IFN Working Paper No. 1083, 2015

Renovatio Monetae: Gesell Taxes in Practice

Roger Svensson and Andreas Westermark

Research Institute of Industrial Economics
P.O. Box 55665
SE-102 15 Stockholm, Sweden
info@ifn.se
www.ifn.se

Renovatio Monetae: Gesell Taxes in Practice*

Roger Svensson[†] and Andreas Westermark[‡]

September 1, 2015

Abstract

Gesell taxes on money holdings have received attention in recent decades as a way of alleviating the zero lower bound on interest rates. Less known is that such a tax was the predominant method used to generate seigniorage in large parts of medieval Europe for around two centuries. When the Gesell tax was levied, current coins ceased to be legal tender and had to be exchanged into new coins for a fee - an institution known as renovatio monetae or periodic re-coinage. This could occur as often as twice a year. Using a cash-in-advance model, prices increase over time during an issue period and falls immediately after the re-coinage date. Agents re-mint coins and the system generates tax revenues if the tax is sufficiently low, if the time period between re-coinages is sufficiently long, and if the probability of being penalized for using illegal coins is sufficiently high.

Keywords: Seigniorage, Gesell tax, periodic re-coinage, cash-in-advance model

JEL classification: E31, E42, E52, N13.

*We would like to thank Martin Allen, Mikael Carlsson, Per Hortlund, Boris Paszkiewics, Karl Walentin and seminar participants at the EEA congress in Mannheim for comments and suggestions. Roger Svensson gratefully acknowledges financial support from the Torsten Söderberg Foundation and the Sven and Dagmar Salén Foundation. The views expressed in this paper are solely the responsibility of the authors and should not be interpreted as reflecting the views of the Executive Board of Sveriges Riksbank.

[†] The Research Institute of Industrial Economics (IFN), P.O. Box 55665, SE-10215 Stockholm, Sweden. Correspondence: roger.svensson@ifn.se

[‡] Research Department, Sveriges Riksbank, SE-103 37, Stockholm, Sweden and Uppsala Center for Labor Studies. Correspondence: andreas.westermark @riksbank.se.

1 Introduction

First proposed by Gesell (1906), the idea of a tax on money holdings has received increasing attention in recent decades. The zero lower bound, which limits the ability of central banks to stimulate the economy through standard interest rate policy, was reached in Japan in the 1990s and in the U.S. and Western Europe after the financial crisis in 2008. Buiter and Panigirtzoglou (1999, 2003), Goodfriend (2000), Mankiw (2009), Buiter (2009) and Menner (2011) have analyzed a periodic tax on money holdings as a way of alleviating this problem. Importantly, the tax breaks the arbitrage condition in standard models that induces savers to hold cash instead of other financial assets when nominal interest rates go below zero, thus allowing for significantly negative nominal interest rates.

Perhaps less known is that a tax on money holdings existed for almost 200 years in large parts of medieval Europe. Gesell taxes were implemented by coins being legal tender for only a limited time period and, at the end of the period the coins had to be exchanged into new coins for a fee - an institution known as renovatio monetae or periodic re-coinage; e.g., see Allen (2012, p.35).[1] In Gesell's original proposal, the holders of money had to buy and attach stamps to bank notes for them to retain their full nominal value. In the system with periodic re-coinage, the monetary authority ensured that the new coins could be distinguished from old coins by altering their physical appearance so that it would be easy to verify that only the new coins were legal tender.

There was substantial variation in the level of Gesell taxes. In Germany, four old coins were usually exchanged for three new coins, and the Gesell tax was 25 percent; in the Teutonic order, the tax was 17 percent, and in Denmark it was up to 33 percent; see Mehl (2011, p. 33), Paszkiewicz (2008) and Grinder-Hansen (2000, p. 85). Note also that, with periodic Gesell taxes, revenues depend not only on the fee charged at the time of the re-coinage but also on the duration of an issue. In specific currency areas, re-coinage could occur up to twice per year and involve annualized rates of up to 44 percent; see Kluge (2007).[2] To generate revenues through seigniorage, the monetary authority benefits from creating an exchange monopoly for the currency. In a system with Gesell taxes and re-mintage, in addition to competing with foreign coin issuers, the monetary authority

[1] Also known as *coin renewals*.

[2] The annualized rate is based on a Gesell tax of 25 percent that was levied twice per year, as in, e.g., Magdeburg; see Mehl (2011, p. 33).

competes with its own older issues. To limit the circulation of illegal coins, the monetary authorities penalized the usage of invalid coins. Furthermore, fees, rents and fines had to be paid with current coins; see Haupt (1974, p. 29), Grinder-Hansen (2000, p. 69) and Hess (2004, p. 16–19). In addition to the system with Gesell taxes, there was also a system with long-lived coins in the High Middle Ages of Europe (1000–1300 A.D.), where the period when coins were legal was not fixed; see Kluge (2007, p. 62–64).[3]

The disciplines of archaeology and numismatics have long been familiar with periodic re-coinage (Kluge, 2007 , Allen, 2012, Bolton, 2012). Although scientific methods in archaeology and numismatics identify the presence of re-coinage, empirical evidence in written sources is scarce on the consequences of re-coinage with respect to prices and people's usage of new and old coins. However, evidence from coin hoards indicates that old (illegal) coins often but not always circulated with new coins; see Allen (2012, p. 520–23) and Haupt (1974, p. 29). In addition, written documents mention complaints against this monetary tax (Grinder-Hansen 2000, p. 51–52 and Hess, 2004, p. 19–20). Despite being common for an extended period of time, this type of monetary system has been seldom if ever analyzed theoretically in the economics or economic history literature.

The purpose of the present study is to fill this void in the literature. We formulate a cash-in-advance model similar to that of Velde and Weber (2000) to capture the implications of Gesell taxation in the form of periodic re-coinage on prices, returns and people's decisions to use new or old coins for transactions. The model includes households, firms and a lord. Households care about the consumption of goods and jewelry, where jewelry captures the commodity property of coins. When trading jewelry and consumption, households face a cash-in-advance constraint. Households can hold both new and old coins, but only the new coins are legal tender. Jewelry can be melted and minted into coins, and coins can be melted and made into jewelry. An issue of coins is only legal for a finite period of time. Old coins must be re-minted at the re-coinage date to be considered legal tender. The lord charges a fee when there is a re-coinage so that for each old coin handed in, the household receives only a fraction in return. Although illegal, old coins can be used for transactions. To deter the use of illegal coins, lord plaintiffs check whether

[3]Sometimes, these coins were valid for the entire duration of the reign of the coin issuer. In these cases, successors occasionally minted variants of the same coin type. These variants are called immobilized types and could be valid for very long time periods - occasionally centuries - and survive through the reigns of several rulers.

legal means of payment are used in transactions. When old coins are discovered in a transaction by the lord plaintiffs, the coins are confiscated and re-minted into new coins. Thus, whether illegal coins circulate is endogenous in the model. The lord's revenues depend on the re-coinage (and mintage) fee, old coin confiscations and the duration of each coin issue. The lord uses the revenues to finance consumption expenditures.

Because re-coinage occurs at a given frequency and not necessarily in each time period, a steady state need not exist. Instead of analyzing steady states, as in Velde and Weber (2000), we analyze a model where re-coinage occurs at fixed (and equal) time intervals. To focus on steady-state-like properties, we analyze cyclical equilibria, i.e., equilibria where the price level, money holdings, consumption, etc., are the same at a given point in different coin issues.

Our key results are that in equilibrium, prices increase over time during an issue period and fall immediately after the re-coinage date. Moreover, the higher the Gesell tax is, the higher the price increases are (as long as the coins are surrendered for re-coinage). Furthermore, in the sense that agents participate in re-minting coins and the system generates tax revenues, the system with Gesell taxes works 1) if the tax is sufficiently low, 2) if the time period between two instances of re-coinage is sufficiently long, and 3) if the probability of being penalized for using old illegal coins is sufficiently high. Additionally, although nominal returns become negative when the Gesell tax is levied (empirical evidence on the tax indicates a negative return of up to -44 percent), real returns are unchanged because the price level adjusts accordingly as a result of the reduction in money holdings.

The paper is organized as follows. In section 2, we provide some stylized facts regarding medieval European coins and the extension of short-lived coinage systems through time and space. Seigniorage and the enforcement of short-lived coinage systems are outlined in section 3. In section 4, we use a cash-in-advance model to analyze the consequences of periodic re-coinage. Finally, section 5 delineates the conclusions.

2 Short-lived coinage systems through time and space

2.1 The basics of medieval money

Money in medieval Europe was overwhelmingly in the form of commodity money, based on silver,[4] fiat money did not exist in its pure form. The control of the coinage, i.e., the right to mint, belonged to the droit de régale, i.e., the king/emperor. In addition to the right to determine, e.g., the design and the monetary standard, the coinage right encompassed the right to use the profits from minting and to decide which coins were legal tender; see Kluge (2007, p. 52). The right to mint for a region could be delegated, sold or pawned to other local authorities (local lords, laymen, churchmen, citizens) for a limited or unlimited time period; see Kluge (2007, p. 53). The size of each currency area was usually smaller than today and could vary substantially. All of England was a single currency area (after 975), whereas Sweden and Denmark each had 2–3 areas. In contrast, in France and Germany, the minting right was delegated to many civil and ecclesiastical authorities, and there were many small currency areas.

A commonly used monetary system in the middle ages was Gesell taxation in the form of periodic re-coinage. The main feature of a re-coinage system is that coins circulate for a limited time, and at the end of the period, the coins must be returned to the monetary authority and re-minted for a fee, i.e., a Gesell tax. Thus, coins can be "short-lived", in contrast to a "long-lived" monetary system in which the coins do not have a fixed period as a legal means of payment.

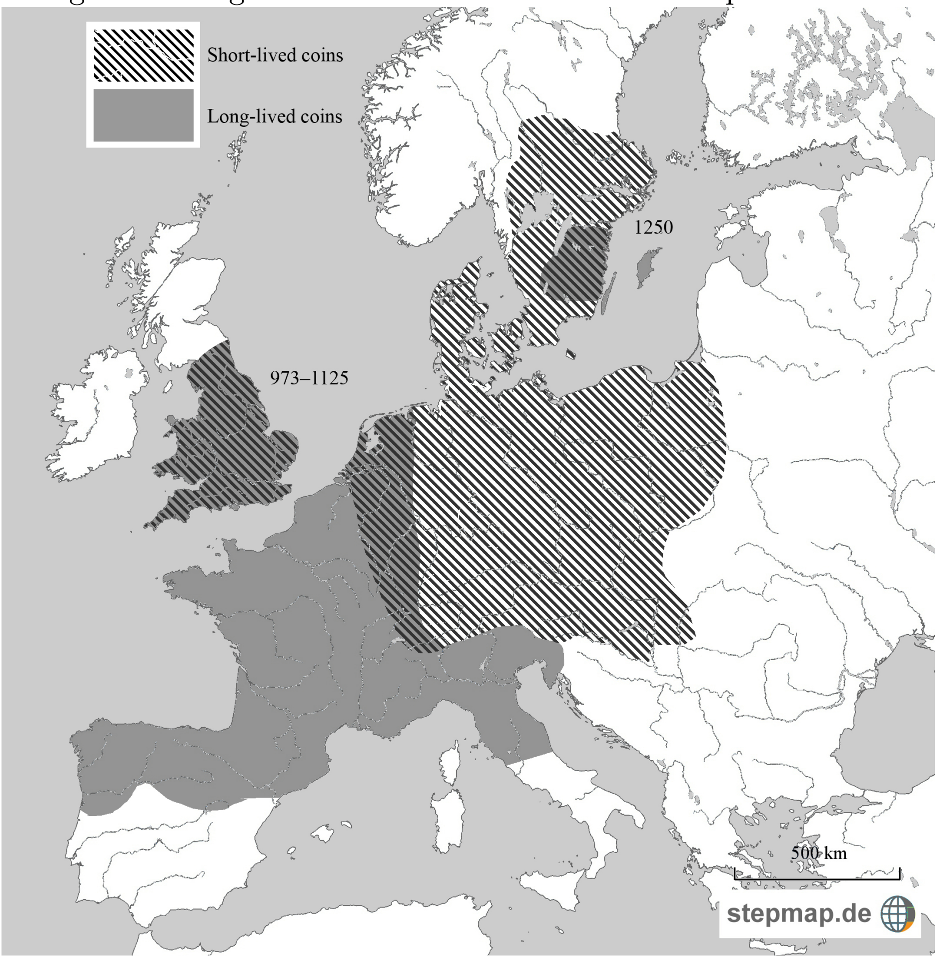
2.2 Geographic extension of short-lived coinage systems

There is a substantial historical and numismatic literature that describes the extent of periodic re-coinage; see, e.g., Kluge (2007), Allen (2012), Bolton (2012) and Svensson (2013). Three methods have been used to identify periodic re-coinage and its frequency, namely, written documents, the number of coin types per ruler and the years and distribution of coin types in hoards (for details, see Appendix A.1). There is a reasonable consensus in determining the extension of long- and short-lived coinage systems through time and space. Long-lived coins were common in northern Italy, France and Christian

[4] The reason for this was the relative abundance of silver mines that lead to a high supply of silver; see Spufford (1988, p.109ff, 119ff).

Spain from 900–1300. This system spread to England when the sterling was introduced during the second half of the 12th century (see Map 1). In France, in the 11th and 12th centuries, long-lived coins dominated in most regions (the southern, western and central parts), and the rights to mint were distributed to many civil authorities. In northern Italy, where towns took over minting rights in the 12th century, long-lived coins likewise dominated; see Kluge (2007, p. 136ff).

Figure 1: Long-lived and short-lived coins in Europe 950-1300.



Note: Eastern Götaland, Sweden, changed from long-lived to short-lived coins ca. 1250. England had periodic re-coinage from 973–1125.

A well-known example where short-lived coinage systems were used is England. Periodic re-coinage was introduced in the English kingdom in approximately 973 and occurred every sixth year until 1035. For approximately one century after 1035, English kings continued to renew their coinage at a higher frequency of 2–3 years. These coins were valid throughout England - a large geographic area; see Spufford (1988, p. 92) and Bolton (2012, p. 87ff).

Short-lived coinage systems were the dominant monetary system in central, northern and eastern Europe from 1000–1300. The eastern parts of France and the western parts of Germany had periodic re-coinage in the 11th and 12th centuries; see Hess (2004, p. 19–20). However, the best examples of short-lived and geographically constrained coins can be found in central and eastern Germany and eastern Europe, where the currency areas were relatively small. Here, periodic re-coinage began in the middle of the 12th century and lasted until approximately 1300 and was especially frequent in areas where uni-faced bracteates were minted,[5] which usually occurred annually but sometimes twice per year; see Kluge (2007, p. 63).

Sweden had periodic re-coinage of bracteates in two of three currency areas (especially in Svealand and to some extent in western Götaland) for more than a century, from 1180–1290. This conclusion is supported by evidence of numerous coin types per reign and the composition of coin hoards; see Svensson (2013, p. 223–24). Denmark introduced periodic re-coinage in all currency areas in the middle of the 12th century, which continued for 200 years with some interruptions; see Grinder-Hansen (2000, p. 61ff). Poland and Bohemia had periodic re-coinage at least once per year in the 12$^{\text{th}}$ and 13$^{\text{th}}$ centuries; see Sejbal (1997, p. 26), Suchodolski (2012) and Vorel (2000, p. 341).

2.3 Other stylized facts and the concept of periodic re-coinage

To obtain revenues from seigniorage, a coin issuer benefits from having an exchange monopoly in both long- and short-lived coinage systems. However, in a short-lived coinage system, the minting authority not only faces competition from other coin issuers but also from its own old issues that it minted. To create a monopoly position for its coins, legal tender laws stated that foreign coins were ipso facto invalid and had to be exchanged for the current local coins with the payment of an exchange fee in an amount determined by the coin issuer. Moreover, only one local coin type was considered legal at a given point in time.[6] The frequency and exchange fee of re-coinage varied (see section 3.1 below). Re-coinage normally occurred on a specific date. Afterward, the new local coins were the

[5] Bracteates are thin, uni-faced coins that were struck with only one die. A piece of soft material, such as leather or lead, was placed under the thin flan. Consequently, the design of the obverse can be seen as a mirror image on the reverse of the bracteates.

[6] The coin issuer therefore has an incentive to ensure that foreign coins are not allowed to circulate. Moreover, to prevent illegal coins from circulating, the minting authority must control both the local market and the coinage; see Kluge (2007, p. 62–63).

only legal tender in the city.[7] In Gesell's original proposal, stamps had to be attached to a bank note for it to retain its full value, which made it easy to verify whether the tax had been paid. Similarly, under periodic re-coinage, the main design of the coin was changed,[8] whereas the monetary standard (e.g., weight, fineness, diameter, shape of the flan, denomination) largely remained unchanged.

Empirical evidence of periodic re-coinages suggests that coins were usually exchanged on recurrent dates at a substantial fee and that coins were only valid for a limited (and ex ante known) time. The withdrawals were systematic and recurrent. Thus, when analyzing periodic re-coinage in section 4 below, we assume that both the exchange fee and the re-coinage dates are known in advance. One may also want to distinguish between periodic re-coinage and coinage reform, which is a distinction that has not necessarily been made explicit by historians and numismatists.[9] When a coinage reform is undertaken, coin validity is not constrained by time. A coinage reform also includes a re-mintage but is announced infrequently, and the validity period of the coins is not (explicitly) known in advance. Moreover, the coin and the monetary standard are generally changed considerably.[10] Note that if the issuer charges a fee at the time of the reform, the coinage reform shares some features of re-coinage, but because the monetary standard is changed, there may be additional effects, e.g., on the price level at the time of the reform.

[7]In 1231, the German king Henry VII (1222–35) published an edict in Worms stating that in towns in Saxony with their own mints, goods could only be exchanged for coins from the local mint; see Mehl (2011, p. 33). However, when this edict was published, the system of coins constrained through time and space had been in force for a century in large parts of Germany.

[8]For example, the portrayed figure is in a different position, or there are different attributes in the hands of the figure.

[9]In fact, historians often use the term re-coinage for both periodic re-coinage and coinage reform.

[10]England had two re-mintings in the 13th century when the coinage was long-lived, but these events had other purposes than to simply charge a gross seigniorage. The short-cross pennies minted in the 12th and 13th centuries were often clipped. A re-minting occurred in 1247. A new penny was introduced ('long-cross') with the cross on the reverse extended to the edge of the coin to help safeguard the coins against clipping. Another coinage reform occurred in 1279. Before 1279, the double-lined cross on the long-cross pennies was used when cutting the coins into halves to obtain small change for the penny. New denominations were introduced in 1279 - all with single-lined crosses on the reverse. In addition to the new penny, groat, halfpence and farthing were also minted.

3 Seigniorage and enforcement of short-lived coinage systems

3.1 Seigniorage and prices in systems with re-coinage

The seigniorage under re-coinage depends not only on the fee charged at the time of the re-coinage but also on the duration of an issue. Given a fee of, for example, 25 percent at each re-coinage, the shorter the duration is, the higher the revenues are, given that money holdings are not affected. Any reduction in money holdings because of a shortening of issue time would move revenues in the other direction.

Table 1: Exchange fees and duration of re-coinage in different areas

Region	Currency area♦	Period	Gesell tax★ (Annualized)	Duration years★	Method/Source†
England	Large	973–1035	n.a.	6	1–3, Spufford (1988)
	Large	1035–1125	n.a.	2–3	2–3, Bolton (2012)
Germany, western✠	Small	ca. 1000–ca. 1300	mostly 25% (4.6%–25%)‡	1–5	1–3, Hess (2004)
Germany, eastern, northern✠	Small	ca. 1140–ca. 1330, sometimes until 15th cent.	mostly 25% (25%–44%)‡	$\frac{1}{2}$ or 1	1–3, Kluge (2007)
Teutonic Order in Prussia	Medium	1237–1364	17% (1.6%)	10	1–3, Paszkiewicz (2008)
Austria	Small	ca. 1200–ca. 1400	n.a.	1	2–3, Kluge (2007)
Denmark	Medium	1140s–1330s.	33% (33%)	1, with inter-ruptions	1–3, Grinder-Hansen (2000)
Sweden, Svealand	Large	1180–1290	n.a.	1–5	2–3, Svensson (2013)
Sweden, Götaland	Large	1180–1290	n.a.	3–7	
Poland	Small	ca. 1100–ca. 1150	n.a.	3–7	1–3, Suchodolski (2012)
	Small	ca. 1150–ca. 1200	n.a.	1	
	Small	ca. 1200–ca. 1300	n.a.	$\frac{1}{3}$ or $\frac{1}{2}$	
Bohemia-Moravia	Large	ca. 1150–1225	n.a.	1	Sejbal (1997) and Vorel (2000)
	Large	1225–ca. 1300	n.a.	$\frac{1}{2}$	

Notes: ♦ We do not use a formal definition of area size. By a large area, we mean a country or a substantial part of a country, such as England or Svealand. A small area is usually a city and its hinterland. A medium-sized area is somewhere in between and is exemplified by the kingdom of Wessex. †As in Appendix A.1. ✠ Various mints and authorities. ‡Annualized rate based on a fee of 25 percent. ★ When known.

There was substantial variation in the level of seigniorage. In England from 973–1035, re-coinage occurred every sixth year. For approximately one century after 1035, English kings renewed their coinage every second or third year; see Spufford (1988, p. 92) and Bolton (2012, p. 99ff). The level of the fee is uncertain.[11]

In other areas in Europe, the duration was often significantly shorter. Austria had annual re-coinage until the end of the 14th century, and Brandenburg had annual re-coinage until 1369 (Kluge (2007, p. 108, 119)). Some individual German mints had bi-annual or annual renewals until the 14th or 15th centuries (e.g., Brunswick until 1412); see Kluge (2007, p. 105). In Denmark, re-coinage was frequent (mostly annual) from the middle of the 12th century and continued for 200 years with some interruptions; see Grinder-Hansen (2000, p. 61ff). Sweden had re-coinage beginning in approximately 1180 that continued for approximately one century; see Svensson (2013, p. 225). In Poland, King Boleslaw (1102–38) began with irregular re-coinages - every third to seventh year, but later, these became far more frequent. At the end of the 12th century, coin renewals were annual, and in the 13th century, they occurred twice per year; see Suchodolski (2012). Bohemia also had re-coinage at least once each year in the 12th and 13th centuries; see Sejbal (1997, p. 83) and Vorel (2000, p. 26). In contrast, the Teutonic Order in Eastern Prussia had periodic re-coinages only every tenth year between 1237 and 1364; see Paszkiewicz (2008).

The exchange fee in Germany was generally four old coins for three new coins, i.e., a Gesell tax of 25 percent; see, e.g., Magdeburg (12 old for 9 new coins, Mehl, 2011 p. 85). In Denmark, the Gesell tax - three old coins for two new coins—was higher, at 33 percent; see Grinder-Hansen (2000, p. 179). The annualized tax in Germany and Denmark could be very high - up to 44 percent.[12] The Teutonic Order in Prussia had a relatively generous exchange fee of seven old coins for six new coins; see Paszkiewicz (2008). This fee represents a tax rate of almost 17 percent, or in annualized terms, 1.6 percent.

Unfortunately, evidence is scarce on the prices in monetary systems with re-coinage. Indeed, finding price indices for the period under discussion is almost impossible. How-

[11] According to Spufford (1988), four old coins were exchanged for three new coins, although this calculation is based on a rather uncertain weight analysis. If the gross seigniorage was 25 percent every sixth year, the annualized rate was almost 4 percent.

[12] The annualized rate is based on a Gesell tax of 25 percent levied twice per year, as in, e.g., Magdeburg; see Mehl (2011, p. 33).

ever, some evidence from the Frankish empire indicates that prices rose during an issue.[13] Specifically, several attempts at price regulations that followed a re-coinage/coinage reform in 793–4 seem to indicate problems with rising prices; see Suchodolski (1983).

3.2 Success, monitoring and enforcement of re-coinage

There was considerable variation in the success of re-coinage. The coin hoards discovered to date can tell us a great deal about the success of re-coinage. Hoards in Germany from this period (1100–1300) usually contain many different issues of the local coinage as well as many issues of foreign coinage, i.e., locally invalid coins; see table 2, which indicates that the monetary authorities had problems enforcing the circulation of their coins. By missing some coin renewals and saving their retired coins, people could accumulate silver or use the old coins illegally. In contrast, hoard evidence from England indicates that the re-coinage systems were partly successful; see Dolley (1983). As shown in table 3, almost all of the coins in hoards are of the last type during the period from 973–1035, when coins were exchanged every sixth year. However, from 1035–1125, only slightly more than half of the coins were of the last type, which indicates that the system worked well up to 1035 but less so after that date. One reason for this result may be that the seigniorage for the later period was higher because of the shorter time period between withdrawals (at an unchanged exchange fee; see table 1).

Table 2: The composition of German coin hoards in Thuringia from 1156–1325 and Upper Lusatia from 1200–1300. Number of coin hoards and share.

Region		Thuringia		Upper Lusatia	
Period		1156–325		1200–1300	
Years between re-coinages		1 year		1 year	
		No. of hoards	Share	No. of hoards	Share
Hoards with	1 type	2	2.4%	0	0.0%
	2 types	3	3.6%	1	3.6%
	3 types	9	10.8%	4	14.3%
	>3 types	69	83.2%	23	82.1%
Total number of coin hoards		83	100.0%	28	100.0%

Notes: Calculations are based on Hävernick (1955:26ff) and Haupt (1954:505ff). Each coin hoard must contain at least 3 coins to be included in the statistics.

[13] The Frankish empire seems to have had a system similar to re-coinage in the 8th and 9th centuries, although the weight of the coins was often changed when they were exchanged in this system.

Table 3: The composition of English coin hoards from 979–1125. Number of coin hoards, number of coins and shares

Period		973–1035		1035–1125	
Years between re-coinages		6 years		2–3 years	
		No. of hoards	Share	No. of hoards	Share
Hoards with	1 type	25	83.3%	19	32.8%
	2 types	2	6.7%	12	20.7%
	3 types	1	3.3%	10	17.2%
	>3 types	2[a]	6.7%	17	29.3%
Total number of coin hoards		30[a]	100.0%	58[b]	100.0%

Period		973–1035		1035–1125	
Years between re-coinages		6 years		2–3 years	
		No. of coins	Share	No. of coins	Share
Coins from	Last issue	886	86.5%	8 771	54.3%
	Second to last issue	137	13.4%	1 724	10.7%
	Third to last issue	1	0.1%	698	4.3%
	Earlier issues	0[a]	0.0%	4 964	30.7%
Total number of coins		1 024[a]	100.0%	16 157[b]	100.0%

Notes: Calculations are based on Allen (2012:520—23). Each coin hoard must contain at least three coins to be included in the statistics. Therefore, five hoards from 973–1035 and eleven hoards from 1035–1125 with only two coins described by Allen (2012) are not included. For some coin hoards, the exact number of coins is not available.
[a] For three hoards, because they have been (partly) lost, a complete distribution of the number of coins over issues cannot be computed. Among these are the large Kingsholm and Cnut hoards, where more than three different types are identified.
[b] For four of the 58 hoards, because they have been (partly) lost (all were discovered before 1845), a complete distribution of the number of coins over issues cannot be computed (these four hoards consist of ca. 1,850 coins).

Because hoards often contain illegal coins, the incentives to try to avoid re-coinage fees appear to occasionally have been rather high. To curb the circulation of illegal coins, monetary authorities used different methods to control the usage of coins. The usage of invalid coins was deemed illegal and was penalized, although the possession of invalid coins was mostly legal.[14] If an inhabitant used foreign coins or old local coins for transactions and was detected, the penalty could be severe. Moreover, sheriffs and other administrators who accepted taxes or fees in invalid coins were penalized; see Haupt (1974, p. 29), Grinder-Hansen (2000, p. 69), and Hess (2004, p. 16). Controlling the usage of current coins was likely easier in cities than in the countryside.[15]

[14] City laws in Germany stated that neither the mint master nor a judge was allowed to enter homes and search for invalid coins.
[15] As noted in sections 2.1 and 2.2, medieval currency areas could be large, such as in England and Sweden, or small, as in Germany and Poland. However, irrespective of the size of the currency area,

The minting authority could also indirectly control the coin circulation in an area. Documents show that fees, rents and fines were to be paid with current coins, in contrast to traditional situations where payment in kind was possible; see Grinder-Hansen (2000, p. 69), and Hess (2004, p. 19).

4 The model

In this section, we outline the model, define equilibria and analyze equilibrium outcomes in terms of how prices evolve. We also analyze under what conditions on re-mintage fees and issue length, old and new coins are used together.

4.1 The economic environment

The economy consists of households, firms and a lord. Households care about goods consumption, c_t, and jewelry consumption, d_t. When trading jewelry and consumption, households face a cash-in-advance constraint. Money holdings consist of new and old coins made of silver.[16] Only new coins are legal tender, but households can use both types in transactions. Thus, whether illegal (old) coins circulate is endogenous in the model. The new coins are withdrawn from circulation every T'th period. Specifically, to be considered legal tender after a withdrawal, coins must be handed in to be re-minted. Any coin that is not returned for re-mintage is not legal tender and is thus treated as an old coin after its withdrawal. Therefore, a given issue of coins is legal tender for T periods. The lord charges a Gesell tax τ at the time of each withdrawal. Specifically, for each coin handed in for re-mintage, each household receives $1 - \tau$ new coins in return, and the lord gets the remainder. Although illegal, old coins can be used for transactions, but because of the possibility of punishment for using illegal coins, it is costly to do so. We model the

systems with short-lived coins as legal tender could often be strictly enforced only in a limited area of the authority's domain, such as in cities. If most trade occurred in cities, this restriction may not be a strong constraint, however. Normally, the city border demarcated the area that included the jurisdiction of the city in the Middle Ages. The use of foreign and retired local coins within the city border was forbidden. This state of affairs is well documented in an 1188 letter from Emperor Friedrich I (1152–90) to the Bishop of Merseburg (Thuringia) regarding an extension of the city. The document plainly states that the market area boundary includes the entire city, not just the physical marketplaces; see Hess (2004, p. 16). A document from Erfurt (1248/51) shows that only current local coins could be used for transactions in the town, whereas retired local coins and foreign coins were allowed for transactions outside of the city border; see Hess (2004, p. 16).

[16] For simplicity, we ignore foreign coins.

punishment for using illegal coins as follows. There are lord plaintiffs that check whether the legal means of payment are used in transactions. If old coins are discovered in a transaction by the lord plaintiffs, they are confiscated by the lord plaintiffs, re-minted (without cost) as new coins and used to fund the lord's expenditures. The lord plaintiffs find old coins with probability $1-\chi$. Because of the confiscation of old coins by the lord plaintiffs, old and new coins need not circulate at par. We let e_t denote the exchange rate between old and new coins. Households can also sell jewelry to the representative firm (mint) in return for new coins. The lord's revenues, i.e., from minting, re-mintage and confiscations, are spent on the lord's consumption, denoted as g_t.

As in Velde and Weber (2000), competitive firms can produce: 1) a consumption good using the endowment; 2) jewelry from melting coins; and 3) new coins by minting jewelry.[17] At the beginning of a period t, households own an endowment ξ_t, jewelry d_t, and a stock of new and old coins, m_t^n and m_t^o, respectively. The stock of silver in the economy is S_t, and the change in the stock $S_t - S_{t-1}$ is added to the household jewelry stock at the beginning of period t. The endowment of the household is sold to the firms in return for a claim on firm profits. Then, shopping begins with households using coin balances to buy consumption and jewelry at competitively determined prices p_t and q_t, respectively. Firms sell the endowment to households and the lord and receive coins in exchange. Moreover, n_t^n coins are minted and μ_t^n new coins and μ_t^o old coins are melted. If coins are minted from jewelry, households pay the same fee as when they return coins on the re-coinage date. Then, the profits are returned to the households in the form of dividends. Finally, on the re-coinage date, households decide on the share s_t^n of coins that is to be handed in to the firm for re-minting into new coins.

4.1.1 The firm

The firm profits are

$$\Pi_t = p_t\left(c_t + g_t\right) + n_t^n - \mu_t^n + q_t h_t - e_t \mu_t^o, \tag{1}$$

[17]A motivation for competitive mints is that, e.g., in the 11th–12th centuries, England had at up to approximately 70 active mints active at some points; see Allen (2012, p. 16 and p. 42f). Moreover, these mints were sometimes farmed out; see Allen (2012, p. 9).

where g_t is lord consumption in period t, μ_t^n and μ_t^o denote melting of new and old coins, n_t^n minting of new coins and h_t jewelry supply. Note that coin holdings m_t^n and m_t^o are determined before firm dividends are disbursed to households and are chosen in period $t-1$ for use in period t. Mintage must be non-negative and melting cannot exceed the stock of new and old coins $m_t^n + \Pi_{t-1}$ and m_t^o, respectively. Moreover, coins are defined by the number b of grams of silver per coin. Thus, the firm faces the following constraints, related to mintage and melting, $n_t^n \geq 0$, $m_t^n + \Pi_{t-1} \geq \mu_t^n \geq 0$, $m_t^o \geq \mu_t^o \geq 0$ and $h_t = b\left(\mu_t^n + \mu_t^o - n_t^n\right)$. The firm maximizes its profits in (1) subject to these constraints and

$$c_t + g_t \leq \xi_t. \tag{2}$$

From the firm's first-order conditions, if

$$\text{if } \frac{1-\tau}{b} \geq q_t \text{ then } n_t^n \geq 0 \tag{3}$$
$$\text{if } \frac{1-\tau}{b} < q_t \text{ then } n_t^n = 0. \tag{4}$$

Thus, as long as the price of jewelry is too high, i.e., $q_t > \frac{1-\tau}{b}$, it is not profitable for the firm to buy jewelry in order to mint new coins. On the other hand, if the jewelry price were lower than $\frac{1-\tau}{b}$, firms would make positive profits on mintage. Due to the constant returns technology, this would lead to an infinite demand for jewelry. Equilibrium then requires that $\frac{1-\tau}{b} \leq q_t$ with equality, whenever $n_t^n > 0$

Firm optimization leads to the following conditions for the melting of new coins;

$$\text{if } \frac{1}{b} > q_t \text{ then } \mu_t^n = 0 \tag{5}$$
$$\text{if } \frac{1}{b} < q_t \text{ then } \mu_t^n = m_t^n + \Pi_{t-1} \tag{6}$$
$$\text{if } \frac{1}{b} = q_t \text{ then } \mu_t^n \in \left(0, m_t^n + \Pi_{t-1}\right). \tag{7}$$

Hence, if the jewelry price is too low, i.e., $\frac{1}{b} > q_t$, it is not profitable for the firm to melt coins and transform them into jewelry. If the price is higher than $\frac{1}{b}$ then the firm makes a positive profit on each new coin that it melts. Once more, due to the constant returns technology, the demand for new coins to be melted by the firm is infinite. Competition then forces the equilibrum jewelry price to $\frac{1}{b}$. Repeating the same for μ_t^o gives

$$\begin{aligned} \text{if } \frac{e_t}{b} &> q_t \text{ then } \mu_t^o = 0 && (8) \\ \text{if } \frac{e_t}{b} &< q_t \text{ then } \mu_t^o = \chi m_t^o. && (9) \end{aligned}$$

Thus, in equilibrium, we have $\frac{e_t}{b} = q_t$ when μ_t^o is interior. The intuition is similar to expressions (5)-(7), with the modification that the cost of buying old coins is e_t instead of one. In equilibrium, the equilibrium jewelry price cannot be higher than $\frac{e_t}{b}$, since competition forces the price to $\frac{e_t}{b}$.

4.1.2 The household

The household preferences are

$$\sum_{t=0}^{\infty} \beta^t \left[u\left(c_t\right) + v\left(d_{t+1}\right)\right]. \tag{10}$$

Both u and v are assumed to be strictly increasing and strictly concave. We impose the standard Inada conditions so that $\lim_{c_t \to 0} u'\left(c_t\right) \to \infty$ and $\lim_{d_t \to 0} v'\left(d_t\right) \to \infty$. Households own an endowment ξ_t of the consumption good and silver S_t. Following Velde and Weber (2000), the endowment is transferred to firms in return for a claim on profits. The household maximizes utility in (10), subject to the law of motion for jewelry

$$d_{t+1} = d_t + h_t + S_{t+1} - S_t \tag{11}$$

the CIA constraint

$$p_t c_t + q_t h_t \leq \left(\left(1 - \mathbb{I}_t\right) + \mathbb{I}_t \left(1 - \tau\right) s_{t-1}^n\right)\left(m_t^n + \Pi_{t-1}\right) + e_t \chi \left(m_t^o + \mathbb{I}_t \left(1 - s_{t-1}^n\right)\left(m_t^n + \Pi_{t-1}\right)\right) \tag{12}$$

where $\mathbb{I}_t = 1$ if $t = T+1, 2T+1, 3T+1$ and 0 otherwise, and where s_{t-1}^n is the share of new coins handed in by households for re-coinage at the time of withdrawal and the budget constraint

$$\begin{aligned} m_{t+1}^{n}+e_{t}m_{t+1}^{o} \quad &\leq \quad \left(\left(1-\mathbb{I}_{t}\right)+\mathbb{I}_{t}\left(1-\tau\right)s_{t-1}^{n}\right)\left(m_{t}^{n}+\Pi_{t-1}\right) \\ &\quad +e_{t}\chi\left(m_{t}^{o}+\mathbb{I}_{t}\left(1-s_{t-1}^{n}\right)\left(m_{t}^{n}+\Pi_{t-1}\right)\right)-p_{t}c_{t}-q_{t}h_{t}. \end{aligned} \tag{13}$$

Also, $c_t \geq 0$, $m_{t+1}^n \geq 0$, $m_{t+1}^o \geq 0$ and $h_t \geq -d_t - (S_{t+1} - S_t)$ and, for $t = T, 2T, \ldots$ $s_t^n \in [0,1]$. Note that the Inada condition with respect to jewelry implies that the last constraint never binds.

We now derive the household Euler equation. Using the first-order condition with respect to c_t and h_t, the first-order condition with respect to d_{t+1} can be written as[18]

$$u'(c_t)\frac{q_t}{p_t} = \beta u'(c_{t+1})\frac{q_{t+1}}{p_{t+1}} + v'(d_{t+1}). \tag{14}$$

As usual, the Euler equation describes the consumption-savings trade-off in the model. However, since the model has commodity money, the expression is slightly different than in standard macro models; see e.g., Gali (2008) equation (7). To get the intuition behind expression (14), consider a consumer that chooses to save some more by reducing consumption today and holding some extra jewelry, in order to increase consumption tomorrow. The decrease in consumption today leads to a decrease in utility through $u'(c_t)$, and is transformed into jewelry at the relative price $\frac{q_t}{p_t}$. When holding some extra jewelry, this gives the consumer a direct payoff effect through $v'(d_{t+1})$ and an indirect effect through an increase in consumption tomorrow. The increase in c_{t+1} is discounted by β and the stored jewelry is sold at the relative price $\frac{q_{t+1}}{p_{t+1}}$. Note that the real interest rate in this model is given by

$$\frac{q_{t+1}/q_t}{p_{t+1}/p_t}, \tag{15}$$

i.e., gross jewelry inflation divided by gross goods inflation.

The first-order conditions are illustrated in Appendix A.2. Here, we describe those used in the analysis below assuming $c_t > 0$ and $p_t > 0$ for all t, which holds in equilibrium. Whether old or new coins are held depend on how exchange rates affect their relative return. Using the first-order conditions with respect to c_t and m_{t+1}^n, if $m_{t+1}^o > 0$ then, if

[18]Note that since jewelry is a consumer durable good, the Euler equation here is similar to Euler equations in such models; see e.g., equation (5) in Barsky, House, and Kimball (2007).

$t \neq T+1, 2T+1$ etc.,

$$\frac{e_{t+1}\chi}{e_t} \geq 1 \tag{16}$$

and, if $t = T+1, 2T+1$ etc.,

$$\frac{e_{t+1}\chi}{e_t} \geq (1-\tau)\, s_t^n + e_{t+1}\chi \left(1 - s_t^n\right), \tag{17}$$

otherwise. Since the consumer holds old coins in period $t+1$, the exchange rates in periods t and $t+1$ have to give the consumer incentives not to only hold new coins. Then, it follows that the exchange rate has to increase by at least $\frac{1}{\chi}$ between adjacent periods, except in the withdrawal period. The appreciation of the exchange rates compensates the consumer for the loss due to confiscations by the lord plaintiff so that the consumer does not lose in value terms by holding an old coin, relative to new coins, for an additional period. The condition is slightly different for the withdrawal period, due to the fact that the return on holding new coins changes for two reasons. First, holding a new coin for an additional period relaxes the cash in advance constraint by $1-\tau$ instead of one. Second, the consumer can choose not to hand in new coins for re-mintage, rendering them old coins in the next period, valued at e_{t+1} and subject to confiscation by the plaintiff at rate $1-\chi$. Between these two options, the consumer optimally chooses the fraction s_t^n to hand in for re-mintage; see equations (20)-(22) below.

If $m_{t+1}^n > 0$, if $t \neq T+1, 2T+1$ etc.,

$$\frac{e_{t+1}\chi}{e_t} \leq 1. \tag{18}$$

Since the consumer now holds new coins in period $t+1$, the exchange rates in period t and $t+1$ have to give the consumer incentives to not only hold old coins. For this to be the case, the exchange rate increase cannot be too large and is bounded above by $\frac{1}{\chi}$. If $t = T+1, 2T+1$ etc.,

$$\frac{e_{t+1}\chi}{e_t} \leq \left((1-\tau)\, s_t^n + e_{t+1}\chi \left(1 - s_t^n\right)\right). \tag{19}$$

Finally, the household also optimally chooses the share of coins to be handed in for re-coinage, s_t^n in periods $t \neq T, 2T$ etc;

$$\text{if } s_t^n \in (0,1) \text{ then } e_{t+1}\chi = 1 - \tau \tag{20}$$

$$\text{if } s_t^n = 0 \text{ then } e_{t+1}\chi \geq 1 - \tau \tag{21}$$

$$\text{if } s_t^n = 1 \text{ then } e_{t+1}\chi \leq 1 - \tau. \tag{22}$$

When choosing how to allocate the new coins in period T to new and old coins in the next period, the household takes into account its relative returns. When handing in a coin for re-mintage, the return is $1 - \tau$, while when not handing it in and letting it become an old coin in the next period, it is valued to e_{t+1} and risks confiscation with probability $1 - \chi$, rendering the return $e_{t+1}\chi$. Thus, if τ is low enough, all new coins are re-minted (when $1 - \tau > e_{t+1}\chi$) and if it is too low, no new coins are re-minted (when $1 - \tau < e_{t+1}\chi$).

4.1.3 The lord

The lord gets revenue from coin withdrawals and confiscation of illegal coins. The lord costlessly re-mints all confiscated old coins into new ones. Letting $m_t^L \geq 0$ denote coins stored by the lord, the lord budget constraint is

$$m_{t+1}^L = \tau \mathbb{I}_t s_{t-1}^n \left(m_t^n + \Pi_{t-1}\right) + \tau n_t^n + (1 - \chi) m_t^o + m_t^L - p_t g_t. \tag{23}$$

Thus, the lord uses revenues from money withdrawals through m_t^n, from new mintage through n_t^n, confiscations through m_t^o and previously stored coins m_t^L to spend on consumption g_t and coins stored to the next period m_{t+1}^L. In equilibrium, government spending is determined by the revenues generated by the Gesell tax τ and the plaintiff confiscation probability $1 - \chi$. Given sequences $\{p_t\}$, $\{m_t^L\}$, $\{m_t^n\}$, $\{\Pi_t\}$, $\{n_t^n\}$, $\{m_t^o\}$, a feasible sequence of government spending $\{g_t\}$ satisfies (23) for all t and that, in the spirit of Leeper (1991),

$$g_t = f(x_t) \tag{24}$$

where x_t are variables that affect spending. Below, when analyzing equilibria, we put additional restrictions on g_t.

4.1.4 Money transition and resource constraints

The household stocks of new and old coins evolve according to

$$m_{t+1}^{n} = \left((1-\mathbb{I}_t)+\mathbb{I}_t(1-\tau)s_{t-1}^{n}\right)(m_t^n+\Pi_{t-1})+(1-\tau)n_t^n-\mu_t^n \tag{25}$$

$$m_{t+1}^{o} = \chi m_t^o-\mu_t^o+\mathbb{I}_t\left(1-s_{t-1}^{n}\right)(m_t^n+\Pi_{t-1}). \tag{26}$$

The holdings of new coins, m_{t+1}^n, in expression (25) depend on the previous stock net the Gesell tax $\left((1-\mathbb{I}_t)+\mathbb{I}_t(1-\tau)s_{t-1}^{n}\right)(m_t^n+\Pi_{t-1})$, net dividends from firms $\Pi_{t-1}=p_{t-1}g$ (noting that the Lord only spends new coins; see section 4.1.3), mintage net of mintage fee $(1-\tau)n_t^n$ and melting μ_t^n. The holdings of old coins, m_{t+1}^o, in expression (26) depend on the previous stock net of plaintiff confiscations χm_t^o, melting μ_t^o and new coins not handed in for re-coinage $\mathbb{I}_t\left(1-s_{t-1}^{n}\right)(m_t^n+\Pi_{t-1})$.

Finally, we have the goods resource constraint

$$c_t+g_t=\xi_t \tag{27}$$

and the silver resource constraint

$$b\left(m_t^n+m_t^L\right)+d_t=S_t. \tag{28}$$

4.2 Equilibria

Definition 1 *An equilibrium is a collection* $\{m_{t+1}^n\}$, $\{m_{t+1}^o\}$, $\{m_{t+1}^L\}$, $\{n_t^n\}$, $\{\mu_t^n\}$, $\{\mu_t^o\}$, $\{c_t\}$, $\{g_t\}$, $\{d_{t+1}\}$, $\{h_t\}$, $\{p_t\}$, $\{q_t\}$ *and* $\{e_t\}$ *such that* $i)$ *the household maximizes (10) subject to (11), (12), (13),* $s_t^n\in[0,1]$, *the boundary constraints and the jewelry constraint;* $ii)$ *the firm maximizes (1) subject to it's constraints and (2);* $iii)$ $c_t+g_t=\xi_t$ *and that (25), (26), (23), (24) and (28) hold.*

For the rest of the analysis, we assume that the endowment is constant; $\xi_t=\xi$. Furthermore, $S_{t+1}=S_t=S$ and hence, the jewelry stock evolves according to

$$d_{t+1}=d_t+h_t. \tag{29}$$

For the lord, we assume e.g. that the Lord keeps a standing army of the same size every period that needs to be sustained by consumption goods. Specifically, we assume that revenues from withdrawals are spread equally across periods and hence $g_t = g$ in every period. Moreover, the budget is balanced over the cycle and $m_t^L < S$ for all t. Thus, summing the lord constraint (23) over $t = 1$ to T

$$\sum_{t=1}^{T} p_t g_t = \tau \left(m_1^n + \Pi_T\right) + \sum_{t=1}^{T} \left(1 - \chi\right) m_t^o. \tag{30}$$

Note that due to the fact that money withdrawals occur infrequently, i.e., every T'th period, a steady state cannot be expected to exist. Therefore, we instead restrict the attention to *cyclical equilibria*, as defined below. As mentioned above, an issue of coins is only legal tender for T periods. Consider an issue with length T where an issue starts just after a withdrawal and ends just before the next withdrawal. Let $L_r^T = \{\tilde{r} : \tilde{r} = nT + r$ for $n \in N^+\}$ denote all time periods corresponding to a given period r in an issue.

Definition 2 *Given that money withdrawals occur every T'th period, an equilibrium is said to be* ***cyclical*** *if it satisfies* $m_{\hat{r}}^n = m_{\bar{r}}^n$, $m_{\hat{r}}^o = m_{\bar{r}}^o$, $m_{\hat{r}}^L = m_{\bar{r}}^L$, $n_{\hat{r}}^n = n_{\bar{r}}^n$, $\mu_{\hat{r}}^n = \mu_{\bar{r}}^n$, $\mu_{\hat{r}}^o = \mu_{\bar{r}}^o$, $d_{\hat{r}} = d_{\bar{r}}$, $h_{\hat{r}} = h_{\bar{r}}$, $p_{\hat{r}} = p_{\bar{r}}$, $q_{\hat{r}} = q_{\bar{r}}$ *and* $e_{\hat{r}} = e_{\bar{r}}$ *for all* $r \in \{1, \ldots, T\}$ *such that* $\hat{r}, \bar{r} \in L_r^T$.

The definition of cyclicality requires that, at the same point in two different issues and, the variables attain the same value, i.e., e.g., $m_{\hat{r}}^n = m_{\bar{r}}^n$. Note that using that government spending is constant over time and since $c_t = \xi - g$, we do not need to put any restrictions on consumption.

4.3 An example

The below example illustrates how to find a cyclical equilibrium when there is a withdrawal of coins every second period. As we will see in section 4.4, many of the results carry over to the general case.

Example 1 *Withdrawals occur every second period and only new coins are held in equilibrium. Also, for simplicity, we set* $m_1^L = 0$. *We first show that minting is zero in equilibrium. Noting that if* $n_1^n > 0$ *then, by cyclicality, we have* $\mu_2^n = n_1^n > 0$, *and hence, using (3) and (7),* $q_1 = \frac{1-\tau}{b}$ *(from competition between firms) and* $q_2 = \frac{1}{b}$. *Thus, using* *the CIA constraint (12) and the money transition equation (25) we have, using cyclicality (i.e., $m_3^n = m_1^n$), $\Pi_2 = p_2 g$ and noting that $c = c_1 = c_2$,*

$$\begin{aligned} p_1 c &= m_2^n \\ p_2 c &= m_1^n \end{aligned} \tag{31}$$

for $t = \{1, 2\}$. A similar result can be established when $n_2^n > 0$ and when $n_1^n = n_2^n = 0$. Note also that $s_2^n = 1$, since no old coins are held.

There are three candidate equilibria; i) $n_1^n > 0$, $n_2^n = 0$ and $\mu_1^n = 0$, $\mu_2^n = n_1^n$; ii) $n_2^n > 0$, $n_1^n = 0$ and $\mu_1^n = n_2^n$, $\mu_2^n = 0$; iii) $n_t^n = \mu_t^n = 0$ for $t = 1, 2$.

First, suppose that $n_1^n > 0$ so that $q_1 = \frac{1-\tau}{b}$ and $q_2 = \frac{1}{b}$. Using the money transition equation (25) and $\Pi_2 = m_1^n \frac{g}{c}$, we have

$$m_2^n = (1-\tau)\left(1+\frac{g}{c}\right) m_1^n + (1-\tau)\, n_1^n > (1-\tau)\left(1+\frac{g}{c}\right) m_1^n. \tag{32}$$

Then, using (31), $p_1 > (1-\tau)\left(1+\frac{g}{c}\right) p_2$ so that the return on jewelry holdings is

$$\frac{q_2/q_1}{p_2/p_1} > (1-\tau)\left(1+\frac{g}{c}\right)\frac{1}{1-\tau} = 1 + \frac{g}{c}. \tag{33}$$

Since $n_1^n > 0$, using $h_t = b\left(\mu_t^n + \mu_t^o - n_t^n\right)$, (11) and (29), we have $d_2 < d_1$ so that $v'(d_2) > v'(d_1)$ and hence, using the Euler equation (14),

$$\frac{q_1}{p_1} - \beta\frac{q_2}{p_2} > \frac{q_2}{p_2} - \beta\frac{q_1}{p_1}. \tag{34}$$

Then, we have $\frac{q_1}{p_1} > \frac{q_2}{p_2}$, a contradiction.

The reason why an equilibrium does not exist is that the high return in (33) implies that households have incentives to save in jewelry in period 1, contradicting $n_1^n > 0$. The equilibrium where $n_2^n > 0$ can also be ruled out.[19] *Thus, the only equilibrium has $n_t^n = \mu_t^n = 0$ for $t = 1, 2$.*

[19] If $n_2^n > 0$ then, using cyclicality, $\mu_1^n > 0$ and hence $q_1 = \frac{1}{b}$ and $q_2 = \frac{1-\tau}{b}$. Then, from the money transition equations, we have $m_1^n > m_2^n$ so that $p_2 > p_1$ from the CIA constraints. Thus, $\frac{q_1}{p_1} > \frac{q_2}{p_2}$. Since $d_2 > d_1$ so that $v'(d_2) < v'(d_1)$ we have, using (34) with the inequality reversed, $\frac{q_1}{p_1} < \frac{q_2}{p_2}$, again a contradiction. Due to the low return of savings in jewelry between periods 1 and 2, as indicated by the fall in the jewelry price (see also (15)), households do not want to transfer coins into jewelry by melting in period 1 and thus, this cannot be an equilibrium.

Since the equilibrium entails neither minting nor melting, using money transition (25) $m_1^n = m_2^n + \Pi_1$, we get that $m_1^n > m_2^n$, in turn implying that prices increase over the cycle (i.e., $p_2 > p_1$) following from a modified quantity theory argument using expression (31).[20]

Example 1 continued. *We now describe prices in equilibria where* $n_t^n = \mu_t^n = 0$ for $t = 1, 2$. *From cyclicality, money transition (25) and the CIA constraint (12)* $m_1^n = \frac{\xi}{\xi - g} m_2^n$. *Moreover, using money transition (25) and (31), we have* $m_2^n = (1-\tau)\frac{\xi}{\xi-g} m_1^n$. *Since, using (31),* $p_1 = (1-\tau)\frac{\xi}{\xi-g}p_2$ *and* $p_2 = \frac{\xi}{\xi-g}p_1$ *we have* $\frac{\xi-g}{\xi} = \sqrt{1-\tau}$ *so that, using (27),* $c = \sqrt{1-\tau}\xi$. *A spending rule (24) implementing this is then* $f = \left(1 - \sqrt{1-\tau}\right)\xi$. *Then, goods prices increase by* $\frac{1}{\sqrt{1-\tau}}$ *between periods* 1 *and* 2*;*

$$p_2 = \frac{1}{\sqrt{1-\tau}} p_1. \tag{35}$$

Finally, the relative jewelry price can be determined by using $d_1 = d_2$ *in the Euler equation, implying that the direct marginal utility payoff from jewelry holdings is constant over the cycle, i.e.,* $v'(d_1) = v'(d_2)$. *Then, we have* $\frac{q_1}{p_1} = \frac{q_2}{p_2}$ *and thus, using (35),* $q_2 = \frac{1}{\sqrt{1-\tau}}q_1$. *Since* $q_2 \leq \frac{1}{b}$, *any combination of jewelry prices such that* $q_1 \in [\frac{1-\tau}{b}, \frac{\sqrt{1-\tau}}{b}]$ *is feasible. Each such jewelry price is associated with a unique level of money holdings via the Euler equation. These equilibria can be Pareto ranked with the equilibrium yielding the highest welfare being associated with the lowest jewelry price.*[21] *For the purpose we are interested in, all equilibriia have the same properties, though. Finally, consider exchange rate restrictions for the equilibrium. Since households hold only new coins and* $s_2^n = 1$, *from cyclicality, (18), (19) and (22), we have* $e_1\chi \leq (1-\tau)e_2$, $e_2\chi \leq e_1$ and $e_1\chi \leq 1-\tau$. *Combining gives the following requirement for households to hold only new coins in equilibrium;*

$$1 - \tau \geq \chi^2. \tag{36}$$

Since there is neither mintage nor melting, household coin holdings increase by $\frac{1}{\sqrt{1-\tau}}$ between period 2 and 1 and decrease by $\sqrt{1-\tau}$ between period 1 and 2. The modified Cash in Advance constraint (31) then implies that prices decrease by $\sqrt{1-\tau}$ at the end of the cycle (between period 2 and 1) and increase by $\frac{1}{\sqrt{1-\tau}}$ during the cycle (between period

[20] Instead of the usual (12).

[21] See the proof of Theorem 1 in the Appendix.

1 and 2). Since jewelry holdings are constant, jewelry relative prices $\frac{q_t}{p_t}$ are constant over the cycle so that the jewelry price changes one to one with goods prices.

Example 1 continued. *We now consider equilibria where both new and old coins are held. From cyclicality and the money transition equation (25), we have*

$$\begin{aligned} p_1 c &= m_2^n + e_1 \left(\chi \left(m_1^o + \left(1 - s_2^n\right)\left(m_1^n + \Pi_2\right)\right) - \mu_1^o\right) \\ p_2 c &= m_1^n + e_2 \left(\chi m_2^o - \mu_2^o\right). \end{aligned} \tag{37}$$

As in the case when only new coins are held, we can show that mintage and melting are always zero so that $n_t^n = \mu_t^n = \mu_t^o = 0$ for $t = 1, 2$; see Lemmata 2 - 3 below for details. Moreover, for cyclicality, we require that $s_2^n < 1$ and $m_1^n > 0$, *since otherwise $m_1^o = m_3^o \leq \chi m_2^o \leq \chi^2 m_1^o$, a contradiction. Using cyclicality, (16) - (18) and (20) - (21), the conditions on exchange rates are $\frac{e_1\chi}{e_2} = (1-\tau)\, s_2^n + (1 - s_2^n)\, e_1\chi$, $e_2\chi = e_1$* and *$e_1\chi \geq 1 - \tau$. If $s_2^n > 0$ then $e_1\chi = 1 - \tau$ and thus $e_2 = 1$* and *$1 - \tau = \chi^2$. If $s_2^n = 0$ so that $e_1\chi \geq 1 - \tau$ we have $1 - \tau \leq \chi^2$,* but again *$e_2 = 1$. Focusing on the case when $1 - \tau < \chi^2$ so that $s_2^n = 0$, using cyclicality, $m_1^o = \chi m_2^o$ and $m_2^o = \chi\left(m_1^o + m_1^n + \Pi_2\right)$, gives $m_1^o = \frac{\chi^2}{1-\chi^2}\left(m_1^n + \Pi_2\right)$. Then, the Cash in Advance constraint in period* 2 *is, using (25), $e_2 = 1$ and $e_1 = \chi$,*

$$p_2 = \frac{1}{\xi\left(1-\chi^2\right) - g} m_1^n. \tag{38}$$

Since, using cyclicality, $m_1^n = m_2^n + p_1 g = m_2^n + \frac{g}{\xi}\frac{1}{1-\chi^2} m_1^n$, the Cash in Advance constraint in period 1 *is,* using $p_2 g = \frac{g}{\xi(1-\chi^2)-g} m_1^n$,

$$p_1 \xi = \frac{\xi - g}{\xi\left(1-\chi^2\right) - g} m_1^n \tag{39}$$

so that $p_1 = \frac{\xi - g}{\xi} p_2$. Akin to the case where only new coins are held, c.f., expression (35), we now find an expression for the price change in terms of parameters of the model. For this purpose, first note that the government revenues over a cycle are $(1-\chi)\left(m_1^o + m_1^n + \Pi_2 + m_2^o\right)$. Using $m_1^o = \frac{\chi^2}{1-\chi^2}\left(m_1^n + \Pi_2\right)$ and *$m_2^o = \frac{\chi}{1-\chi^2}\left(m_1^n + \Pi_2\right)$, the revenues are $\frac{\xi\left(1-\chi^2\right)}{\xi(1-\chi^2)-g} m_1^n$. Using the Cash in Advance constraints (38) and (39), government spending is*

$$p_1 g + p_2 g = \frac{g}{\xi}\frac{\xi}{\xi\left(1-\chi^2\right) - g}\left(2 - \frac{g}{\xi}\right) m_1^n. \tag{40}$$

For revenues to equal spending, we require that $\chi = 1 - \frac{g}{\xi}$ *and thus* $p_2 = \frac{1}{\chi}p_1$*.Once more, prices increase during the cycle and fall at the start of a new cycle. Then, from the Euler equation (14) we have, using* $p_1 = \chi p_2$*, (38) and (39), and that jewelry holdings are constant over the cycle, i.e.,* $d_1 = d_2$, $q_2 = \frac{1}{\chi}q_1$*. Since* $q_2 \leq \frac{1}{b}$*, any combination of jewelry prices such that* $q_1 \in [\frac{1-\tau}{b}, \frac{\chi}{b}]$ *is feasible. Note that both new and old coins are held in equilibrium, since from (25), we have* $m_1^n = \Pi_2 + m_2^n > 0$.

4.4 The general case

This section shows the existence of and analyzes properties of equilibria in the general case. By using money transition (25) in the CIA constraint (12), we can derive the following Lemma, akin to expression (31) in example 1.

Lemma 1 *The CIA constraint (12) is, when* $t \neq T+1$

$$p_t c = m_{t+1}^n + e_t \left(\chi m_t^o - \mu_t^o\right) \tag{41}$$

and, when $t = T+1$

$$p_t c = m_{t+1}^n + e_t \left(\chi \left(m_t^o + \left(1 - s_T^n\right)\left(m_t^n + \Pi_{t-1}\right)\right) - \mu_t^o\right). \tag{42}$$

Proof: See the appendix. ■

We now show that there is neither minting nor melting in equilibrium. First, we show that there can only be minting in the first period of a cycle.

Lemma 2 *Mintage can be positive only in period* 1*.*

To see this, suppose that only new coins are held so that $m_t^o = \mu_t^o = 0$ for all t. It is convenient to rearrange the Euler equation (14) as, using that consumption is constant,

$$p_t = Q_t \left(q_t, q_{t-1}, d_t, p_{t-1}\right) p_{t-1} \tag{43}$$

where

$$Q_t \left(q_t, q_{t-1}, d_t, p_{t-1}\right) = \beta \frac{q_t u'(c)}{q_{t-1} u'(c) - v'(d_t) p_{t-1}}. \tag{44}$$

To get a clearer idea about expression (44), it might be advantageous to think about the intertemporal consumption choice in terms of calculus of variation. The denominator in (44) is the (negative of the) nominal change in payoff when decreasing consumption at $t-1$ by increasing the jewelry holdings and the numerator is the change in payoff from the resulting increase in tomorrow's consumption that follows if future jewelry holdings d_{t+s} and consumption decisions c_{t+1+s} are unchanged for $s > 0$.

Now, let us look at why the mintage must be zero, except at the initial period of the cycle. If mintage is positive in some period $t > 1$, i.e., $n_t^n > 0$, then the jewelry price q_t is equal to $\frac{1-\tau}{b}$ and is thus low in that period, c.f. equation (33) in example 1. Then $\mu_t^n = 0$. Moreover, using money transition (25) and Lemma 1, we have $m_{t+1}^n > \frac{\xi}{\xi-g} m_t^n$ and then, by Lemma 1, prices increase so that $Q_t > 1$. Since prices increase $(p_t > p_{t-1})$, jewelry holdings decrease $(d_{t+1} < d_t)$ so that $v'(d_t) < v'(d_{t+1})$ and jewelry prices weakly increase $(q_{t+1} \geq q_t)$ in period $t+1$ households have even stronger incentives to postpone consumption and, using (44), we have $Q_{t+1} > Q_t$. Then, prices in the next period increase even more. Money transition (25) and Lemma 1 then imply that there is positive mintage also in the next period. Induction then establishes that mintage is positive in all periods, thus violating cyclicality. Since the Gesell tax is collected in the first period, this argument does not work starting at $t = 1$, since even if we have positive minting in the first period, then (25) and Lemma 1 only imply that $m_2^n > (1-\tau) \frac{\xi}{\xi-g} m_1^n$ so that m_2^n can be lower than m_1^n, in turn allowing p_2 to be lower than p_1.[22]

The next lemma shows that there is no melting of coins during a cycle. As a corollary, it then follows by cyclicality that there cannot be minting in the first period of a cycle.

Lemma 3 *There is no melting of either new or old coins.*

Proof: See the appendix. ■

To see this, note first, akin to models with durable consumption goods, that we can rewrite the relative jewelry price, by repeatedly using future Euler equations (14) as the

[22] Note that this initial period argument does not affect the induction argument when starting at $n_t^n > 0$ for $t > 1$. The reason for this is twofold. First, since we only use the the Cash in Advance constraint from Lemma 1 at the start of the induction argument, we must have $m_{t+1}^n > \frac{\xi}{\xi-g} m_t^n$ for that period and thus $Q_t > 1$ (instead of $Q_t > (1-\tau) \frac{\xi}{\xi-g}$). Second, the rest of the induction argument is based on the Euler equation as described in (44), which does not depend on the Gesell tax τ.

discounted value of future jewelry holdings, measured in monetary terms[23]

$$\frac{q_t}{p_t} = \frac{1}{1-\beta^T} \sum_{s=0}^{T} \beta^s \frac{v'(d_{t+s+1})}{u'(\bar{c})}. \tag{46}$$

We here focus on the case where only new coins are held during the entire cycle (so that $s_T^n = 1$). Assume that $\mu_t^n > 0$ for some t. Then, by cyclicality, we must have $n_1^n > 0$ from Lemma 3. Since there can be minting only in the first period, using (11), we have that jewelry holdings are increasing over the cycle; $d_2 \leq d_3 \leq \ldots \leq d_{T+1}$ where the inequality is strict for some t. Then, using strict concavity of v in (46), we have $\frac{q_{T+1}}{p_{T+1}} > \frac{q_T}{p_T}$, since jewelry holdings are valued higher in period $T+1$ than in period T. In fact, as is shown in the proof of Lemma 3, for $t = 2, \ldots, T$

$$\frac{q_{t+1}}{p_{t+1}} > \frac{q_t}{p_t}. \tag{47}$$

Thus, the real return, as defined in (15), increases during a cycle. Since mintage is positive in the first period, we have $q_1 = \frac{1-\tau}{b}$. Then, using money transition (25) and rewriting $\frac{\xi}{\xi-g}$, we have $m_2^n > \frac{1-\tau}{1-\frac{g}{\xi}} m_1^n$ and thus, using Lemma 1 and that the relative goods price between T and $T+1$, $\frac{p_T}{p_{T+1}}$ is smaller than $\frac{1-\frac{g}{\xi}}{1-\tau}$. Then, using $q_{T+1} = \frac{1-\tau}{b}$ and that the real return is high, as can be seen by (47), q_T is bounded from above;

$$q_T < \left(1 - \frac{g}{\xi}\right) \frac{1}{b}. \tag{48}$$

Thus, since the relative price between T and $T+1$ changes by less than $\frac{1-\frac{g}{\xi}}{1-\tau}$ and $q_{T+1} = \frac{1-\tau}{b}$, it follows that q_T has to be smaller than $\frac{1}{b}$, in turn making it unprofitable to melt coins by (5); $\mu_T^n = 0$. From money transition $m_{T+1}^n = \frac{\xi}{\xi-g} m_T^n$, in turn implying that $p_T \left(1 - \frac{g}{\xi}\right) = p_{T-1}$ from Lemma 1, so that prices increase by $\frac{1}{1-\frac{g}{\xi}}$. Then, again using that real returns are increasing, i.e., we have $q_T \frac{p_{T-1}}{p_T} > q_{T-1}$ from (47), and repeating a similar argument we then have $q_{T-1} < \left(1 - \frac{g}{\xi}\right)^2 \frac{1}{b}$ and thus, $\mu_{T-1}^n = 0$. Induction then

[23] In general, we have

$$\frac{q_t}{p_t} = \sum_{s=0}^{\infty} \beta^s \frac{u'(c_{t+s})}{u'(c_t)} \frac{v'(d_{t+s+1})}{u'(c_{t+s})}. \tag{45}$$

However, using cyclicality and the fact that consumption is constant, we can write the relative price as functions of jewelry holdings during a cycle as in (46).

establishes that $q_t < \left(1 - \frac{g}{\xi}\right)^{T-t+1} \frac{1}{b}$, in turn implying that melting is zero for all t by (5). The argument when old coins are held is slightly more complicated and is dealt with in the appendix.

Combining Lemmata 2 - 3 implies that mintage must be zero in period 1 as well. The reason is that, since coins are never melted, for cyclicality to hold, there cannot be any mintage in any period. We then have the following theorem.

Theorem 1 *A cyclical equilibrium exists and entails $n_t^n = \mu_t^n = \mu_t^n = 0$ for all t. If $1 - \tau > \chi^T$ $(1 - \tau < \chi^T)$, in any cyclical equilibrium, only new (both new and old) coins are held. If $1 - \tau = \chi^T$ either only new or both new and old coins are held. In any equilibrium, prices increase during an issue, i.e., $p_t > p_{t-1}$ for $t = 2, \ldots, T$ and drop between periods T and $T + 1$. If $1 - \tau \geq \chi^T$ prices increase at the rate $(1 - \tau)^{-\frac{1}{T}}$ during a cycle and if $1 - \tau < \chi^T$ prices increase at the rate χ^{-1} and no coins are handed in for re-coinage.*

Proof: See the appendix. ■

Suppose that only new coins are held. The results for increasing prices follow from the fact that money transition (25) implies that household money holdings increase over the cycle, due to the fact that firm dividends from government consumption increase household money holdings, so that, using a quantity theory argument and Lemma 1, prices increase. A modification of this argument establishes a similar result when also old coins are held. As long as only new coins are held, price increases are higher the higher is the Gesell tax, since a higher Gesell tax leads to higher government spending and, in turn, a higher increase in household money holdings during a cycle. When $1 - \tau < \chi^T$ so that old coins are also held, price increases only depend on the plaintiff confiscation rate χ. The reason is that since no coins are handed in for re-coinage, the only source of government revenues is the confiscation of illegal coins and thus, χ is the sole determinant of government spending and hence, of the increase in money holdings during a cycle. Since the nominal return is $\frac{q_1}{q_T} = 1 - \tau$ when the Gesell tax is levied, nominal returns can be substantially negative - empirical evidence on the tax indicate that the implied yearly returns is as low as -44 percent at the date of tax collection. However, since goods prices fall simultaneously, due to the reduction in money holdings, real returns are unchanged.

The cutoff values for whether old coins are held depend on χ and τ. The reason for these cutoffs is that, assuming that both types are held, using (16) and (18), the exchange rate must appreciate at rate $\frac{1}{\chi}$ so that $e_{t+1}\chi = e_t$ and, from (17), (19) together with (20) when s_T^n is interior, that $e_T = 1$. We then have

$$e_1 = \frac{1}{\chi} e_2 = \cdots = \frac{1}{\chi^{T-1}}. \tag{49}$$

Since not all new coins are handed in for re-coinage, households must weakly prefer not to hand in new coins and hence $e_1\chi \geq 1 - \tau$. Thus, $1 - \tau \leq \chi^T$. When only new coins are held, appreciation is bounded above by $\frac{1}{\chi}$, implying that $1 - \tau \geq \chi^T$.

4.5 Relationship to empirical evidence

The empirical evidence in section 3.2 indicates that new coins almost exclusively circulated in England during a period when withdrawals occurred rather infrequently (973–1035). After 1035, the intervals became shorter, which tightened the condition that $1 - \tau > \chi^T$, and if the fee was unchanged, the shorter intervals also increased the implied yearly fee. When fees increase, old coins tend to be found much more frequently in hoards, which indicates that both old and new coins circulated together. Before 1035, hoards that contain only the last issue dominate - 83 percent of the hoards have only the last type—whereas after 1035, 33 percent of the hoards contain only the last type. Regarding the number of coins from different issues in the hoards, the pattern is similar. Before 1035, the share of the last type is 86.5 percent, and after 1035, the share drops to 54.3 percent. In Germany, where re-coinage could occur as often as twice per year, hoards contain old coins even more frequently, which again indicates that old coins tended to circulate with new coins. Specifically, in Thuringia, where the tax was 25 percent and withdrawals occurred every year, the coin hoards usually contain several types; see table 2. The share of hoards that contain only the last type is 2.4 percent (and zero in Upper Lusatia), whereas the vast majority - more than 80 percent - contains three types or more.

Regarding prices, the evidence is scarce. However, some evidence of price regulation from the Frankish empire in the late 8th century seems to indicate that prices rose during a cycle, which is consistent with Theorem 1 (see also section 3.1).

5 Conclusions

A frequent method for generating revenues from seigniorage in the Middle Ages was to use Gesell taxes through periodic re-coinage. Under re-coinage, coins are legal tender only for a limited period of time. In such a short-lived coinage system, old coins are declared invalid and exchanged for new coins at publicly announced dates and exchange fees, which is similar to Gesell taxes. Empirical evidence based on several methods shows that re-coinage could occur as often as twice per year in a currency area during the Middle Ages. In contrast, in a long-lived coinage system, coins did not have a fixed period as the legal means of payment. Long-lived coins were common in western and southern Europe in the High Middle Ages, whereas short-lived coins dominated in central, northern and eastern Europe. Although the short-lived coinage system defined legal tender for almost 200 years in large parts of medieval Europe, it has seldom if ever been mentioned or analyzed in the literature of economics.

The main purpose of this study is to discuss the evidence for and analyze the consequences of short-lived coinage systems. In a short-lived coinage system, only one coin type may circulate in the currency area, and different coin types that reflect various issues must be clearly distinguishable for everyday users of the coins. The coin-issuing authority had several methods to monitor and enforce a re-coinage. First, there were exchangers and other administrators in the city markets. Second, the payment of any fees, taxes, rents, tithes or fines had to be made with the new coins. Although only new coins were allowed to be used for transactions, the evidence from coin hoards indicates that agents often also used illegal coins.

A cash-in-advance model is formulated to capture the implications of this monetary institution. The model includes households, firms and a lord, where households care about goods and jewelry consumption, and the firms care about profits. The lord uses seigniorage to finance consumption. When trading jewelry and consumption, households face a cash-in-advance constraint. Households can hold both new and old coins so that the equilibrium choice of which coins to hold is endogenous.

Our key results are that in equilibrium, prices increase over time during an issue period and fall immediately after the re-coinage date. Moreover, the higher the Gesell tax is, the greater the price increases are (as long as coins are returned for re-coinage). Furthermore, in the sense that agents participate in re-minting coins and the system generates tax revenues, the system with Gesell taxes works 1) if the tax is sufficiently low, 2) if the time period between two instances of re-coinage is sufficiently long, and 3) if the probability of being penalized for using old illegal coins is sufficiently high. Additionally, although nominal returns become negative when the Gesell tax is levied (and empirical evidence on the tax indicates substantially negative returns), real returns are unchanged because the price level adjusts accordingly as a result of the reduction in money holdings.

References

ALLEN, M. (2012): *Mints and money in medieval England.* Cambridge University Press, New York.

BARSKY, R., C. HOUSE, AND M. KIMBALL (2007): "Sticky-Price Models and Durable Goods," *American Economic Review*, 97, 984–998.

BOLTON, J. (2012): *Money in the medieval English economy: 973-1489.* Manchester University Press, Manchester and New York.

BUITER, W. (2009): "Negative nominal interest rates: three ways to overcome the zero lower bound," NBER Working Paper 15118.

BUITER, W., AND N. PANIGIRTZOGLOU (2003): "Overcoming the Zero Bound on Nominal Interest Rates with Negative Interest on Currency: Gesell's Solution," *The Economic Journal*, 113, 723–46.

BUITER, W., AND N. PANIGIRZOGLOU (1999): "Liquidity Traps: How to Avoid Them and How to Escape Them," NBER Working Paper 7245.

DOLLEY, M. (1983): "Aelfred the Great's abandonment of the concept of periodic recoinage," in *Studies in numismatic method*, ed. by C. Brooke, B. Stewart, J. Pollard, and T. Volk, pp. 153–160. Cambridge University Press.

GALI, J. (2008): *Monetary Policy, Inflation, and the Business Cycle.* Princeton University Press, Princeton and Oxford.

GESELL, S. (1906): *The natural economic order, Translated from German by Philip Pye 1958.* Peter Owen, London.

GOODFRIEND, M. (2000): "Overcoming the Zero Bound on Interest Rate Policy.," *Journal of Money, Credit and Banking*, pp. 1007–1035.

GRINDER-HANSEN, K. (2000): *Kongamagtens krise. Det danske møntvæsen 1241-1340*. Museum Tusculanums Forlag, Copenhagen.

HAUPT, W. (1974): *Sächsische Münzkunde*. Deutsche Verlag der Wisssenschaften, Berlin.

HESS, W. (2004): "Münzverrufungen der späten Pfennigzeit: Besonders in den Brakteatengebieten Mittel- und Südwestdeutschland," in *Dirham und Pfennige 2. Mittelalterliche Münzprägung in Südwestdeutschland. Zeitschrift für Archäologie des Mittelalters. Beiheft 19*, ed. by L. Ilisch, L. Sönke, and H. Steuer, pp. 11–22. Dr Rudolf Habelt GmbH, Bonn.

KLUGE, B. (2007): *Numismatik des Mittelalters: Handbuch und Thesaurus Nummorum Medii Aevi*. Münzkabinett Staatlichen Museen zu Berlin and Verlag der Österreichischen Akademie der Wissenschaften, Berlin and Vienna.

LEEPER, E. (1991): "Equilibria under 'active'and 'passive' monetary and fiscal policies," *Journal of Monetary Economics*, 27, 129–147.

MANKIW, G. (2009): "Reloading Weapons of Monetary Policy," http://gregmankiw.blogspot.se/2009/03/reloading-weapons-of-monetary-policy.html.

MEHL, M. (2011): *Münz- und Geldgeschichte des Erzbistums Magdeburg im Mittelalter. Part I and II*. Verlag Manfred Mehl, Hamburg.

MENNER, M. (2011): "Gesell tax and efficiency of monetary exchange," IVIE working paper 2011-26, Department of Economics, University of Alicante.

PASZKIEWICZ, B. (2008): "A Chronology of the Teutonic Prussian Bracteates," in *Magister Monetae. Studies in Honour of Jørgen Steen Jensen*, ed. by C. Väterlein, U. Schneider, and H. Klaiber, pp. 171–186. National Museum, Copenhagen.

SEJBAL, J. (1997): *Zaklady penezniho vyvoje*. Masarykova univerzita, Brno.

SPUFFORD, P. (1988): *Money and Its Use in Medieval Europe*. Cambridge University Press, Cambridge.

Suchodolski, S. (1983): “On the rejection of good coin in Carolingian Europe,” in *Studies in numismatic method*, ed. by C. Brooke, B. Stewart, J. Pollard, and T. Volk, pp. 147–152. Cambridge University Press.

——— (2012): “Polityka mennicza a wydarzenia polityczne w Polsce we wczesnym sredniowieczuİ (Minting policy and political events in Poland in the early Middle Ages),” in *Numizmatyka sredniowieczna. Moneta zrodlem archeologicznym, historycznym i ikonograficznym*, ed. by S. Suchodolski, pp. 341–348. Wydawnictwo Trio, Warsaw.

Svensson, R. (2013): *Renovatio Monetae: Bracteates and coinage policies in medieval Europe*. Spink and Son, London.

Sydsaeter, K. (1990): *Matematisk analyse, Bind II.* Gjovik Trykkeri AS, Gjovik.

Velde, F., and W. Weber (2000): “A Model of Bimetallism,” *Journal of Political Economy*, pp. 1210–1234.

Vorel, P. (2012): *Od prazskeho grose ke korune ceske. Pruvodce dejnami penez v ceskych zemich.* Rybka, Praha.

A Appendix

A.1 Methods to identify re-coinage

From archaeology and numismatics, there are three basic methods used to identify re-coinage. In table 4, we have ranked the methods by level of confidence. The most confident way of identifying re-coinage is through written documents that contain explicit information regarding the dates of re-coinage and/or exchange fees. However, for most currency areas and mints, there are no written sources concerning recurrent re-coinage. Therefore, other methods must be used.

By classifying different coin types as originating from a specific coin issuer and mint, it is relatively straightforward to establish whether re-coinage occurred. When there is only one type of coin per reign, the coinage system is likely to be long-lived.[24] However, when there are as many coin types as years of a specific reign, the evidence indicates that

[24] When coins were long-lived as in e.g, the post-medieval period, several different coin types could be used as legal tender simultaneously. For example, in 19th century England, rulers issued new coin types, whereas older coin types continued to be legal tender.

renewals were annual. If the number of coin types exceeds (falls short of) the number of years of a specific reign, then the indications are that the renewals were more (less) frequent.

Table 4: Methods for identifying short-lived and long-lived coinage systems

Method	Long-lived coins	Short-lived coins	Confidence of method
1. Written documents	-	-	Very strong
2. Coin types per reign and currency area	One	At least two	Strong
3. Distribution of coin types in hoards	One or a few types from each mint	Many types from each mint, but a few late dominate	Medium

A third method for identifying re-coinage involves carefully analyzing the concentration and distribution of coin types in hoards. Coin hoards from the Middle Ages may contain a few or many issues from each mint represented in a hoard. If (a successful) re-coinage has occurred, one would expect many coin types in a hoard from a specific mint but only a few types to strongly dominate the composition of the hoard. The coin types in such cases would be relatively young, whereas the older types should have a more sparse representation. In cases where there are several coin hoards from a specific coin issuer, one can expect the coin types that exist in many hoards to be older and the coin types that are found in only a few hoards to be younger.

A.2 Household optimization

Using the first-order conditions with respect to c_t and m^n_{t+1}, if $m^o_{t+1} > 0$ then, if $t \neq T, 2T$ etc.,

$$\left(\frac{e_{t+1}\chi}{e_t} - 1\right)\frac{u'\left(c_{t+1}\right)}{p_t} \geq 0 \tag{A.1}$$

and

$$\left(\frac{e_{t+1}\chi}{e_t} - \left(\left(1-\tau\right)s^n_t + e_{t+1}\chi\left(1-s^n_t\right)\right)\right)\frac{u'\left(c_{t+1}\right)}{p_t} \geq 0, \tag{A.2}$$

otherwise.

If $m^n_{t+1} > 0$, if $t \neq T, 2T$ etc.

$$\beta\left(\frac{e_{t+1}\chi}{e_t} - 1\right)\left(\frac{u'\left(c_{t+1}\right)}{p_t}\right) \leq 0. \tag{A.3}$$

Since the consumer now holds new coins in period $t+1$, the exchange rates in period t and $t+1$ have to give the consumer incentives to not only hold old coins. For this to be the case, the exchange rate increase cannot be too large and is bounded above by $\frac{1}{\chi}$. If $t = T, 2T$ etc.,

$$\beta \left(\frac{e_{t+1}\chi}{e_t} - \left((1-\tau) s_t^n + e_{t+1}\chi \left(1 - s_t^n\right) \right) \right) \left(\frac{u'(c_{t+1})}{p_t} \right) \leq 0. \tag{A.4}$$

Furthermore, using the first-order condition with respect to m_{t+1}^n gives, if $t \neq T, 2T$ etc.

$$\beta \max \left\{ \frac{u'(c_{t+1})}{p_{t+1}}, \frac{e_{t+1}\chi}{e_t} \frac{u'(c_{t+1})}{p_{t+1}} \right\} \leq \frac{u'(c_t)}{p_t} \tag{A.5}$$

and

$$\beta \max \left\{ \left((1-\tau) s_t^n + e_{t+1}\chi \left(1 - s_t^n\right) \right) \frac{u'(c_{t+1})}{p_{t+1}}, \frac{e_{t+1}\chi}{e_t} \frac{u'(c_{t+1})}{p_{t+1}} \right\} \leq \frac{u'(c_t)}{p_t}. \tag{A.6}$$

The conditions hold with equality only if the cash in advance constraint does not bind. One way of thinking about the above expression is the following. Consider a relaxation of the cash in advance constraint today and a corresponding tightening tomorrow. The relaxation today yields an increase in utility (readjusted for the relative price) of $\frac{u'(c_t)}{p_t}$ or $e_t \frac{u'(c_t)}{p_t}$, while the strengthening tomorrow leads to a decrease of either $\beta \frac{u'(c_{t+1})}{p_{t+1}}$ or $\beta e_{t+1}\chi \frac{u'(c_{t+1})}{p_{t+1}}$, depending on whether it is new or old coins that are used. To ensure that this deviation is not optimal, the above inequality must hold.

Finally, the household also optimally chooses the share of coins to be handed in for re-coinage, s_t^n. If $s_t^n \in (0,1)$ then

$$\left(-(1-\tau) + e_{t+1}\chi \right) \left(m_{t+1}^n + \Pi_t \right) = 0. \tag{A.7}$$

If $s_t^n = 0$ then

$$\left(-(1-\tau) + e_{t+1}\chi \right) \left(m_{t+1}^n + \Pi_t \right) \geq 0. \tag{A.8}$$

Finally, if $s_t^n = 1$ and hence

$$\left(-(1-\tau) + e_{t+1}\chi \right) \left(m_{t+1}^n + \Pi_t \right) \leq 0. \tag{A.9}$$

A.3 Proofs

Before proceeding with the proofs, we state and prove the following Lemma.

Lemma 4 *In a cyclical equilibrium,* $\frac{1-\tau}{b} \leq q_t \leq \frac{1}{b}$.

Proof

Suppose first that $q_t > \frac{1}{b}$. Then, using (6), $m^n_{t+1} = 0$. Moreover, we have $\frac{e_t}{b} \geq q_t$, since otherwise, using (9), $\mu^o_t = \chi m^o_t$ so that $m^o_{t+1} = 0$ and hence $p_t = 0$. Selling a small amount of jewelry allows the consumer infinite consumption, contradicting the resource constraint. Hence, if $t = T$ then $e_T > 1$. Suppose $T > t$. As long as $m^o_s > 0$ for $s > t$, using (16) $e_s\chi \geq e_{s-1}$ we have $e_s > 1$. Thus, if $t < T$ and $m^o_s > 0$ for all $s \in \{t+1, \ldots, T\}$, we have $e_T > 1$. From (17), using that holding of old coins requires $s^n_T < 1$ and hence $e_{T+1}\chi \geq 1 - \tau$ from (20) and (21) with equality if s^n_T is interior, we have

$$\frac{e_{T+1}\chi}{(1-\tau)\, s^n_T + e_{T+1}\chi\,(1 - s^n_T)} = 1 \geq e_T, \tag{A.10}$$

a contradiction. If $m^o_s = 0$ for some $s \in \{t+2, \ldots, T\}$ then $\frac{e_{s-1}}{b} \leq q_{s-1}$ and $e_r\chi \geq e_{r-1}$ for all $t + 1 < r < s - 1$ so that $e_{s-1} > 1$. Then $q_{s-1} > \frac{1}{b}$, implying that $m^n_s = 0$ and thus $p_{s-1} = 0$, a contradiction.

Suppose now that $q_t < \frac{1-\tau}{b}$. Then, from firm optimization, $n^n_t = d_t$ so that $d_{t+1} = 0$. The Inada conditions then establish a contradiction.■

Proof of Lemma 1:

Case 1**.** First, suppose that $t \neq T + 1$.

Suppose that $\mu^o_t = 0$. If $n^n_t > 0$ then $h_t = -bn^n_t$ and $q_t = \frac{1-\tau}{b}$ from (3) and thus, $\mu^n_t = 0$. Using that the Inada conditions imply that (12) holds with equality, the CIA constraint (12) is,

$$p_t c_t + \frac{1-\tau}{b}\left(-bn^n_t\right) = m^n_t + \Pi_{t-1} + e_t\chi m^o_t. \tag{A.11}$$

Using money transition (25), we get

$$p_t c_t + \frac{1-\tau}{b}\left(-bn^n_t\right) = m^n_{t+1} - (1-\tau)\, n^n_t + e_t\chi m^o_t \tag{A.12}$$

establishing that $p_t c_t = m^n_{t+1} + e_t\chi m^o_t$. A similar argument holds if $n^n_t = \mu^n_t = 0$.

Suppose that $\mu_t^n > 0$ so that $q_t = \frac{1}{b}$ from (6) - (7) and Lemma 4. Then, $h_t = b\mu_t^n$ and

$$p_t c_t + \frac{1}{b} b\mu_t^n = m_t^n + \Pi_{t-1} + e_t \chi m_t^o. \tag{A.13}$$

Using money transition (25) we get $p_t c_t = m_{t+1}^n + e_t \chi m_t^o$. A similar argument holds if $\mu_t^o > 0$.

Case 2**.** Now, suppose that $t = T + 1$.

Suppose that $\mu_t^o = 0$. If $n_t^n > 0$**,** we can proceed as in Case 1 to establish

$$p_t c_t + \frac{1-\tau}{b} \left(-bn_t^n\right) = (1-\tau)\, s_T^n \left(m_t^n + \Pi_{t-1}\right) + e_t \chi \left(m_t^o + \left(1 - s_T^n\right)\left(m_t^n + \Pi_{t-1}\right)\right). \tag{A.14}$$

Again using money transition (25) establishes that

$$p_t \xi = m_{t+1}^n + e_t \left(\chi m_t^o + \left(1 - s_T^n\right)\left(m_t^n + \Pi_{t-1}\right)\right). \tag{A.15}$$

A similar argument holds if $\mu_t^n > 0$, if $\mu_t^o > 0$ and if $\mu_t^n = n_t^n = 0$.■

Lemma 5 *Suppose that $m_s^o > 0$ for some s. Then, in a cyclical equilibrium, $m_{T+1}^n > 0$ and $s_T^n < 1$.*

Proof: Suppose $s_T^n = 1$ or $m_{T+1}^n = 0$. Then, using money transition (26), $m_{s+T}^o = \chi^T m_s^o$, and, since $s \in L_1^T$ and $s + T \in L_1^T$, cyclicality is violated. Thus, $m_{T+1}^n > 0$ and $s_T^n < 1$.

Lemma 6 *Suppose that $m_{T+1}^o > 0$. Then, $e_T = 1$.*

Proof:

From Lemma 5, we have $m_{T+1}^n > 0$ and $s_T^n < 1$.

If $s_T^n = 0$ then, using (17) and (19) so that $\frac{e_{T+1}\chi}{e_T} = e_{T+1}\chi$, we have $e_T = 1$.

If $s_T^n \in (0,1)$, then, from (20) we have $e_{T+1}\chi = 1 - \tau$. From (17) and (19), we have

$$\frac{e_{T+1}\chi}{e_T} = \left((1-\tau)\, s_T^n + e_{T+1}\chi \left(1 - s_T^n\right)\right) = e_{T+1}\chi, \tag{A.16}$$

and thus $e_T = 1$.■

Proof of Lemma 2

We prove this by contradiction. Suppose that $n_s^n > 0$ for some $2 \le s \le T$.

Step 1. Finding a relationship between current and tomorrows money holdings and showing $\mu_s^o = 0$ whenever $m_s^o > 0$ and $s < T+1$.

Since $n_s^n > 0$ we have, from (3) and Lemma 4, that $q_s = \frac{1-\tau}{b}$. Using that in case $m_s^o > 0$ (requiring $s_T^n < 1$ for cyclicality to be satisfied; see Lemma 5) we have, $m_r^o > 0$ for $r < s$ and, from (20), (21) and (16) that $e_1\chi \ge 1-\tau$ and $e_r\chi \ge e_{r-1}$ so that $e_r > 1-\tau$ for $r \le s$ and hence $\frac{e_s}{b} > q_s$. Then, using (8), we have $\mu_s^o = 0$. Moreover, $m_{s+1}^o = \chi m_s^o - \mu_s^o = \chi m_s^o$. Then, $e_s m_{s+1}^o = e_s \chi m_s^o \ge e_{s-1} m_s^o$. Using (25),

$$m_{s+1}^n + e_s\chi m_s^o = m_{s+1}^n + e_s m_{s+1}^o = m_s^n + e_s m_{s+1}^o + \Pi_{s-1} + (1-\tau)\, n_s^n > m_s^n + e_{s-1} m_s^o + \Pi_{s-1} \tag{A.17}$$

and, using that $\Pi_{s-1} = p_{s-1} g$ and that we from Lemma 1 have $p_{s-1}c = m_s^n + e_{s-1}m_s^o$, gives

$$m_{s+1}^n + e_s m_{s+1}^o > \frac{\xi}{\xi - g}\left(m_s^n + e_{s-1} m_s^o\right) \tag{A.18}$$

so that, using Lemma 1, we have $p_s > \frac{\xi}{\xi-g} p_{s-1}$ and hence $Q_s > \frac{\xi}{\xi-g}$.

Since $q_{s-1} \ge \frac{1-\tau}{b}$ and $n_s^n > 0$ implies $d_{s+1} < d_s$ we have, using concavity of v,

$$b q_{s-1} u'\left(\bar{c}\right) - b v'\left(d_s\right) p_{s-1} > b q_s u'\left(\bar{c}\right) - b v'\left(d_{s+1}\right) p_s. \tag{A.19}$$

Finally, since $q_{s+1} \ge \frac{1-\tau}{b}$ we have, using (44), that

$$Q_{s+1} > Q_s > \frac{\xi}{\xi - g}, \tag{A.20}$$

implying that $p_{s+1} > \frac{\xi}{\xi-g} p_s$.

From Lemma 1, using that, when $s < T$, $m_{s+1}^o = \chi m_s^o - \mu_s^o$, and when $s = T$, $m_{s+1}^o = \chi\left(m_s^o + \left(1 - s_T^n\right)\left(m_s^n + \Pi_{s-1}\right)\right) - \mu_s^o$, we have

$$m_{s+2}^n + e_{s+1} m_{s+2}^o > \frac{\xi}{\xi - g}\left(m_{s+1}^n + e_s m_{s+1}^o\right) \tag{A.21}$$

and hence, following similar steps as above, using Lemma 1 and that $\Pi_s = p_s g = \frac{g}{\xi-g}\left(m_{s+1}^n + e_s m_{s+1}^o\right)$, we have

$$m_{s+2}^n + e_{s+1} m_{s+2}^o > m_{s+1}^n + e_s m_{s+1}^o + \Pi_s. \tag{A.22}$$

Step 2. Showing that $n^n_{s+1} > 0$ when $s < T$.

Case 1. Suppose that $m^o_{s+2} = 0$.

If $m^o_{s+2} = 0$ then, from (A.22),

$$m^n_{s+2} > m^n_{s+1} + e_s m^o_{s+1} + \Pi_s > m^n_{s+1} + \Pi_s. \tag{A.23}$$

From money transition (25), we have

$$m^n_{s+2} - \left(m^n_{s+1} + \Pi_s\right) = (1-\tau)\, n^n_{s+1} - \mu^n_{s+1} > 0, \tag{A.24}$$

and hence, since $\mu^n_{s+1} \geq 0$, it follows that $n^n_{s+1} > 0$.

Case 2. Suppose that $m^o_{s+2} > 0$.

Note first that, since $s + 2 \leq T + 1$, we have, using (26), $m^o_{s+2} \leq \chi m^o_{s+1}$.

First, suppose that $m^n_{s+2} = 0$. Then, using that $n^n_s > 0$ implies that $m^n_{s+1} > 0$, we have $\mu^n_{s+1} > 0$ so that, using Lemma 4, $q_{s+1} = \frac{1}{b}$. If $m^o_{s+i} > 0$ for all $T - s + 1 \geq i > 2$ then, from Lemma 6, $e_T = 1$ and by (16), $e_{s+i}\chi \geq e_{s+i-1}$ for all i such that $s + i \leq T$ and thus $e_{s+1} < 1$. Then, from (9), we have $q_{s+1} = \frac{1}{b} > \frac{e_{s+1}}{b}$ so that $\mu^o_{s+1} = \chi m^o_{s+1}$ implying that $m^o_{s+2} = 0$, a contradiction. If $m^o_{s+i} = 0$ (and $m^o_{s+i-1} > 0$) for some i such that $T - s + 1 \geq i > 2$, then, using (9), $\frac{e_{s+i-1}}{b} \leq q_{s+i-1}$ and $\mu^o_{s+i-1} = \chi m^o_{s+i-1}$ and, for all k such that $i - 1 > k \geq 2$, since $m^o_{s+i-1} > 0$, $m^o_{s+k} > 0$ and, using (8), $\frac{e_{t+k}}{b} \geq q_{s+k}$. Since $q_{s+1} = \frac{1}{b} \leq \frac{e_{s+1}}{b}$ we get $e_{s+1} \geq 1$. Moreover, since $m^o_{s+k} > 0$ for all $i - 1 > k \geq 2$ then, by (16), $e_{s+k}\chi \geq e_{s+k-1}$, implying that $e_{s+i-1} > 1$ and thus $q_{s+i-1} \geq \frac{e_{s+i-1}}{b} > \frac{1}{b}$ in turn implying that $\mu^n_{s+i-1} = m^n_{s+i-1} + p_{s+i-2}g$ so that $m^n_{s+i} = 0$. Then, using Lemma 1, we have $p_{s+i} = 0$, a contradiction.

Second, suppose that $m^n_{s+2} > 0$. Thus, by (16) and (18), we have $e_{s+2}\chi = e_{s+1}$ and, using (26),

$$e_{s+1} m^o_{s+2} = e_{s+1}\left(\chi m^o_{s+1} - \mu^o_{s+1}\right) \leq e_s m^o_{s+1} \tag{A.25}$$

and thus, using (A.22),

$$m^n_{s+2} + e_s m^o_{s+1} > m^n_{s+2} + e_{s+1} m^o_{s+2} > m^n_{s+1} + e_s m^o_{s+1} + \Pi_s. \tag{A.26}$$

Then, using money transition (25), we have

$$m^n_{s+2} - \left(m^n_{s+1} + \Pi_s\right) = (1-\tau)\, n^n_{s+1} - \mu^n_{s+1} > 0 \tag{A.27}$$

and hence, since $\mu^n_{s+1} \geq 0$, it follows that $n^n_{s+1} > 0$.

Step 3. Showing that $n^n_{s+1} > 0$ when $s = T$.

From

Case 1. If $m^o_{T+2} = 0$ then, from (A.22) and Lemma 1, using $\Pi_T = p_T g$,

$$m^n_{T+2} > \left(\frac{\xi - g}{\xi - g} + \frac{g}{\xi - g}\right)\left(m^n_{T+1} + e_T m^o_{T+1}\right) \geq m^n_{T+1} + e_T m^o_{T+1} + \Pi_T \geq (1-\tau)\, s^n_T m^n_{T+1} + \Pi_T. \tag{A.28}$$

From money transition (25), we have

$$m^n_{T+2} - \left((1-\tau)\, s^n_T m^n_{T+1} + \Pi_T\right) = (1-\tau)\, n^n_{T+1} - \mu^n_{T+1} > 0 \tag{A.29}$$

and hence, since $\mu^n_{T+1} \geq 0$, it follows that $n^n_{T+1} > 0$.

Case 2. If $m^o_{T+2} > 0$ then, using (26),

$$e_{T+1} m^o_{T+2} = e_{T+1}\left(\chi\left(m^o_{T+1} + (1 - s^n_T)\left(m^n_{T+1} + \Pi_T\right)\right) - \mu^o_{T+1}\right) \leq e_{T+1}\chi\bigl(m^o_{T+1} + (1 - s^n_T)\left(m^n_{T+1} + \Pi_T\right) \tag{A.30}$$

and thus, using (25), we can write (A.22) as

$$\begin{aligned} m^n_{T+2} &> m^n_{T+1} + e_T m^o_{T+1} - e_{T+1} m^o_{T+2} + \Pi_T \\ &\geq m^n_{T+1} + e_T m^o_{T+1} - e_{T+1}\left(\chi\left(m^o_{T+1} + (1 - s^n_T)\left(m^n_{T+1} + \Pi_T\right)\right)\right) + \Pi_T. \end{aligned} \tag{A.31}$$

Suppose that $s^n_T > 0$ so that, using (20), we have $e_{T+1}\chi = 1 - \tau$. Then, using $\tau < 1$ and, whenever $m^o_{T+1} > 0$, from Lemma 6, $e_T = 1$,

$$\begin{aligned} m^n_{T+2} &> \left(1 - e_{T+1}\chi\left(1 - s^n_T\right)\right) m^n_{T+1} + \left(e_T - e_{T+1}\chi\right) m^o_{T+1} + \Pi_T - e_{T+1}\chi\left(1 - s^n_T\right)\Pi_T \\ &> (1-\tau)\, s^n_T \left(m^n_{T+1} + \Pi_T\right). \end{aligned} \tag{A.32}$$

We claim that $e_1\chi \leq 1$ when $s^n_T = 0$. First, consider the case when $m^o_1 > 0$. If $m^o_1 > 0$ then $m^o_t > 0$ for all t and, from Lemma 5, $m^n_1 > 0$. Thus, using (16), we have

$e_{t+1}\chi \geq e_t$, and, from Lemma 6, $e_T = 1$. Then, since $m_1^n > 0$ we have $\chi e_1 \leq \chi^T e_T < 1$ from (16). Now suppose that $m_1^o = 0$ and $e_1\chi > 1$. If $m_1^o = 0$ then, since $s_T^n = 0$, using (26), $\mu_t^o = \chi m_t^o > 0$ for some t. Then, using (9), $q_t \geq \frac{e_t}{b}$. Since, for any $s < t$, we have $m_s^o > 0$ and, using (16), $e_{s+1}\chi \geq e_s$ and thus, using (16), $e_2 \geq \frac{1}{\chi}e_1 > 1$ so that $e_t > 1$ and thus $q_t > \frac{1}{b}$. Then $\mu_t^n = m_t^n + \Pi_{t-1}$ so that $m_{t+1}^n = 0$, in turn implying that $p_t = 0$ from Lemma 1, a contradiction. Thus, we have $e_1\chi \leq 1$.

Then, since $e_1\chi \leq 1$, a similar argument as when $s_T^n > 0$ in (A.32) above establishes that

$$m_{T+2}^n > (1-\tau)\, s_T^n \left(m_{T+1}^n + \Pi_T\right). \tag{A.33}$$

From the above cases, using (25),

$$m_{T+2}^n - \left((1-\tau)\, s_T^n m_{T+1}^n + \Pi_T\right) = (1-\tau)\, n_{T+1}^n - \mu_{T+1}^n > 0 \tag{A.34}$$

and hence, since $\mu_{T+1}^n \geq 0$, it follows that $n_{T+1}^n > 0$.

Step 4. Induction.

By induction we have $n_t^n > 0$ for all $t \geq 1$, contradicting cyclicality.■

Proof of Lemma 3.

Preliminaries

Note first that we cannot have $\mu_1^n > 0$, since then $q_1 = \frac{1}{b}$, in turn implying that $n_1^n = 0$, violating cyclicality by noting that Lemma 2 implies $n_s^n = 0$ for all s.

From money transition (25), we have, except when $t = 1$, using Lemma 1,

$$m_{t+1}^n = m_t^n + \left(m_t^n + e_{t-1}\left(\chi m_{t-1}^o - \mu_{t-1}^o\right)\right)\frac{g}{c} - \mu_t^n \tag{A.35}$$

and hence

$$\left(1 + \frac{g}{c}\right) m_t^n = m_{t+1}^n - e_{t-1}\left(\chi m_{t-1}^o - \mu_{t-1}^o\right)\frac{g}{c} + \mu_t^n. \tag{A.36}$$

We have $m_1^n > 0$. Too see this, note that if $m_1^n = 0$ then, if $m_1^o = 0$ we have, using Lemma 1, $p_T = 0$, a contradiction. Also, if $m_1^o > 0$ we require $m_1^n > 0$ from Lemma 5.

Using Lemma 1 gives

$$
\begin{aligned}
\frac{p_t}{p_{t-1}} &= \frac{m_{t+1}^n + e_t\left(\chi m_t^o - \mu_t^o\right)}{m_t^n + e_{t-1}\left(\chi m_{t-1}^o - \mu_{t-1}^o\right)} = \frac{\left(1+\frac{g}{c}\right) m_t^n + e_{t-1}\left(\chi m_{t-1}^o - \mu_{t-1}^o\right)\frac{g}{c} - \mu_t^n + e_t\left(\chi m_t^o - \mu_t^o\right)}{m_t^n + e_{t-1}\left(\chi m_{t-1}^o - \mu_{t-1}^o\right)} \\
&= \left(1+\frac{g}{c}\right) + \frac{e_t\left(\chi m_t^o - \mu_t^o\right) - e_{t-1}\left(\chi m_{t-1}^o - \mu_{t-1}^o\right) - \mu_t^n}{m_t^n + e_{t-1}\left(\chi m_{t-1}^o - \mu_{t-1}^o\right)}.
\end{aligned}
$$

If $m_t^o > 0$ and $m_t^n > 0$, then, using money transition (26) and that we have $e_t\chi = e_{t-1}$, we get

$$
\frac{p_t}{p_{t-1}} = \frac{\xi}{\xi - g} - \frac{e_t\mu_t^o + \mu_t^n}{m_t^n + e_{t-1}\left(\chi m_{t-1}^o - \mu_{t-1}^o\right)}. \tag{A.37}
$$

If $m_t^o = 0$ so that $\mu_t^o = 0$ and $m_t^n > 0$ then, using $\chi m_{t-1}^o - \mu_{t-1}^o = 0$, establishes the same expression. If $m_t^o > 0$ and $m_t^n = 0$ then, since $m_t^n = 0$ requires $\mu_{t-1}^n > 0$ due to dividend disbursement, we have $q_{t-1} = \frac{1}{b}$ and $\frac{e_{t-1}}{b} \geq q_{t-1}$. Using $e_t\chi \geq e_{t-1}$ from (16), we have $\frac{e_s}{b} > q_s$ for $s \geq t$ and thus $\mu_s^o = 0$ and $m_{s+1}^o > 0$ and thus $m_{T+1}^o > 0$. Then, since $e_s \geq \chi^{-(s-t+1)} e_{t-1} > 1$ contradicting Lemma 6. Thus, $\frac{p_t}{p_{t-1}} \leq \frac{\xi}{\xi - g}$ where the inequality is strict if $e_t\mu_t^o + \mu_t^n > 0$. Hence,

$$
\frac{p_{t-1}}{p_t} \geq 1 - \frac{g}{\xi} \tag{A.38}
$$

where the inequality is strict whenever $e_t\mu_t^o + \mu_t^n > 0$.

Step 1. Jewelry prices.

Since, using that we from Lemma 2 have $n_t^n = 0$ for $t = 2, \ldots, T$, we have $d_2 \leq d_3 \leq \ldots \leq d_T \leq d_1$ and thus, using (14) and the properties of v,

$$
\frac{q_1}{p_1} - \beta\frac{q_2}{p_2} \geq \frac{q_2}{p_2} - \beta\frac{q_3}{p_3} \geq \ldots \geq \frac{q_T}{p_T} - \beta\frac{q_1}{p_1}. \tag{A.39}
$$

Let $y_i = \frac{bv'(d_{i+1})}{\xi u'(\xi - g)}$ for $i = 1, \ldots, T-1$ and $y_T = \frac{bv'(d_1)}{\xi u'(\xi - g)}$. Note that, using $d_2 \leq d_3 \leq \ldots \leq d_T \leq d_1$ and the properties of v, we have

$$
y_1 \geq y_2 \geq \ldots \geq y_T. \tag{A.40}
$$

Then, using (A.40),

$$
\frac{q_1}{p_1} - \frac{q_T}{p_T} \geq \frac{(1-\beta) + \beta(1-\beta) + \ldots + (1-\beta)\beta^{T-2} + \left(\beta^{T-1} - 1\right)}{1 - \beta^T} y_T = 0, \tag{A.41}
$$

with strict inequality if $y_t > y_T$ for some t. Note that, if $y_s = y_t$ for all s,t, then, from (14), $\frac{q_s}{p_s} = \frac{q_t}{p_t}$ for all s,t.

Moreover, from the Euler equations (A.39) for periods $T-1$ and T,

$$\frac{q_T}{p_T} - \beta\frac{q_1}{p_1} \leq \frac{q_{T-1}}{p_{T-1}} - \beta\frac{q_T}{p_T} \iff \frac{q_{T-1}}{p_{T-1}} \geq \frac{q_T}{p_T} - \beta\left(\frac{q_1}{p_1} - \frac{q_T}{p_T}\right). \tag{A.42}$$

For remaining time periods, using (A.40),

$$\frac{q_{T-j}}{p_{T-j}} \geq \frac{q_{T-j+1}}{p_{T-j+1}} - \beta\left(\frac{q_{T-j+2}}{p_{T-j+2}} - \frac{q_{T-j+1}}{p_{T-j+1}}\right) \tag{A.43}$$

again with equality if $y_s = y_t$ for all s,t.

Furthermore, if $T > 2$ and $\mu_t^o = \mu_t^n = 0$ for $T > t \geq s \geq 2$ then $d_s = \ldots = d_1$ so that

$$\frac{q_s}{p_s} - \beta\frac{q_{s+1}}{p_{s+1}} = \ldots = \frac{q_T}{p_T} - \beta\frac{q_1}{p_1}. \tag{A.44}$$

Then, using (A.41), expression (A.43) holds with equality so that,

$$\frac{q_s}{p_s} = \frac{q_{s+1}}{p_{s+1}} - \beta\left(\frac{q_{s+2}}{p_{s+2}} - \frac{q_{s+1}}{p_{s+1}}\right). \tag{A.45}$$

By (A.41) and using induction, we have, if $y_T < y_1$, and $\mu_r^n = \mu_r^o = 0$ for $r \geq t$,

$$\frac{q_t}{p_t} < \frac{q_{t+1}}{p_{t+1}}. \tag{A.46}$$

The above expression holds with equality if $y_T = y_1$.

If $T = 2$ and $\mu_T^o = \mu_T^n = 0$ then, by cyclicality, $n_1^n = 0$ (and, when $m_s^o > 0$ so $s_T^n < 1$, since $e_1\chi \geq 1-\tau$ from (20) and (21) so that $e_1 > 1-\tau$, we have $\mu_1^o = 0$) and thus, using (A.41), expression (A.43) again holds with equality.

Step 2. Showing $\mu_t^n = 0$ for all t.

If $n_1^n = 0$ cyclicality requires that $\mu_t^n = 0$ and $\mu_t^o = 0$ for all t.

Suppose now that $n_1^n > 0$. From (3) and Lemma 4, we have $q_1 = \frac{1-\tau}{b}$ and, from (5), $\mu_1^n = 0$.

Case 1. $m_t^o = 0$ for all t. Then, for some s, $d_s < d_1$ and $y_1 > y_s$ and thus, from (A.41), $\frac{q_1}{p_1} > \frac{q_T}{p_T}$. Then, if only new coins are held, using the Cash in Advance constraints

(12), money transition (25) and that $m_t^o = 0$ for all t implies that $s_T^n = 1$, we have

$$\begin{aligned} p_T c &= m_1^n \\ p_1 c &= m_2^n = (1-\tau)\left(m_1^n + p_T g\right) + (1-\tau)\, n_1^n. \end{aligned} \tag{A.47}$$

Then

$$m_2^n = (1-\tau)\left(\frac{\xi}{\xi - g} m_1^n + n_1^n\right) > (1-\tau)\,\frac{\xi}{\xi - g} m_1^n. \tag{A.48}$$

and thus, using $q_1 = \frac{1-\tau}{b}$ and (A.41),

$$q_1 \frac{m_1^n}{(1-\tau)\,\frac{\xi}{\xi-g} m_1^n} > q_1 \frac{p_T}{p_1} > q_T \iff q_T < \left(1 - \frac{g}{\xi}\right)\frac{1}{b} \tag{A.49}$$

so that $\mu_T^n = 0$.

For $T = 2$ the conclusion follows immediately, since $\mu_T^n = \mu_T^o = 0$ requires $n_1^n = 0$ for cyclicality to be satisfied.

Suppose $T \geq 3$. Using expression (5), $\mu_T^n = 0$ and $d_T = d_1$ so that, using (12), (25) and, from (A.38), $p_{T-1} = \left(1 - \frac{g}{\xi}\right) p_T$, we have $m_1^n \left(1 - \frac{g}{\xi}\right) = m_T^n$. Then, using (A.46) and (A.38),

$$q_{T-1} < q_T \frac{p_{T-1}}{p_T} = \left(1 - \frac{g}{\xi}\right) q_T < \left(1 - \frac{g}{\xi}\right)^2 \frac{1}{b}. \tag{A.50}$$

Hence, from (5) - (7), $\mu_{T-1}^n = 0$. Using (A.46) and Lemma 1 repeatedly, we have

$$q_{T-j} < \left(1 - \frac{g}{\xi}\right)^{j+1} \frac{1}{b} \tag{A.51}$$

and $\mu_{T-j}^n = 0$. Thus, there is no melting for $t \geq 2$, contradicting cyclicality.

Case 2. $m_t^o > 0$ for some t. Since $m_t^o > 0$ for some t, we have from Lemma 5 that $m_1^n > 0$ and $s_T^n < 1$ and, if $m_1^o > 0$ by Lemma 6, $e_T = 1$. Using (20) and (21), $e_1 \chi \geq 1 - \tau$. Then $\frac{e_1}{b} > \frac{1-\tau}{b} = q_1$ so that, from (8), $\mu_1^o = 0$. If $m_1^o = 0$ then $e_T \geq 1$ by (19).

We have, using (25) and Lemma 1,

$$\begin{aligned} p_T c &= m_1^n + e_T \left(\chi m_T^o - \mu_T^o\right) = m_1^n + m_1^o \\ p_1 c &= m_2^n + e_1 \chi \left(m_1^o + \left(1 - s_T^n\right)\left(m_1^n + \Pi_T\right)\right). \end{aligned} \tag{A.52}$$

Since $s_T^n < 1$ and $e_1 \chi \geq 1 - \tau$ we have, using (25),

$$\begin{aligned} p_1 c &\geq (1-\tau)\, s_T^n \left(m_1^n + \Pi_T\right) + (1-\tau)\left(m_1^o + \left(1 - s_T^n\right)\left(m_1^n + \Pi_T\right)\right) + (1-\tau)\, n_1^n \\ &= (1-\tau)\left(m_1^n + \Pi_T\right) + (1-\tau)\, m_1^o + (1-\tau)\, n_1^n > (1-\tau)\, m_1^n + (1-\tau)\, m_1^o \end{aligned} \tag{A.53}$$

so that, using (A.46) and that, using Lemma 4, $n_1^n > 0$ implies that $q_1 = \frac{1-\tau}{b}$, we have

$$q_1 \frac{m_1^n + m_1^o}{(1-\tau)\left(m_1^o + m_1^n\right)} > q_1 \frac{p_T}{p_1} > q_T \iff q_T < \frac{1}{b}, \tag{A.54}$$

implying $\mu_T^n = 0$. Moreover, since $q_T < \frac{1}{b} \leq \frac{e_T}{b}$ we have $\mu_T^o = 0$ and thus $d_T = d_1$.

For $T = 2$ the conclusion follows immediately, since $\mu_T^n = \mu_T^o = 0$ requires $n_1^n = 0$ for cyclicality to be satisfied.

Suppose that $T \geq 3$.

First, suppose that $\mu_t^o = 0$ for all t. Then, using (A.38) and (A.46), we have $q_t < \left(1 - \frac{g}{\xi}\right)^{T-t} \frac{1}{b}$ so that $q_t < \frac{1}{b}$, inductively establishing $\mu_t^n = 0$ for all t.

Second, suppose that $\mu_{\hat{t}-1}^o > 0$ for some $\hat{t}$ and $\mu_s^o = 0$ for $s \geq \hat{t}$.

If $\hat{t} - 1 = T$ then, proceeding as above using (A.41) establishes $q_T < \left(1 - \frac{g}{\xi}\right) \frac{1}{b}$ so that $\mu_T^n = 0$. Moreover, if $m_1^o = 0$ then, from the argument at the start of Case 2, $e_T \geq 1$, and, if $m_1^o > 0$ then, from Lemma 6, $e_T = 1$. We then have $\frac{e_T}{b} > q_T$ so that, from (8), $\mu_T^o = 0$, a contradiction.

Now suppose that $\hat{t} - 1 < T$. We first prove that $\mu_t^n = 0$ for $t \geq \hat{t}$.

From $\mu_{\hat{t}}^o > 0$ we have, using (9), that $\frac{e_{\hat{t}}}{b} \leq q_{\hat{t}}$. Moreover, since $\mu_s^o = 0$ for $s > \hat{t}$, we have, using (A.38), (A.46) and (A.54), for $t \geq \hat{t}$,

$$q_t < \left(1 - \frac{g}{\xi}\right)^{T-t} \frac{1}{b} \tag{A.55}$$

so that $q_t < \frac{1}{b}$, inductively establishing $\mu_t^n = 0$ for $t \geq \hat{t}$.

We now show that $\mu_t^n = 0$ for all t. Suppose first that $\hat{t} < T$ so that $\mu_{T-1}^o = 0$.

For all $t > \hat{t}$ we have $\mu_t^o = 0$ so that $d_{t+1} = d_{t+2} = \ldots = d_{T+1}$. Then, using (A.45), we have

$$\frac{q_t}{p_t} - \frac{q_{t+1}}{p_{t+1}} = \beta \left(\frac{q_{t+1}}{p_{t+1}} - \frac{q_{t+2}}{p_{t+2}}\right) = \beta^s \left(\frac{q_{t+s}}{p_{t+s}} - \frac{q_{t+s+1}}{p_{t+s+1}}\right) = \ldots = \beta^{T-t} \left(\frac{q_T}{p_T} - \frac{q_{T+1}}{p_{T+1}}\right). \tag{A.56}$$

Then, using (A.38)

$$q_t - \left(1 - \frac{g}{\xi}\right) q_{t+1} = \left(1 - \frac{g}{\xi}\right) \beta \left(q_{t+1} - q_{t+2} \left(1 - \frac{g}{\xi}\right)\right). \tag{A.57}$$

Then, computing the solution gives, using Sydsæter (1990, p. 294),

$$m_{1,2} = \begin{cases} \left(1 - \frac{g}{\xi}\right)^{-1} \\ \frac{1}{\beta}\left(1 - \frac{g}{\xi}\right)^{-1} \end{cases} \tag{A.58}$$

where

$$q_t = C_1 m_1^{t-\hat{t}} + C_2 m_2^{t-\hat{t}}. \tag{A.59}$$

We have two boundary conditions; $q_{\hat{t}} = C_1 + C_2$ and $q_T = C_1 m_1^{T-\hat{t}} + C_2 m_2^{T-\hat{t}}$. Then, using $C_1 = q_{\hat{t}} - C_2$, we have

$$C_1 = \frac{q_{\hat{t}} (\beta)^{-\left(T-\hat{t}\right)} - q_T \left(1 - \frac{g}{\xi}\right)^{\left(T-\hat{t}\right)}}{(\beta)^{-\left(T-\hat{t}\right)} - 1} \tag{A.60}$$

$$C_2 = \frac{q_T \left(1 - \frac{g}{\xi}\right)^{\left(T-\hat{t}\right)} - q_{\hat{t}}}{\left((\beta)^{-\left(T-\hat{t}\right)} - 1\right)}. \tag{A.61}$$

Note that, using (A.38) and (A.46), $C_2 > 0$. Also, by repeatedly using (14), we have $\frac{q_{\hat{t}}}{p_{\hat{t}}} > \beta^{T-\hat{t}} \frac{q_T}{p_T}$ and hence, using that (A.38) holds with equality for t such that $T \geq t > \hat{t}$, we have

$$q_{\hat{t}} > \beta^{T-\hat{t}} q_T \left(1 - \frac{g}{\xi}\right)^{\left(T-\hat{t}\right)}. \tag{A.62}$$

Then $C_1 > 0$.

We also have, using that $d_{\hat{t}} < d_{\hat{t}+1}$, (A.43) and that $\frac{p_{\hat{t}-1}}{p_{\hat{t}}} > 1 - \frac{g}{\xi}$ from (A.38),

$$q_{\hat{t}-1} > \left(1 - \frac{g}{\xi}\right) \left((1 + \beta) q_{\hat{t}} - q_{\hat{t}+1} \beta \left(1 - \frac{g}{\xi}\right)\right). \tag{A.63}$$

Note that, for $s \geq \hat{t}$ and using (A.59),

$$\frac{q_{s+1}}{q_s} - \frac{q_s}{q_{s-1}} = C_1 C_2 \left(\frac{1}{\beta}\right)^{s+1-\hat{t}} \frac{(1-\beta)^2}{\left(C_1 + C_2\left(\frac{1}{\beta}\right)^{s-\hat{t}}\right)\left(C_1 + C_2\left(\frac{1}{\beta}\right)^{s-1-\hat{t}}\right)} \left(1 - \frac{g}{\xi}\right)^{-1}. \tag{A.64}$$

Thus $\frac{q_{s+1}}{q_s} > \frac{q_s}{q_{s-1}}$. Moreover, $\frac{q_T}{q_t} = \frac{q_T}{q_{T-1}} \cdot \ldots \cdot \frac{q_{t+1}}{q_t}$. Since $\frac{q_{s+1}}{q_s} > \frac{q_s}{q_{s-1}}$, we have $\frac{q_T}{q_t} > \left(\frac{q_{t+1}}{q_t}\right)^{T-t}$. Moreover, from (A.54)

$$\frac{1}{b} > q_T > \left(\frac{q_{t+1}}{q_t}\right)^{T-t} q_t \tag{A.65}$$

and $q_{\hat{t}} \geq \frac{e_{\hat{t}}}{b} \geq \frac{\chi^{T-\hat{t}}}{b}$, where the last equality follows from $m_t^n > 0$ for $t > \hat{t}$, $e_T \geq 1$ and, using (18), $\frac{e_{t+1}\chi}{e_t} \leq 1$. Then, using (A.65),

$$\left(\frac{q_{\hat{t}+1}}{q_{\hat{t}}}\right)^{T-\hat{t}} q_{\hat{t}} \geq \left(\frac{q_{\hat{t}+1}}{q_{\hat{t}}}\right)^{T-\hat{t}} \frac{\chi^{T-\hat{t}}}{b} \Rightarrow \left(\frac{q_{\hat{t}+1}}{q_{\hat{t}}}\right)^{T-\hat{t}} \chi^{T-\hat{t}} < 1, \tag{A.66}$$

implying that $\chi < \left(\frac{q_{\hat{t}+1}}{q_{\hat{t}}}\right)^{-1}$. Let the right-hand side of (A.63) be denoted $q^*_{\hat{t}-1}$. From (A.63), $q_{\hat{t}-1} > q^*_{\hat{t}-1}$. We have, using (A.59),

$$q^*_{\hat{t}-1} = C_1 m_1^{-1} + C_2 m_2^{-1} \tag{A.67}$$

so that, proceeding as in (A.64) above

$$\frac{q_{\hat{t}}}{q^*_{\hat{t}-1}} < \frac{q_{\hat{t}+1}}{q_{\hat{t}}}. \tag{A.68}$$

Then, using $\chi < \left(\frac{q_{\hat{t}+1}}{q_{\hat{t}}}\right)^{-1}$,

$$q^*_{\hat{t}-1}\frac{q_{\hat{t}+1}}{q_{\hat{t}}} > q_{\hat{t}} > \frac{\chi^{T-\hat{t}}}{b} \iff q^*_{\hat{t}-1} > \frac{\chi^{T-\hat{t}}}{b}\left(\frac{q_{\hat{t}+1}}{q_{\hat{t}}}\right)^{-1} > \frac{\chi^{T-\hat{t}+1}}{b}, \tag{A.69}$$

and thus $q_{\hat{t}-1} > \frac{\chi^{T-\hat{t}+1}}{b}$. Since $m^o_{\hat{t}-1} > 0$ we have $e_{\hat{t}-1} \leq \chi e_{\hat{t}}$ so that $q_{\hat{t}-1} > \frac{e_{\hat{t}-1}}{b}$ and hence $\mu^o_{\hat{t}-1} = \chi m^o_{\hat{t}-1}$, a contradiction.

Hence, $\mu^o_{\hat{t}} = 0$. Moreover, since $q_{\hat{t}} < \left(1 - \frac{g}{\xi}\right)^{T-\hat{t}} \frac{1}{b}$, we have $\mu^n_{\hat{t}} = 0$. Induction then establishes $\mu^n_s = \mu^n_s = 0$ for all s.

Consider now the case when $\hat{t} = T$ so that $\mu^o_{T-1} > 0$. Following similar arguments as in (A.55), we have $\mu^n_{T-1} = 0$. Moreover, using that we require that $m^n_1 > 0$ for old coins to be held so that, from (18), we have $e_T\chi \leq e_{T-1}$ and thus, using that by Lemma 6 and (19), $e_T \geq 1$, we have $\frac{\chi}{b} \leq \frac{e_{T-1}}{b} \leq q_{T-1} < \left(1 - \frac{g}{\xi}\right)\frac{1}{b}$ where the last inequality follows from (A.55), using $\mu^o_T = 0$. Thus, $\chi < 1 - \frac{g}{\xi}$. Moreover, since $\frac{e_T}{b} \geq \frac{1}{b} \geq q_T$ and $q_{T-1} \geq \frac{\chi}{b}$ we have $\frac{q_{T-1}}{q_T} \geq \chi$. Consider period $T-2$. Then, since $\mu^o_{T-1} > 0$, using that the inequality in (A.43) is strict and that, from (A.38), $\frac{p_{T-2}}{p_{T-1}} > 1 - \frac{g}{\xi}$ and $\frac{p_{T-1}}{p_T} = 1 - \frac{g}{\xi}$, we have

$$q_{T-2} > \frac{p_{T-2}}{p_{T-1}}\left((1+\beta)\, q_{T-1} - \beta q_T \frac{p_{T-1}}{p_T}\right). \tag{A.70}$$

Define q^*_{T-2} as the value of q_{T-2} corresponding to $\mu^o_{T-1} = 0$ (with $\frac{p_{T-2}}{p_{T-1}} = 1 - \frac{g}{\xi}$), given the jewelry prices q_{T-1} and q_T. Thus,

$$q^*_{T-2} = \left(1 - \frac{g}{\xi}\right)\left((1+\beta)\, q_{T-1} - \beta q_T\left(1 - \frac{g}{\xi}\right)\right). \tag{A.71}$$

This expression defines a difference equation. Proceeding as in (A.64) and (A.68) above, using $\frac{q_{T-1}}{q_T} > \chi$, establishes that $q^*_{T-2} > q_{T-1}\frac{q_{T-1}}{q_T} > \chi q_{T-1}$. Then, since $m^o_{T-1} > 0$, using (A.70) and that we from (16) have $e_{T-2} \leq \chi e_{T-1}$, we get

$$q_{T-2} > q^*_{T-2} > \chi q_{T-1} \geq \chi\frac{e_{T-1}}{b} \geq \frac{e_{T-2}}{b} \tag{A.72}$$

so that $\mu^o_{T-2} = \chi m^o_{T-2}$, a contradiction.

Once more, induction then establishes $\mu^n_s = \mu^n_s = 0$ for all s. ■

Proof of Theorem 1.

From Lemma 3, $n^n_t = 0$, $\mu^n_t = 0$ and $\mu^o_t = 0$ for all t. Thus, $d_t = \bar{d}$ for all t and thus, from (A.41) and that (A.46) holds with equality when $y_1 = y_T$, we have $\frac{q_s}{p_s} = \frac{q_t}{p_t}$ for all s, t.

Case 1. $m^o_{t+1} = 0$ for all t.

Step 1. Since $s^n_T = 1$ we have, from (18), (19) and (22), that $e_1\chi \leq e_T(1-\tau)$, $e_{t+1}\chi \leq e_t$ and $e_1\chi \leq 1-\tau$ and hence

$$e_1\chi \geq e_2\chi^2 \geq \ldots \geq e_T\chi^T \geq \frac{e_1\chi}{1-\tau}\chi^T \iff 1-\tau \geq \chi^T. \tag{A.73}$$

Step 2. Prices.

We have, using Lemma 1, (25) and that (A.38) holds with equality, for $t \neq T+1$,

$$m_{t+1}^n = m_1^n \left(1 - \frac{g}{\xi}\right)^{T-t+1} \tag{A.74}$$

and, using (23),

$$\tau\left(m_1^n + \Pi_T\right) = \sum_{t=1}^{T} p_t g = \sum_{t=1}^{T} m_{t+1}^n \frac{g}{c} = \frac{g}{c}\sum_{t=2}^{T+1}\left(1 - \frac{g}{\xi}\right)^{T-t+1} m_1^n \tag{A.75}$$

so that, using $\Pi_T = p_T g = m_1^n \frac{g}{c}$,

$$\tau\frac{\xi}{\xi - g} = \frac{g}{\xi - g}\sum_{t=2}^{T+1}\left(1 - \frac{g}{\xi}\right)^{T-t+1} = \frac{\xi}{\xi - g}\left(1 - \left(1 - \frac{g}{\xi}\right)^{T}\right) \tag{A.76}$$

and hence

$$1 - \tau = \left(1 - \frac{g}{\xi}\right)^{T} \iff \left(1 - \frac{g}{\xi}\right) = (1-\tau)^{\frac{1}{T}}. \tag{A.77}$$

Then, using Lemma 1 and (25),

$$\left(1 - \frac{g}{\xi}\right) m_{t+1}^n = m_t^n \tag{A.78}$$

so that, using that (A.38) holds with equality, for $t = 2, \ldots, T$, we have $(1-\tau)^{\frac{1}{T}} p_t = p_{t-1}$ and thus, using $\frac{q_t}{p_t} = \frac{q_{t-1}}{p_{t-1}}$, $(1-\tau)^{\frac{1}{T}} q_t = q_{t-1}$, and hence $q_1 = (1-\tau)^{\frac{T-1}{T}} q_T$. Since $q_T \leq \frac{1}{b}$, using Lemma 4, any $q_1 \in [\frac{1-\tau}{b}, \frac{(1-\tau)^{\frac{T-1}{T}}}{b}]$ is possible, implying that $q_T \in [\frac{(1-\tau)^{\frac{1}{T}}}{b}, \frac{1}{b}]$.

Step 3. Finding m_1^n.

Using that $c = (1-\tau)^{\frac{1}{T}}\xi$ from (A.77), $\frac{q_1}{p_1} = \frac{q_T}{p_T}$ from $(1-\tau)^{\frac{1}{T}} p_t = p_{t-1}$ and $(1-\tau)^{\frac{1}{T}} q_t = q_{t-1}$, the Cash in Advance constraint $p_T c = m_1^n$ and the silver market clearing condition $d_1 = \ldots = d_T = S - b\left(m_1^n + \Pi_T + m_1^L\right) = S - b\left(\frac{\xi}{\xi - g} m_1^n + m_1^L\right)$, we can write the Euler equation (14) as, letting $B = (1-\tau)^{-\frac{1}{T}}$,

$$q_T\xi = \frac{1}{1-\beta}\frac{1}{u'\left((1-\tau)^{\frac{1}{T}}\xi\right)}(1-\tau)^{-\frac{1}{T}} m_1^n v'\left(S - b\left(Bm_1^n + m_1^L\right)\right). \tag{A.79}$$

The right-hand side is continuous and increasing in m_1^n due to the concavity of v. Moreover, we have $\lim_{m_1^n \to 0} m_1^n v'\left(S - b\left(Bm_1^n + m_1^L\right)\right) = 0$ and $\lim_{m_1^n \to \frac{S - bm_1^L}{bB}} m_1^n v'\left(S - b\left(Bm_1^n + m_1^L\right)\right) =$

∞. Then, for each $q_T \in [\frac{(1-\tau)^{\frac{1}{T}}}{b}, \frac{1}{b}]$, there is a unique m_1^n that satisfies the Euler equation. Furthermore, by differentiating the Euler equation, we have $\frac{dq_T}{dm_1^n} > 0$.[25]

Case 2. $m_{t+1}^o > 0$ for all t.

Step 1. Exchange rates.

Using that $\mu_t^o = 0$ from Lemma 3 and that $e_t\chi = e_{t-1}$ from (16) and (18) and $e_T = 1$ from Lemma 6, we have $e_t = \chi^{T-t}$. Moreover, if $s_T^n \in (0,1)$ then $e_1\chi = 1-\tau$. Combining this and $e_t = \chi^{T-t}$ establishes that $\chi^T = 1-\tau$ whenever $s_T^n \in (0,1)$.

If $s_T^n = 0$ then $e_1\chi \geq 1-\tau$ so that $\chi^T \geq 1-\tau$.

Step 2. Showing $\chi = 1 - \frac{g}{\xi}$.

Note that, using $\mu_t^o = 0$ for all t, we have $m_t^o = \chi m_{t-1}^o$. Then, using that we from (26) have $m_2^o = \chi\left(m_1^o + \left(1 - s_T^n\right)\left(m_1^n + \Pi_T\right)\right)$ and $m_t^o = \chi m_{t-1}^o$ we have

$$m_{t+1}^o = \chi^t\left(m_1^o + \left(1 - s_T^n\right)\left(m_1^n + \Pi_T\right)\right), \tag{A.80}$$

by repeatedly using $m_t^o = \chi m_{t-1}^o$, and thus, setting $t = T$ above, $m_1^o = \frac{\chi^T}{1-\chi^T}\left(1 - s_T^n\right)\left(m_1^n + \Pi_T\right)$ and hence,

$$m_{t+1}^o = \frac{\chi^t}{1-\chi^T}\left(1 - s_T^n\right)\left(m_1^n + \Pi_T\right). \tag{A.81}$$

Government revenues during a cycle are, in terms of new coins, recalling that old confiscated coins are re-minted costlessly by the lord as new coins and also using (A.81),

$$\begin{aligned}
&\tau s_T^n\left(m_1^n + \Pi_T\right) + (1-\chi)\left(m_1^o + \left(1 - s_T^n\right)\left(m_1^n + \Pi_T\right)\right) + \sum_{t=2}^{T}(1-\chi)\, m_t^o \\
= \;& \tau s_T^n\left(m_1^n + \Pi_T\right) + \left(1 - s_T^n\right)\left(m_1^n + \Pi_T\right).
\end{aligned} \tag{A.82}$$

To find government expenditures, using Lemma 1, $e_{t-1} = \chi e_t$ and $m_t^o = \chi m_{t-1}^o$ from (16), (18) and (26), we can write

$$p_t(\xi - g) = m_{t+1}^n + e_1\chi\left(m_1^o + \left(1 - s_T^n\right)\left(m_1^n + \Pi_T\right)\right). \tag{A.83}$$

[25] Noting that consumption is independent of q_T and that jewelry holdings are decreasing in money holdings, the equilibrium yielding the highest welfare then has $q_T = \frac{(1-\tau)^{\frac{1}{T}}}{b}$.

The expenditures are, using (A.83) and that $\Pi_t = p_t g$

$$\begin{aligned}\sum_{t=1}^{T} p_t g &= \frac{g}{\xi}\left(\sum_{t=1}^{T}\left(m_t^n + \Pi_{t-1}\right) + Te_1\left(\chi\left(m_1^o + \left(1 - s_T^n\right)m_1^n\right)\right)\right) \\ &= \frac{g}{\xi}\frac{\xi}{\xi - g}\left(\sum_{t=1}^{T} m_t^n + Te_1\left(\chi\left(m_1^o + \left(1 - s_T^n\right)m_1^n\right)\right)\right).\end{aligned} \tag{A.84}$$

Using money transition (25),

$$m_{t+1}^n = m_t^n + p_{t-1}g = m_t^n + \frac{g}{\xi - g}\left(m_t^n + e_1\chi\left(m_1^o + \left(1 - s_T^n\right)\left(m_1^n + \Pi_T\right)\right)\right). \tag{A.85}$$

Solving the above expression for m_t^n and repeatedly substituting gives

$$m_t^n = \left(1 - \frac{g}{\xi}\right)^{T-t+1} m_1^n - e_1\chi\left(m_1^o + \left(1 - s_T^n\right)\left(m_1^n + \Pi_T\right)\right)\left(1 - \left(1 - \frac{g}{\xi}\right)^{T-t+1}\right).$$

Then

$$\begin{aligned}\sum_{t=1}^{T} m_t^n &= \frac{1 - \left(1 - \frac{g}{\xi}\right)^T}{\frac{g}{\xi}} m_1^n - \left(T - 1\right)e_1\chi\left(m_1^o + \left(1 - s_T^n\right)\left(m_1^n + \Pi_T\right)\right) \\ &\quad + \frac{\left(1 - \frac{g}{\xi}\right) - \left(1 - \frac{g}{\xi}\right)^T}{\frac{g}{\xi}} e_1\chi\left(m_1^o + \left(1 - s_T^n\right)\left(m_1^n + \Pi_T\right)\right).\end{aligned} \tag{A.86}$$

Since expenditures equal revenues, using (A.82), (A.84) and the above expression, we get

$$\begin{aligned}&\tau s_T^n\left(m_1^n + \Pi_T\right) + \left(1 - s_T^n\right)\left(m_1^n + \Pi_T\right) \\ &= \left(1 - \left(1 - \frac{g}{\xi}\right)^T\right)\frac{\xi}{\xi - g}\left(m_1^n + e_1\chi\left(m_1^o + \left(1 - s_T^n\right)\left(m_1^n + \Pi_T\right)\right)\right).\end{aligned} \tag{A.87}$$

Using that $\Pi_T = p_T g = \frac{g}{c}\left(m_1^n + e_1\chi\left(m_1^o + \left(1 - s_T^n\right)\left(m_1^n + \Pi_T\right)\right)\right)$, we have

$$\frac{\xi}{\xi - g}\left(m_1^n + e_1\chi\left(m_1^o + \left(1 - s_T^n\right)\left(m_1^n + \Pi_T\right)\right)\right) = m_1^n + \Pi_T + e_1\chi\left(m_1^o + \left(1 - s_T^n\right)\left(m_1^n + \Pi_T\right)\right), \tag{A.88}$$

that $e_1 = \chi^T$ and that, from Lemma 6, $e_T = 1$, we have, from (A.81), that

$$\begin{aligned} & \tau s_T^n \left(m_1^n + \Pi_T\right) + \left(1 - s_T^n\right)\left(m_1^n + \Pi_T\right) \\ = & \left(1 - \left(1 - \frac{g}{\xi}\right)^T\right)\left(m_1^n + \Pi_T + \frac{\chi^T}{1-\chi^T}\left(1 - s_T^n\right)\left(m_1^n + \Pi_T\right)\right). \end{aligned} \tag{A.89}$$

Suppose that $s_T^n = 0$. Then $1 - \chi^T = \left(1 - \left(1 - \frac{g}{\xi}\right)^T\right)$ so that

$$\chi = 1 - \frac{g}{\xi}. \tag{A.90}$$

Suppose that $s_T^n > 0$. Then, using (20), $e_{T+1}\chi = 1 - \tau$ and, using (16) and (18), $e_{t-1} = \chi e_t$. Moreover, from Lemma 6 and (A.81), $e_T = 1$ so that $1 - \tau = \chi^T$. Then

$$\begin{aligned} & \left(1 - \chi^T\right) s_T^n \left(m_1^n + \Pi_T\right) + \left(1 - s_T^n\right)\left(m_1^n + \Pi_T\right) \\ = & \left(1 - \left(1 - \frac{g}{\xi}\right)^T\right)\left(s_T^n\left(m_1^n + \Pi_T\right) + \frac{1}{1-\chi^T}\left(1 - s_T^n\right)\left(m_1^n + \Pi_T\right)\right). \end{aligned}$$

Once more, we get

$$\chi = 1 - \frac{g}{\xi}. \tag{A.91}$$

Step 3. Prices.

From money transition (25), for $t = 2, \ldots, T$, we have $\chi m_{t+1}^n = m_t^n$ so that $\chi p_t = p_{t-1}$ and thus $\chi q_t = q_{t-1}$, and hence, $q_1 = \chi^T q_T$. Since, using Lemma 4, $q_T \leq \frac{1}{b}$ any $q_1 \in \left[\frac{1-\tau}{b}, \frac{\chi^T}{b}\right]$ is possible.

Step 4. Finding m_1^n.

Fix s_T^n. Using Lemma 1, expression (A.81) and that expression (A.81) when $t = T$ and Lemma 6 imply $e_T = 1$, we can write, using $\Pi_T = p_T g$,

$$p_T\left(\xi - g\right) = m_1^n + e_T \chi m_T^o = m_1^n + \chi m_T^o = m_1^n + \frac{\chi^T}{1-\chi^T}\left(1 - s_T^n\right)\left(m_1^n + p_T g\right) \tag{A.92}$$

or, using $g = (1-\chi)\,\xi$ to define

$$Z\left(\chi, s_T^n, \xi\right) = \frac{1 + \frac{\chi^T}{1-\chi^T}\left(1 - s_T^n\right)}{\xi\left(1 - (1-\chi)\left(1 + \frac{\chi^T}{1-\chi^T}\left(1 - s_T^n\right)\right)\right)}, \tag{A.93}$$

we get

$$p_T = Z(\chi, s_T^n, \xi)\, m_1^n. \tag{A.94}$$

Moreover, using (A.81), we have

$$\begin{aligned} m_1^n + \Pi_T + m_1^o &= \left(1 + \frac{\chi^T}{1-\chi^T}(1-s_T^n)\right)(m_1^n + Z(\chi, s_T^n, \xi)\, m_1^n g) \qquad (A.95) \\ &= \left(1 + \frac{\chi^T}{1-\chi^T}(1-s_T^n)\right)(1 + Z(\chi, s_T^n, \xi)(1-\chi)\xi)\, m_1^n. \end{aligned}$$

Letting $B = \left(1 + \frac{\chi^T}{1-\chi^T}(1-s_T^n)\right)(1 + Z(\chi, s_T^n, \xi)(1-\chi)\xi)$ and using that $c = \chi\xi$ from (27) and (A.91), $\frac{q_1}{p_1} = \frac{q_T}{p_T}$ from $\chi p_t = p_{t-1}$ and $\chi q_t = q_{t-1}$, the Cash in Advance constraint (A.94) and the silver market clearing condition $d_1 = \ldots = d_T = S - b\left[Bm_1^n + m_1^L\right]$ we can, using (A.81), (A.94) and (A.95), write the Euler equation (14) as,

$$q_T = \frac{1}{1-\beta}\frac{1}{u'(\chi\xi)} Z(\chi, s_T^n, \xi)\, m_1^n v'\left(S - b\left[Bm_1^n + m_1^L\right]\right). \tag{A.96}$$

The right-hand side is continuous and increasing in m_1^n due to concavity of v. Moreover, $\lim_{m_1^n \to 0} m_1^n v'\left(S - b\left[Bm_1^n + m_1^L\right]\right) = 0$ and $\lim_{m_1^n \to \frac{S-bm_1^L}{bB}} m_1^n v'\left(S - b\left[Bm_1^n + m_1^L\right]\right) = \infty$. Then, for each $q_T \in [\frac{(1-\tau)\chi^{-T}}{b}, \frac{1}{b}]$ there is a unique m_1^n that satisfies the Euler equation. Furthermore, by differentiating the Euler equation, we have $\frac{dq_T}{dm_1^n} > 0$. [26]

Suppose that $\chi^T > 1-\tau$ so that $s_T^n = 0$. Then, using (A.96), m_1^n solves

$$\begin{aligned} q_T &= \frac{1}{1-\beta}\frac{1}{u'(\chi\xi)} Z(\chi, 0, \xi)\, m_1^n \qquad (A.97) \\ &\cdot v'\left(S - b\left[\frac{1}{1-\chi^T}\left(1 + Z(\chi, 0, \xi)\frac{1-\chi}{\xi}\right) m_1^n + m_1^L\right]\right). \qquad (A.98) \end{aligned}$$

Suppose that $\chi^T = 1-\tau$. Then, for any $s_T^n \in [0,1]$, the equilibrium value of m_1^n solves (A.96). [27] ■

[26] Noting that consumption is independent of q_T and that jewelry holdings are decreasing in money holdings, the equilibrium yielding the highest welfare then has $q_T = \frac{(1-\tau)\chi^{-T}}{b}$.

[27] Pareto optimality now also requires $s_T^n = 1$.