha)} \sum_{i=1}^{m} i(\frac{\lambda}{\lambda+\alpha})^i = \frac{1}{\lambda}[G_m(\lambda) - H_m(\lambda)] > 0,$$

$$H_m(\lambda) = m(\frac{\lambda}{\lambda+\alpha})^{m+1}$$

$$G_m''(\lambda) = -\frac{(m+1)m}{\lambda^2} \cdot \frac{\alpha}{\lambda+\alpha} (\frac{\lambda}{\lambda+\alpha})^{m+1} < 0$$

$$\lim_{\lambda\to\infty} G_m(\lambda) = m \qquad G_m'(0) = \frac{1}{\alpha}$$

Equipped with these results we return to

$$V(n,w) = \frac{pa-w}{\alpha}[n + G_{N-n}(\lambda)]$$

and observe that by virtue of the properties of $G_m(\lambda)$ with respect to m, V(n,w) is increasing and concave in n.

We form

$$\alpha \cdot \frac{dV(n,w)}{dw} = \alpha \cdot V'(n,w) =$$

$$= -[n + G_{N-n}(\lambda)] + (pa-w) \cdot \lambda' \cdot G_{N-n}'(\lambda), \tag{3}$$

where $\lambda' = \gamma \cdot f(w) \geqslant 0$.

V'(n,w) is strictly decreasing in n, because $n + G_{N-n}(\lambda)$ is strictly increasing in n and $G_{N-(n+1)}' - G_{N-n}' = -\frac{\alpha}{\lambda(\lambda+\alpha)} \cdot (N-n)(\frac{\lambda}{\lambda+\alpha})^{N-n} < 0$. This property of V'(n,w) is essential for demonstrating the optimality of an unchanged wage level over future employment states (Schager (1986)).

At this stage of the analysis we must consider the properties of $\lambda(w) = \gamma \cdot F(w)$, i.e., the properties of the reservation wage distribution. Up to now we have just assumed that F(·) is defined over some interval $[\underline{v}, \bar{v}]$, $\underline{v} \geqslant 0$, $\bar{v} \leqslant \infty$, in which f(·)

is continuous. To this we will now add the weak assumptions that $f(\cdot)$ is strictly positive in the interior of this interval and differentiable with $f'(\cdot)$ continuous, so that we can form

$$\alpha \cdot V''(n,w) =$$

$$= G'_{N-n}(\lambda) \cdot [(pa-w)\lambda'' - 2\lambda'] + (pa-w) \cdot \lambda' \cdot G''_{N-n}(\lambda) \cdot \lambda', \tag{4}$$

$$\lambda'' = \gamma \cdot f'(w).$$

It is clear that if $f'(\cdot) \leq 0$ everywhere, i.e., if $f(\cdot)$ is non-increasing in its argument, $V''(n,w)$ will be strictly negative for every w, $\underline{v} \leq w < \min\{\bar{v}, pa\}$. (It should be obvious why $w \geq pa$ or $w > \bar{v}$ can never be an optimal choice.) $V'(n,w)$ will thus be strictly falling in w and reach its highest value at $w = \underline{v}$. If $V'(n,\underline{v}) > 0$ we know that $V(n,w)$ reaches its maximum in the interior of $[\underline{v}, pa]$ at the unique $w = w_{max}$ where $V'(n,w) = 0$; if $V'(n,\underline{v}) \leq 0$ the maximum of $V(n,w)$ is attained at $w = \underline{v}$.

The conclusions in the preceding paragraph apply when w may vary within the whole interval $[\underline{v}, \infty]$. But w is not allowed to be chosen below w_n. There is a possibility (although not too relevant), that $w_n < \underline{v}$, but the complications that arise in such a case are more satisfactorily handled by changing the assumptions about $F(\cdot)$ as we shall shortly do. If we suppose that the condition $w_n \geq \underline{v}$ is fullfilled, we can identify the following cases of the character of the optimal wage w^*, when $f'(\cdot) \leq 0$ everywhere:

$V'(n,\underline{v}) \leq 0 \quad \Rightarrow \quad w^* = w_n$
$V'(n,\underline{v}) > 0;\ w_n < w_{max} \Rightarrow w^* = w_{max}$
$V'(n,\underline{v}) > 0;\ w_n \geq w_{max} \Rightarrow w^* = w_n$

However, the assumption that the probability density function $f(\cdot)$ of the reservation wage distribution is falling in its argument is not very satisfactory. Little is known of such distributions in practice but the scanty evidence supports the conjecture that reservation wages will be distributed according to an unimodal pdf, presumably reaching its peak rather rapidly and tapering off at higher wage levels (see Lancaster-Chesher (1983)). Such distributions are often successfully represented by a gamma or a lognormal distribution.

While keeping the analysis on a general level but avoiding unnecessary complications, we will add the following assumptions about $f(\cdot)$: $f(\cdot)$ is defined on $[\underline{v},\infty]$, strictly increasing on $[\underline{v},\tilde{v}]$, and strictly falling on $[\tilde{v},\infty]$. $f(\cdot)$ approaches $\underline{v}$ 'smoothly', i.e., $f(\underline{v}) = f'(\underline{v}) = 0$ and $\lim_{v\to\underline{v}} f(v)/f'(v) = 0$. We sacrifice nothing essential by also assuming that $w_n \geq \underline{v}$.

Let us return to $V''(n,w)$ as given by (4). It does not any longer hold that the term $(pa-w)\lambda'' - 2\lambda' < 0$ for all $w > \underline{v}$. On the contrary it must be positive (and increasing) for w in a neighborhood of $\underline{v}$. As the term is negative at $w = pa$, it must be zero at least at one point on $[\underline{v},pa]$. While it is possible to proceed without specifying the (uneven) number of zeros, the analysis will be more tractable if we can limit our attention to the case where only one zero exists. If $F(\cdot)$ can be assumed to be one of the standard distributions of statistical theory, in fact it holds that $(pa-w)\lambda'' - 2\lambda'$ has one unique zero on $[\underline{v},pa]$.

To show this we write

$$(pa-w) \cdot \lambda'' - 2\lambda' = \gamma \cdot f(w) \left[(pa-w) \frac{f'(w)}{f(w)} - 2 \right]$$

The ratio $\frac{f'(w)}{f(w)}$ is the fundamental criterion in the Pearson system of distributions; if $F(\cdot)$ belongs to this system, we have, transfering the origin to the start of the distribution (and hence with a slight abuse of notation)

$$\frac{f'(x)}{f(x)} = \frac{\tilde{x} - x}{x(b+cx)}; \; x = w - \underline{v}; \; \tilde{x} = \tilde{v} - v; \; b \geqslant 0; \; c \geqslant 0$$

(b and c not both = 0).

If $c = 0$, $F(\cdot)$ is the gamma-distriution; if $c > 0$, $F(\cdot)$ is the very flexible beta-distribution (see Kendall and Stuart (1977)).

Consequently, we get

$$(pa-w)\lambda'' - 2\lambda' = \frac{\gamma \cdot f(x)}{x(b+cx)} \left[(d-x)(\tilde{x}-x) - 2x(b+cx) \right],$$

$d = pa-\underline{v}$.

The sign of this expression is determined by the polynomial of second degree in x within brackets, which clearly has one and one zero only on $[0,d]$.

The lognormal distribution does not belong to the Pearson system, but here (again transfering to the origin) it holds that

$$\frac{f'(x)}{f(x)} = \frac{\log \tilde{x} - \log x}{b \cdot x}, \qquad b > 0$$

and substitution into $(pa-w)\lambda'' - 2\lambda'$ yields the same result that there exists one and only one zero for $x \in [0,d]$ or $w \in [\underline{v},pa]$.

So we conclude that we suffer little loss in generality by assuming that $(pa-w)\lambda'' - 2\lambda'$ will be positive for all w on $[\underline{v},\tilde{w}]$, zero for $w = \tilde{w}$ and negative for all w on $[\tilde{w},pa]$ in the case where reservation wages are distributed according to an unimodal pdf.

As a consequence of the behavior of $(pa-w)\lambda'' - 2\lambda'$, we conclude that as w increases

$$(pa-w)\lambda' - \lambda = \int_{\underline{v}}^{w} [(pa-v)\lambda''(v) - 2\lambda'(v)]dv$$

is first increasing from zero up to a maximum, then decreasing to zero and finally increasingly negative up to $w = pa$.

We can rewrite (3) as

$$\alpha \cdot V'(n,w) = \frac{G_{N-n}(\lambda)}{\lambda}\{(pa-w)\cdot\lambda'\cdot[1-K_{N-n}(\lambda)] - \lambda\} - n \qquad (5)$$

where K_{N-n} is defined as

$$K_{N-n}(\lambda) = \frac{H_{N-n}(\lambda)}{G_{N-n}(\lambda)} = \frac{N-n}{\sum_{i=1}^{N-n}(1+\frac{\alpha}{\lambda})^i}$$

so that $0 < K_{N-n}(\lambda) < 1$ and monotonically increasing from zero as w increases from $\underline{v}$.

The factor $\frac{G_{N-n}(\lambda)}{\lambda}$ will just have the effect of reducing the value of the expression within curly brackets as w increases; the factor is equal to $1/\alpha$ at $\underline{v}$ and it decreases monotonically at the rate $-H_{N-n}/\lambda^2$.

The expression within curly brackets determines the qualitative behavior of $\alpha.V'(n,w)$ and we see that it is essentially the same as that of

$(pa-w)\lambda' - \lambda$. The only difference is that $(pa-w)\lambda'$ is reduced by a fraction K_{N-n}, which is increasing in w, so that the subinterval on $[\underline{v}, pa]$ where the expression is non-negative is smaller compared to $(pa-w)\lambda' - \lambda$. Consequently it will have its zero at a lower value of w.

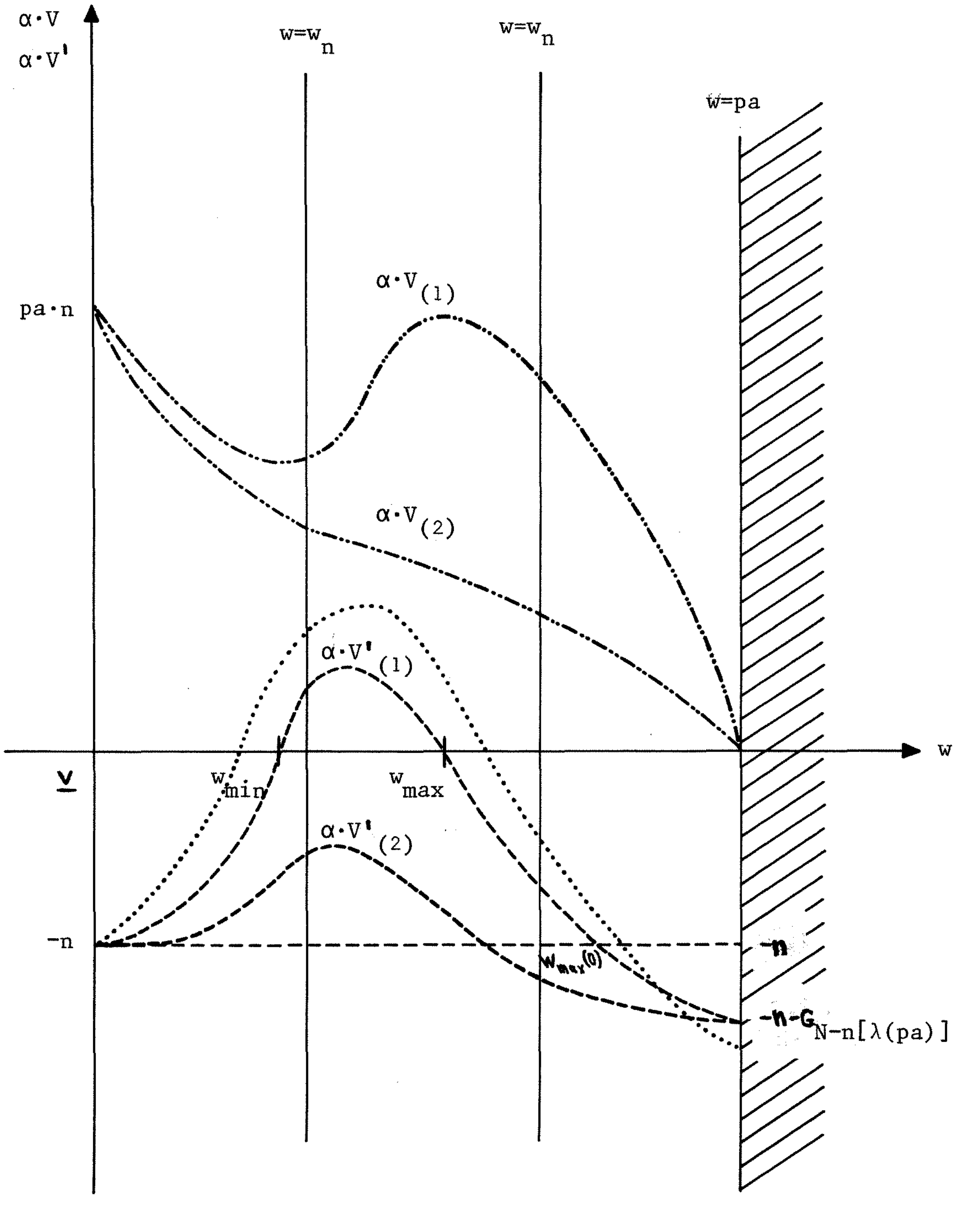
The fact that the expression $(pa-w)\lambda' - \lambda$ plays such an important role for the behavior of $V'(n,w)$ is of course no coincidence. The reader can easily verify that $\lim_{N\to\infty} V'(n,w) = \frac{1}{\alpha}[(pa-w)\lambda' - \lambda] - n$ so that any modification of this expression is just a result of the limited capacity assumption.

The results of the analysis of the case with an unimodal reservation wage distribution are summarized in the diagram. If we first look at $\alpha \cdot V'(n,w)$ with $n=0$, the situation is fairly straightforward. In the diagram $\alpha \cdot V'(0,w)$ is visualized by the $\alpha \cdot V'$-curve with the $-n$-line regarded as the w-axis. $\alpha \cdot V'(0,w)$ will start from zero and increase to a maximum, fall to zero and remain negative afterwards as w goes from zero to pa. At the only interior zero value, $w_{max}(0)$, of $V'(0,w)$, $V(0,w)$ attains its global maximum on $[\underline{v}, pa]$.

If we let $n>0$, this corresponds to a downward shift in $\alpha \cdot V'(0,w)$ by the amount $-n$. There will at most exist one local maximum in the interior of $[\underline{v}, pa]$; this situation is represented by the $\alpha \cdot V'_{(1)}$-curve in the diagram. However, for n large enough $V'(n,w)$ will remain negative on the whole interval $[\underline{v}, pa]$ and no interior maximum obtains; the $\alpha \cdot V'_{(2)}$-curve is drawn to represent this case.

So far the situation is similar to the former one where $\lambda''(w)$ was assumed non-positive for all w. But now there is no need for a local maximum,

Diagram



····· indicates the curve representing $\frac{1}{\alpha}[(pa-w)\lambda'-\lambda]-n$

situated at $w = w_{max}$, to constitute a global maximum, even if $w_{max} > w_n$. If a w_{max} exists, so does a w_{min}, and $V(n,w)$ will be falling on $[\underline{v}, w_{min}]$ and increasing on $[w_{min}, w_{max}]$ to the result that a $w_n \in [\underline{v}, w_{min}]$ might very well yield a global maximum on $[w_n, pa]$.

To summarize we will have the optimal wage $w^* = w_n$ not only when $V'(n,w) < 0$ on $[w_n, pa]$ (which in turn occurs when no w_{max} exists or when $w_n > w_{max}$). We also have $w^* = w_n < w_{max}$ when

$$\int_{w_n}^{w_{max}} V'(n,v)dv = \int_{w_n}^{w_{min}} V'(n,v)dv + \int_{w_{min}}^{w_{max}} V'(n,v)dv < 0.$$

The reversed inequality is necessary and sufficient to establish that $w^* = w_{max}$. It is bound to hold when w_n is located on $[w_{min}, w_{max}]$.

The Dynamics of the Decision Process

We have shown that the solution to the firm's decision problem is to choose a wage level which is unique and keep it fixed for all future time. The optimal wage w^* is unambiguously determined by n and w_n and the decision process is thus a Markov process with states defined by n and w_n.

This Markov decision process has simple properties. The intensity of the process is constant and equal to $\lambda(w^*)$, irrespective of future realized employment states. Hence the number of new employees $i=0,1,\ldots,N-n-1$ during a period of time $[0,t]$ will be Poisson-distributed with parameter $\lambda(w^*) \cdot t$; as the process stops when $i=N-n$, the capacity limit will be reached during $[0,t]$ with a probability equal to the cumulated Poisson probabilities from N-n to infinity.

In the preceding analysis w_n was introduced as an arbitrary initial condition at the decision instant. However, w_n is the result of earlier wage decisions. As a matter of fact, if w_n is not optimal when employment state n is entered the optimal policy rule requires that the firm makes an immediate jump to the higher wage level w^*. In other words, if a state $[n,w_n]$ is to exist more than momentarily, $w_n = w^*(n,w_n)$.

Consequently, if the firm has lived under static conditions with respect to the parameters of the decision problem during a time period $[0,t]$ since it entered employment state n the following holds. With a probability equal to $e^{-\lambda(w^*)t}$ the firm is still in employment state n; the wage level $w^*(n,w_n) = w_n$ might have been established by a strictly positive wage increase at the entrance of that employment state, in which case $V'(n,w_n) = 0$.

With probabilities, the sum of which is equal to $1-e^{-\lambda(w^*)t}$, the firm will be in employment states $n + i$, $i = 1,\dots,N-n$; the wage level is still $w^*(n,w_n) = w_n$, but it must hold that $V'(i,w_n) < 0$ and that $V'(i+1,w_n) < V'(i,w_n)$.

As every point of time is conceptually a decision instant to the firm, we see that for a firm that has recruited under static conditions the solution to $\max_{w \geq w_i} \{V(i,w)\}$ is always to be found at $w^*(i,w_i) = w_i$

with $V'(i,w^*) < 0$. The situation is easily visualized in the diagram as the aquisition of a new employee corresponds to a shift downwards of the V'-curve with one unit.

It is an important property of our model that a successfully recruiting firm will find itself in 'disequilibrium' with respect to its wage level and increasingly so as recruitment proceeds. This implication of downward wage rigidity merits more attention, as it should have considerable impact on observed wage dynamics. Clearly, the comparative statics results of the present model, to which we now turn, is not insensitive to the fact that $V'[i,w^*(i,w_i)]$ may be strongly negative in the initial position.

Comparative Statics Results

We will in this section analyze how changes in the different parameters in $V(n,w)$ will affect the value of w^* and $V(n,w^*) = V^*(n)$. "Changes" are here to be interpreted in the comparative static sense, i.e. we look at the effects when a static parameter takes on a new value.

We have already heavily exploited the effects of changes in n, the number of employees at the decision instant. For the sake of completeness, we repeat that $V'(n,w)$ is monotonically declining in n and hence w^* is non-increasing in n. Clearly $V^*(n)$ is increasing in n.

Changes in capacity limit

A change in the capacity limit (or in the sales constraint) N will produce a change of the same sign in $\alpha.V'(n,w)+n$ in the interval where it is non-negative. This is obvious from (5) and the fact that K_{N-n} is decreasing in N.

So $V'(n,w|N+1) - V'(n,w|N) > 0$ in the region $[\underline{v}, w_{max}(0)]$ and this is sufficient to establish that w* is non-decreasing in N. A larger N will also mean an increase in V*(n), as is immediately obvious from (2).

Changes in value added

A change in value added per employee, pa, i.e. a change in product price or in labor productivity, obviously produces a change in V(n,w) and V'(n,w) in the same direction on the whole interval $[\underline{v}, pa]$. Thus we have V*(n) increasing and w* non-decreasing in pa. *

It is worth noting that an increase in value added, which is bound to increase the present value V*(n) of the production opportunities, may have quite different effects on the optimal profit margin pa-w*, depending on the character of the optimal solution. To illustrate this point - which is of relevance when large changes in other parameters are considered - let us suppose that $V'(n,w_1^*|(pa)_1) < 0$ and $V'(n,w_2^*|(pa)_2) = 0$, $(pa)_2 > (pa)_1$.

Conceptually, we can decompose the increase from $(pa)_1$ to $(pa)_2$ in three parts. The first part will preserve the condition $V'(n,w^*) < 0$ so that up to this intermediate pa-level, the value of w* is unaffected. Consequently, this part of the value added increase will be completely transmitted into the profit margin. The second increase in pa, which is infinitesimally small, will cause the condition for $V'(n,w^*) = 0$ to be fullfilled. The interior optimal solution at this critical pa-value may or may not coincide with w_1^*; in the latter case there occurs an upward jump in w^* and hence a sharp reduction in the profit margin. The third part of the increase in pa up to $(pa)_2$ is consistent with a preserved interior solution, increasing in pa. Whether or not the profit margin increases at this stage cannot be ascertained without further information not specified in the model.

So we conclude that the profit margin of the firm - in practice often used as a profitability measure - will respond ambiguously in our model to better business conditions as represented by a higher value added per employee.

Changes in the reservation wage distribution and the effect of a general inflation

A change in the reservation wage distribution $F(\cdot)$ can clearly take on many forms. An extensive discussion is hardly worthwhile, unless we are prepared to specify $F(\cdot)$. Here we will confine ourselves to one case, which has received much attention in the economics literature, namely when all reservation wages change proportionately. Let the proportional factor be β. Then we have the following relations between the original distribution function and its pdf and the new ones, which we denote $F_\beta(\cdot)$ and $f_\beta(\cdot)$ respectively:

$$F_\beta(v) = F(\frac{v}{\beta}); \quad f_\beta(v) = \frac{1}{\beta} f(\frac{v}{\beta})$$

(Both the mean and the standard deviation of the distribution is multiplied by β by this transformation.)

To simplify the presentation, let us suppose that $\beta > 1$. Then we have $F_\beta(w) < F(w)$ and from (2) we immediately have that $V^*(n)$ will decrease as a result of this upward proportional shift in all reservation wages $\left(\text{as } \lambda_\beta(w) = \gamma \cdot F_\beta(w) < \gamma \cdot F(w) = \lambda(w)\right)$.

More specifically we have

$\lambda_\beta(w) = \lambda(\frac{w}{\beta})$ and $\lambda_\beta'(w) = \lambda'(\frac{w}{\beta})/\beta$.
After the proportional increase (3) reads

$$\alpha \cdot V'(n,w|F_\beta) =$$

$$= (pa-w) \cdot \lambda_\beta' \cdot G_{N-n}'(\lambda_\beta) - G_{N-n}(\lambda_\beta) - n,$$

and by substitution,

$$\alpha \cdot V'(n,w|F_\beta) = (pa-w) \cdot \frac{1}{\beta} \cdot \lambda'(\frac{w}{\beta}) \cdot G_{N-n}'\left[\lambda(\frac{w}{\beta})\right] - G_{N-n}\left[\lambda(\frac{w}{\beta})\right] - n.$$

This is almost the expression (3) before the change, evaluated at w/β instead of w. Let us for a while assume that pa is multiplied by β too. It represents a situation when all prices and wages that face the firm externally are inflated at the rate $\beta-1$. In such a case it holds that

$$V'(n,w|F_\beta,\beta \cdot pa) = V'(n,\frac{w}{\beta})$$

Let us consider this result. A general inflation will affect the firm as if it had lowered its own wage from w to w/β along an unchanged $V'(n,w)$-curve. Consequently it seems to be the natural response of the firm to restore its position by increasing its wage from w_1^* to $\beta \cdot w_1^*$; this will indeed be the optimal solution under unrestricted optimization, i.e. when an interior w^*-solution obtains both before and after the change. The firm follows the general wage and price increase, its relative position is unchanged and the recruitment process is unaltered (as $\lambda_\beta(\beta \cdot w) = \lambda(w)$).

However, when a corner solution is obtained either before or after the change, this simple result does not hold. It is easy to verify that $V'(n,w_1^*) < 0$ implies $w_1^* \leqslant w_2^* < \beta \cdot w_1^*$. In terms of the diagram the general inflation does not only produce an effect equivalent to a move from w_1^* to w_1^*/β along the unchanged $V'(n,w)$-curve. To the firm the restriction line $w_n (=w_1^*)$ moves in effect to w_n/β. As the condition $V'(n,w_1^*) < 0$ reflects that the wage is 'too high', the firm will happily accept the opportunity of making no or just a partial adjustment of its own wage to the general inflation.

As there is also a possibility that $V'(n,w_2^*) < 0$ (so that $w_2^* = w_n$) at the same time as w_1^* constitutes an interior solution, $w_1^* > w_2^*$, we cannot even conclude that the optimal wage level is non-decreasing in a proportional increase in price and all reservation wages. To remove this inconclusiveness one has to assume that $f(v)$ is non-increasing in its argument for all $v \geqslant \underline{v}$.

It is illuminating to see that if w_n is also increased by the factor β together with the product price and the reservation wages, we indeed obtain $w_2^* = \beta \cdot w_1^*$, regardless of the character of the optimal solution. (Such an increase is most easily envisaged as a result of wage negotiations or as an enacted change in legal minimum wages.) So with the lower bound on the feasible region of w included in the vector of external "prices" of the decision problem, we get the familiar result that the optimal solution is homogeneous of degree one in that vector with respect to the "price" (wage) level (and of degree zero with respect to any "real" variable, such as the recruitment process).

We started this subsection with the case, in which only the reservation wages change. We can now utilize the fact that such a change can be looked upon as consisting of two parts: an equiproportional change in pa and all reservation wages and a corresponding change in pa in the opposite direction. We recall that a change in pa will always shift the $V'(n,w)$-curve in the same direction. Thus we conclude that an increase in all reservation wages with the factor $\beta > 1$ will never result in an increase of the optimal wage with as much as a factor β. In other words, the firm's relative wage must decrease as an optimal response to such a wage distribution change.

On the other hand, we cannot in general say how the optimal money wage behaves. A taxonomy of the different cases, similar to what has just been given for the general inflationary change, can easily be constructed, though. As the structure of the results are closely related to each other for both types of changes, we do not repeat the details, but leave them to the interested reader.

Changes in contact intensity and in discount rate

The rate at which contacts between the firm and job-applicants occur, γ, is an important parameter as it reflects labor market conditions and can be regarded as a measure of the degree of labor scarcity facing the firm (at a given relative wage). Unfortunately, changes in γ do not seem to have any clear-cut effects on the optimal wage policy.

There is no ambiguity as to the effect of changes in γ on $V(n,w)$. From (2) it is immediately clear that an increase in γ increases $V^*(n)$.

The effect on $V'(n,w)$ is easily obtained from (5):

$$\gamma \cdot \alpha \cdot \frac{dV'(n,w)}{d\gamma} = G'(\lambda)\{(pa-w)\lambda' \cdot [1-K(\lambda)] - \lambda\} - $$
$$- (pa-w)\lambda' \cdot G(\lambda) \cdot K'(\lambda)$$

or

$$\gamma \cdot \alpha \cdot \frac{dV'(n,w)}{d\gamma} = G'(\lambda)\{(pa-w)\lambda' \cdot [1- \frac{H'(\lambda)}{G'(\lambda)}] - \lambda\}$$

As $K'(\lambda) > 0$, we can at least conclude that $\frac{dV'}{d\gamma} < 0$ for $w \geqslant w_{max}(0)$. However, in order to say anything about the impact on the optimal wage we need to know how changes in γ affect $V'(n,w)$ on the interval $[\underline{v}, w_{max}(0)]$, where unfortunately the sign of $\frac{dV'}{d\gamma}$ is ambiguous. For small values of λ the factor $[1 - \frac{H'(\lambda)}{G'(\lambda)}]$ is positive, so we can conclude that $\frac{dV'}{d\gamma} > 0$ for w in a neighbourhood of $\underline{v}$, but as its sign is reversed as w approaches $w_{max}(0)$ we remain ignorant about the total effect of changes in γ on $V'(n,w)$ and hence on w^*.

We may note that when $(N-n)$ is very large, $V'(n,w)$ is increasing in γ on almost the whole interval $[\underline{v}, w_{max}(0)]$, so we would then expect w^* to be non-decreasing in γ for $n > 0$. Loosely stated there seems to be a tendency that if the firm has not been able to exploit fully its recruitment opportunities in the initial position an increase in γ produces an increase in the optimal wage and vice versa. This tentative suggestion clearly requires more strict elaboration in order to yield useful results; they do not appear to be within easy reach.

Turning to the effect of a change in the discount rate α we first note that $V^*(n)$ is clearly decreasing in α. As to the effect on $V'(n,w)$, there is a close connection between changes in α and in γ as it holds

$$\alpha \cdot \frac{dV'(n,w)}{d\alpha} + \gamma \cdot \frac{dV'(n,w)}{d\gamma} = -V'(n,w)$$

Consequently the lack of definite results concerning the effect of changes in γ on the optimal wage carries over to changes in α.

This finding may appear surprising as the related models analyzed in Deshmukh and Chitke (1976), Deshmukh and Winston (1979) and Lippman (1980) show a clear-cut effect on the optimal policy of a change in the discount rate. By analogy one might have expected that in our model the optimal wage should have been non-increasing in the discount rate. The reason for this difference is that in (an analogue of) these models a higher wage level would imply a higher cost only momentarily and a higher (expected) revenue in the future, while in our model it implies higher costs both now and in the future as well as higher (expected) future revenues. Consequently the wage level does not have the unambiguous effect on the balance between current and future gains that is required to obtain a simple relationship between its optimal level and the discount rate.

Summary and Conclusions

In this paper we have analyzed a simple stochastic recruitment process, controlled by the firm's wage offer. What makes the model deviate from other applications with the same structure (but with different interpretations) is (a) the unimodality of the pdf of the job applicants' reservation wages and (b) the downward rigidity of the wage level control, both assumptions having the character of 'stylized facts' of the labor market.

Assumption (a) makes the dependence of the recruitment process on the wage level control somewhat complicated. The recruitment intensity is monotonically increasing but not concave in the wage level. Hence the value function itself is not concave in the wage level. However, enough restriction is laid on the reservation wage distribution to establish the existence of at most one local maximum in the interior of the permissible range of wage levels. The main complication of the lack of concavity is that the existence of an interior local maximum does not imply global maximum. The optimal wage level may very well be situated at the lower bound of the permissible wage level range. Hence the optimal wage may have a point of discontinuity with respect to changes in the parameters of the problem.

Assumption (b) is introduced because of relevance but it is not to deny that it makes some aspects of the model, which appear in this paper easier to handle (as opposed to the problem of finding the structure of the optimal wage policy, treated in (Schager 1985)). The main advantage of introducing (b) is not of technical nature, however. It saves us from the effort of analysing a decreasing structure of the optimal wage level in terms of employment states during expansion, a rather futile exercise if the aim is to model observed wage behavior. At the same time assumption (b) makes it impossible to ignore the fact that once a wage level has been established by the firm, it has effectively limited the range of possible future wage actions. Thus optimal solutions situated at the lower bound of the permissible wage level range must be paid due attention as being a perfectly normal and presumably very frequent case. They occur not only as a result of realized employment expansion itself, but can also result from an unexpected worsening in business conditions. It is not a bold guess that many firms in the industrial world have been in such 'disequilibrium' situations in recent years.

The comparative statics analysis of our model produces mixed results. Increases in the number of employees imply a non-increasing optimal wage, increases in capacity limit and in value productivity a non-decreasing optimal wage. A general wage-price inflation has a non-increasing effect on the firm's optimal <u>relative</u> wage, but keeps the optimal relative wage constant only under special conditions. On the other hand, the effects of changes in labor market conditions (the contact intensity) and the discount factor on the optimal wage have defied closer specification.

Turning to possible extensions of the present model, the most interesting development would be the inclusion of quits. In such a model the firm controls both the arrival rates of new employees and the departure rates of its already aquired employees through its wage level. The arrival process may still be Poisson, while the departure process may preferably be modelled as a linear death process. Allowing also for labor hoarding (i.e. more employees than corresponds to productive capacity), the existence of which might be profitable, one could specify the ideal model in queueing theory terms as the control of a M/M/s/N-queue, where the wage level operates on the arrival and departure rates as indicated, s is to be interpreted as the number of productive job positions (here not intended to be controlled) and N-s is the number of hoarded employees (to be controlled). The tractability of such a decision process is an open question, but if results could be obtained as to the structure of the optimal policy it would mean a significant contribution.

Keeping within the confinements of the pure recruitment model, one natural extension is to postulate a production function that is concave in the number of employees. That is an assumption in accordance with traditional economic theory, regularly used in the deterministic control counterparts to the present analysis. In our model this generalization is best accomplished by introducing a piece-wise constant productivity over different intervals of employment, exhibiting a downward jump when transition into a higher employment interval occurs. This assumption will in effect imply that the firm is not only choosing a wage policy, but also a capacity limit within a set indicated by the highest employment level in each productivity interval. Such an approach is indeed tractable and it can be shown that the optimal structure of a non-decreasing wage policy is again a constant wage level over future employment states.

The recruitment model can also be modified as to represent recruitment out of a limited source of job-applicants, i.e. the recruitment intensity is falling linearly in the number of hires. The structure of the optimal policy seems to be unaffected by this change in the character of the recruitment process. It is interesting to note that the structure of the model now shows strong resemblence to the structure of a deterministic model with both hires and quits, where the hire rate is independent of and the quit rate proportional to the volume of employment, provided that the firm has chosen a wage level consistent with employment expansion. This observation points to the conclusion that our stochastic recruitment model may incorporate more of the contents of standard deterministic control models than would appear plausible at first glance.

In the last few years there have been attempts to free the deterministic optimal control models of the firm's employment and wage policy from the limitations imposed by the assumptions of a static environment. As such models (as well as the present one) seem to have their greatest relevance in depicting short run dynamic behavior, it is natural to place them in a short run economic context such as the business cycle. Contributions in that direction are notably Leban (1982a,b). As a last indication of interesting future research opportunities one should thus point to the possibility of letting the parameters of the decision problem change their values in a known way at specific points of time in the future. Such an extension of the present model, where the states of the firm are uniquely ordered in time, seems quite possible.

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