Price Instability in Multi-Unit Auctions

Edward Anderson and Pär Holmberg

Price instability in multi-unit auctions[*]

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Abstract

We consider a procurement auction, where each supplier has private costs and submits a stepped supply function. We solve for a Bayesian Nash equilibrium and show that the equilibrium has a price instability in the sense that a minor change in a supplier's cost sometimes result in a major change in the market price. In wholesale electricity markets, we predict that the bid price of the most expensive production unit can change by 1-10% due to price instability. The price instability is reduced when suppliers have more steps in their supply functions for a given production technology. In the limit, as the number of steps increases and the cost uncertainty decreases, the Bayesian equilibrium converges to a pure-strategy NE without price instability, the Supply Function Equilibrium (SFE).

Key words: Multi-unit auctions, indivisible unit, price instability, Bayesian Nash equilibria, supply function equilibria, convergence of Nash equilibria, wholesale electricity markets

JEL Classification C62, C72, D43, D44, L94

1 Introduction

Each year multi-unit auctions trade divisible-goods worth trillions of dollars, for example in wholesale electricity markets and treasury bond auctions. Most multi-unit auctions are organized as uniform-price auctions, where each accepted bid is transacted at the clearing price. The bidding format is typically such that each bidder submits a set of bids that together form a stepped bid function. Von der Fehr and Harbord (1993) showed that this type of bidding format introduces price instability; in procurement auctions there are large changes in bid prices for small realized changes in the production cost of a supplier. They therefore argued that approximating Nash equilibria (NE) of stepped supply functions by smooth supply function equilibria (SFE), as in Green and Newbery (1992), is inappropriate since this approximation cannot capture price instability. However, suppliers in von der Fehr and Harbord (1993) can only choose one bid price for their whole production capacity and Newbery (1998) conjectured that the price instability would become smaller with more steps in the bid functions and disappear in the limit when the number of steps increases towards infinity. In this paper we analyse the price stability issue in detail and we are the first to prove that Newbery's conjecture is correct. However, our results also predict that the price instability can be significant in practice where the number of steps is finite. We focus on procurement auctions, but results are analogous for sales auctions.

The mechanism behind von der Fehr and Harbord's result is similar to the Bertrand-Edgeworth game (Edgeworth, 1925). A supplier would find it optimal to undercut a competitor's bid unless the bid price is sufficiently uncertain or unless its mark-up is non-positive.[1] If the cost uncertainty is small, the bid price variation therefore needs to be much higher than the cost variation in equilibrium, i.e. price instability. In the limit where the cost uncertainty is zero, price instability corresponds to a mixed-strategy NE as in von der Fehr and Harbord (1993).[2] We consider cases where each producer submits a set of bids that together form a stepped supply function. Thus the quantity increment of each bid becomes smaller when each bidder submits supply functions with more steps. This decreases the incentive to undercut one of the competitor's bid prices, because there is less to gain from undercutting if it has less impact on sales. This explains our convergence result: the bid price uncertainty of each bid can be smaller if bidders make more bids with smaller quantity increments.

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[1] Von der Fehr and Harbord (1993) consider uniform-pricing, which is different to the Bertrand model. Unlike the Bertrand model, there are cases where asymmetric pure-strategy Nash equilibria with positive mark-ups can exist in von der Fehr and Harbord's model. When demand is sufficiently certain and one producer is sufficiently dominating, then there is a pure-strategy equilibrium where the dominating supplier makes an offer at the reservation price which is accepted. The competitor bids at a sufficiently low price to avoid that the dominating supplier undercuts. Fabra et al. (2006) generalize this type of NE to cases with multiple bids per bidder that form stepped supply functions.

[2] This is in accordance with Harsanyi's (1973) purification theorem, which generally proves that a mixed strategy NE corresponds to a Bayesian NE where small changes in an agent's private payoff has a large impact on its chosen equilibrium action.

We generalize von der Fehr and Harbord's (1993) model by considering a procurement auction where each supplier sells a number of homogenous indivisible units. Indivisible units could arise from restrictions in the bidding format, which we are interested in, but also from constraints and non-convexities in the production technology. Each supplier offers each of its indivisible units at a unit price. Suppliers have private and independent costs; each supplier receives a signal with full information about its own production costs, but is generally imperfectly informed of the competitors' costs. A supplier makes a higher bid for a higher signal (cost), which gives rise to a bid range for each indivisible unit of the supplier, as illustrated in Fig. 1.

We show that if the cost uncertainty is sufficiently small and indivisible units are sufficiently small, then there exists a Bayesian Nash equilibrium where the bid ranges for the different units of a supplier will not overlap, as illustrated in Fig. 1. We call this property *step separation*. In this case we can explicitly solve for equilibria in the multi-unit auction. We prove that this is the unique equilibrium if the auctioneer's demand is uncertain and sufficiently evenly distributed. We also show that the size of the bid range for a supplier's unit n (the n'th cheapest unit of a supplier) is approximately given by $(I - 1)(p_n - c_n)/n$, where I is the number of symmetric producers and $p_n - c_n$ is the approximate mark-up for the unit. From this formula and stylized facts of wholesale electricity markets, we predict that a minor change in the cost of the most expensive production unit could change its bid by between 1% and 10%. We predict that price instability will be less pronounced for less expensive units in wholesale electricity markets.

As conjectured by Newbery (1998), it follows from our results that the price instability would decrease if each supplier was allowed to submit more bids. In our model this corresponds to a reduction in the size of indivisible units. In the limit where costs are common knowledge and units are infinitesimally small, our Bayesian Nash equilibrium converges to Klemperer and Meyer's (1989) smooth, pure-strategy supply function equilibrium (SFE), which does not have this type of price instability. This gives support to the use of smooth supply function equilibria to approximate stepped supply function equilibria in wholesale electricity markets (Anderson and Hu, 2008; Green and Newbery, 1992; Holmberg and Newbery, 2010) and treasury auctions (Wang and Zender, 2002). This type of equilibrium convergence also appears to occur in practice. Empirical studies of the wholesale electricity market in Texas (ERCOT) show that offers of the two to three largest firms in this market, who submit many bids per supplier, roughly match Klemperer and Meyer's first-order condition for continuous supply functions, while the fit is worse for small producers with fewer bids per supplier (Hortaçsu and Puller, 2008; Sioshansi and Oren, 2007). Thus, the smooth supply-function approximation is rough in some circumstances, and models that consider details in the bidding format are sometimes preferable in practice (Kastl, 2011; Kastl, 2012; Wolak, 2007).

A supplier's optimal set of bids for its indivisible units is determined from the characteristics of its residual demand curve.[3] When submitting its set of bids, a

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[3]The residual demand at a specific price is given by demand at that price less competitors'
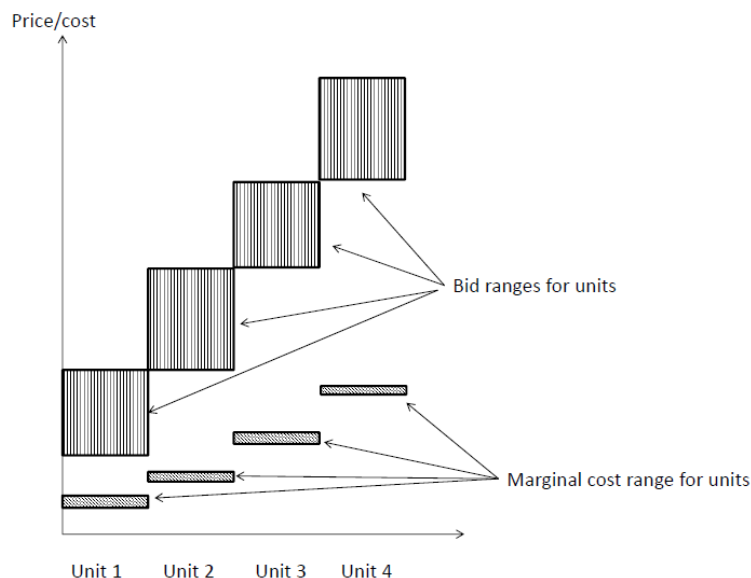
Figure 1: Illustration of price instability where small variations in costs introduces large changes in the bid prices of a supplier. Bid and marginal cost ranges of its indivisible units are non-overlapping in this example. In addition, the maximum bid of one unit equals the minimum bid of the next unit. We refer to this as *step-separation without gaps.*

supplier's residual demand curve is uncertain. This is due to uncertainties in the auctioneer's demand and/or uncertainties in competitors' stepped supply functions, because of imperfect knowledge of competitors' costs or because they randomize their bids. A methodological contribution in this paper is that we develop a discrete version of Anderson and Philpott's (2002b) *market distribution function* and Wilson's (1979) probability distribution of the market price to characterize the uncertainty of a producer's residual demand. We also derive optimality conditions for the best response of a producer facing a stochastic residual demand process that has been characterized by a discrete market distribution function. This new tool should be of general interest for theoretical and empirical studies of auctions with multiple indivisible units. Kastl (2012) and Wolak (2007) present related necessary optimality conditions; our main contribution is that we establish sufficient conditions for global optimality of stepped offers, which are crucial when Nash equilibria are constructed in multi-unit auctions with indivisible units.

Similar to Holmberg et al. (2013), our equilibrium convergence proof partly relies on the convergence theory for Ordinary Differential Equations (ODEs). However, note that Holmberg et al. (2013) prove convergence for the case when quantities are chosen from a continuous set and bid prices are chosen from a discrete set; a setting for which price instability is not an issue.[4] In our model, as in von

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sales as that price.

[4]Pure-strategy NE exists even if supply functions are stepped and costs are common knowledge.

der Fehr and Harbord (1993), bid prices are chosen from a continuous set, while permissible quantities are discrete, which introduces price instability.

Except for Holmberg et al. (2013), our purpose, methodology and results significantly depart from previous convergence studies of equilibria in multi-unit auctions by for example Reny (1999) and McAdams (2003). These approaches use existence of Nash equilibria for a discrete strategy space and equilibrium convergence to establish that there exists equilibria in the limit game with a continuous strategy space. Unlike them, we use a constructive approach to establish existence by deriving expressions for the Nash equilibria. Because of the controversy raised by von der Fehr and Harbord (1993) and Newbery (1998), we are interested in the properties of the Nash equilibria in multi-unit auctions and the extent to which they exhibit price instability.

There are a number of papers (Hortaçsu and McAdams, 2010; Rostek and Weretka, 2012; Vives, 2011; Wilson, 1979) that analyse multi-unit auctions with divisible-goods where bidders have private information and submit smooth bid functions. Anwar (2006) and Ausubel et al. (2014) are more similar to our study in that they analyse Nash equilibria of stepped bid functions. However, our main contributions do not overlap with their results, because they do not explicitly solve for Nash equilibria in this setting, quantify the price instability or prove equilibrium convergence. In particular, the analysis of stepped bid functions in Ausubel et al. (2014) is limited to cases where each bidder submits two bids.

We introduce the model in Section 2. In Section 3 we derive necessary and sufficient conditions, explicitly solve for Bayesian NE and prove equilibrium convergence for our baseline model with a symmetric duopoly. Section 4 extends this to multiple firms for the special case when costs are common knowledge. Section 4.1 uses stylized facts to predict price instability in wholesale electricity markets. Section 5 concludes. All proofs are in the Appendix.

2 The model

In our baseline model, $I = 2$ producers compete in a single-shot game by bidding in a uniform-price auction. Each firm has N production units of equal size h with a total production capacity $\overline{q} = Nh$.[5] Producers have private and independent costs. The cost of each firm i is decided by a private signal α_i, which is chosen by nature and which is not observed by the competitor.[6] There is no loss in generality in assuming that the range of α_i values is uniformly distributed on $[0, 1]$, so that the

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[5]Production capacities in our procurement setting corresponds to purchase constraints in sales auctions. As an example, the U.S. Treasury auction has a 35% rule, which prevents anyone from buying more than 35% of the auctioneer's supply. This is to avoid a situation where a single bidder can corner the market.

[6]Our results would change if suppliers' costs were dependent. It is less critical whether a producer receives a one-dimensional or multi-dimensional signal, such as a vector with individual cost information for each of its production units. The Bayesian NE with step separation that we solve for would still be the same as long as each unit has the same probability distribution for its costs, even if the production costs of a supplier's units are imperfectly correlated.

probability distribution of a signal is $G(\alpha_i) = \alpha_i$.[7] We assume that suppliers are symmetric ex-ante; the marginal cost for the n'th unit of firm i is given by $c_n(\alpha_i)$. We suppose that $c_n(\alpha_i)$ is weakly and continuously increasing in α_i and strictly increasing in n. In the special case where costs do not depend on signals, i.e. costs are common knowledge among producers, independent signals effectively act as randomization devices that help producers to independently randomize their strategies in a mixed-strategy NE, as in the purification theorem by Harsanyi (1973). We write $C_n(\alpha_i) = h \sum_{m=1}^n c_m(\alpha_i)$ for the total cost for producer i of supplying an amount nh. We assume that the highest marginal cost, $c_N(1)$, is strictly smaller than the reservation price $\bar{p}$. In Section 4, we extend the duopoly model to multiple suppliers.

We consider Bayesian Nash equilibria, where each producer i first observes its signal α_i and then chooses an optimal bid price $p_n^i(\alpha_i)$ for each unit $n \in \{1, \ldots, N\}$. We consider cases where $p_n^i(\alpha_i)$ is a continuous, piece-wise smooth, strictly increasing function of its signal α_i and strictly increasing with respect to the unit number n. Thus outcomes where a sharing rule is needed to clear the auction can be neglected.

Given a value of α_i, we can calculate the stepped supply of firm i as a function of price as follows:
$$s_i(p, \alpha_i) = h \sup\{n : p_n^i(\alpha_i) \leq p\}.$$

Note that $s_i(p, \alpha_i)$ is a weakly decreasing function of α_i and weakly increasing with respect to p.

Similar to von der Fehr and Harbord (1993), we assume that demand is uncertain and inelastic up to the reservation price. The demand shock β is realized after producers have submitted their bids and is independent of producers' signals. In wholesale electricity markets the shock could correspond to uncertainty in consumers' demand (including own production, e.g. solar power) and uncertainty in the output of renewable power (e.g. wind power) or must-run plants from non-strategic competitors.[8]

We consider a similar discreteness on the demand side, which is natural if the discreteness has been imposed by the bidding format. We assume that the demand shock β can take values on the set $\mathcal{Q}(h) = \{0, h, 2h, 3h..., 2Nh\}$, where each element in the set $\mathcal{Q}(h)$ occurs with a positive probability. We let $F(\beta)$ be the probability distribution of the demand shock β, i.e. $F(b) = \Pr(\beta \leq b)$ with $b \in \mathcal{Q}(h)$, and let $f(\beta)$ be the probability mass function $f(b) = \Pr(\beta = b) > 0$ for $b \in \mathcal{Q}(h)$.

The auctioneer clears the market at the lowest price where supply is weakly

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[7]Note that we are free to choose the cost parameterization to achieve this. Assume that there is some signal $\widetilde{\alpha}$ with the probability distribution $\widetilde{G}(\widetilde{\alpha})$ and cost function $\widetilde{c}_n(\widetilde{\alpha})$, for which this is not true. Then we can always define a new signal $\alpha = \widetilde{G}(\widetilde{\alpha})$ and define a new cost function $c_n(\alpha) = \widetilde{c}_n(\widetilde{G}^{-1}(\alpha))$, which would satisfy our assumptions.

[8]There is an analogous supply shock in many multi-unit sales auctions. In treasury auctions there is often an uncertain amount of non-competitive bids from many small non-strategic investors (Wang and Zender, 2002; Rostek et al., 2010). Thus the remaining amount of treasury securities available to the large strategic investors is uncertain.

larger than demand.

$$p = \inf \{r : \beta \leq s_1(r, \alpha_1) + s_2(r, \alpha_2)\}.$$

We consider a uniform-price auction, so all accepted offers are paid the clearing price p. Thus, the payoff of a producer i selling n units at price p is:

$$\pi_i = pnh - C_n(\alpha_i).$$

3 Analysis

3.1 Optimality conditions

We start by deriving the best response of a producer to a stochastic residual demand. As a characterization of the residual demand, we let $\Psi_i(n, p)$ be the probability that the offer of the nth unit of producer i is rejected if offered at the price p. This probability depends on the random demand shock β and the cost signal α_j of the competitor $j \neq i$. Thus

$$\Psi_i(n, p) = \Pr(\beta - s_j(p, \alpha_j) < nh).$$

It follows on our assumptions that $\Psi_i(n, p)$ will be continuous and piecewise smooth as a function of p. This probability is a discrete version of Anderson and Philpott's (2002b) market distribution function, which corresponds to Wilson's (1979) probability distribution of the market price. We can now show that

Lemma 1

$$
\begin{aligned}
\frac{\partial \pi_i(r_1, r_2, ...r_N, \alpha_i)}{\partial r_n} &= nh\left(\Psi_i(n+1, r_n) - \Psi_i(n, r_n)\right) \quad (1) \\
&\quad - \frac{\partial \Psi_i(n, r_n)}{\partial r_n} h\left(r_n - c_n(\alpha_i)\right),
\end{aligned}
$$

provided that $\frac{\partial \Psi_i(n, r_n)}{\partial r_n}$ exists.

In the case where the left and right derivatives of $\frac{\partial \Psi_i(n, r_n)}{\partial r_n}$ do not match, then it is easy to see that (1) will still hold provided we choose either left or right derivatives consistently. The result in (1) can be interpreted as follows. Assume that firm i increases the offer price of its unit n, then there are two counteracting effects on the expected pay-off. The revenue increases for outcomes when the offer for unit n is price-setting, which occurs with the probability $\Psi_i(n+1, r_n) - \Psi_i(n, r_n)$. Thus the first term in (1) corresponds to a price-effect; the marginal gain from increasing the bid price of the nth unit if acceptance was unchanged. On the other hand, a higher bid price means that there is a higher risk that the offer of the nth unit is rejected. This is the quantity effect. The marginal loss in profit is given by the increased rejection probability $\frac{\partial \Psi_i(n, r_n)}{\partial r_n}$ for the nth unit

times the pay-off from this unit when it is on the margin of being accepted. Thus $\frac{\partial \pi_i(r_1, r_2, \ldots r_N, \alpha_i)}{\partial r_n}$ equals the price effect minus the loss related to the quantity effect.

We can identify the right-hand side of (1) as being a discrete version of Anderson and Philpott's (2002b) Z function for uniform-price auctions.[9] Thus we define:

Definition 1

$$Z_i \left(n, r_n, \alpha_i\right) = nh \left(\Psi_i \left(n+1, r_n\right) - \Psi_i \left(n, r_n\right)\right) - \frac{\partial \Psi_i \left(n, r_n\right)}{\partial r_n} h \left(r_n - c_n(\alpha_i)\right), \quad (2)$$

where we take the right hand derivative of Ψ_i if left and right derivatives do not match.

Hence, $Z_i \left(n, r_n, \alpha_i\right)$ is the right hand derivative of π_i with respect to r_n, which is independent of other bid prices r_m, where $m \neq n$. We use $Z_i^- \left(n, r_n, \alpha_i\right)$ to denote the left hand derivative of $\pi_i \left(r_1, r_2, \ldots r_N, \alpha_i\right)$ in the following result.

Lemma 2 *A set of bids $\{r_n^*\}_{n=1}^N$ is globally optimal for producer i for signal α_i if:*

$$Z_i \left(n, r_n, \alpha_i\right) \begin{array}{c} \leq \\ \geq \end{array} \begin{array}{cc} 0 & \text{for } r_n \geq r_n^* \\ 0 & \text{for } r_n < r_n^*. \end{array} \quad (3)$$

If π_i is differentiable at r_n^* then a necessary condition for bids $\{r_n^*\}_{n=1}^N$ to be optimal for signal α_i is

$$Z_i \left(n, r_n^*, \alpha_i\right) = 0.$$

In case, the left and right derivatives differ at r_n^*, the necessary condition generalizes to

$$Z_i \left(n, r_n^*, \alpha_i\right) \leq 0 \text{ and } Z_i^- \left(n, r_n^*, \alpha_i\right) \geq 0.$$

Intuitively, we can interpret Lemma 2 as follows: an offer $r_n^* \left(\alpha_i\right)$ is optimal for unit n for signal α_i if the quantity effect dominates for all prices above $r_n^* \left(\alpha_i\right)$ and the price effect dominates for all prices below this price.

3.2 Necessary properties of an equilibrium

In this subsection, we use the optimality conditions to derive necessary properties of an equilibrium. We will show that in many cases the equilibrium must have the property that the lowest bid in the bid range of unit n is at exactly the same price as the highest offer in the bid range for the previous unit $n - 1$. We first prove that there are no gaps between the bid price ranges of successive units of a supplier.

Lemma 3 *In a Bayesian Nash equilibrium, $p_{k-1}^i \left(1\right) \geq p_k^i \left(0\right)$ for each $k \in \{2, \ldots, N\}$, i.e. there are no gaps between the bid ranges of successive units of a supplier $i \in \{1, 2\}$. Moreover, for the highest realized cost, the highest bid of supplier $i \in \{1, 2\}$ is at the reservation price, i.e. $p_N^i \left(1\right) = \bar{p}.$*

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[9]Note that we have chosen our Z function to have a sign opposite to Anderson and Philpott (2002b). Thus there is also a corresponding change in our optimality conditions.

This result is established by showing that if player i has a gap with $p_{k-1}^i(1) < p_k^i(0)$ so there is no offer in this range of prices, then player $j \neq i$ can always improve an offer in this range by increasing it. Thus, in equilibrium, there will also be a matching gap in the offer of player j. But this implies that player i will gain from increasing the bid $p_{k-1}^i(1)$. This contradicts the optimality of player i's bids.

Next, we will show that price ranges of successive units do not overlap; a property we call *step separation*. However, proving this requires additional conditions on costs and the demand uncertainty. The first condition we make use of is related to the cost functions where we require successive units of supplier $i \in \{1,2\}$ not to have overlapping ranges for their marginal costs, as illustrated in Fig. 1.

Assumption 1: *For all $n = 2...N$ and $\alpha \in (0,1)$*

$$c_{n-1}(\alpha) < c_n(0). \tag{4}$$

In order to characterize the uncertainty of the auctioneer's demand, we find it useful to introduce:

Definition 2
$$\tau_m = \frac{f(mh) - f((m-1)h)}{f((m-1)h)}. \tag{5}$$

Note that both $f(mh)$ and $f((m-1)h)$ are non-negative, so $\tau_m \geq -1$.
Assumption 2: *Demand is sufficiently evenly distributed so that*

$$|\tau_m| < 1/(3m) \tag{6}$$

for $m \in \{2, \ldots, 2N\}$.

Lemma 4 *Under Assumption 1 and 2, bid ranges for successive units of supplier $i \in \{1,2\}$ do not overlap in an equilibrium, i.e. $p_{k-1}^i(1) \leq p_k^i(0)$ for $k \in \{2, \ldots, N\}$.*[10]

It follows from Lemma 3 and Lemma 4 that equilibria must necessarily have step separation without gaps, as illustrated in Fig. 1, if cost uncertainty is sufficiently small and the demand density is sufficiently even. In this case, we can also prove that the Bayesian equilibrium must be symmetric.

Lemma 5 *Under Assumption 1 and 2, the Bayesian NE must be symmetric, i.e. $p_n^1(\alpha) = p_n^2(\alpha)$ for $\alpha \in [0,1]$ and $n \in \{1, \ldots, N\}$.*

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[10]Lemma 10 in the Appendix proves that this statement would also hold for a less stringent, but also more complex inequality than (6).

3.3 Existence results

In the previous section we derived necessary properties for an equilibrium - symmetry and step separation without gaps - for cases where the cost uncertainty is sufficiently small and the demand shock is sufficiently evenly distributed. In this section we establish existence of such an equilibrium for a weaker assumption on the demand uncertainty by explicitly solving for a symmetric Bayesian Nash equilibrium with step separation and no gaps. If the stricter condition in (6) is satisfied, then we can prove that this equilibrium is the unique Bayesian NE.

Proposition 1 *The set of solutions $\{p_n(\alpha)\}_{n=1}^N$ as defined by the end-conditions*

$$
\begin{aligned}
p_N(1) &= \bar{p} \\
p_n(1) &= p_{n+1}(0) \ \ \forall n \in \{1, \ldots, N-1\}
\end{aligned}
$$

and

$$
p_n(\alpha) = p_n(1) \frac{(\alpha\tau_{2n}+1)^{1/(n\tau_{2n})}}{(\tau_{2n}+1)^{1/(n\tau_{2n})}}
$$
$$
+ (\alpha\tau_{2n}+1)^{1/(n\tau_{2n})} \int_\alpha^1 \frac{c_n(u)(u\tau_{2n}+1)^{-1/(n\tau_{2n})-1}}{n} du \tag{7}
$$

constitutes a symmetric Bayesian Nash equilibrium, if Assumption 1 is satisfied and

$$
|m\tau_{m+\hat{n}} - (m-1)\tau_{m-1+\hat{n}}| \le 1 \tag{8}
$$

for all $(m, \hat{n}) \in \{1, \ldots, N\} \times \{1, \ldots, N\}$. The equilibrium is unique if, in addition, Assumption 2 is satisfied. Mark-ups are strictly positive in equilibrium, $p_n(\alpha) > c_n(\alpha)$ for $\alpha \in [0, 1]$.

Note that the condition in (8) is always satisfied if indivisible production units are sufficiently small. In the special case when demand is uniformly distributed ($\tau_{2n} \to 0$), then (7) can be simplified to:

$$
p_n(\alpha) = p_n(1) e^{\frac{\alpha-1}{n}} + \int_\alpha^1 \frac{c_n(u) e^{\frac{\alpha-u}{n}}}{n} du.
$$

Proposition 1 simplifies as follows when costs are common knowledge. In this limit of our model, the private signals do not influence costs; they are simply used as randomization devices by the producers when choosing their bids. Thus our symmetric Bayesian Nash equilibrium corresponds to a symmetric mixed-strategy Nash equilibrium.

Corollary 1 *The set of solutions $\{p_n(\alpha)\}_{n=1}^N$ as defined by the end-conditions*

$$
\begin{aligned}
p_N(1) &= \bar{p}, \\
p_n(1) &= p_{n+1}(0) \ \ \forall n \in \{1, \ldots, N-1\},
\end{aligned}
$$

and

$$p_n(\alpha) = (p_n(1) - c_n) \left(\frac{\alpha \tau_{2n} + 1}{\tau_{2n} + 1} \right)^{1/(n\tau_{2n})} + c_n, \qquad (9)$$

constitutes a symmetric mixed-strategy Nash equilibrium, if costs are common knowledge and

$$|m\tau_{m+n} - (m-1)\tau_{m-1+n}| \le 1 \qquad (10)$$

for all $(m, n) \in \{1, \dots, N\} \times \{1, \dots, N\}$. The equilibrium is unique if, in addition, Assumption 2 is satisfied.

3.4   Equilibrium convergence

The supply function equilibrium (SFE) is a pure-strategy Nash equilibrium of smooth supply functions when costs are common knowledge and units are perfectly divisible. We know from Holmberg (2008) and Anderson (2013) that there is a unique symmetric supply function equilibrium for production capacities $\bar{q}$ when inelastic demand has support in the range $[0, 2\bar{q}]$. We let $\tilde{C}'(Q)$ be the marginal cost of the divisible output Q. The unique, symmetric SFE for duopoly markets can be determined from Klemperer and Meyer's (1989) differential equation

$$P'(Q) = \frac{P(Q) - \tilde{C}'(Q)}{Q} \qquad (11)$$

and the boundary condition $P(\bar{q}) = \bar{p}$. Note that we have rewritten the differential equation so that it is on the standard form. In this section we will refer to the solution of this differential equation as the continuous solution.

In this subsection we consider the special case where costs of the indivisible units are common knowledge; this is in line with standard SFE models and the model in von der Fehr and Harbord (1993). Hence, it follows from Corollary 1 and $p_{n-1}(1) = p_n(0)$ that equilibrium bids for the highest signal can be determined from the following difference equation:

$$p_{n-1}(1) = (\tau_{2n} + 1)^{-1/(n\tau_{2n})} (p_n(1) - c_n) + c_n. \qquad (12)$$

A solution to this difference equation is referred to as a discrete solution.

In the equilibrium convergence proof below, we consider a sequence of auctions with successively smaller units h and larger $N = \frac{\bar{q}}{h}$, such that the exogenous shock distribution has a well-defined continuous probability density in the limit $\bar{f}(b) = \lim_{h \to 0} \frac{f(b)}{h}$ and where $\bar{f}'(\beta)/\bar{f}(\beta)$ is bounded in the interval $[0, 2\bar{q}]$. Below we will show that mixed-strategy NE of auctions in this sequence converges to the SFE model in the limit as the size of the indivisible production units h decreases towards 0. Our convergence proof goes through similar steps as in the proof of convergence of stepped pure-strategy NE by Holmberg et al. (2013).

We first show that the difference equation in (12) converges to Klemperer and Meyer's (1989) differential equation. This type of convergence is referred to as consistency in the numerical analysis of differential equations (Le Veque, 2007).

Lemma 6 *The difference equation*

$$p_{n-1}(1) = (\tau_{2n} + 1)^{-1/(n\tau_{2n})} (p_n(1) - c_n) + c_n \qquad (13)$$

can be approximated by

$$p_n(1) - p_{n-1}(1) = \frac{p_n(1) - c_n}{n} - \frac{\tau(p_n(1) - c_n)}{2n} + O(h^3)$$

and is consistent with the differential equation in (11) if

$$
\begin{aligned}
nh &\rightarrow Q \\
c_n &\rightarrow \tilde{C}'(Q)
\end{aligned}
$$

and

$$c_n < p_n(1)$$

when $h \rightarrow 0$.

If we let the error be the difference between the continuous and discrete solutions, then the convergence of the differential and difference equations (consistency) ensures that the local error that is introduced over a short price interval is reduced as the size of the production units, h, becomes small. However, this does not ensure that the discrete solution will converge to the continuous solution when $h \rightarrow 0$, because accumulated errors may still grow at an unbounded rate along a fixed interval as h becomes smaller and the number of production units increases. Hence another step in the convergence analysis is to establish stability, i.e. that small changes in $p_n(1)$ does not drastically change $p_n(0)$. This is verified in the proposition below. The proposition also shows that the converging discrete solution is a mixed-strategy equilibrium.

Proposition 2 *Let $P(Q)$ be the unique pure-strategy continuous supply function equilibrium for divisible units, then there exists a corresponding mixed-strategy NE for indivisible units with properties as in Corollary 1, which converges to the continuous supply function equilibrium in the sense that $p_n(\alpha_i) \rightarrow P(nh)$ when $h \rightarrow 0$ and $c_n \rightarrow \tilde{C}'(nh)$. If $\frac{|\bar{f}'(x)|}{\bar{f}(x)} < \frac{1}{3x}$, i.e. the slope of the probability density of the demand shock is relatively small in the limit, then this ensures that the mixed-strategy NE is the unique equilibrium in the auction with indivisible units when $h \rightarrow 0$.*

As shown by Klemperer and Meyer (1989), the SFE for divisible units is ex-post optimal and it does not depend on the demand shock distribution. This is different to our model for indivisible units where equilibrium bids do depend on the demand uncertainty when $h > 0$. If indivisible production units are sufficiently small or if demand shocks are sufficiently uniformly distributed, then τ_m is close to zero, and second-order effects are negligible. For such cases, Lemma 7 below establishes that the bid price p_n increases when the difference $f(2nh) - f((2n-1)h)$, and hence τ_{2n}, increases. Recall that we consider a symmetric duopoly with step separation, so the bid price of a supplier's unit n, p_n, is influenced by the properties of the demand shock distribution near $2n$ units, which is captured by τ_{2n}.

Figure 2: Bids for the unique, symmetric mixed-strategy NE in a duopoly market with uniformly distributed demand where each producer has 5 indivisible units, each with the size 2.

Lemma 7 $\left.\frac{\partial p_n(\alpha_i)}{\partial \tau_{2n}}\right|_{\tau_{2n}=0} \geq 0$ *for every* $n \in \{1, \ldots, N\}$ *and* $\alpha_i \in [0,1]$. *The inequality is strict unless* $n = N$ *and* $\alpha_i = 1$.

We end this section with some simple examples. When the demand is uniformly distributed (so $\tau_{2n} = 0$), we obtain from (9) that:

$$p_n(\alpha) = (p_n(1) - c_n) e^{\frac{\alpha-1}{n}} + c_n.$$

We illustrate this formula in Figure 2 and Figure 3 for 5 and 20 units per firm, respectively. In Figure 3 we also compare our equilibrium for indivisible units with the supply function equilibrium for divisible goods. Formulas for the latter can be found in Anderson and Philpott (2002a) and Holmberg (2008). The comparison illustrates that the supply function equilibrium approximation works well in this example.

4 Extension: Multiple firms

In this section we generalize results to $I \geq 2$ firms for the special case when costs are common knowledge. We use $K = I - 1$ to denote the number of competitors of a supplier.
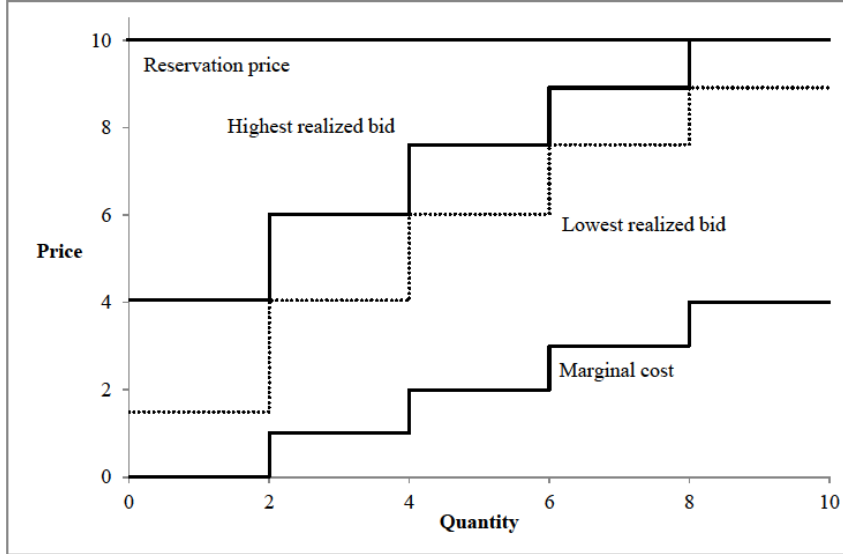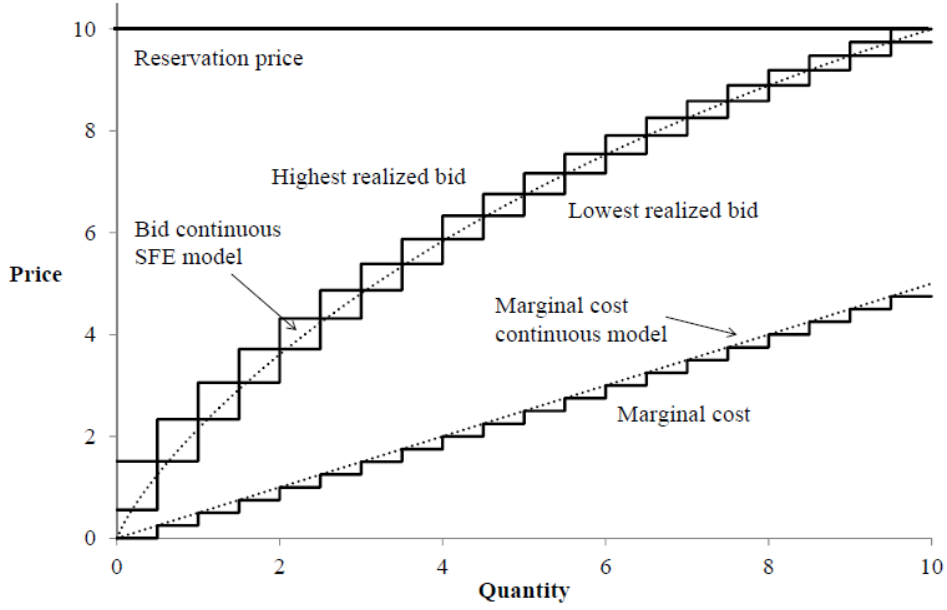
Figure 3: Bids for the unique, symmetric mixed-strategy NE in a duopoly market with uniformly distributed demand where each producer has 20 indivisible units, each with the size 0.5. The equilibrium is compared with an SFE for divisible-goods.

Proposition 3 *In an oligopoly market with $I = K + 1 \geq 2$ symmetric producers with costs that are common knowledge, the set of solutions $\{p_n(\alpha)\}_{n=1}^{N}$ as defined by the end-conditions*

$$
\begin{aligned}
p_N(1) &= \bar{p} \\
p_n(1) &= p_{n+1}(0) \quad \forall n \in \{1, \ldots, N-1\}
\end{aligned}
$$

and

$$
p_n(\alpha) = c_n + (p_n(1) - c_n) e^{-\int_\alpha^1 g(u)du/n}, \tag{14}
$$

where α is a random variable that is uniformly distributed in the interval $[0,1]$ and

$$
g(u) = \frac{\displaystyle\sum_{v=0}^{K-1} \frac{K!}{(K-1-v)!v!} u^v (1-u)^{K-1-v} f((n+v+K(n-1))h)}{\displaystyle\sum_{v=0}^{K} \frac{K!}{v!(K-v)!} u^v (1-u)^{K-v} f((n+v+K(n-1))h)}, \tag{15}
$$

constitutes a symmetric mixed-strategy NE, if

$$
|m\tau_{m+Kn} - (m-1)\tau_{m-1+Kn}| \leq 1 \tag{16}
$$

for all $(m,n) \in \{1, \ldots, N\} \times \{1, \ldots, N\}$.

(16) is satisfied when demand is sufficiently close to a uniform distribution or when units are sufficiently small. In case demand is uniformly distributed, we

14

have from Proposition 3 and the binomial theorem that

$$
\begin{aligned}
g\left(u\right) &= \frac{K\sum_{v=0}^{K-1}\frac{(K-1)!}{(K-1-v)!v!}u^{v}\left(1-u\right)^{K-1-v}}{\sum_{v=0}^{K}\frac{K!}{v!(K-v)!}u^{v}\left(1-u\right)^{K-v}} \\
&= \frac{K\left(u+1-u\right)^{K-1}}{\left(u+1-u\right)^{K}} = K.
\end{aligned}
$$

This gives the result below, where the approximation in (18) follows from a Taylor expansion of (14).

Corollary 2 *In an oligopoly market with uniformly distributed demand and $I = K + 1 \geq 2$ symmetric producers with costs that are common knowledge among bidders, the set of solutions $\{p_n\left(\alpha\right)\}_{n=1}^{N}$ as defined by the end-conditions*

$$
\begin{aligned}
p_N\left(1\right) &= \bar{p} \\
p_n\left(1\right) &= p_{n+1}\left(0\right) \quad \forall n \in \{1, \ldots, N-1\}
\end{aligned}
$$

and

$$
p_n\left(\alpha\right) = c_n + \left(p_n\left(1\right) - c_n\right)e^{K(\alpha-1)/n}, \tag{17}
$$

where α is a random variable that is uniformly distributed in the interval $[0,1]$, constitutes a symmetric mixed-strategy NE. The bid range of the mixed-strategy NE can be approximated from:

$$
p_n\left(1\right) - p_{n-1}\left(1\right) = \frac{\left(p_n\left(1\right) - c_n\right)K}{n} + O\left(h^2\right). \tag{18}
$$

It follows from Corollary 2 that equilibrium bids for the highest signal can be determined from the following difference equation:

$$
p_{n-1}\left(1\right) = p_n\left(0\right) = c_n + \left(p_n\left(1\right) - c_n\right)e^{-K/n}. \tag{19}
$$

Similar to the proof of Lemma 6 in the Appendix, we can use (18) to prove that this difference equation is consistent with the first-order condition of an SFE for multiple firms. It follows from Corollary 2 that

$$
\frac{\partial p_{n-1}\left(1\right)}{\partial p_n\left(1\right)} = e^{-K/n} \in [0,1], \tag{20}
$$

which ensures that the discrete solution is numerically stable also for multiple firms. Thus we can use an argument similar to the proof of Proposition 2 to show that the mixed-strategy NE in Corollary 2 converges to a pure-strategy SFE also for $I \geq 2$ firms.

4.1 Price instability in wholesale electricity markets

In Figure 4, we illustrate Corollary 2 for 6 firms and 20 indivisible units per firm. Each indivisible unit has the size 0.5, so that each firm has the maximum output 10. As expected from the theory on smooth SFE (Holmberg, 2008), this graph also illustrates that mark-ups tend to be convex with respect to output in oligopoly markets. This becomes even more pronounced in a more competitive market, where mark-ups are essentially zero far below the production capacity and then take off near the production capacity. This is consistent with hockey-stick bidding that has been observed in practice, i.e. observed bid prices and estimated mark-ups become drastically larger near the total production capacity of the market (Hurlbut et al., 2004; Holmberg and Newbery, 2010). In this case, mark-ups will be largest and the bid range will be widest for units with the highest marginal cost. Typically wholesale electricity markets have reservation prices in the range \$1.000-\$100.000/MWh (Holmberg et al., 2013; Stoft, 2002), which normally is significantly higher than the marginal cost of the most expensive unit. Market concentration in wholesale electricity markets as measured by the Herfindahl Hirschman Index (HHI) is typically in the range 1000-2000, both in Europe (Newbery, 2009) and U.S. (Bushnell et al., 2008). This degree of market concentration can be represented by 5-10 symmetric firms. From the discussion in Holmberg et al. (2013) and Green and Newbery (1992), it is reasonable to assume that each representative firm submits 50-500 bids each. From these stylized facts, it follows from (18) that the bid range of the most expensive unit is significant, 1-10% of the reservation price. However, it is rare that bids from the most expensive units are accepted in wholesale electricity markets. Typically this only occurs 0.1% of the time or even more seldom. The price instability is expected to be lower for less expensive units that typically have smaller mark-ups. In practice producers have sold some of their output in advance with forward contracts. As shown by Newbery (1998) and Holmberg (2011) this lowers mark-ups in electricity markets, and accordingly it should also mitigate price-instability.

5   Conclusions

We consider a procurement multi-unit auction where each supplier makes a bid for each of its homogenous indivisible units. This set-up can represent restrictions in the bidding format, which is our main interest, but also restrictions in the production technology, such as production constraints or non-convex costs. Producers have private information on their costs. A producer submits a higher bid for a unit when its costs are higher. This gives a bid range for each unit of the supplier. We show that if the cost uncertainty is sufficiently small and units are sufficiently small, then there exists a Bayesian Nash Equilibrium in which the bid ranges for the different units of a supplier do not overlap. We call this property *step separation.* In this case we can explicitly solve for equilibria in the multi-unit auction. We prove that this is the unique equilibrium if the auctioneer's demand is uncertain and sufficiently evenly distributed. This Bayesian NE has a price in-
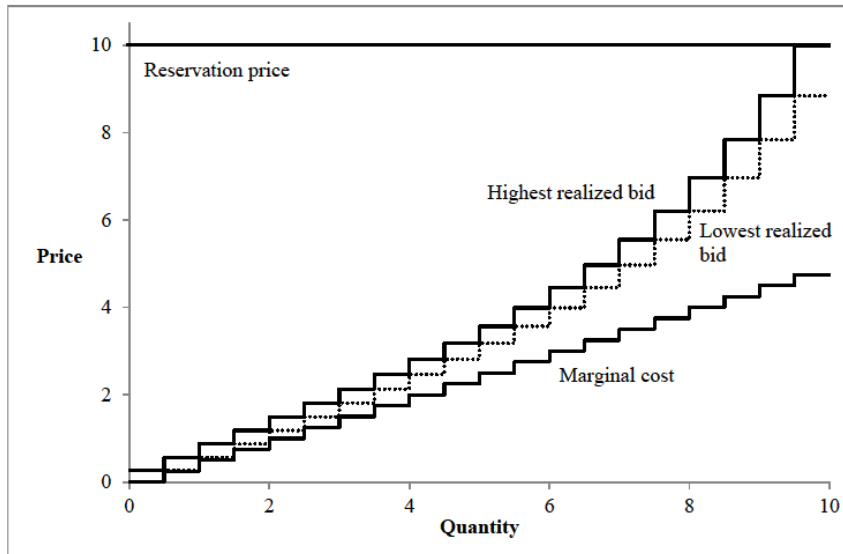
Figure 4: Bids for the unique, symmetric mixed-strategy NE bids in an oligopoly market with 6 firms and uniformly distributed demand.

stability. A small cost change for a unit can have a much larger impact on the bid price of the unit. The length of the bid range for a production unit n (which is the n'th cheapest unit of a supplier) is approximately given by $(I - 1)(p_n - c_n)/n$, where I is the number of producers and $p_n - c_n$ is the approximate mark-up for unit n. We estimate that minor cost changes can change the bid price of the most expensive unit in wholesale electricity markets by 1-10%. We predict that the price instability will be lower for cheaper units in electricity markets, and when producers hedge their output by selling forward contracts.

As price instability worsens welfare for risk averse market participants, there are advantages with market designs that mitigate price instability. The price instability decreases when producers are allowed to submit more bids. We prove that the Bayesian NE of our set-up converges to a pure-strategy, smooth supply function equilibrium (SFE) without price instability when both the unit size and cost uncertainty decreases towards zero. This result gives support to the use of smooth SFE to approximate bidding with stepped supply functions in wholesale electricity markets, as in Green and Newbery (1992). Previously, the convergence of stepped SFE to smooth SFE has been established by Holmberg et al. (2013) for cases where ticks-sizes are large and the size of indivisible units is small, so that price instability is not an issue. We also believe that price instability can be avoided in market designs with piece-wise linear supply functions, as in the wholesale electricity markets in the Nordic countries, Nord Pool, and France, Power Next.

The paper also contributes by introducing a discrete version of Anderson and Philpott's (2002b) market distribution function and Wilson's (1979) probability

distribution of the market price. We use this tool to characterize the uncertainty in the residual demand of a producer. We also derive conditions for the globally best response of a producer facing a given discrete market distribution function. These conditions can be used in both empirical and theoretical studies of auctions with multiple indivisible units.

Similar to SFE (Anderson, 2013; Holmberg, 2008) our uniqueness result relies on the auctioneer's demand varying in a sufficiently wide range. More equilibria would occur if this demand range was reduced. If demand was certain and production capacities non-restrictive, then there is likely to be a continuum of similar equilibria for indivisible units, as is the case for continuous SFE of divisible units (Klemperer and Meyer, 1989). In this case, it has been popular in the literature that analyses bidding formats with stepped bid functions to select the equilibrium that does not have price instability. When bidders' costs/values are common knowledge this corresponds to selecting a pure-strategy NE, as in von der Fehr and Harbord (1993), Kremer and Nyborg (2004a;2004b) and Fabra et al. (2006). These NE are rather extreme, the market price is either at the marginal cost or reservation price. Similar bidding behaviour has been observed in the capacity market of New York's electricity market, which is dominated by one supplier and where the demand variation is small (Schwenen, 2012). Still, we are not convinced that NE without price instability will always be selected when the demand variation is small. Especially as the price instability becomes smaller if bidders are allowed to choose more steps in their supply functions. With sufficiently many steps, a market participant would not be able to notice the price instability in markets or experiments, it would become negligible in comparison to erratic behaviour of the agents and other types of noise. Under such circumstances it does not seem likely that price instability would be a reliable criterion for equilibrium selection.

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Appendix A: Optimality conditions

In the proofs, we find it convenient to identify the values of α that give rise to bid prices for given units n. We define

$$\widehat{\alpha}_i(p, n) = \sup(\alpha_i : s_i(p, \alpha_i) \geq nh), \ n = 1, 2..., N,$$

with $\widehat{\alpha}_i(p, n) = 0$ if $s_i(p, 0) < nh$. Thus $\widehat{\alpha}_i$ is the probability that firm i will sell at least n units at the clearing price p. It follows from our assumptions for $p_n^i(\alpha_i)$ that $\widehat{\alpha}_i$ is increasing, continuous and piece-wise smooth with respect to the price and decreasing with respect to n. To simplify our equations, we set $\widehat{\alpha}_i(p, 0) = 1$

and $\widehat{\alpha}_i(p, N + 1) = 0$. We let

$$\Delta\widehat{\alpha}_i(p, n) = \widehat{\alpha}_i(p, n) - \widehat{\alpha}_i(p, n + 1) \geq 0.$$

Thus $\Delta\widehat{\alpha}_i(p, n)$ is the probability that an α_i value is chosen by nature such that an agent will sell exactly nh units at price p.

Lemma 8 *The expected profit of producer i for a set of bids $\{r_n\}_{n=1}^N$ and a signal α_i is given by:*

$$
\begin{aligned}
\pi_i\left(r_1, r_2, ... r_N, \alpha_i\right) = &\sum_{n=1}^N \left(\Psi_i\left(n + 1, r_n\right) - \Psi_i\left(n, r_n\right)\right)\left(nhr_n - C_n(\alpha_i)\right) \\
&+ \sum_{n=1}^N \int_{r_n}^{r_{n+1}} \frac{\partial \Psi_i(n+1, p)}{\partial p}\left(nhp - C_n(\alpha_i)\right)dp,
\end{aligned}
\tag{21}
$$

where we choose $\frac{\partial \Psi_i(n+1, p)}{\partial p}$ to equal the right hand derivative of Ψ_i at the points where left and right derivatives do not match.

Proof. We first calculate the probability that producer i sells exactly n units. This can occur under two different circumstances. In the first case, r_n is price-setting. This occurs when the nth unit of producer i is accepted and the competitor's last accepted unit has a bid below r_n. The probability for this event is $\Psi_i\left(n + 1, r_n\right) - \Psi_i\left(n, r_n\right)$.

Next we consider the case where producer i sells exactly n units at a price $p \in (r_n, r_{n+1})$, which is set by the residual demand of producer i. This occurs when the nth unit of producer i is accepted and the competitor's last accepted unit has a bid in the interval (r_n, r_{n+1}). The probability that the competitor has its last accepted bid in an interval $[p, p + \Delta p]$ is given by $\Psi_i\left(n + 1, p + \Delta p\right) - \Psi_i\left(n + 1, p\right)$. When the derivative exists, this approaches $\frac{\partial \Psi_i(n+1, p)}{\partial p}\Delta p$ as $\Delta p \to 0$. Thus we can write the total expected profit of the firm as in (21). By assumption $\Psi_i\left(n + 1, p\right)$ has only isolated points where it is non-smooth, and consequently the choice of derivative value at these points will not affect the integral. ■

Proof. (Lemma 1) We have from (21) that

$$
\begin{aligned}
\frac{\partial \pi_i\left(r_1, r_2, ... r_N, \alpha_i\right)}{\partial r_n} = &\left(\frac{\partial \Psi_i\left(n + 1, r_n\right)}{\partial r_n} - \frac{\partial \Psi_i\left(n, r_n\right)}{\partial r_n}\right)\left(nhr_n - C_n(\alpha_i)\right) \\
&+ nh\left(\Psi_i\left(n + 1, r_n\right) - \Psi_i\left(n, r_n\right)\right) \\
&+ \frac{\partial \Psi_i\left(n, r_n\right)}{\partial r_n}\left((n - 1)hr_n - C_{n-1}(\alpha_i)\right) \\
&- \frac{\partial \Psi_i\left(n + 1, r_n\right)}{\partial r_n}\left(nhr_n - C_n(\alpha_i)\right),
\end{aligned}
$$

which can be simplified to (1). ■

Proof. (Lemma 2) Suppose (3) is satisfied and that there is another set of bids $\{r_n\}_{n=1}^N$, such that $\pi_i\left(r_1^*, r_2^*, ... r_N^*, \alpha_i\right) < \pi_i\left(r_1, r_2, ... r_N, \alpha_i\right)$. This leads to a contradiction, because it follows from (3) and (2) that $\frac{\partial \pi_i(r_1, r_2, ... r_N, \alpha_i)}{\partial r_n} \leq 0$ for

almost all $r_n \geq r_n^*$ and that $\frac{\partial \pi_i(r_1, r_2, \dots r_N, \alpha_i)}{\partial r_n} \geq 0$ for almost all $r_n \leq r_n^*$. Thus the profit must (weakly) decrease as the set of bids is changed from $\{r_n^*\}_{n=1}^N$ to $\{r_n\}_{n=1}^N$. We realize that this change can be done in steps without violating the constraint that bids must be monotonic with respect to n. Hence, the necessary conditions follow straightforwardly from the definition in (1). $\blacksquare$

We can derive the following result for the duopoly market that we are studying. This shows how Ψ_i and Z_i can be expressed in terms of $\widehat{\alpha}_j$ values.

Lemma 9 *In a duopoly market*

$$\Psi_i(n, p) = \sum_{m=0}^{N} \Delta \widehat{\alpha}_j(p, m) F\left((n + m - 1) h\right), \tag{22}$$

and

$$
\begin{aligned}
Z_i(n, r_n, \alpha_i) &= nh \sum_{m=0}^{N} \Delta \widehat{\alpha}_j(r_n, m) f((n + m) h) \\
&\quad - h(r_n - c_n(\alpha_i)) \sum_{m=0}^{N-1} \frac{\partial \widehat{\alpha}_j(r_n, m + 1)}{\partial r_n} f((n + m) h) \tag{23}
\end{aligned}
$$

where we interpret $\frac{\partial \widehat{\alpha}_j}{\partial r_n}$ as a right derivative where left and right derivatives differ.

Proof. An offer of n units at price p by producer i is rejected if the competitor $j \neq i$ offers exactly m units at the price p, which occurs with probability $\Delta \widehat{\alpha}_j(p, m)$, when demand at this price is less than $n + m$ units, which has the probability $F\left((n + m - 1) h\right)$. We get $\Psi_i(n, p)$ in (22) by summing across all $m \in \{0, \dots, N\}$. It now follows from Lemma 1 and (22) that

$$
\begin{aligned}
Z_i(n, r_n, \alpha_i) &= nh \left(\sum_{m=0}^{N} \Delta \widehat{\alpha}_j(r_n, m) f((n + m) h) \right) \\
&\quad - h(r_n - c_n(\alpha_i)) \sum_{m=0}^{N} \left(\frac{\partial \widehat{\alpha}_j(r_n, m)}{\partial r_n} - \frac{\partial \widehat{\alpha}_j(r_n, m + 1)}{\partial r_n} \right) F\left((n + m - 1) h\right) \\
&= nh \left(\sum_{m=0}^{N} \Delta \widehat{\alpha}_j(r_n, m) f((n + m) h) \right) \\
&\quad - h(r_n - c_n(\alpha_i)) \sum_{m=-1}^{N-1} \frac{\partial \widehat{\alpha}_j(r_n, m + 1)}{\partial r_n} F\left((n + m) h\right) \\
&\quad + h(r_n - c_n(\alpha_i)) \sum_{m=0}^{N} \frac{\partial \widehat{\alpha}_j(r_n, m + 1)}{\partial r_n} F\left((n + m - 1) h\right).
\end{aligned}
$$

which can be simplified to (23), because $\frac{\partial \widehat{\alpha}_j(p, 0)}{\partial r_n} = \frac{\partial \widehat{\alpha}_j(p, N+1)}{\partial r_n} = 0$. $\blacksquare$

Appendix B: Necessary properties of a duopoly equilibrium

Proof. (Lemma 3) Assume to the contrary that $p_{k-1}^i(1) < p_k^i(0)$ for some $k \in \{2, \ldots, N\}$. Thus for any price in the range $\left(p_{k-1}^i(1), p_k^i(0)\right)$ agent i sells exactly $(k-1)h$ units. In other words $\Delta\widehat{\alpha}_i(p, k-1) = 1$ and $\Delta\widehat{\alpha}_i(p, m) = 0$ for $p \in \left(p_{k-1}^i(1), p_k^i(0)\right)$ and $m \neq k-1$.

Now, first suppose that producer j makes some offer in this price range. Hence there is some $\widetilde{n}, \widetilde{\alpha}$ with $\widetilde{p} = p_{\widetilde{n}}^j(\widetilde{\alpha}) \in \left(p_{k-1}^i(1), p_k^i(0)\right)$. Then $\frac{\partial\widehat{\alpha}_i(p,m)}{\partial p} = 0$ at $\widetilde{p}$, and so, from Lemma 9 and Definition 1,

$$
\begin{aligned}
Z_j(\widetilde{n}, \widetilde{p}, \widetilde{\alpha}) &= \frac{\partial\pi_j(r_1, r_2, \ldots r_N, \widetilde{\alpha})}{\partial r_{\widetilde{n}}} = \widetilde{n}h\sum_{m=0}^{N}\Delta\widehat{\alpha}_i(\widetilde{p}, m)f((\widetilde{n}+m)h) \\
&= \widetilde{n}hf((\widetilde{n}+k-1)h) > 0
\end{aligned}
$$

because $f((\widetilde{n}+k-1)h) > 0$ (we assume that every value of demand up to $2N$ is possible). Hence producer j would gain from increasing its bid for unit $\widetilde{n}$ when observing signal $\widetilde{\alpha}$. This cannot occur in equilibrium, and so we deduce that there is no offer from producer j in the range $\left(p_{k-1}^i(1), p_k^i(0)\right)$. This implies that $\frac{\partial\widehat{\alpha}_j(p,m)}{\partial p} = 0$ for p in this range. With a similar argument as above, it now follows that

$$
\begin{aligned}
Z_i(k-1, p, 1) &= \frac{\partial\pi_j(r_1, r_2, \ldots r_N, 1)}{\partial r_{k-1}} = (k-1)h\sum_{m=0}^{N}\Delta\widehat{\alpha}_j(p, m)f((k-1+m)h) \\
&= (k-1)hf((2k-2)h) > 0,
\end{aligned}
$$

for $p \in \left(p_{k-1}^i(1), p_k^i(0)\right)$. Hence, producer i will gain by increasing its bid for unit $k-1$ when observing signal $\alpha_i = 1$. Hence the strategy is not optimal for producer i and again we have a contradiction. We can use the same argument to rule out that $p_N^i(1) < \overline{p}$. ∎

Lemma 10 *Under Assumption 1, bid ranges for successive units of supplier $i \in \{1, 2\}$ do not overlap in an equilibrium, i.e. $p_{k-1}^i(1) \leq p_k^i(0)$ for $k \in \{2, \ldots, N\}$, if*

$$f((n-2)h) < \Gamma_n f((n-1)h) \quad \text{and} \quad f((n-1)h) < \Gamma_n f(nh) \tag{24}$$

where

$$\Gamma_n = \left(\frac{n-1}{n-2}\right)\frac{\min\left[f((n-3)h), f((n-2)h)\right]}{\max\left[f((n-2)h), f((n-1)h)\right]}$$

for $n \in \{3, \ldots, N\}$.

Proof. We let p_Z^i be the highest price at which there is an overlap for i, thus we have $p_Z^i = p_{k_i-1}^i(1) > p_{k_i}^i(0)$ for some k_i, and $p_{k-1}^i(1) \leq p_k^i(0)$ for $k > k_i$. Without loss of generality we can assume that $p_Z^i \geq p_Z^j$ and we will need to deal separately with the two cases $p_Z^i = p_Z^j$ and $p_Z^i > p_Z^j$.

First we take the case that they are equal. By assumption, bid prices are strictly increasing with respect to the number of units for a given signal, so $p_{k_i}^i(1) > p_Z^i > p_{k_i-2}^i(1)$ and $p_Z^i > p_{k_j-2}^j(1)$. Thus we may find p_0 with $p_Z^i > p_0 > \max\{p_{k_i-2}^i(1), p_{k_j-2}^j(1), p_{k_i}^i(0), p_{k_j}^j(0)\}$. In this case, we can identify signals α_X^j, α_Y^j, α_X^i, α_Y^i in the range $(0,1)$, such that $p_0 = p_{k_j-1}^j(\alpha_X^j) = p_{k_j}^j(\alpha_Y^j) = p_{k_i-1}^i(\alpha_X^i) = p_{k_i}^i(\alpha_Y^i)$ in the equilibrium. By assumption $p_n^i(\alpha)$ is strictly increasing with respect to n and α below the reservation price. Thus $\alpha_X^i > \alpha_Y^i$ and $\alpha_X^j > \alpha_Y^j$. Moreover,

$$\widehat{\alpha}_j(p_0, n) = \begin{cases} 1 & \text{for } n \leq k_j - 2 \\ \alpha_X^j & \text{for } n = k_j - 1 \\ \alpha_Y^j & \text{for } n = k_j \\ 0 & \text{for } n \geq k_j + 1, \end{cases}$$

and

$$\Delta\widehat{\alpha}_j(p_0, n) = \begin{cases} 0 & \text{for } n \leq k_j - 1 \\ 1 - \alpha_X^j & \text{for } n = k_j \\ \alpha_X^j - \alpha_Y^j & \text{for } n = k_j \\ \alpha_Y^j & \text{for } n = k_j + 2 \\ 0 & \text{for } n \geq k_j + 3. \end{cases}$$

It now follows from Lemma 9 that :

$$\begin{aligned} Z_i\left(k_i, p_0, \alpha_Y^i\right) &= k_i h (1 - \alpha_X^j) f((k_i + k_j - 2)h) \\ &\quad + k_i h\left(\alpha_X^j - \alpha_Y^j\right) f\left((k_i + k_j - 1)h\right) \\ &\quad + k_i h \alpha_Y^j f\left((k_i + k_j)h\right) \\ &\quad - h\left(p_0 - c_{k_i}(\alpha_Y^i)\right) \sum_{m=k_j-1}^{k_j} \frac{\partial\widehat{\alpha}_j(p_0, m)}{\partial p} f\left((k_i + m - 1)h\right), \end{aligned} \tag{25}$$

$$\begin{aligned} Z_i\left(k_i - 1, p_0, \alpha_X^i\right) &= (k_i - 1) h (1 - \alpha_X^j) f((k_i + k_j - 3)h) \\ &\quad + (k_i - 1) h\left(\alpha_X^j - \alpha_Y^j\right) f\left((k_i + k_j - 2)h\right) \\ &\quad + (k_i - 1) h \alpha_Y^j f\left((k_i + k_j - 1)h\right) \\ &\quad - h\left(p_0 - c_{k_i-1}(\alpha_X^i)\right) \sum_{m=k_j-1}^{k_j} \frac{\partial\widehat{\alpha}_j(p_0, m)}{\partial p} f\left((k_i + m - 2)h\right). \end{aligned} \tag{26}$$

From (24) we observe that

$$\begin{aligned} \eta &= \left(\alpha_X^j - \alpha_Y^j\right)\left(\Gamma_{k_i+k_j} f\left((k_i + k_j - 1)h\right) - f\left((k_i + k_j - 2)h\right)\right) \\ &\quad + \alpha_Y^j\left(\Gamma_{k_i+k_j} f\left((k_i + k_j)h\right) - f\left((k_i + k_j - 1)h\right)\right) \\ &> 0 \end{aligned} \tag{27}$$

We will write

$$\begin{aligned} f_{\max} &= \max\left[f\left((k_i + k_j - 2)h\right), f\left((k_i + k_j - 1)h\right)\right], \\ f_{\min} &= \min\left[f\left((k_i + k_j - 3)h\right), f\left((k_i + k_j - 2)h\right)\right]. \end{aligned}$$

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Then

$$\Gamma_{k_i+k_j}\left(p_0 - c_{k_i}(\alpha_Y^i)\right) \sum_{m=k_j-1}^{k_j} \frac{\partial \widehat{\alpha}_j\left(p_0,m\right)}{\partial p} f\left((k_i+m-1)h\right)$$

$$= \binom{k_i+k_j-1}{k_i+k_j-2}\left(f_{\min}\left(p_0 - c_{k_i}(\alpha_Y^i)\right) \sum_{m=k_j-1}^{k_j} \frac{\partial \widehat{\alpha}_j\left(p_0,m\right)}{\partial p} \underbrace{\frac{f\left((k_i+m-1)h\right)}{f_{\max}}}_{\leq 1}\right)$$

$$\leq \binom{k_i+k_j-1}{k_i+k_j-2}\left(f_{\min}\left(p_0 - c_{k_i-1}(\alpha_X^i)\right) \sum_{m=k_j-1}^{k_j} \frac{\partial \widehat{\alpha}_j\left(p_0,m\right)}{\partial p} \underbrace{\frac{f\left((k_i+m-2)h\right)}{f_{\min}}}_{\geq 1}\right)$$

So we have from (25) and (27):

$$\begin{aligned}
&\Gamma_{k_i+k_j} Z_i\left(k_i,p_0,\alpha_Y^i\right)\\
\geq\;& k_i h \Gamma_{k_i+k_j}(1-\alpha_X^j)f((k_i+k_j-2)h)\\
&+ k_i h\left(\left(\alpha_X^j - \alpha_Y^j\right)f\left((k_i+k_j-2)h\right) + \alpha_Y^j f\left((k_i+k_j-1)h\right)\right) + k_i h \eta\\
&- h\binom{k_i+k_j-1}{k_i+k_j-2}\left(\left(p_0 - c_{k_i-1}(\alpha_X^i)\right)\sum_{m=k_j-1}^{k_j}\frac{\partial\widehat{\alpha}_j\left(p_0,m\right)}{\partial p}f\left((k_i+m-2)h\right)\right).
\end{aligned}$$

Now $\binom{k_i+k_j-1}{k_i+k_j-2} \leq \binom{k_i}{k_i-1}$ and so $k_i \geq \binom{k_i+k_j-1}{k_i+k_j-2}(k_i-1)$. We deduce from (26) that

$$\begin{aligned}
\Gamma_{k_i+k_j} Z_i\left(k_i,p_0,\alpha_Y^i\right) \geq\;& k_i h \Gamma_{k_i+k_j}(1-\alpha_X^j)\left(f((k_i+k_j-2)h) - f((k_i+k_j-3)h)\right)\\
&+ \binom{k_i+k_j-1}{k_i+k_j-2} Z_i\left(k_i-1,p_0,\alpha_X^i\right) + k_i h \eta. \qquad (28)
\end{aligned}$$

It follows from our assumptions that $\widehat{\alpha}_j\left(p_0,m\right)$ is piece-wise differentiable. We consider a presumed equilibrium. Hence, provided that we do not choose p_0 where some $\widehat{\alpha}_j\left(p_0,m\right)$ is non-smooth, we deduce from the necessary conditions in Lemma 2 that

$$Z_i\left(k_i-1,p_0,\alpha_X^i\right) = 0 \qquad (29)$$

and

$$Z_i\left(k_i,p_0,\alpha_Y^i\right) = 0. \qquad (30)$$

By assumption $p_n^i\left(\alpha\right)$ is continuous with respect to α. Thus by choosing p_0 below and sufficiently close to p_Z^i, and thereby α_X close enough to 1, we can ensure that the right hand side of (28) is strictly greater than zero, which would contradict (30). This leads to the conclusion that $p_Z^i = p_Z^j$ cannot occur in equilibrium.

The next step is to consider the case where $p_Z^i > p_Z^j$. We choose p_0 with $p_Z^i > p_0 > \max\{p_{k_i-2}^i\left(1\right), p_{k_i}^i\left(0\right), p_Z^j\}$. In this case, we can identify signals α_X^i, α_Y^i

in the range $(0, 1)$, such that $p_0 = p^i_{k_i-1}(\alpha^i_X) = p^i_{k_i}(\alpha^i_Y)$. Moreover from Lemma 3 we can deduce the existence of m_j and α^j for which $p_0 = p^j_{m_j}(\alpha^j)$. Thus

$$
\widehat{\alpha}_j(p_0, n) = \begin{cases} 1 & \text{for } n \leq m_j - 1 \\ \alpha^j & \text{for } n = m_j \\ 0 & \text{for } n \geq m_j + 1, \end{cases}
$$

and

$$
\Delta\widehat{\alpha}_j(p_0, n) = \begin{cases} 0 & \text{for } n \leq m_j - 1 \\ 1 - \alpha^j & \text{for } n = m_j \\ \alpha^j & \text{for } n = m_j + 1 \\ 0 & \text{for } n \geq m_j + 2. \end{cases}
$$

Now we can use Lemma 9 to show

$$
\begin{aligned}
Z_i\left(k_i, p_0, \alpha^i_Y\right) &= k_i h\left((1 - \alpha^j)f((k_i + m_j - 1)h) + \alpha^j f\left((k_i + m_j)h\right)\right) \\
&\quad - h\left(p_0 - c_{k_i}(\alpha^i_Y)\right)\frac{\partial\widehat{\alpha}_j(p_0, m_j)}{\partial p}f\left((k_i + m_j - 1)h\right),
\end{aligned}
$$

$$
\begin{aligned}
Z_i\left(k_i - 1, p_0, \alpha^i_X\right) &= (k_i - 1)h\left((1 - \alpha^j)f((k_i + m_j - 2)h) + \alpha^j f\left((k_i + m_j - 1)h\right)\right) \\
&\quad - h\left(p_0 - c_{k_i-1}(\alpha^i_X)\right)\frac{\partial\widehat{\alpha}_j(p_0, m_j)}{\partial p}f\left((k_i + m_j - 2)h\right).
\end{aligned}
$$

The rest of the proof follows from a contradiction achieved using the same argument as above, with m_j instead of k_j and a single term $\frac{\partial\widehat{\alpha}_j(p_0, m_j)}{\partial p}$. ∎

Proof. (Lemma 4) Below we show that (6) is sufficient to satisfy the conditions for Lemma 10. We can deduce immediately from Definition 2 that $|\tau_n| < 1/(3n)$ implies that:

$$
f((n - 1)h) > \left(\frac{3n}{3n + 1}\right)f(nh), \tag{31}
$$

and

$$
f(nh) > \left(\frac{3n - 1}{3n}\right)f((n - 1)h). \tag{32}
$$

There are two inequalities in (24) that we need to establish. The first inequality can be written

$$
\begin{aligned}
&(n - 1)f\left((n - 1)h\right)\min\left[f\left((n - 3)h\right), f\left((n - 2)h\right)\right] \\
&> (n - 2)f\left((n - 2)h\right)\max\left[f\left((n - 2)h\right), f\left((n - 1)h\right)\right].
\end{aligned}
$$

We have to check four cases.
(a) $(n-1)f\left((n - 1)h\right)f\left((n - 3)h\right) > (n-2)f\left((n - 2)h\right)f\left((n - 2)h\right)$: From (31) applied at $n - 2$ we have

$$
f\left((n - 3)h\right) > \left(\frac{3n - 6}{3n - 5}\right)f((n - 2)h) \tag{33}
$$

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and with (32) applied at $n-1$ we have

$$f((n-1)h) > \left(\frac{3n-4}{3n-3}\right) f((n-2)h). \tag{34}$$

Since
$$(n-1)\left(\frac{3n-6}{3n-5}\right)\left(\frac{3n-4}{3n-3}\right) = \left(\frac{3n-4}{3n-5}\right)(n-2) > n-2,$$

we establish the inequality we require.

(b) $(n-1)f\left((n-1)\,h\right)f\left((n-2)\,h\right) > (n-2)f\left((n-2)\,h\right)f\left((n-2)\,h\right)$: From (34) we can deduce

$$(n-1)f\left((n-1)\,h\right) > \left(n-\frac{4}{3}\right) f((n-2)h) > (n-2)f((n-2)h),$$

which immediately implies the inequality.

(c) $(n-1)f\left((n-1)\,h\right)f\left((n-3)\,h\right) > (n-2)f\left((n-2)\,h\right)f\left((n-1)\,h\right)$: From (33) we obtain

$$(n-1)f((n-3)h) > (n-1)\left(\frac{3n-6}{3n-5}\right) f((n-2)h) = (n-2)\left(\frac{3n-3}{3n-5}\right) f((n-2)h),$$

which immediately implies the inequality.

(d) $(n-1)f\left((n-1)\,h\right)f\left((n-2)\,h\right) > (n-2)f\left((n-2)\,h\right)f\left((n-1)\,h\right)$: This is immediate.

Now we turn to the other inequality, we need to establish:

$$(n-1)f\left(nh\right)\min\left[f\left((n-3)\,h\right),f\left((n-2)\,h\right)\right]$$
$$> (n-2)f\left((n-1)\,h\right)\max\left[f\left((n-2)\,h\right),f\left((n-1)\,h\right)\right].$$

Again we have four cases to check.

(a) $(n-1)f\left(nh\right)f\left((n-3)\,h\right) > (n-2)f\left((n-1)\,h\right)f\left((n-2)\,h\right)$: Using (32) and (33) we have

$$(n-1)f\left(nh\right)f\left((2n-3)\,h\right) \tag{35}$$
$$> (n-1)\left(\frac{3n-1}{3n}\right)\left(\frac{3n-6}{3n-5}\right) f((2n-1)h)f((n-2)h). \tag{36}$$

Since $(n-1)\,(3n-1) - n(3n-5) = n+1 > 0$ we have $(n-1)\left(\frac{3n-1}{n}\right)\left(\frac{1}{3n-5}\right) > 1$ which is enough to show the inequality we require.

(b) $(n-1)f\left(nh\right)f\left((n-2)\,h\right) > (n-2)f\left((n-1)\,h\right)f\left((n-2)\,h\right)$: Since $(n-1)\,(3n-1) - 3n(n-2) = 2n+1 > 0$, we deduce that $(n-1)\left(\frac{3n-1}{3n}\right) > n-2$. Then the inequality follows from (32).

(c) $(n-1)f\left(nh\right)f\left((n-3)\,h\right) > (n-2)f\left((n-1)\,h\right)f\left((n-1)\,h\right)$: From (31) at $n-1$ we see that

$$f((n-2)h) > \left(\frac{3n-3}{3n-2}\right) f((n-1)h). \tag{37}$$

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Together with (35) we deduce that

$$(n-1)f(nh)f((n-3)h)$$
$$> \frac{(n-1)}{n}\left(\frac{3n-1}{3n-5}\right)\left(\frac{3n-3}{3n-2}\right)(n-2)f((n-1)h)f((n-1)h).$$

Since $(n-1)(3n-1)(3n-3) - n(3n-5)(3n-2) = 5n - 3 > 0$, we have $\frac{(n-1)}{n}\left(\frac{3n-1}{3n-5}\right)\left(\frac{3n-3}{3n-2}\right) > 1$ and the result is established.

(d)$(n-1)f(nh)f((n-2)h) > (n-2)f((n-1)h)f((n-1)h)$: But from (32) and (37) we deduce

$$(n-1)f(nh)f((n-2)h) > (n-1)\left(\frac{3n-1}{3n}\right)\left(\frac{3n-3}{3n-2}\right)f((n-1)h)f((n-1)h).$$

However since $(n-1)(3n-1)(3n-3) - 3n(3n-2)(n-2) = 3n^2 + 3n - 3 > 0$ we have $(n-1)\left(\frac{3n-1}{3n}\right)\left(\frac{3n-3}{3n-2}\right) > n - 2$, and the inequality follows. ∎

Lemma 11 *Consider a duopoly market where each producer has step separation without gaps in its bidding strategy and consider a price p where there is a unique unit $\widehat{n}(p)$ such that $\widehat{\alpha}_j(p,\widehat{n}) \in (0,1)$, where producer $j \neq i$ is the competitor of producer i. In this case,*

$$\begin{aligned}
Z_i(n, r_n, \alpha_i) &= nh\widehat{\alpha}_j(r_n, \widehat{n}(r_n))\left(f((n+\widehat{n}(r_n))h) - f((n+\widehat{n}(r_n)-1)h)\right) \\
&\quad + nhf((n+\widehat{n}(r_n)-1)h) \\
&\quad - h(r_n - c_n(\alpha_i))\frac{\partial\widehat{\alpha}_j(r_n, \widehat{n}(r_n))}{\partial r_n}(r_n)f((n+\widehat{n}(r_n)-1)h) \quad (38)
\end{aligned}$$

The first-order condition of a symmetric Bayesian Nash equilibrium, so that $\widehat{n}(r_n) = n$ and $\widehat{\alpha}_i(r_n, n) = \widehat{\alpha}_j(r_n, n) = \widehat{\alpha}(r_n, n)$ is given by:

$$\left(\widehat{\alpha}(r_n, n)\tau_{2n} + 1\right)n = \left(r_n - c_n(\widehat{\alpha}(r_n, n))\right)\frac{\partial\widehat{\alpha}(r_n, n)}{\partial r_n}. \quad (39)$$

Proof. For $m < \widehat{n}(p)$ we have $\widehat{\alpha}_j(p, m) = 1$ and for $m > \widehat{n}(p)$ we have $\widehat{\alpha}_j(p, m) = 0$. Thus it follows from Lemma 9 that

$$\begin{aligned}
Z_i(n, r_n, \alpha_i) &= nh\left(\widehat{\alpha}_j(r_n, \widehat{n}(r_n)) - 0\right)f((n+\widehat{n}(r_n))h) \\
&\quad + nh\left(1 - \widehat{\alpha}_j(r_n, \widehat{n}(r_n))\right)f((n+\widehat{n}(r_n)-1)h) \\
&\quad - h(r_n - c_n(\alpha_i))\frac{\partial\widehat{\alpha}_j(r_n, \widehat{n}(r_n))}{\partial r_n}f((n+\widehat{n}(r_n)-1)h),
\end{aligned}$$

which gives (38). In a symmetric equilibrium $\widehat{n}(r_n) = n$, which yields (39), because the first-order condition is that $Z_i(n, r_n, \alpha_i) = 0$ and we can divide all terms by $hf((2n-1)h)$. ∎

Proof. (Lemma 5) It follows from the necessary first-order condition implied by (38) and $Z_i(n, r_n, \alpha_i) = 0$ that $p_n^i(\alpha_i) > c_n(\alpha_i)$ for $\alpha_i \in [0,1]$ and

$n \in \{1, \dots, N\}$. Using a similar argument as in the proof of Lemma 3, it can be shown that firms must submit identical bids for the lowest cost realization, i.e. $p_1^1(0) = p_1^2(0)$. The two firms also have identical costs ex-ante, before signals have been realized. Thus it follows from the Picard-Lindelöf theorem that the solution of the symmetric system of differential equations implied by (38) can only have a symmetric solution, such that $p_1^1(\alpha) = p_1^2(\alpha)$ for $\alpha \in [0, 1]$. Under the stated assumptions, it follows from Lemma 3 and Lemma 4 that steps are necessarily separated without gaps, so that $p_k^i(1) = p_{k+1}^i(0)$. Thus, we can repeat this argument $N-1$ times to show that $p_1^1(\alpha) = p_1^2(\alpha)$ for $\alpha \in [0, 1]$ and $n \in \{1, \dots, N\}$.
∎

Appendix C: Existence results

Lemma 12 *The first-order conditions for the symmetric Bayesian equilibrium when $c_n(1) < p_n(1) = p_{n+1}(0)$ has a unique symmetric solution for unit n:*

$$p_n(\alpha) = p_n(1) \frac{(\alpha\tau_{2n} + 1)^{1/(n\tau_{2n})}}{(\tau_{2n} + 1)^{1/(n\tau_{2n})}}$$

$$+ (\alpha\tau_{2n} + 1)^{1/(n\tau_{2n})} \int_\alpha^1 \frac{c_n(u)(u\tau_{2n} + 1)^{-1/(n\tau_{2n})-1}}{n} du$$

$$> c_n(\alpha),$$

where

$$p_n'(\alpha) > 0. \tag{40}$$

Proof. In order to solve (39) we write the bid price as a function of the signal. We have $\frac{\partial \widehat{\alpha}(r_n, n)}{\partial r_n} = \frac{1}{p_n'(\alpha)}$ where $r_n = p_n(\alpha)$ and $\alpha = \widehat{\alpha}(r_n, n)$. Thus

$$(\alpha\tau_{2n} + 1)\, n = \frac{(p_n(\alpha) - c_n(\alpha))}{p_n'(\alpha)}$$

$$p_n'(\alpha) - \frac{p_n(\alpha)}{(\alpha\tau_{2n} + 1)\, n} = -\frac{c_n(\alpha)}{(\alpha\tau_{2n} + 1)\, n}. \tag{41}$$

Next we multiply both sides by the integrating factor $(\alpha + 1/\tau_{2n})^{-1/(n\tau_{2n})}$.

$$p_n'(\alpha)(\alpha + 1/\tau_{2n})^{-1/(n\tau_{2n})} - \frac{p_n(\alpha)(\alpha + 1/\tau_{2n})^{-1/(n\tau_{2n})-1}}{n\tau_{2n}}$$

$$= -\frac{c_n(\alpha)(\alpha + 1/\tau_{2n})^{-1/(n\tau_{2n})}}{(\alpha + 1/\tau_{2n})\, n\tau_{2n}},$$

so

$$\frac{d}{d\alpha}\left(p_n(\alpha)(\alpha + 1/\tau_{2n})^{-1/(n\tau_{2n})}\right) = -\frac{c_n(\alpha)(\alpha + 1/\tau_{2n})^{-1/(n\tau_{2n})-1}}{n\tau_{2n}}.$$

Integrating both sides from α to 1 yields:

$$p_n(1)\left(1 + 1/\tau_{2n}\right)^{-1/(n\tau_{2n})} - p_n(\alpha)\left(\alpha + 1/\tau_{2n}\right)^{-1/(n\tau_{2n})}$$

$$= \int_\alpha^1 -\frac{c_n(u)\left(u + 1/\tau_{2n}\right)^{-1/(n\tau_{2n})-1}}{n\tau_{2n}}du,$$

so

$$p_n(\alpha) = p_n(1)\frac{\left(\alpha + 1/\tau_{2n}\right)^{1/(n\tau_{2n})}}{\left(1 + 1/\tau_{2n}\right)^{1/(n\tau_{2n})}}$$

$$+ \left(\alpha + 1/\tau_{2n}\right)^{1/(n\tau_{2n})}\int_\alpha^1 \frac{c_n(u)\left(u + 1/\tau_{2n}\right)^{-1/(n\tau_{2n})-1}}{n\tau_{2n}}du,$$

which can be immediately written in the form of the Lemma statement. Thus

$$p_n(\alpha) \geq p_n(1)\frac{\left(\alpha + 1/\tau_{2n}\right)^{1/(n\tau_{2n})}}{\left(1 + 1/\tau_{2n}\right)^{1/(n\tau_{2n})}} +$$

$$\left(\alpha + 1/\tau_{2n}\right)^{1/(n\tau_{2n})}c_n(\alpha)\left[-\left(u + 1/\tau_{2n}\right)^{-1/(n\tau_{2n})}\right]_\alpha^1$$

$$= \left(p_n(1) - c_n(\alpha)\right)\frac{\left(\alpha + 1/\tau_{2n}\right)^{1/(n\tau_{2n})}}{\left(1 + 1/\tau_{2n}\right)^{1/(n\tau_{2n})}} + c_n(\alpha) > c_n(\alpha).$$

Finally, it follows from (41) that $p_n'(\alpha) > 0$, because we know from Definition 2 that $\alpha\tau_{2n} + 1 \geq 1 - \alpha \geq 0$. $\blacksquare$

Proof. (Proposition 1) The solution is given by the end-conditions and Lemma 12 above. We will now verify that this is an equilibrium using the optimality conditions in Lemma 2.

i) Prove that $Z_i(m, p, \alpha_i) \leq 0$ if $p \in (p_m(\alpha_i), p_m(1))$. It is known from Lemma 12 that the first-order solutions are monotonic. Thus $\widehat{\alpha}_i(p, m) \geq \alpha_i$ and so $c_m(\widehat{\alpha}_i(p, m)) \geq c_m(\alpha_i)$. Thus, it follows from Lemma 11 that

$$Z_i(m, p, \alpha_i) \leq mh\widehat{\alpha}_j(p, m)\left(f(2mh) - f((2m-1)h)\right) + mhf((2m-1)h)$$
$$-h\left(p - c_m(\widehat{\alpha}_i(p, m))\right)\frac{\partial\widehat{\alpha}_j(p,m)}{\partial p}f((2m-1)h) = Z_i(m, p, \widehat{\alpha}_i(p, m)) = 0. \tag{42}$$

ii) Prove that $Z_i(m, p, \alpha_i) \leq 0$ if $p \in (p_n(0), p_n(1))$ where $n > m$. For any price $p \in (p_n(0), p_n(1))$, we can find some signal $\widetilde{\alpha}_i$ such that $p_n(\widetilde{\alpha}_i) \leq p$. It follows from the argument above that

$$\frac{Z_i(n, p, \widetilde{\alpha}_i)}{f((2n-1)h)h} = n\widehat{\alpha}_j(p, n)\tau_{2n} + n - \left(p - c_n(\widetilde{\alpha}_i)\right)\frac{\partial\widehat{\alpha}_j(p,n)}{\partial p} \leq 0. \tag{43}$$

Now consider unit $m < n$ with signal α_i at the same price p.

$$\frac{Z_i(m,p,\alpha_i)}{f((n+m-1)h)h} = m\widehat{\alpha}_j(p, n)\tau_{n+m} + m - \left(p - c_m(\alpha_i)\right)\frac{\partial\widehat{\alpha}_j(p,n)}{\partial p}$$
$$\leq m\widehat{\alpha}_j(p, n)\tau_{n+m} + m - \left(p - c_m(1)\right)\frac{\partial\widehat{\alpha}_j(p,n)}{\partial p} = \frac{Z_i(m,p,1)}{f((n+m-1)h)h}. \tag{44}$$

We have from the inequality in (8) that $\alpha\left((m+1)\tau_{n+m+1}-m\tau_{n+m}\right)+1\geq 0$ for $\alpha\in(0,1)$, so

$$\frac{Z_i(m,p,\alpha_i)}{f((n+m-1)h)h}\leq\frac{Z_i(m,p,1)}{f((n+m-1)h)h}\leq(m+1)\widehat{\alpha}_j(p,n)\tau_{n+m+1}+m+1$$
$$-\left(p-c_{m+1}(\widetilde{\alpha}_i)\right)\frac{\partial\widehat{\alpha}_j(p,n)}{\partial p}=\frac{Z_i(m+1,p,\widetilde{\alpha}_i)}{f((n+m)h)h}. \tag{45}$$

Starting with (43), we can use the expression above, to recursively prove that

$$Z_i(m,p,\alpha_i)\leq 0,$$

if $m\leq n-1$.

iii) Prove that $Z_i(m,p,\alpha_i)\geq 0$ if $p\in(p_m(0),p_m(\alpha_i))$. It is known from Lemma 12 that the first-order solutions are monotonic. Thus $\widehat{\alpha}_i(p,m)\leq\alpha_i$ and so $c_m(\widehat{\alpha}_i(p,m))\leq c_m(\alpha_i)$. Thus, it follows from Lemma 11 that

$$Z_i(m,p,\alpha_i)\geq mh\widehat{\alpha}_j(p,m)\left(f(2mh)-f((2m-1)h)\right)+mhf((2m-1)h)$$
$$-h\left(p-c_m(\widehat{\alpha}_i(p,m))\right)\frac{\partial\widehat{\alpha}_j(p,m)}{\partial p}f((2m-1)h)=Z_i(m,p,\widehat{\alpha}_i(p,m))=0.$$

iiii) Prove that $Z_i(m,p,\alpha_i)\geq 0$ if $p\in(p_n(0),p_n(1))$ where $m>n$. For any price $p\in(p_n(0),p_n(1))$, we can find some signal $\widetilde{\alpha}_i$ such that $p_n(\widetilde{\alpha}_i)\geq p$. It follows from an argument similar to (42) that

$$\frac{Z_i(n,p,\widetilde{\alpha}_i)}{f((2n-1)h)h}=n\widehat{\alpha}_j(p,n)\tau_{2n}+n-\left(p-c_n(\widetilde{\alpha}_i)\right)\frac{\partial\widehat{\alpha}_j(r_n,n)}{\partial p}\geq 0. \tag{46}$$

Now consider unit $m>n$ with signal α_i at the same price p.

$$\frac{Z_i(m,p,\alpha_i)}{f((n+m-1)h)h}=m\widehat{\alpha}_j(p,n)\tau_{n+m}+m-\left(p-c_m(\alpha_i)\right)\frac{\partial\widehat{\alpha}_j(p,n)}{\partial p}$$
$$\geq m\widehat{\alpha}_j(p,n)\tau_{n+m}+m-\left(p-c_m(0)\right)\frac{\partial\widehat{\alpha}_j(p,n)}{\partial p}=\frac{Z_i(m,p,0)}{f((n+m-1)h)h}. \tag{47}$$

We have from (8) that $\alpha\left(-(m-1)\tau_{n+m-1}+m\tau_{n+m}\right)+1\geq 0$ for $\alpha\in(0,1)$, so

$$\frac{Z_i(m,p,\alpha_i)}{f((n+m-1)h)h}\geq\frac{Z_i(m,p,0)}{f((n+m-1)h)h}\geq(m-1)\widehat{\alpha}_j(p,n)\tau_{n+m-1}+m-1$$
$$-\left(p-c_{m-1}(\widetilde{\alpha}_i)\right)\frac{\partial\widehat{\alpha}_j(p,n)}{\partial p}=\frac{Z_i(m-1,p,\widetilde{\alpha}_i)}{f((n+m-2)h)h}. \tag{48}$$

Starting with (46), we can use the expression above, to recursively prove that

$$Z_i(m,p,\alpha_i)\geq 0,$$

if $m\geq n+1$.

Finally, the uniqueness result follows from the necessary conditions in Lemma 3, Lemma 4 and Lemma 5. We can rule out symmetric equilibria where $p_N(\alpha)=\overline{p}$ for some $\alpha\in[0,1)$. This would imply that events where both firms have offers at the reservation price at the same time would occur with a positive probability. Due to the properties of the sharing rule, a firm would then find it profitable to unilaterally deviate by slightly undercutting $\overline{p}$ for signal α. ∎

Appendix D: Equilibrium convergence

Proof. **(Lemma 6)** A discrete approximation of an ordinary differential equation is consistent if the local truncation error is infinitesimally small when the step length is infinitesimally small (LeVeque, 2007). The local truncation error is the discrepancy between the continuous slope and its discrete approximation when values $\bar{p}_n$ in the discrete system are replaced with samples of the continuous solution $P(nh)$. The difference equation can be written as follows:

$$p_{n-1} = (\tau_{2n} + 1)^{-1/(n\tau_{2n})} (p_n - c_n) + c_n$$

We have $\tau_{2n} = \frac{f(2nh) - f((2n-1)h)}{f((2n-1)h)}$, so $\tau_{2n} \to \frac{f'(2nh)h}{f(2nh)}$ when $h \to 0$. By assumption $\frac{f'(2nh)}{f(2nh)}$ is bounded, so $\tau_{2n} = O(h)$. By means of a MacLaurin series expansion we obtain:

$$p_{n-1} = c_n + (1 - 1/n)\left(1 + \frac{\tau}{2n}\right)(p_n - c_n) + O(h^3)$$

$$p_{n-1} = c_n + \left(1 - 1/n + \frac{\tau}{2n}\right)(p_n - c_n) + O(h^3)$$

$$\frac{p_n - p_{n-1}}{h} = (p_n - c_n)/(nh) - \frac{\tau(p_n - c_n)}{2nh} + O(h^2).$$

This gives a discrete estimate of the slope of the continuous solution if we replace values in the discrete system are replaced with samples of the continuous solution $P(nh)$. To calculate the local truncation error, υ^n, subtract this discrete estimate of the slope from the slope of the continuous solution, so

$$\upsilon^n = \frac{P(nh) - C'(nh)}{nh} - (\bar{p}_n - c_n)/(nh) + O(h), \tag{49}$$

Hence, it follows from our assumptions that $\lim_{h \to 0} \upsilon^n = 0$. ∎

Proof. (Proposition 2) Lemma 6 states that the discrete difference equation is a consistent approximation of the continuous differential equation. To show that the discrete solution converges to the continuous solution, it is necessary to prove that the stepped solution exists and is numerically stable, i.e., the error grows at a finite rate over the finite interval $[a, b]$, where $a = \tilde{C}'(0)$ and $b = \bar{p}$. The proof is inspired by LeVeque's (2007) convergence proof for general one-step methods. It follows from Lemma 6 that

$$\frac{dp_{n-1}(1)}{dp_n(1)} = (\tau_{2n} + 1)^{-1/(n\tau_{2n})} = \frac{1}{(\tau_{2n} + 1)^{1/(n\tau_{2n})}} \in [0, 1],$$

because $\tau_{2n} + 1 \geq 1$ when $\tau_{2n} \geq 0$ and $\tau_{2n} + 1 \leq 1$ when $\tau_{2n} \leq 0$. Thus we can introduce a Lipschitz constant $\lambda = 1$ (LeVeque, 2007) that uniformly bounds $\left|\frac{dp_{n-1}(1)}{dp_n(1)}\right|$ for each h. Define the global error at the quantity nh by $E^n =$

$p_n (1) - P (nh)$. It follows that the global error satisfies the following inequality at the unit $N - 1$:

$$\left| E^{N-1} \right| = \left| p_{N-1} (1) - P (nh) \right| \leq \lambda \left| E^N \right| + h \left| v^N \right|, \tag{50}$$

where v^N is the local truncation error as defined by (49). Similarly

$$\left| E^k \right| = \left| p_k (1) - P (kh) \right| \leq \lambda \left| E^{k+1} \right| + h \left| v^{k+1} \right|. \tag{51}$$

Let $v_{\max} \geq \left| v^k \right|$ for $k \in \{1, \ldots, N\}$. From the inequality in (51) and $\lambda = 1$, we can show by induction:

$$\begin{aligned}
\left| E^k \right| &\leq \lambda^{N-k} \left| E^N \right| + \sum_{m=k+1}^{N} h \left| v^m \right| \lambda^{m-k-1} \\
&\leq \left| E^N \right| + (N - k - 1) h v_{\max} \\
&\leq \left| E^N \right| + N h v_{\max} = \left| E^N \right| + v_{\max} \overline{q}.
\end{aligned} \tag{52}$$

The consistency property established in Lemma 6 ensures that the truncation error $v_{\max}$ can be made arbitrarily small by decreasing h. Moreover, $p_N (1) = P (\overline{q}) = \overline{p}$, so $\left| E^N \right| = 0$. Thus from (52), $\left| E^k \right| \to 0$ when $h \to 0$, proving that the discrete solution converges to the continuous one.

We note that the inequality (8) is satisfied when h is sufficiently small. Hence, it follows from Corollary 1 that the discrete solution corresponds to a mixed-strategy NE. The uniqueness condition corresponds to Assumption 2 when $h \to 0$. ∎

Proof. (Lemma 7) We have from Corollary 1 that

$$p_n (\alpha_i) = (p_n (1) - c_n) \left(\frac{\alpha_i \tau_{2n} + 1}{\tau_{2n} + 1} \right)^{1/(n \tau_{2n})} + c_n$$

It can be shown that

$$\left. \frac{\partial p_n (\alpha_i)}{\partial \tau_{2n}} \right|_{\tau_{2n}=0} = (p_n (1) - c_n) \frac{e^{\frac{1}{n}(\alpha_i - 1)}}{2n} \left(1 - \alpha_i^2 \right) \geq 0.$$

Thus for the same or higher $p_n (1)$, a higher τ_{2n} will result in a higher $p_n (\alpha_i)$ if h is sufficiently small, so that $\tau_{2n} \to 0$. We have $p_N (1) = \overline{p}$ irrespective of τ_{2n}. Thus for symmetric mixed-strategy equilibria with step separation without gaps, so that $p_{n-1} (1) = p_n (0)$, and sufficiently small h, a weakly higher τ_{2n} will weakly increase $p_n (\alpha_i)$ for $n \in \{1, \ldots, N\}$. ∎

Appendix E: Multiple firms

Consider an oligopoly market with I producers. Let $\varphi_{-i} (p, m)$ be the probability that the $K = I - 1$ competitors of firm i together offer at least m units at price p. We also introduce

$$\Delta \varphi_{-i} (p, m + 1) = \varphi_{-i} (p, m) - \varphi_{-i} (p, m + 1),$$

the probability that competitors of firm i together offer exactly m units at price p.

Lemma 13 *In an oligopoly market*

$$Z_i\left(n, p, \alpha_i\right) = nh \sum_{m=0}^{N} \Delta\varphi_{-i}\left(p, m+1\right) f\left(\left(n+m\right)h\right)$$
$$-h\left(p - c_n(\alpha_i)\right) \sum_{m=0}^{N} \frac{\partial \Delta\varphi_{-i}(p, m+1)}{\partial p} F\left(\left(n+m-1\right)h\right), \tag{53}$$

where $Z_i\left(n, p, \alpha_i\right)$ is defined.

Proof. (Lemma 13) An offer of n units at price p by producer i is rejected if the competitors together offer exactly m units at the price p when demand is at most $n + m - 1$ units. Thus

$$\Psi_i\left(n, p\right) = \sum_{m=0}^{N} \Delta\varphi_{-i}\left(p, m+1\right) F\left(\left(n+m-1\right)h\right). \tag{54}$$

(53) now follows from Definition 1, Lemma 1 and (54). ∎

Consider an outcome with price p, where the K competitors of producer i together sell m units. In case of step separation this means that one of the competitors will sell either

$$\widehat{m} = \left\lceil \frac{m}{K} \right\rceil$$

units or $\widehat{m} - 1$ units at this price.

Lemma 14 *Consider step-separated offers that are symmetric ex-ante (before private signals have been observed) from K competitors, then $\Delta\varphi_{-i}\left(p, m+1\right)$ can be determined from a binomial distribution.*

$$\Delta\varphi_{-i}\left(p, m+1\right)$$
$$= \binom{K}{m - K(\widehat{m}-1)} \left(\widehat{\alpha}\left(p, \widehat{m}\right)\right)^{m - K(\widehat{m}-1)} \left(1 - \widehat{\alpha}\left(p, \widehat{m}\right)\right)^{K\widehat{m}-m}$$

if $\left(\widehat{m} - 1\right)K \leq m \leq \widehat{m}K$ and $\widehat{\alpha}(p, \widehat{m}) \in (0, 1)$. Moreover,

$$\Delta\varphi_{-i}\left(p, m+1\right) = 0, \tag{55}$$

if $m < \left(\widehat{m} - 1\right)K$ and $m > \widehat{m}K$.

Proof. $\Delta\varphi_{-i}\left(p, m+1\right)$ is the probability that competitors together sell exactly m units at price p. Competitors' offers are symmetric ex-ante and have step separation. Moreover, the price is such that $\widehat{\alpha}(p, \widehat{m}) \in (0, 1)$. It follows that each producer sells either $\widehat{m}$ units (with probability $\widehat{\alpha}\left(\widehat{m}, p\right)$) or $\widehat{m} - 1$ units at price p (with probability $1 - \widehat{\alpha}\left(\widehat{m}, p\right)$). This immediately gives (55). It also follows that $m - K(\widehat{m} - 1)$ producers are selling exactly $\widehat{m}$ units and the other producers are selling exactly $\widehat{m} - 1$ units. There are $\binom{K}{m - K(\widehat{m} - 1)}$ such outcomes each

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occurring with a probability $(\widehat{\alpha}(\widehat{m},p))^{m-K(\widehat{m}-1)}(1-\widehat{\alpha}(p,\widehat{m}))^{K\widehat{m}-m}$, which gives the result. ∎

Note that the binomial coefficient in Lemma 14 is defined as follows:

$$\begin{pmatrix} K \\ m - K(\widehat{m}-1) \end{pmatrix} = \frac{K!}{(m-K(\widehat{m}-1))!\,(K\widehat{m}-m)!}. \tag{56}$$

It follows from Lemma 13 and Lemma 14 that:

Corollary 3 *In an oligopoly market where producer i has $K = I - 1$ competitors that submit step-separated offers that are symmetric ex-ante (before private signals have been observed), then*

$$Z_i(n,p,\alpha_i)$$

$$= nh \sum_{m=K(\widehat{m}-1)}^{K\widehat{m}} \begin{pmatrix} K \\ m - K(\widehat{m}-1) \end{pmatrix} (\widehat{\alpha}(p,\widehat{m}))^{m-K(\widehat{m}-1)}(1-\widehat{\alpha}(p,\widehat{m}))^{K\widehat{m}-m} f((n+m)h)$$

$$-h(p-c_n(\alpha_i)) \sum_{m=K(\widehat{m}-1)}^{K\widehat{m}} \begin{pmatrix} K \\ m - K(\widehat{m}-1) \end{pmatrix} \frac{\partial\left((\widehat{\alpha}(p,\widehat{m}))^{m-K(\widehat{m}-1)}(1-\widehat{\alpha}(p,\widehat{m}))^{K\widehat{m}-m}\right)}{\partial p} F((n+m-1)h)$$

for p, such that $\widehat{\alpha}(p,\widehat{m}) \in (0,1)$, and $\alpha_i \in (0,1)$.

Proof. (Proposition 3) In an equilibrium that is symmetric ex-ante, we have $n = \widehat{m}$. Hence, Corollary 3 gives the following symmetric first-order condition:

$$n \sum_{m=K(n-1)}^{Kn} w(\widehat{\alpha}, m - K(n-1)) f((n+m)h)$$

$$= (p-c_n)\frac{\partial}{\partial p}\left(\sum_{m=K(n-1)}^{Kn} w(\widehat{\alpha}, m - K(n-1)) F((n+m-1)h)\right),$$

where we write

$$w(u,t) = \begin{pmatrix} K \\ t \end{pmatrix} u^t (1-u)^{T-t}$$

for the probability of t out of K to be chosen where each component has a probability u of being chosen. So after differentiation

$$\frac{n}{p-c_n} \sum_{m=K(n-1)}^{Kn} w(\widehat{\alpha}, m - K(n-1)) f((n+m)h)$$

$$= \frac{\partial\widehat{\alpha}}{\partial p} \sum_{m=K(n-1)+1}^{Kn} \frac{w(\widehat{\alpha}, m - K(n-1))}{\widehat{\alpha}} (m - K(n-1)) F((n+m-1)h)$$

$$-\frac{\partial\widehat{\alpha}}{\partial p} \sum_{m=K(n-1)}^{Kn-1} \frac{w(\widehat{\alpha}, m - K(n-1))}{1-\widehat{\alpha}} (Kn - m) F((n+m-1)h).$$

The differential equation above can be written in the following form:

$$\frac{n}{p-c_n} = g(\alpha)\frac{d\alpha}{dp}, \tag{57}$$

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where

$$
\begin{aligned}
g\left(u\right) \;=\; & \frac{\sum\limits_{m=K(n-1)+1}^{Kn} w\left(u, m-K(n-1)\right)\left(m-K(n-1)\right) F\left(\left(n+m-1\right)h\right)}{u \sum\limits_{m=K(n-1)}^{Kn} w\left(u, m-K(n-1)\right) f\left(\left(n+m\right)h\right)} \\
& -\frac{\sum\limits_{m=K(n-1)}^{Kn-1} w\left(u, m-K(n-1)\right)\left(Kn-m\right) F\left(\left(n+m-1\right)h\right)}{(1-u)\sum\limits_{m=K(n-1)}^{Kn} w\left(u, m-K(n-1)\right) f\left(\left(n+m\right)h\right)}.
\end{aligned}
$$

We can simplify this to:

$$
\begin{aligned}
g\left(u\right) \;=\; & \frac{\sum\limits_{v=0}^{K-1} w\left(u, v+1\right)\left(v+1\right) F\left(\left(n+v+K(n-1)\right)h\right)}{u\sum\limits_{v=0}^{K-1} w\left(u, v\right) f\left(\left(n+v+K(n-1)\right)h\right)} \\
& -\frac{\sum\limits_{v=0}^{K-1} w\left(u, v\right)\left(K-v\right) F\left(\left(n+v+K(n-1)-1\right)h\right)}{(1-u)\sum\limits_{v=0}^{K-1} w\left(u, v\right) f\left(\left(n+v+K(n-1)\right)h\right)}.
\end{aligned}
$$

Next, we use (56) to simplify this further.

$$
\begin{aligned}
g\left(u\right) \;=\; & \frac{\sum\limits_{v=0}^{K-1} \frac{K!}{v!(K-v-1)!}u^{v}\left(1-u\right)^{K-1-v} F\left(\left(n+v+K(n-1)\right)h\right)}{\sum\limits_{v=0}^{K-1} w\left(u, K, v\right) f\left(\left(n+v+K(n-1)\right)h\right)} \\
& -\frac{\sum\limits_{v=0}^{K-1} \frac{K!}{v!(K-v-1)!}u^{v}\left(1-u\right)^{K-1-v} F\left(\left(n+v+K(n-1)-1\right)h\right)}{\sum\limits_{v=0}^{K-1} w\left(u, K, v\right) f\left(\left(n+v+K(n-1)\right)h\right)},
\end{aligned}
$$

which can be simplified to (15). The differential equation in (57) can be separated as follows:

$$
\frac{ndp}{p-c_n} = g\left(u\right) du.
$$

We integrate the left hand side from $p_n\left(\alpha\right)$ to $p_n\left(1\right)$ and the right-hand side from α to 1. Hence,

$$
n\ln\left(\frac{p_n\left(1\right)-c_n}{p_n\left(\alpha\right)-c_n}\right) = \int_{\alpha}^{1} g\left(u\right) du,
$$

which gives (14). Using Corollary 3 in Appendix E, it can be verified that this is an equilibrium with a similar argument as in the proof of Proposition 1. ∎