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COMPARATIVE STATICS IN DYNAMIC PROGRAMMING MODELS WITH AN APPLICATION TO JOB SEARCH

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Abstract:

This paper presents a technique for qualitative comparative statics analysis in dynamic programming models. Let the value function v be the fixed point of a contraction mapping which depends differentiably on some exogenous parameter θ. Then the derivative of v with respect to θ exists and is also the fixed point of a contraction mapping. Since this derivative is the fixed point of a contraction mapping its qualitative properties can be investigated using mathematical induction. This comparative statics methodology is illustrated with an application to a model of job search.

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1. Introduction

This paper presents a general method for deducing qualitative comparative statics in dynamic programming models and applies that method to a model of individual job search. The model uses a continuous-time Markovian set-up to analyze choice with respect to (i) how much effort to devote to job search and (ii) whether or not to accept new job offers.

The methodological problem we consider is as follows. The first-order conditions for the choice variables in a dynamic programming problem typically involve the value functions associated with the states of the model. To carry out comparative statics one needs to know how these value functions vary with the exogenous parameters of the model. We show that if the equations defining the value functions are contraction mappings (ensuring the consistency of the model), then, under very weak conditions, these value functions are differentiable with respect to the exogenous parameters of the model. Further, the equations defining these derivatives are themselves contraction mappings. This means it is possible to use mathematical induction to derive qualitative properties of the derivatives, and often this is enough to establish qualitative comparative statics properties of the model.

The model of job search we present is designed with the empirical analysis of the sources of variation across individuals in unemployment duration in mind.[1] It is very close to the two-state model presented in

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[1]Preliminary empirical results based on this model are presented in Albrecht, Holmlund, and Lang (1989). The objective of our empirical work is to disentangle the roles of "choice" and "chance" in determining individual unemployment durations (Mortensen and Neumann (1984)). That is, can longer durations be understood primarily in terms of job search behavior (low search intensity and/or high reservation wage) or are exogenous environmental conditions (an unfavorable offer arrival rate and/or wage offer distribution) more to blame? And, to the extent that variations in job search behavior explain variation in unemployment duration, what factors determine the choice of search effort and a reservation wage?

Burdett and Mortensen (1978), a remarkable paper, which has something of the status of a neglected classic. The results presented in their paper are important, but it seems fair to say that the difficulty of the methods they used to derive their results has deterred those interested in this area. Relative to the techniques used in Burdett and Mortensen, our comparative statics methodology allows straightforward and unified derivations.

The organization of our paper is as follows. In the next section we state and prove our comparative statics result on the differentiability of contraction mappings and explain how to apply this result. In Section 3 we present our model of job search, and in Section 4 we illustrate the use of our differentiability result to deduce the qualitative comparative statics properties of the model. In the final section we conclude.

2. Comparative Statics in Dynamic Programming Models

Consider a dynamic programming problem of the form:

$$v(x) = \max_{s \in K(x)} T(s,\theta,v)(x), \tag{1}$$

where v is the "value function," s a vector of choice parameters belonging to some compact subset K(x) of R^n, θ a parameter vector, and $T(s,\theta,\cdot)$ a contraction mapping with modulus $\beta < 1$ taking bounded, continuous functions of x into new bounded, continuous functions of x. These conditions ensure the existence of a unique solution v, and we assume that T is increasing in v so that the maximization maximizes v (Bellman's Principle).

For comparative statics analysis, the differentiability properties of (1) are of interest. Proposition 1 below states that the value function v is differentiable with respect to the exogenous parameters θ. Further, the differentiated function is itself the fixed point of a contraction mapping. This last fact is very useful for deriving comparative statics properties of the derivative v_θ: Proposition 2 below shows how this fact can be used.

Proposition 1: Assume that the mapping T is continuously differentiable with respect to θ and v,[2] that the compact-valued correspondence K(x) is continuous, and that the maximization problem in (1) has a unique solution $s = s^*(x)$. Then $v = v(\theta;x)$ is differentiable with respect to θ, the continuous derivative v_θ is the unique solution to

$$v_\theta(x) = T_\theta(s^*,\theta,v)(x) + T_v(s^*,\theta,v;v_\theta)(x),$$

and the right-hand side defines a contraction mapping in v_θ.

Remark: As v_θ is defined by a contraction mapping, it too is continuously differentiable with respect to θ. Indeed, Proposition 1 implies that $v(\theta;x)$ is continuously differentiable to arbitrary order in θ, so long as T is differentiable in v and θ to the requisite orders. Note that Proposition 1 says nothing about the differentiability of v with respect to x. Results on this question are given in Benveniste and Scheinkman (1979).

To prove Proposition 1 we use the following lemma:

Lemma: Assume that no choice parameter s is present, so that we consider the equation

$$v(x) = T(\theta,v)(x).$$

Then $v = v(\theta;x)$ is differentiable with respect to θ, the continuous derivative v_θ is the unique solution to

$$v_\theta(x) = T_\theta(\theta,v)(x) + T_v(\theta,v;v_\theta)(x),$$

and the right-hand side defines a contraction mapping in v_θ.

Proof: The differentiability of v with respect to θ follows from the implicit function theorem for mappings between Banach spaces so long as the derivative of $v - T(\theta,v)$ with respect to v, ie, $I - T_v(\theta,v;\cdot)$, where I is the identity map, is invertible (see, eg, Dieudonné (1960)). But, $T_v(\theta,v;\cdot)$ is a linear map (by definition of derivative in this context) and since T by assumption satisfies a Lipschitz condition with Lipschitz constant β, the modulus of T_v

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[2]Let B be a Banach space with norm $\|\cdot\|$, and let $T:B \to B$. Let $T_v:B\times B \to B$ be linear in its second argument and be such that $\|T(v+h)-T(v)-T_v(v,h)\| = o(\|h\|)$. Then T_v is the derivative of T with respect to v. See, eg, Dieudonné (1960) on the differentiability of mappings between Banach spaces.

is at most $\beta < 1$. It follows that $I - T_v(\theta,v;\cdot)$ is invertible, the inverse being $\sum_{n=0}^{\infty} T_v(\theta,v;\cdot)^n$. The fact that the modulus of T_v is at most β proves the last statement of the lemma. QED.

Proof of the Proposition: The "maximum theorem" implies that the function $s^*(\theta;x)$ is continuous in θ and x. Define $g(\theta',\theta;x)$ by the equation

$g(\theta',\theta;x) = T(s^*(\theta';\cdot),\theta,g)(x)$.

It follows from the Lemma that $g(\theta',\theta;x)$ is differentiable with respect to θ for fixed θ' and x and that $g_\theta(\theta',\theta;x)$ is continuous in θ', θ, and x. By Bellman's Principle

$v(\theta;x) \geq g(\theta',\theta;x)$

with equality for $\theta' = \theta$. Hence by the Mean Value Theorem, for some $\mu \in (0,1)$

$$v(\theta+d\theta;x) - v(\theta;x) - g_\theta(\theta,\theta;x)d\theta \leq g(\theta+d\theta,\theta+d\theta;x) - g(\theta+d\theta,\theta;x) - g_\theta(\theta,\theta;x)d\theta$$
$$= [g_\theta(\theta+d\theta,\theta+\mu d\theta;x) - g_\theta(\theta+d\theta,\theta;x)]d\theta = o(d\theta).$$

(Here g_θ denotes differentiation with respect to the second argument.)

On the other hand,

$$v(\theta+d\theta;x) - v(\theta;x) - g_\theta(\theta,\theta;x)d\theta \geq g(\theta,\theta+d\theta;x) - g(\theta,\theta;x) - g_\theta(\theta,\theta;x)d\theta$$
$$= o(d\theta).$$

These two relations show that $v(\theta;x)$ is differentiable with respect to θ with $v_\theta(\theta;x) = g_\theta(\theta,\theta;x)$. The rest of the theorem now follows from the Lemma. QED

The fact that the derivative of the value function with respect to the exogenous parameters is the fixed point of a contraction mapping allows the following standard result to be exploited:

Proposition 2: Let $T(f)$, $f \in B$, be a contraction mapping on the Banach space B. The solution $g = T(g)$ is obtained as $g = \lim_{n\to\infty} f^n$, where $f^0 \in B$ is chosen arbitrarily and $f^{n+1} = T(f^n)$. Let P be some property that is preserved by T (ie, f^n has property P implies $f^{n+1} = T(f^n)$ has property P) and also preserved by limits in B. Then the unique solution $g = T(g)$ also has the property P.

By Proposition 1, the derivative of the value function with respect to the exogenous parameters is the contraction mapping

$$v_\theta(x) = T_\theta(s^*,\theta,v)(x) + T_v(s^*,\theta,v;v_\theta)(x). \qquad (2)$$

By Proposition 2, properties of v_θ can be established inductively. That is, if one wants to show that v_θ has some property P, then it suffices to show that some v_θ^0 has property P and that property P is preserved by the right-hand side of (2) and in the limit. Note, incidentally, that Proposition 1 does not follow from Proposition 2 since differentiability is a property that need not be preserved in the limit.

The idea that something like Propositions 1 and 2 might be used for comparative statics analysis in dynamic optimization problems has some precedent in the literature. Mortensen (1986, p. 875) uses Proposition 2 to inductively establish a property of the derivative of a value function with respect to an exogenous parameter.[3] He recognizes that if v can be differentiated with respect to θ, then at least some such derivatives are themselves fixed points of contraction mappings. What Proposition 1 establishes is that such differentiation is "always" legitimate and that "all" such derivatives are fixed points.

Our result is also related to Araujo and Scheinkman (1977), the standard comparative dynamics reference in the optimal growth literature. Using a calculus of variations set-up, Araujo and Scheinkman provide sufficient conditions for the differentiability of the policy function with respect to the exogenous parameters of the problem. That is, in the notation of our equation (1), they provide sufficient conditions for the differentiability of s with respect to θ. The difficulty is that it does not seem possible to check these sufficient conditions (the "dominant diagonal" conditions of their Assumption 2, p. 608) in problems that cannot be easily fit into the calculus of variations framework. Araujo and Scheinkman's dominant diagonal conditions are required to ensure that the implicit function theorem can be applied in an infinite-dimensional setting. Likewise, the key to the proof of our

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[3]We thank Dale Mortensen for alerting us to this reference.

Proposition 1 is also the use of the implicit function theorem in infinite-dimensional spaces. The virtue to the dynamic programming formulation of the optimization problem is that the applicability of the implicit function theorem falls out immediately.

In the next two sections we present our model of job search and apply our comparative statics methodology. In these sections we focus on the model as an object of interest in its own right; however, in concluding this "methodological" section we want to make clear that the techniques we will use are certainly not limited to models of job search. Propositions 1 and 2 are applicable to any optimization problem that can be formulated as a dynamic program, optimal growth and stochastic dynamic macroeconomics included.

3. The Model of Job Search

We consider an individual who at any instant is either employed (e) or not employed (n). This individual derives utility from the rates at which he or she consumes (c) and enjoys leisure (ℓ) with a concave instantaneous utility indicator $u(c,\ell)$; that is, the utility enjoyed over an interval of time of length Δt is given by $u(c,\ell)\Delta t + o(\Delta t)$. The function $u(c,\ell)$ satisfies $u_c > 0$, $u_\ell > 0$, $u_{cc} < 0$, $u_{\ell\ell} < 0$, and $u_{c\ell} \geq 0$. The individual has an infinite horizon and discounts the future at the rate $\frac{1}{1+\rho\Delta t}$.

If employed at wage w, this individual's rate of consumption is $c = wh+y$. The fraction of time spent working, h, and the non-wage income flow, y, are exogenous. If the individual is not employed, he or she also receives the non-wage income y: in addition, benefits specific to non-employment are received at the rate b. These benefits are received by all non-employed, irrespective of how that state was entered and of behavior in that state. Finally, income and consumption are identical; ie, lending and borrowing are precluded by assumption.

An individual with a job can search for a better alternative. The fraction of time, s, spent searching for a better job (search intensity) is a choice variable constrained by ℓ = 1-h-s; ie, the fraction of time available for leisure equals the fraction of time spent neither working nor searching. An individual without a job is similarly constrained by ℓ = 1-s.

Random events of two types may induce changes in utility flows. First, an individual may receive a job offer, which can either be accepted or rejected. Offers come from a known distribution, F(w), with support $[\underline{w},\bar{w}]$, and arrive according to Poisson processes with arrival rates $\alpha_i(s)$, i=e,n. That is, there are two offer arrival processes, one relevant while employed, the other while non-employed. The arrival rates are assumed to be bounded, increasing, and concave in s; and search off the job is assumed to be at least as efficient as search while employed, both in absolute terms and on the margin. (That is, $\alpha_n(s) \geq \alpha_e(s)$ and $\alpha'_n(s) \geq \alpha'_e(s) \geq 0$, for all $s \in [0,1-h]$.) Second, an employed worker may lose his or her job. Shocks that induce separations arrive according to a Poisson process with parameter γ. Whereas the individual has the option to accept or reject job offers, separations cannot be avoided.

Summarizing the notation:

$u(c,\ell)$	instantaneous utility indicator
$c = wh+y$	consumption rate when employed at wage w
$c = b+y$	consumption rate when non-employed
$\ell = 1-h-s$	leisure when employed
$\ell = 1-s$	leisure when non-employed
$\alpha_e(s)$	arrival rate of job offers when employed
$\alpha_n(s)$	arrival rate of job offers when non-employed
$F(w)$	wage offer distribution with support $[\underline{w},\bar{w}]$
γ	separation rate
ρ	discount rate

The individual's objective is to maximize expected lifetime utility discounted over the infinite horizon by choosing search intensities and an acceptance rule for job offers. This decision problem will be characterized by two value functions - V(w), the value (ie, expected discounted lifetime utility) of being employed at wage w, and U, the value of non-employment. We begin by developing expressions for these values conditional on any given search intensities and acceptance rules.

For an interval of time of length Δt the value function for an individual employed at wage w and searching with intensity s_e is

$$V(w) = \frac{1}{1+\rho\Delta t}\{ u(wh+y,1-h-s_e)\Delta t + \alpha_e(s_e)\Delta t E\max[V(w'),V(w)] + \gamma\Delta t U + (1-(\alpha_e(s_e)+\gamma)\Delta t)V(w) + o(\Delta t) \}. \quad (3)$$

The analogous value of non-employment, conditional on a search intensity s_n, is

$$U = \frac{1}{1+\rho\Delta t}\{u(b+y,1-s_n)\Delta t + \alpha_n(s_n)\Delta t E\max[V(w'),U] + (1-\alpha_n(s_n)\Delta t)U + o(\Delta t)\}. \quad (4)$$

In both expressions the expectation is taken with respect to the distribution of the prospective new wage offer, w′, and the acceptance rule is presumed to be optimal relative to the given search intensities, s_e and s_n.

The interpretation of (3) is as follows. The "instantaneous utility" of being employed at wage w is proportional to the length Δt of the time interval. With probability $\alpha_e(s_e)\Delta t$ a new job offer will be received, and that offer will be accepted if the value of the new job exceeds that of the current job. This reflects the assumption that the acceptance rule is optimal relative to the given s_e . With probability $\gamma\Delta t$ a separation will occur with the associated value U. With probability $1 - (\alpha_e(s_e)+\gamma)\Delta t$ the worker will neither receive a new offer nor lose his current job. In this case he or she retains the value V(w). The remainder term $o(\Delta t)$ reflects the assumptions that as $\Delta t \to 0$ any non-proportionality of utility to the length of the time interval goes to zero at an even faster rate. Likewise, the probability of receiving more than one job offer or more than one separation-inducing shock goes to zero at an even faster rate. Finally, all of the above is discounted at the rate $\frac{1}{1+\rho\Delta t}$. The interpretation of (4) is analogous.

It is easier to treat the individual's decision problem in continuous time. Re-arranging, dividing through by Δt, and taking limits as $\Delta t \to 0$ gives

$$V(w) = \frac{1}{\rho}\{u(wh+y,1-h-s_e)+\alpha_e(s_e)E\max[V(w'),V(W)]+\gamma U-(\alpha_e(s_e)+\gamma)V(w)\} \quad (5)$$

$$U = \frac{1}{\rho}\{ u(b+y,1-s_n) + \alpha_n(s_n)E\max[V(w'),U] - \alpha_n(s_n)U \}. \quad (6)$$

It is convenient to re-write these equation slightly. Choose any constant $M > 0$ such that $M > \alpha_e(s)+\gamma$ and $M > \alpha_n(s)$ for all $s\in[0,1]$. Multiplying (5) and (6) by ρ, adding MV(w) (resp., MU) to both sides, and dividing through by $M+\rho$ gives

$$V(w) = \frac{1}{M+\rho} \{u(wh+y,1-h-s_e) + \alpha_e(s_e)E\max[V(w'),V(w)] + \gamma U + (M-\alpha_e(s_e)-\gamma)V(w)\} \quad (7)$$

$$U = \frac{1}{M+\rho} \{u(b+y,1-s_n) + \alpha_n(s_n)E\max[V(w'),U] + (M-\alpha_n(s_n))U \}. \quad (8)$$

For short we write these equations as

$V = \phi(V,U)$

$U = \psi(V,U)$.

Since (i) $V_2(w) \geq V_1(w)$ and $U_2 \geq U_1$ implies $\phi(V_2,U_2) \geq \phi(V_1,U_1)$ and $\psi(V_2,U_2) \geq \psi(V_1,U_1)$ (monotonicity) and (ii) for any constant $c > 0$, $\phi(V+c,U+c) \leq \phi(V,U) + \beta c$ and $\psi(V+c,U+c) \leq \psi(V,U) + \beta c$, where $\beta = \frac{M}{M+\rho} < 1$ (discounting), we have that ϕ,ψ defines a contraction mapping on the Banach space C×R, where C denotes continuous functions on $[\underline{w},\bar{w}]$ and R the real numbers, with norm given by $\|(f,m)\| = \max[\sup_{\underline{w}\leq w\leq \bar{w}} |f(w)|, |m|]$. The proof is a slight generalization of Blackwell (1965).

Equations (7) and (8) define a contraction mapping for each fixed pair of search intensities, s_e and s_n, along with the corresponding acceptance rule with modulus independent of s_e and s_n. Now consider

$$V(w) = \max_{s_e\in[0,1-h]} \frac{1}{M+\rho} \{u(wh+y,1-h-s_e) + \alpha_e(s_e)E\max[V(w'),V(w)] + \gamma U + (M-\alpha_e(s_e)-\gamma)V(w)\} \qquad (9)$$

$$U = \max_{s_n\in[0,1]} \frac{1}{M+\rho} \{u(b+y,1-s_n) + \alpha_n(s_n)E\max[V(w'),U] + (M-\alpha_n(s_n))U\}. \qquad (10)$$

Bellman's Principle ensures that equations (9) and (10) define both the value functions V(w) and U and the "strategies" ($s_e^* = s_e^*(w)$ and s_n^* plus the corresponding optimal acceptance rule) that maximize V(w) and U.[4]

Since equations (9) and (10) define a contraction mapping, we can use Proposition 2 to establish that V(w) is increasing in w. Regarding U as fixed, equation (9) is a contraction mapping for V(w). The Banach space is that of continuous functions defined on $[\underline{w},\bar{w}]$ and normed by $\|f\| = \sup_{\underline{w}\leq w\leq \bar{w}} |f(w)|$. Thus, the optimal acceptance rule for an employed worker is simply to take any job

[4]The results of Sharma (1987) could also be used to show that (9) and (10) define a contraction mapping.

offering a higher wage. Likewise, the optimal acceptance rule for an individual without a job is a simple reservation wage rule: Accept any job offering a wage w iff $w \geq r$, where the reservation wage r is defined by $V(r) = U$.

This allows us to re-write the value functions once again, this time in a way that is useful for deriving the necessary conditions for the optimal search intensities and reservation wage:

$$V(w) = \max_{s_e} \frac{1}{M+\rho}\{u(wh+y,1-h-s_e)+\alpha_e(s_e)\int_w^{\bar{w}}[V(w')-V(w)]dF(w')+\gamma U+(M-\gamma)V(w)\} \qquad (11)$$

$$U = \max_{s_n, r} \frac{1}{M+\rho} \{u(b+y,1-h-s_n) + \alpha_n(s_n)\int_r^{\bar{w}}[V(w')-U]dF(w') + MU\}. \qquad (12)$$

Thus, the first-order condition for the optimal search intensity while employed is

$$-u_\ell(wh+y,1-h-s_e^*) + \alpha_e'(s_e^*)\int_w^{\bar{w}}[V(w')-V(w)]dF(w') \leq 0 \text{ ("=" if } s_e^* > 0\text{).} \qquad (13)$$

If the optimal search intensity while employed, s_e^*, is positive, then s_e^* is such that the marginal utility of leisure is equated to the expected gain from search at the margin. The second-order condition (given $s_e^* > 0$) is that the LHS of (13) be decreasing in s_e. The concavity of u and of α_e ensure that this will always be satisfied.

An individual without a job chooses both a search intensity and a reservation wage. From (12) the first-order conditions for the optimal off-the-job search intensity, s_n^*, and reservation wage, r^*, are

$$-u_{\ell}(b+y,1-s_n^*) + \alpha_n'(s_n^*)\int_{r^*}^{\bar{w}} [V(w')-U]dF(w') \le 0 \text{ ("=" if } s_n^* > 0\text{)}^5 \tag{14}$$

$$U - V(r^*) = 0. \tag{15}$$

An individual without a job, if he or she searches at all, again does so to equate the marginal utility of leisure to the expected gain from search at the margin. The reservation wage is chosen to equate the value of the marginal job with the value of unemployment. The second-order condition for s_n^* is ensured by the concavity of u and of α_n, and the second-order condition for r* is ensured by the fact that V(w) is increasing in w.

4. Comparative Statics Results: An Illustration of the Method

Using the first-order conditions we can now address the standard comparative statics questions of how the choice variables vary with respect to the exogenous parameters of the model. Specifically, we examine how s_e^*, s_n^*, and r* vary with respect to changes in γ, ρ, y, b, θ_w, θ_n, and θ_e. The three heretofore undefined parameters, θ_w, θ_n, and θ_e, refer to improvements in the wage offer distribution and in the two offer arrival functions. More precisely, θ_w is any parameter such that $\frac{\partial F(w;\theta_w)}{\partial \theta_w} \le 0$ for all w; similarly, θ_n and θ_e are such that $\frac{\partial \alpha_n(s_n;\theta_n)}{\partial \theta_n} \ge 0$, for all s_n and $\frac{\partial \alpha_e(s_e;\theta_e)}{\partial \theta_e} \ge 0$, for all s_e, respectively. In addition, we examine how s_e^* varies with respect to a change in w.[6]

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[5]We are implicitly ruling out the solutions $s_e^* = 1-h$ and $s_n^* = 1$ *a priori*. This could be justified by the natural assumption that $\lim_{\ell \to 0} u_{\ell}(c,\ell) = +\infty$.

[6]Throughout this section we are assuming that s_e^* and s_n^* are positive, ie, that (13) and (14) hold as equalities.

To carry out a comparative statics analysis we need to take into account that the first-order conditions involve not only the endogenous variables and the exogenous parameters but also the value functions. In considering the effect of a variation in an exogenous parameter account needs to be taken of both the direct effect of the parameter change and the indirect effect via the change in the value functions.

Propositions 1 and 2 handle this complication. We illustrate the idea by working through the comparative statics of an improvement in the wage offer distribution on s_e^*. We now show $\frac{\partial s_e^*}{\partial \theta_w} \geq 0$.

The first-order condition for s_e^* (equation (13)) is of the form $G(s_e^*, \theta_w) = 0$. The parameter θ_w enters both directly through the distribution function $F(w;\theta_w)$ and indirectly through the value function, $V(w)$. Differentiating the first-order condition with respect to θ_w gives $\frac{\partial s_e^*}{\partial \theta_w} = \frac{-\partial G(s_e^*,\theta_w)/\partial \theta_w}{\partial G(s_e^*,\theta_w)/\partial s_e}$. But, $\frac{\partial G(s_e^*,\theta_w)}{\partial s_e} < 0$ by the second order condition, so the sign of $\frac{\partial s_e^*}{\partial \theta_w}$ is determined by the sign of $\frac{\partial G(s_e^*,\theta_w)}{\partial \theta_w}$. That is, the sign of $\frac{\partial s_e^*}{\partial \theta_w}$ is the same as that of

$$\alpha_e'(s_e^*)\{ \int_{w}^{\bar{w}} [V_\theta(w') - V_\theta(w)]\ dF(w') + \int_{w}^{\bar{w}} [V(w') - V(w)]\ dF_\theta(w') \}.$$

Here, of course, subscripting by θ denotes differentiation with respect to θ_w.

Differentiate $V(w)$ (equation (11)) with respect to θ_w (Proposition 1 allows us to do this), substituting in the maximizing values of s_e^*, s_n^*, and r^* in advance. This gives

$$V_\theta(w) = \frac{1}{M+\rho}\{ \alpha_e(s_e^*)EV_\theta(\max[w',w]) + \alpha_e(s_e^*)\int_{w}^{\bar{w}}[V(w') - V(w)]\ dF_\theta(w') + \gamma U_\theta + (M - \alpha_e(s_e^*) - \gamma)V_\theta(w) \} \tag{16}$$

Note from equation (16) that

$$\alpha_e(s_e^*)\{ \int_w^{\bar{w}} [V_\theta(w')-V_\theta(w)]\ dF(w')+ \int_w^{\bar{w}} [V(w')-V(w)]\ dF_\theta(w')\ \} = (\rho+\gamma)V_\theta(w) - \gamma U_\theta .$$

Both $\alpha_e(s_e^*)$ and $\alpha_e'(s_e^*)$ are positive, so all that needs to be shown is that $(\rho+\gamma)V_\theta(w)-\gamma U_\theta \geq 0$. This is done by appealing to Proposition 1 to show that (16) is a contraction mapping for any fixed value of U_θ so that Proposition 2 can then be used to inductively establish the result.

The application of Proposition 2 is as follows. Choose any $V_\theta^0(w)$ such that $(\rho+\gamma)V_\theta^0(w) \geq \gamma U_\theta$ for all w. We need to show that $(\rho+\gamma)V_\theta^n(w) \geq \gamma U_\theta$ implies $(\rho+\gamma)V_\theta^{n+1}(w) \geq \gamma U_\theta$, where

$$V_\theta^{n+1}(w) = \frac{1}{M+\rho}\{\ \alpha_e(s_e^*)EV_\theta^n(\max[w',w]) + \alpha_e(s_e^*)\int_w^{\bar{w}}[V(w')-V(w)]\ dF_\theta(w')$$
$$+\gamma U_\theta + (M-\alpha_e(s_e^*)-\gamma)V_\theta^n(w)\ \}$$

Multiplying through by $(\gamma+\rho)$ and using the inductive hypothesis,

$$(\gamma+\rho)V_\theta^{n+1}(w) \geq \frac{1}{M+\rho}\{\ \alpha_e(s_e^*)\gamma U_\theta + (\gamma+\rho)\alpha_e(s_e^*)\int_w^{\bar{w}}[V(w')-V(w)]\ dF_\theta(w')$$
$$+ (\gamma+\rho)\gamma U_\theta + (M-\alpha_e(s_e^*)-\gamma)\gamma U_\theta\ \}.$$

Multiplying both sides by $(M+\rho)$ and cancelling some common terms gives

$$(M+\rho)(\gamma+\rho)V_\theta^{n+1}(w) \geq (\gamma+\rho)\alpha_e(s_e^*)\int_w^{\bar{w}}[V(w')-V(w)]\ dF_\theta(w') + (M+\rho)\gamma U_\theta .$$

Finally, note that if $\varphi(w')$ is any positive, non-decreasing function of w',

then $\int_w^{\bar{w}}\varphi(w')dF_\theta(w') \geq 0$. (Proof: Integration by parts.) Hence

$\int_w^{\bar{w}}[V(w')-V(w)]\ dF_\theta(w') \geq 0$, ie, $(\rho+\gamma)V_\theta^{n+1}(w) \geq \gamma U_\theta$.

Using Proposition 2 we have shown $(\gamma+\rho)V_\theta(w) \geq \gamma U_\theta$. QED.

The remaining comparative statics calculations use similar methods. The results of these calculations are presented in Table 1; the calculations themselves are available from the authors on request. We are able to sign 27 out of the 28 effects using our qualitative comparative statics methodology.

[Table 1 goes about here]

The comparative statics results presented in Table 1 have straightforward interpretations. For example, individuals with high values of γ, ie, those who can expect short employment durations, will search less intensely, both on- and off-the-job, but will be less selective about which wage offers to accept, than will otherwise equivalent individuals with low values of γ. Note that the effect of an increase in γ on expected duration in non-employment is ambiguous since the hazard from non-employment to employment is $\alpha_n(s_n^*)[1-F(r^*)]$ and a change in γ affects s_n^* and r^* in the same direction. The effects of an improvement in the wage offer distribution and of an increase in the marginal efficiency of search while non-employed are similarly ambiguous. The net effects of these changes must be determined by the data.

Table 2 presents Burdett and Mortensen's comparative statics results in our notation.

[Table 2 goes about here]

The specification they used for their comparative statics analysis differs from ours in three ways. First, they assumed the search technology to be the same in both states and the average and marginal efficiencies of search to be identical; ie, Burdett and Mortensen assumed $\alpha_e(s) = \alpha_n(s) = \alpha s$. They gave comparative statics results for "an improvement in the worker's labor market"; in Table 2 this is the amalgamated parameter $\frac{\alpha}{\rho+\gamma}$. Their comparative statics results are sensitive to the assumed search technology. Second, Burdett and Mortensen did not allow for any unemployment compensation in their specification. This does not seem to drive any of their other comparative statics results. Finally, hours are determined endogenously in Burdett and Mortensen versus exogenously in our model. Exogenous hours allow us to derive a result that they could not, namely, $\partial s_e^*/\partial y \leq 0$; on the other hand, endogenous hours allowed Burdett and Mortensen to derive a sharper result for $\partial s_n^*/\partial y$ than the one we obtain. However, these differences are minor. There is basic agreement between our comparative statics results and those of Burdett and Mortensen.

5. Conclusion:

In this paper we have made two contributions. First, we have presented a technique for carrying out qualitative comparative statics analysis in dynamic programming models. The basic idea is that if a value function v, which depends on some exogenous parameter θ, is defined by a contraction mapping, then the derivative of v with respect to θ exists and is also defined by a contraction mapping. Since this derivative is defined by a contraction mapping its qualitative properties can be investigated using mathematical induction. This technique should find useful application in models of job search, optimal growth, stochastic dynamic macroeconomics, etc.

We illustrated our comparative statics methodology with an application to a model of job search. The model is a variant of one developed by Burdett and Mortensen (1978), and our second contribution has been to rederive their results in a unified methodological framework. From the point of view of empirical analysis, an important moral of both our analysis and that of Burdett and Mortensen is that models of individual job search that treat the reservation wage as the only decision variable are likely to be misleading. Changes in the labor market environment that have unambiguous effects on the reservation wage often have offsetting effects on search intensity. Our comparative statics methodology has allowed us to illustrate this point in a straightforward fashion.

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Table 1 -- Comparative Statics Results

	w	γ	ρ	y	b	θ_w	θ_e	θ_n
s_e^*	-	-	-	$-^*$	0	+	**	0
s_n^*		-	-	$-^*$	-	+	+	***
r^*		-	-	?	+	+	-	+

Notes: * These results require $u_{c\ell} = 0$.

** Let $\alpha_e(s) = \theta_e + b_e s$. Then $\frac{\partial s_e^*}{\partial \theta_e} \leq 0$.

Let $\alpha_e(s) = a_e + \theta_e s$. Then $\frac{\partial s_e^*}{\partial \theta_e} \geq 0$.

*** Let $\alpha_n(s) = \theta_n + b_n s$. Then $\frac{\partial s_n^*}{\partial \theta_n} \leq 0$.

Let $\alpha_n(s) = a_n + \theta_n s$. Then $\frac{\partial s_n^*}{\partial \theta_n} \geq 0$.

Table 2

Burdett/Mortensen Comparative Statics Results

	w	y	θ_w	$\frac{\alpha}{\rho+\gamma}$
s_e^*	-	?	+	+
s_n^*		-	+	+
r^*		?	+	+