

The Role of Social Networks

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DOI/Link: <https://doi.org/10.1016/j.jue.2016.11.004>

Reference: Verdier, Thierry and Yves Zenou (2017). "The Role of Social Networks in Cultural Assimilation". *Journal of Urban Economics*, 97(January), 15–39.



The role of social networks in cultural assimilation[☆]



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ARTICLE INFO

Article history:

Received 12 September 2015

Revised 16 November 2016

Available online 1 December 2016

JEL Classification:

D85

J15

Z13

Keywords:

Assimilation

Majority individuals

Ethnic minorities

Network centrality

Network formation

ABSTRACT

We develop a model where, in the first stage, minority individuals have to decide whether or not they want to assimilate to the majority culture while, in the second stage, all individuals (both from the majority and the minority group) embedded in a network have to decide how much effort they exert in some activity (say education). We show that the more central minority agents are located in the social network, the more they assimilate to the majority culture. We also show that denser networks tend to favor assimilation so that, for example, it is easier to assimilate in a complete network than in a star-shaped network. We show that the subgame-perfect equilibrium is not optimal because there is not enough activity and assimilation. We then endogenize the network and show under which condition the ethnic minorities either assimilate to or separated themselves from the majority group.

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1. Introduction

An intense political and intellectual debate is taking place in Europe around migration issues. Rather than being centered on the economic costs and benefits of such inflows, the debate has instead focused on the perceived costs and benefits of cultural diversity. The attention paid to this issue is relatively novel in Europe and does represent a departure from the long-standing debate which has tended to emphasize racial discrimination as the key explanation of ethnic disadvantage. This is well illustrated by the hot debate in Europe about the veiling among Muslim women and the recent votes in Switzerland against the construction of Muslim mosques and against immigrants. Many European countries are concerned about the *cultural integration* and the (*lack of*) *assimilation* of immigrants, that is whether the basic norms and values of the majority society are adopted by existing minority

groups.¹ For example, recently, the European eastern countries such as Hungary or Poland have clearly expressed their reluctance to welcome Syrian refugees fleeing war, who are mostly Muslims, arguing that their integration and assimilation to the European culture will be very difficult (see e.g. Mangin and Zenou, 2016).

The *assimilation* outcomes of second generation youths have also been hotly debated amongst scholars, especially in the United States (Alba et al., 2011; Haller et al., 2011) where the immigrant population's growth in recent decades has raised questions about whether and how their children, the second generation, will integrate into American society. Current perspectives on second generation integration have evolved and are varied. Some scholars adhere to the segmented assimilation framework in which the second generation will assimilate into different segments of American society based on structural barriers and prejudices (Portes and Rumbaut, 2001; Portes and Zhou, 1993). Other scholars believe that the outcome between immigrant and mainstream culture is less dichotomous, and that immigrants and the American mainstream will eventually coalesce as lifestyles and patterns gradually become similar over time (Alba and Nee, 2003). Different studies have shown distinct significant influences on the assimilation process for immigrants: the quality of immigrant cohorts (Borjas, 1985),

^{*} We thank the editor, Gilles Duranton, two anonymous referees as well as Marc Sommer for very helpful comments. This research was undertaken under the project TECTACOM. Financial support from the European Research Council (ERC) grant number 324004 is gratefully acknowledged. Yves Zenou gratefully acknowledges the financial support from the French National Research Agency grant ANR-13-JSH1-0009-01.

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¹ For an overview of these issues in Europe, see Kahanec and Zimmermann (2011).

country of origin (e.g. Beenstock et al., 2010; Borjas, 1987; 1992; Chiswick and Miller, 2011), ethnic concentration (e.g. Edin et al., 2003; Lazear, 1999) and personal English skill (e.g. Chiswick and Miller, 1995; 1996; Dustmann and Fabbri, 2003; McManus et al., 1983).

An often overlooked structural factor of assimilation of ethnic minorities is the role of *immigrant networks* in the assimilation process. This is what we study in this paper.

To be more precise, we develop a model where all individuals, native and minorities (or immigrants), are embedded in a network. The network we are interested in is the one that forms, in the host country, between immigrants and natives but also between immigrants. As a result, we are not modeling the migration process and the role of network in migration (for such a model, see, e.g. Giulietti et al., 2014). Instead, we examine the role of networks in the host country on assimilation choices and outcomes. In this network, immigrants and natives possess different positions in the network and have different productivities.

There are two stages. In the first stage, minority individuals (immigrants) have to decide whether or not they want to assimilate to the majority culture while, in the second stage, all individuals (both from the majority and the minority group) embedded in a network² have to decide how much effort they exert in some activity (say education or work on a job). Here both minority and majority workers belong to the same network. In this network, links can represent social or working relationships between people. There is a trade off for minority workers. If they choose assimilation, they will be more productive (in terms of education or in terms of skills on the job) because they are more adapted to the social norms of the host country (they know better the language and the habits of the host country). However, they need to pay a fixed cost of assimilation because it is costly, for example, to learn a new language. On the other hand, if they choose not to assimilate, they do not need to pay this cost but end up with a lower productivity, which decreases their outcomes (for example, their education level or their wage). As a result, the incentives for an individual belonging to the minority group to assimilate and adopt the culture of the majority are directly related to the expected gains and costs that such a strategy implies. We consider a model where efforts are strategic complements. If, for example, we think of education, then if someone that I'm linked to studies hard then I enjoy more utility to study hard myself. Similarly, if we think of productivity on the job, then if I'm linked to someone who works hard, then I enjoy more utility from working hard.

We show that the more central agents in the network tend to have higher productivity than the less central ones and thus, the ethnic minorities who are more central in the network tend to assimilate more than those who are less central because the gains of assimilation are higher. We also show that, when the strength of interactions in the network increases, social interactions become more valuable and, because it is costly not to assimilate (in terms of productivity), more people choose to assimilate. This highlights the fact that endogenous assimilation choices affect the contribution to equilibrium efforts. For example, when the cost of assimilation decreases, more agents choose to assimilate to the majority culture, which, in turn, increases social interactions in the network and thus equilibrium productive efforts and outcomes. Furthermore, we find that denser networks tend to favor assimilation so that, for example, it is easier to assimilate in a complete network than in a star-shaped network.

We also show that the subgame-perfect equilibrium is not optimal because there is not enough activity and assimilation. To re-

store the first best, we find that it is optimal for the planner to give higher effort subsidies (to all individuals) to more central agents in the social network. We also consider different communities that are not linked to each other and show that bridging them is always good for assimilation and also for total welfare.

We then extend our model to include network formation and, instead of having a binary choice of assimilation, we have a more “continuous” definition of assimilation, which is endogenously determined by the percentage of friends from the majority group each minority worker has. In this framework, the cost of forming links differ between intra-ethnic and inter-ethnic relationships. Due to cultural or language barriers, it is indeed always more costly to interact with someone from the other community than from own community. We assume that the cost of inter-ethnic relationships of two individuals negatively depends on the rate of exposure of these two agents to the other community. Indeed, the more an ethnic minority is “exposed” to the majority culture (by having an increasing fraction of friends from the majority group), the easier it will be for her to communicate and interact with other persons from the majority group and the lower will be the interaction costs between them. Using the concept of pairwise stability for equilibrium networks, we give conditions under which the two communities fully interact with each other (totally integrated communities) and when they do not interact at all (totally separated communities). We also provide some conditions for specific networks to be pairwise stable.

If we interpret the network as a city, then we have some interesting implications for urban economics. First, we show that the denser is the city, the more assimilated are the ethnic minorities and the more productive they are. Second, if a star network could be interpreted as a monocentric city while the complete network could be viewed as a polycentric city, then we show that for a monocentric city to emerge, the cost of interactions c and the labor-market opportunities α should have intermediate values while c has to be low enough and α has to be high enough for a polycentric city to emerge.

The rest of the paper unfolds as follows. In the next section, we relate our model to the relevant literatures and highlight our contribution. In Section 3, we expose the basic model, derive the two stages and characterize the subgame-perfect Nash equilibrium of this game. In Section 4, we illustrate our results with two specific networks: the star-shaped and the complete network. Section 5 is devoted to the welfare analysis and the subsidy policies aiming at restoring the first best. In Section 6, we consider separated communities and analyze how bridging them affects the outcomes of the individuals. Section 7 is devoted to network formation and assimilation choices. Finally, Section 8 concludes. In Appendix 1, we propose an alternative formulation of the utility function. In Appendix 2, we define the Katz–Bonacich centrality measure and give its value for the complete network. Appendix 3 provides all the proofs.

2. Related literature

2.1. Cultural transmission

There is a literature related to the assimilation of ethnic minorities and immigrants that has looked at the *cultural transmission*, in particular, at the transmission of ethnic identity from parents and peers to children. Indeed, based on some works on anthropology and sociology (see, in particular, Boyd and Richerson, 1985; Cavalli-Sforza and Feldman, 1981; Phinney, 1990), there is a literature in economics initiated by Bisin and Verdier (2000); 2001³ ar-

² The economics of networks is a growing field. See Jackson (2008), Ioannides (2012), Jackson and Zenou (2015), Jackson et al. (2017) for recent overviews.

³ For an overview, see Bisin and Verdier (2011).

guing that the transmission of a particular trait (religion, ethnicity, identity, etc.) is the outcome of a socialization inside (parents) and outside the family (peers and role models). Parents directly make various socialization choices, e.g., the rules and beliefs the family conforms to and how much time they spend with their children. This clearly affects the assimilation process of immigrants.

2.2. Ethnic identity

The role of parents in the socialization of their own children is nonetheless limited by the children's pro-active role in choosing who to imitate and learn from, thereby directly shaping their own cultural identity. This is what Akerlof and Kranton (2010) have studied by developing the concept of identity economics, which aims to incorporate into economic theory the fact that people do not only pursue economic objectives, but also exert effort to gain and/or retain acceptance into a social group with which they identify. Many other papers have investigated the role of identity in the assimilation patterns of ethnic minorities (see, e.g. Bisin et al., 2011a; 2011b; Panebianco, 2014; Verdier and Zenou, 2015; 2016). In a series of papers, Zimmermann et al. (2007), Constant and Zimmermann (2008), Constant et al. (2009) have proposed a new measure of the ethnic identity of migrants by modeling its determinants and explores its explanatory power for various types of their economic performance. They have proposed the *ethnosizer*, a measure of the intensity of a person's ethnic identity, which is constructed from information on language, culture, societal interaction, history of migration, and ethnic self-identification.

There is also an important literature that studies the concept of *oppositional cultures* among ethnic minorities. In this literature, ethnic groups may "choose" to adopt what are termed "oppositional" identities, that is, some actively reject the dominant ethnic (e.g., white) behavioral norms (they are oppositional) while others totally assimilate to it (see, in particular, Ainsworth-Darnell and Downey, 1998). Studies in the United States (but also in Europe for ethnic minorities) have found, for example, that African American students in poor areas may be ambivalent about learning standard English and performing well at school because this may be regarded as "acting white" and adopting mainstream identities (Fordham and Ogbu, 1986; Wilson, 1987; Delpit, 1995; Ogbu, 1997; Bisin et al., 2011b; Battu and Zenou, 2010; Fryer and Torelli, 2010; Patacchini and Zenou, 2016). On the theoretical side, Akerlof (1997), Austen-Smith and Fryer (2005), Selod and Zenou (2006), Battu et al. (2007), Bisin et al. (2011a), De Marti and Zenou (2017) have proposed different models analyzing how oppositional identities affect the outcomes of ethnic minorities.

Compared to these two literatures, our main contribution is to put forward the role of social networks in the choice of assimilation of ethnic minorities and immigrants.

2.3. Networks and assimilation

There is a very small literature that looks at the impact of social networks on the assimilation choices of immigrants and ethnic minorities. Zhou and Bankston (1998) suggest that participating in ethnic religious institutions promotes upward assimilation through instilling an ethnic identity onto youths. Nguyen Le (2010) studies how do peer networks formed at youth groups affect assimilation trajectory for Vietnamese Americans. He shows that participation in Buddhist youth groups instills a Vietnamese-American identity on youths in Seattle and, in turn, this ethnic identity can lead to upward assimilation only if the individual is part of a peer network that promotes normative values. Peer networks formed at youth group shape individual behaviors because the peer networks' ties are remarkably strong. Members can relate to each other better than they can relate to their family members and school friends,

since they share life experiences common amongst second generation Vietnamese Americans. Gang and Zimmermann (2000) show that ethnic network size has a positive effect on educational attainment, and a clear pattern is exhibited between countries-of-origin and education even in the second generation. Using the 2000 U.S. Census, Furtado and Theodoropoulos (2010) study whether having access to native networks, as measured by marriage to a native, increases the probability of immigrant employment. They show that, indeed, marriage to a native increases immigrant employment rates. They present several pieces of evidence suggesting that networks obtained through marriage play an important part in explaining the relationship between marriage decisions and employment. Mouw et al. (2014) use a unique binational data on the social network connecting an immigrant sending community in Guanajuato, Mexico, to two destination areas in the United States. They test for the effect of respondents' positions in cross-border networks on their migration intentions and attitudes towards the United States using data on the opinions of their peers, their participation in cross-border and local communication networks. They find evidence of network clustering consistent with peer effects. Assuming that assimilation is taking place in the acquisition of education in Australia, Maani et al., (2015) show the ethnic capital network effects on immigrants' earnings assimilation are significant so that immigrants in Australia benefit from being spatially concentrated.

None of these studies, however, look at the exact network structure and the role of the position of the network in the assimilation process.

3. The model

There are $q + n$ individuals in the economy, q individuals belonging to the majority group (type W) and n individuals belonging to the minority group (type B). The network g is a set of individuals $Q + N$, where $Q = \{1, \dots, q\}$, $q \geq 2$ (set of majority individuals) while $N = \{1, \dots, n\}$, $n \geq 2$ (set of minority individuals), and a set of *links* or *direct connections* between them. The adjacency matrix $\mathbf{G} = [g_{ij}]$ keeps track of the *direct connections* in the network. By definition, agents i and j are directly connected if and only if $g_{ij} = 1$; otherwise, $g_{ij} = 0$. We assume that if $g_{ij} = 1$, then $g_{ji} = 1$, so the network is *undirected*. By convention, $g_{ii} = 0$.

The timing of the model is as follows. In the first stage, each ethnic minority i , embedded in a social network, decides whether she wants to keep her culture of origin or assimilate to the culture of the majority group of the host country. Individuals from the majority group do not need to make this choice since, by definition, they are assimilated to their own culture. In the second stage, each individual (ethnic minorities and individuals from the majority group) has to decide how much effort to put in some activity, say education. The network, which keeps track of how individuals influence each other, has a key impact on how effort level is chosen. We assume that more assimilated minorities are more productive (in terms of acquiring education or in terms of productivity on the job) than less assimilated minorities because the latter lack the exact skills required for jobs in the host country (for example, they do not speak fluently the language of the host country, they have different ways of working, etc.; see e.g. Lazear, 1999; De Marti and Zenou, 2017 and our discussion in Section 2).⁴ To be more precise, *assimilated* minority individuals have a productivity of $\alpha_1 = \alpha$ but have a cost of assimilation equal to c while *oppositional* minority individuals have a productivity of $\alpha_0 = \alpha - t$ but have no

⁴ See Meng (2005) who shows that intermarried (i.e. more assimilated) immigrants earn significantly higher incomes than endogamously married immigrants, even after human capital endowments and endogeneity of intermarriage are taken into account.

cost of assimilation. In other words, c is the *assimilation cost* (because of peer pressure from the ethnic group or simply because it is costly to learn the new culture and language of the majority group) while t is the *productivity cost* of being non-assimilated (because of poorer language skills that affect the productivity of education or on-the-job skill). We have $\alpha_i \in \{\alpha - t, \alpha\}$, $\forall i = 1, 2, \dots, n$. As usual, we solve the model backwards.

3.1. Second stage: choosing education effort

Let us solve the second stage where individuals choose simultaneously their education efforts.

3.1.1. Preferences

The preferences of individual i in network g is given by:

$$U_i(x_i, \mathbf{x}_{-i}, g) = \alpha_i x_i - \frac{1}{2} x_i^2 + \theta \sum_{j=1}^{q+n} g_{ij} x_i x_j, \quad (1)$$

where x_i is the education effort⁵ that individual i exerts, \mathbf{x}_{-i} is the corresponding vector of efforts for the other $q+n-1$ agents in the network. The parameter $\theta > 0$ measures the synergy of social interactions between linked individuals. When θ is positive, an increase in x_j , the education effort of individual j who is directly linked to i , increases the marginal utility for individual i of exerting own effort x_i . For the majority group (type W), by definition, their productivity is $\alpha_i = \alpha$, $\forall i = 1, 2, \dots, q$, while, for the minority group (type B), $\alpha_i = \alpha - t$ for those who assimilate to the majority group while $\alpha_i = \alpha$ for those who do not assimilate (i.e. oppositional). We assume $\alpha > t$, so that $\alpha_i > 0$, $\forall i = 1, 2, \dots, n$. The utility function (1) has two parts. The first one, $\alpha_i x_i - \frac{1}{2} x_i^2$, is the utility of exerting x_i when there is no interaction with other individuals. The second part, $\theta \sum_{j=1}^{q+n} g_{ij} x_i x_j$, is what is obtained by interacting with direct friends in the network.

Observe that, in this model with exogenous network, α_i measures the *individual* marginal productivity, which is not necessarily affected by the interactions with others. In other words, the effects of social networks on individual's returns from effort (the third term of right-hand side of (1)) depend only on the effort of an individual who is directly linked to the individual and does not depend on the level of assimilation (i.e., α_i). For example, an unskilled immigrant arrives in a new country and then learns a new specific technology. Assume that this person comes from Iraq and has a very specific competence (e.g. electrician) and then moves to Sweden. Because being an electrician in Sweden requires different skills than in Iraq, this person needs to learn them and she will learn them better if she is assimilated. For example, learning the Swedish language necessary to be an electrician is easier if you interact with Swedish people than with Iraqis. This does not necessarily affect your interactions on the job, captured by $\frac{\partial^2 U_i(x_i, \mathbf{x}_{-i}, g)}{\partial x_i \partial x_j}$, which means that the more your direct friends exert effort on the job, the higher is your utility of exerting effort on the job. We view these two aspects as different: the first one is learning and improving own productivity and the second one is peer effects or peer pressure, not necessary improving your skills.

However, when an individual is not assimilated, she has lower returns from own effort level because of e.g., poor language skills that affect the productivity of education or on the job skill. Then, we could also assume that she has lower benefits from links to majority individuals than when she is assimilated. In Appendix 1, we consider this case and assume a utility function, which is given by (25). This specification implies that a minority individual has

⁵ We interpret our model in terms of education effort but, of course, it can be interpreted in terms of skill or productive effort.

stronger spillover effects from majority individuals when she is assimilated than when she is not. In Appendix 1, we show that the results are similar to the ones obtained with utility (1) and given by Proposition 1.

In the second stage, each agent i chooses x_i to maximize (1) taking the structure of the network and the effort choices of other agents as given. In Appendix 2, we define the Katz–Bonacich centrality, which will be useful in the characterization of the Nash equilibrium.

3.1.2. Nash equilibrium in education efforts

For each $i = 1, \dots, n$, the first-order condition with respect to x_i is given by:

$$x_i^*(\alpha_i, \boldsymbol{\alpha}_{-i}, g) = \alpha_i + \theta \sum_{j=1}^{q+n} g_{ij} x_j^* \quad \forall i = 1, 2, \dots, n+q. \quad (2)$$

The optimal education effort choice of agent i is a linear function of the education efforts of the agents to whom i is directly connected in the network. Solving for the $(q+n \times 1)$ vector \mathbf{x} and using (33) gives the Nash equilibrium visit vector \mathbf{x}^* :

$$\mathbf{x}^* = (\mathbf{I} - \theta \mathbf{G})^{-1} \boldsymbol{\alpha} = \mathbf{M} \boldsymbol{\alpha} \quad (3)$$

where $\mathbf{M} \equiv (\mathbf{I} - \theta \mathbf{G})^{-1}$ is a $(q+n \times q+n)$ matrix. Denote by $\rho(\mathbf{G})$ the spectral radius of the adjacency matrix \mathbf{G} . Then, we have the following result:

Proposition 1 (Equilibrium efforts). *If $\theta \rho(\mathbf{G}) < 1$, there exists a unique, interior Nash equilibrium in effort choices, which, for each individual i , is given by:*

$$x_i^*(\alpha_i, \boldsymbol{\alpha}_{-i}, g) = b_{\alpha_i}(g, \theta). \quad (4)$$

The proof of this proposition can be found in Ballester et al. (2006) and Calvo-Armengol et al. (2009). The Nash equilibrium number of efforts $x_i^*(\alpha_i, \boldsymbol{\alpha}_{-i}, g)$ depends on the position in the social network and the assimilation choices of the minority workers. Proposition 1 implies that an agent who is more central in the social network, as measured by her Katz–Bonacich centrality, will make more (education) effort in equilibrium. Intuitively, agents who are better connected have more to gain from interacting with others and so exert higher effort for any vector of assimilation choices. It is straightforward to verify that $x_i^*(\alpha_i, \boldsymbol{\alpha}_{-i}, g)$ increases with α , the degree of assimilation, and θ , and the intensity of social interactions but decreases with the productivity cost t .

3.2. First stage: choosing the degree of assimilation

As stated above, there is an exogenous cost differential $c > 0$ associated with assimilation. We consider any network with $q+n$ agents but only focus on the choice of the ethnic minority population given by $N = \{1, \dots, n\}$. We assume that the q individuals from the majority group occupy the most central positions (in terms of Katz–Bonacich centrality) in the network (for historical reasons and because of old-boy networks). In fact, this assumption is not necessary for the results of this section but will facilitate the characterization of equilibrium.

Agents from the minority group (type B) choose the α_i that maximizes their net utility, taking the efforts of all other agents (including the majority individuals) as given. We look at subgame-perfect equilibria. We have seen in the previous section (Proposition 1) that, if $\theta \rho(\mathbf{G}) < 1$, there exists a unique effort level for each individual i given by: $x_i^*(\alpha_i, \boldsymbol{\alpha}_{-i}, g) = b_{\alpha_i}(g, \theta)$. Using the best-response function (2) and plugging it into (1), we can write the equilibrium utility level of individual $i \in N$ as:

$$U_i^*(x_i^*, \mathbf{x}_{-i}^*, g) = \frac{1}{2} [x_i^*(\alpha_i, \boldsymbol{\alpha}_{-i}, g)]^2 = \frac{1}{2} [b_{\alpha_i}(g, \theta)]^2 \quad (5)$$

where $x_i^*(\alpha - t, \mathbf{x}_{-i}, g)$ and $x_i^*(\alpha, \mathbf{x}_{-i}, g)$ are the equilibrium effort of individual i if she is *oppositional* or not assimilated (so that she chooses $\alpha_i^* = \alpha - t$) and if she is *assimilated* (so that she chooses $\alpha_i^* = \alpha$),⁶ respectively. We need now to solve the first stage of the game, i.e. the assimilation choices.

Define \mathcal{A} as the set of *assimilated* individuals (i.e. all individuals from the minority group who choose $\alpha_i^* = \alpha$ and all individuals from the majority group) and \mathcal{O} as the set of *oppositional* individuals (i.e. all minority individuals who choose $\alpha_i^* = \alpha - t$). If individual i choose to assimilate ($\alpha_i^* = \alpha$), her equilibrium utility is equal to:

$$U_i^*(x_i^*(\alpha, \mathbf{x}_{-i}, g), \mathbf{x}_{-i}^*, g) = \frac{1}{2} \left[\sum_{j \in \mathcal{A} - \{i\}} \sum_{k=0}^{+\infty} \theta^k g_{ij}^{[k]} \alpha + \sum_{j \in \mathcal{O} - \{i\}} \sum_{k=0}^{+\infty} \theta^k g_{ij}^{[k]} (\alpha - t) + \sum_{k=0}^{+\infty} \theta^k g_{ii}^{[k]} \alpha \right]^2 - c$$

We have here decomposed the Katz–Bonacich centrality $b_{\alpha_i}(g, \theta)$ into self-loops ($m_{ii} = \sum_{k=0}^{+\infty} \theta^k g_{ii}^{[k]}$)⁷ and non self-loops ($m_{ij} = \sum_{k=0}^{+\infty} \theta^k g_{ij}^{[k]}$) and give different weights to these paths depending if agents are assimilated (weight α) or oppositional (weight $\alpha - t$). If individual i chooses to be oppositional ($\alpha_i^* = \alpha - t$), her equilibrium utility is given by:

$$U_i^*(x_i^*(\alpha - t, \mathbf{x}_{-i}, g), \mathbf{x}_{-i}^*, g) = \frac{1}{2} \left[\sum_{j \in \mathcal{A} - \{i\}} \sum_{k=0}^{+\infty} \theta^k g_{ij}^{[k]} \alpha + \sum_{j \in \mathcal{O} - \{i\}} \sum_{k=0}^{+\infty} \theta^k g_{ij}^{[k]} (\alpha - t) + \sum_{k=0}^{+\infty} \theta^k g_{ii}^{[k]} (\alpha - t) \right]^2$$

Thus, individual i will assimilate if and only if $U_i^*(x_i^*(\alpha, \mathbf{x}_{-i}, g), \mathbf{x}_{-i}^*, g) > U_i^*(x_i^*(\alpha - t, \mathbf{x}_{-i}, g), \mathbf{x}_{-i}^*, g)$.

We would now like to deal with the issues of existence and uniqueness of the subgame-perfect equilibrium assimilation-effort. Since all the q individuals from the majority group are the most central agents (in terms of Katz–Bonacich centrality), we can rank them starting with agent 1, who is the most central agent from the majority group, $m_{11} = \max_i m_{ii}$, $\forall i \in Q$, then agent 2, who has the next highest centrality, etc. until we reach agent q who has the lowest centrality in the network among the majority individuals, i.e. $m_{q,q} = \min_i m_{ii}$, $\forall i \in Q$. Then, we start with the agent who has the highest centrality among the minority individuals, i.e. $m_{q+1,q+1} = \max_i m_{ii}$, $\forall i \in N$. Then, we have the second minority agent, who has the next highest centrality among the minority group, etc., until we reach agent n who has the lowest centrality in the network among the majority individuals, i.e. $m_{q+n,q+n} = \min_i m_{ii}$, $\forall i \in N$. Define each *minority* individual by her type ω , where the type of an agent is her Katz–Bonacich centrality (or her m_{ii}). Since two minority agents can have the same centrality, there are $\omega \leq n$ types in each network of $q+n$ agents. Observe that $\omega = n$ if there are no minority individuals who have the same position (or centrality) in the network. For example, in a star network with 10 minority individuals (and no majority individuals), $\omega = 2$ and $n = 10$ while, in a complete network with the same number of individuals, $\omega = 1$ and $n = 10$. For the characterization of the subgame-perfect equilibrium, the types of the *majority* individuals do not matter as they are all assimilated.

⁶ When we say that a minority worker i is assimilated if $\alpha_i = \alpha$ and oppositional if $\alpha_i = \alpha - t$, we mean that the assimilated and oppositional worker chooses productivity level $\alpha_i = \alpha$ and $\alpha_i = \alpha - t$, respectively. This is because there is a one-to-one relation between assimilation and non-assimilation of individuals and their productivity level.

⁷ See Appendix 2 for the interpretations of the m_{ii} s and m_{ij} s.

To characterize the equilibria, we will use a technique similar to the one developed in Helsley and Zenou (2014). Denote by

$$\Phi^{\mathcal{A}}(m_{i,i}) \equiv t(2\alpha - t)(m_{i,i})^2 + 2t \left(\alpha \sum_{j=1, j \neq i}^{q+n} m_{i,j} \right) m_{i,i}, i = 1, \dots, n \tag{6}$$

where all the m_{ii} s and m_{ij} s are defined by the cells of the $(q+n \times q+n)$ matrix $\mathbf{M} = [\mathbf{I} - \theta \mathbf{G}]^{-1}$. In (6), $\Phi^{\mathcal{A}}(m_{i,i})$ is the incentive function for a given ethnic individual $i \in N$ to choose to assimilate or not when all other minority (and, of course, majority) agents are assimilated. We have a first intuitive result:

Proposition 2. Assume $\theta \rho(\mathbf{G}) < 1$ and consider any network of $q+n$ agents with $\omega \leq n$ types for the minority individuals. In any equilibrium, two minority workers with the same Katz–Bonacich centrality make the same assimilation choice and agents with higher Katz–Bonacich centrality cannot be oppositional if agents with lower Katz–Bonacich centrality are assimilated. Moreover, the number of equilibria is equal to the number of types of minority individuals plus one, i.e. $\omega + 1$.

Let us now derive our existence and uniqueness result:

Proposition 3 (Existence and uniqueness of equilibrium assimilation behaviors). Assume $\theta \rho(\mathbf{G}) < 1$ and consider any network of $q+n$ agents with $\omega \leq n$ types for the minority individuals. If the number of types of minority individuals is the same as the number of minority individuals, i.e. $\omega = n$, we can characterize the subgame-perfect equilibria as follows:

- (i) If $2c < \Phi^{\mathcal{A}}(m_{q+n,q+n})$ there exists a unique Assimilation equilibrium where all agents choose to assimilate, i.e. $\mathcal{A} = Q + N$ and $\mathcal{O} = \emptyset$.
- (ii) If $\Phi^{\mathcal{A}}(m_{q+n,q+n}) < 2c < \Phi^{\mathcal{A}}(m_{q+n-1,q+n-1}) - 2t^2 m_{q+n-1,q+n} m_{q+n-1,q+n-1}$ there exists a unique Assimilation-Oppositional equilibrium such that $\mathcal{A} = Q + N - \{n\}$ and $\mathcal{O} = \{n\}$.
- (iii) If $\Phi^{\mathcal{A}}(m_{q+n-1,q+n-1}) - 2t^2 m_{q+n-1,q+n} m_{q+n-1,q+n-1} < 2c < \Phi^{\mathcal{A}}(m_{q+n-2,q+n-2}) - 2t^2 (m_{q+n-2,q+n-1} + m_{q+n-2,q+n}) m_{q+n-2,q+n-2}$ there exists a unique Assimilation-Oppositional equilibrium such that $\mathcal{A} = Q + N - \{n-1, n\}$ and $\mathcal{O} = \{n-1, n\}$.
- (iv) If $\Phi^{\mathcal{A}}(m_{q+n-2,q+n-2}) - 2t^2 (m_{q+n-2,q+n-1} + m_{q+n-2,q+n}) m_{q+n-2,q+n-2} < 2c < \Phi^{\mathcal{A}}(m_{q+n-3,q+n-3}) - 2t^2 \left(\sum_{j \in \mathcal{O}} m_{q+n-3,q+j} \right) m_{q+n-3,q+n-3}$ there exists a unique Assimilation-Oppositional equilibrium such that $\mathcal{A} = Q + N - \{n-2, n-1, n\}$ and $\mathcal{O} = \{n-2, n-1, n\}$.
- (v) etc. until we arrive at agent 1 who has the highest centrality among the minority group. Then,
- (vi) If

$$\Phi^{\mathcal{A}}(m_{q+1,q+1}) - 2t^2 \left(\sum_{j \in \mathcal{O} - \{1\}} m_{q+1,q+j} \right) m_{q+1,q+1} < 2c$$

there exists a unique Oppositional equilibrium where all minority individuals choose not to assimilate, i.e. $\mathcal{A} = Q$ and $\mathcal{O} = N$.

If the number of types is less than the number of minority individuals, i.e. $\omega < n$, then each step described above has to be made by type and not by individual so that each subscript refers to types and not to individuals.

This proposition characterizes the (subgame-perfect Nash) equilibrium assimilation choices and shows that there always exists a unique equilibrium within each interval. The key object is $\Phi^A(m_{ii})$, which is the “incentive function” when there is an Assimilated equilibrium, i.e. when all minority individuals choose to assimilate. Indeed, when, for all i , $\Phi^A(m_{ii}) > 2c$, all minority individuals choose to assimilate because the cost of assimilation is low enough compared to the gain from assimilation in terms of productivity and synergies from other individuals (both from the majority and minority groups). In that case, there is a unique Assimilation equilibrium. Then, when we start to increase the cost of assimilation, c , and change the decision of minority workers from assimilation to oppositional, the weight in the Katz–Bonacich centrality changes from α (when assimilated) to $\alpha - t$ (when oppositional). This corresponds to the terms of both the right-hand side and left-hand side of each inequality since this is what is needed to be compensated for the agents who are oppositional compared to the Assimilation equilibrium where these agents are all assimilated. Observe that there cannot be multiple equilibria within the same set of parameters.

This result is related to the literature on ethnic identity highlighted in Section 2. In this literature, ethnic groups may “choose” to adopt “oppositional” identities, that is, some actively reject the dominant ethnic (e.g., white) behavioral norms (they are oppositional) while others totally assimilate to it. Proposition 3 shows that these choices, especially the assimilation choice, negatively depends on the cost c of assimilating to the majority culture, positively on t , the productivity cost of being non-assimilated, and positively on the position (or centrality) of the individual in the network.

The impact of α on assimilation choices is interesting since it shows how labor market opportunities determine the assimilation choices. Indeed, it is easily verified that $\Phi^A(m_{ii})$ is increasing in α and thus, when α increases, there is more assimilation. The parameter α can capture labor-market opportunities so that when the labor market is favorable, the ethnic minorities tend to assimilate more.

The impact of c on assimilation choices is well-documented. Indeed, studying the assimilation of Muslims in France, Adida et al. (2014) show that if the cost of assimilation is sufficiently high (possibly due to taste-based discrimination by the dominant group), it then becomes rational for members of the minority to eschew assimilation. This is consistent with Proposition 3 but also with other theoretical models (see, in particular, Laitin, 1995; Akerlof and Kranton, 2010; Fang and Loury, 2005; Bisin et al., 2011a). Also, focusing on Mexican immigrants in the United States, Mouw et al. (2014) show that respondents who speak English poorly are more likely to report that they would be happier living in Mexico (a measure of non-assimilation). Finally, Dustmann (1996) documents that immigrants from more distant cultural backgrounds (e.g., Turks, Greeks) exhibit a lower assimilation tendency into German society. Language proficiency and education are other important determinants of cultural assimilation, as is labor market integration (Dustmann, 1994; 1996). In these studies, discrimination, language skills, education and cultural distance are a measure of c , the cost of assimilating to the majority culture.

If we now consider the impact of t on assimilation choices, there is also plenty of evidence. For example, Constant et al. (2009), Zimmermann (2007), Constant and Zimmermann (2008) show that there are positive and significant impacts of assimilation with the host culture on economic behavior such

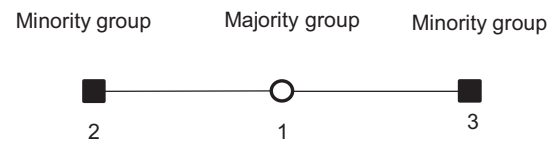


Fig. 1. A star network with 3 individuals from different communities.

as work probability, and earnings. In particular, Constant and Zimmermann (2009) find that assimilation with German culture significantly increases the probability of work for male immigrants, relative to those identified as separated or marginalized. More recently, Gorinas (2014), shows that immigrants who share social norms with the majority experience significantly better employment outcomes, particularly first-generation immigrant women. These results are also consistent with Proposition 3 where t can capture workers’ productivity but also the probability of finding a job.

Finally, the most original result in Proposition 3 is the role of social networks, in particular, the role of the centrality of each individual in the assimilation process. For example, if the cost of assimilation c is relatively large (as in case (iv) in the proposition), then the more (less) central are the agents, the more (less) likely they will (not) assimilate to the majority culture. This is because the benefits of assimilation (better outcomes in terms of education or employment) outweigh the costs and the more central are the individuals, the more externalities in terms of productivity they obtain from being assimilated. As stated in Section 2, there are very few papers that study the role of networks on assimilation of immigrants. Some papers, such as the ones by Gang and Zimmermann (2000), Furtado and Theodoropoulos (2010), Mouw et al. (2014) and Maani et al., 2015) have examined the impact of immigrant networks on outcomes and find positive effects. An interesting paper is that of Furtado and Theodoropoulos (2010) who show that having access to native networks increases the probability of immigrant employment. This is what is found in Proposition 3 where some minorities are linked to some majorities workers and this helps them become assimilated and improve their productivity and thus their situation in the labor market.

4. Examples

4.1. Star-shaped network

To illustrate the previous results, consider the star-shaped network g depicted in Fig. 1.

In this figure, circle (white) nodes correspond to the majority group while square (black) nodes correspond to the minority group. There are three agents (i.e. $q + n = 3$) and agent 1 holds a central position whereas agents 2 and 3 are peripherals. We see that individual 1 belongs to the majority group while individuals 2 and 3 belong to the minority group, i.e. $Q = \{1\}$ and $N = \{2, 3\}$. Since there is only one type for the minority workers, i.e. $\omega = 1$, there will be only two possible equilibria. Denote $\bar{c}(t, \theta, \alpha) \equiv \frac{t(1-\theta)(1+\theta)^2[2\alpha-(1-\theta)t]}{2(1-2\theta^2)^2}$.

Proposition 4 (Assimilation choices in a star-shaped network). Consider the star-shaped network depicted in Fig. 1 where individual 1 belongs to the majority group and individuals 2 and 3 are minority workers. Assume $\theta < 1/\sqrt{2}$.

- (i) If $c < \bar{c}(t, \theta, \alpha)$, there exists a unique Assimilation equilibrium where all agents are assimilated, i.e. $A = \{1, 2, 3\}$ and $O = \emptyset$, and

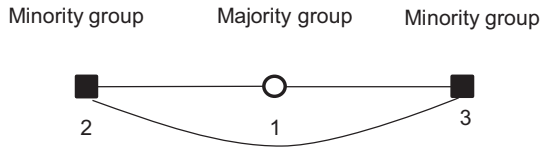


Fig. 2. A complete network with three individuals from different communities.

where the equilibrium efforts are given by:

$$\begin{pmatrix} x_1^* \\ x_2^* \\ x_3^* \end{pmatrix} = \frac{\alpha}{(1 - 2\theta^2)} \begin{pmatrix} 1 + 2\theta \\ 1 + \theta \\ 1 + \theta \end{pmatrix}$$

- (ii) If $c > \bar{c}(t, \theta, \alpha)$, there exists a unique *Oppositional equilibrium* where all minority workers are oppositional, i.e. $\mathcal{A} = \{1\}$ and $\mathcal{O} = \{2, 3\}$, and where the equilibrium efforts are equal to:

$$\begin{pmatrix} x_1^* \\ x_2^* \\ x_3^* \end{pmatrix} = \frac{1}{(1 - 2\theta^2)} \begin{pmatrix} \alpha(1 + 2\theta) - 2t\theta \\ \alpha(1 + \theta) - t \\ \alpha(1 + \theta) - t \end{pmatrix}$$

This proposition shows the role of c and t in the assimilation choices. For fixed values of α , t and θ , when we increase c , we switch from an Assimilation equilibrium to an Oppositional equilibrium. As stated above, this is what was found by Adida et al. (2014) who study the assimilation of Muslims in France. In their paper, they measure the cost of assimilation c by the taste-based discrimination of Muslims by the majority group. Similar results are obtained when t decreases. In our model, t is the productivity cost of being non-assimilated because, for example, of poorer language skills that affect the productivity of education or on-the-job skill.

Interestingly, for fixed values of α , t and c , when we decrease θ , the degree of social interactions, we obtain the same types of result because an increase in θ means that social interactions are more valuable in terms of outcomes and thus tend to induce people to assimilate more. Let us give some parameter values for which each condition is satisfied given that $\theta < 1/\sqrt{2} = 0.707$. For example, for Proposition 4, if we set $\alpha = 6$, $t = 1$ and $\theta = 0.2$, then: (i) if $c < 7.62$, there exists a unique Assimilation equilibrium where $\mathcal{A} = \{1, 2, 3\}$ and $\mathcal{O} = \emptyset$ and where efforts are given by: $x_1^* = 9.13$ and $x_2^* = x_3^* = 7.83$; (ii) if $c > 7.62$, there exists a unique Oppositional equilibrium where $\mathcal{A} = \{1\}$ and $\mathcal{O} = \{2, 3\}$ and where efforts are given by: $x_1^* = 8.70$ and $x_2^* = x_3^* = 6.74$.

4.2. Complete networks

Let us now consider a complete network where, as in the previous example, we set $q + n = 3$. The complete network is displayed in the Fig. 2.

Since there is necessary only one type of minority worker, there will always only be the two following equilibria: an Assimilation equilibrium and an Oppositional equilibrium. Assume that agent 1 belongs to the majority group and individuals 2 and 3 to the minority group. Denote $\underline{c}(t, \theta, \alpha) \equiv \frac{t(1-\theta)(1+\theta)^2[2\alpha-(1-\theta)t]}{2(1-2\theta^2)^2}$.

Proposition 5 (Assimilation choices in a complete network). Consider the complete network with 3 agents where individual 1 belongs to the majority group and 2 and 3 are minority workers. Assume that $\theta < 1/2$.

- (i) If $c < \underline{c}(t, \theta, \alpha)$, there exists a unique *Assimilation equilibrium* where all minority agents assimilate, i.e. $\mathcal{A} = \{1, 2, 3\}$ and $\mathcal{O} = \emptyset$.
- (ii) If $c > \underline{c}(t, \theta, \alpha)$, there exists a unique *Oppositional equilibrium* where all minority agents are oppositional, i.e. $\mathcal{A} = \{1\}$ and $\mathcal{O} = \{2, 3\}$.

This proposition completely characterizes the equilibrium configuration for a complete network. We can give parameter values for which each condition is satisfied given that $\theta < 1/2 = 0.5$. For example, if take exactly the same parameters as for the star network, i.e. $\alpha = 6$, $t = 1$ and $\theta = 0.2$, then: (i) if $c < 21.61$, there exists a unique Assimilation equilibrium where $\mathcal{A} = \{1, 2, 3\}$ and $\mathcal{O} = \emptyset$; (ii) if $c > 21.61$, there exists a unique Oppositional equilibrium where $\mathcal{A} = \{1\}$ and $\mathcal{O} = \{2, 3\}$.

When we compare the star network and the complete network with 3 agents, where individual 1 belongs to the majority group and 2 and 3 are minority workers, we see that there is *much more assimilation among minority workers in the complete network than in the star-shaped network*. Indeed, if we again consider the parameters $\alpha = 6$, $t = 1$ and $\theta = 0.2$, then when $7.62 < c < 21.61$, all minority workers are assimilated in the complete network while they all oppositional in the star network. This is because there are much more interactions in the complete than in the star network because, in the former, everybody interact directly with everybody while, in the latter, agents 1 and 2 interact directly with the star (agent 1) but only indirectly with each other. This is in fact a general result, which is straightforward to prove, which says that denser networks will have more assimilated individuals than less dense networks. To summarize,

Proposition 6 (Aggregate interactions). Assume $\theta\rho(\mathbf{G}) < 1$. There is more assimilation in denser networks than in less dense networks.

This result is due to the fact that, because of (local) complementarities in effort in the utility function, aggregate interactions as well as the entire vector of individual interactions increase with the density of the network.⁸ As a result, minority individuals find it more beneficial to assimilate in denser networks. As White et al. (1976) remind us, social networks, by themselves, do not produce a uniformly simple effect, nor are they simply the conduit of contextual influence. Rather, they can facilitate or inhibit assimilation by structuring interactions between initiates and members and by tying both into the larger social structure. At each stage, then, networks constrain or facilitate contact with members, condition members' reactions to initiates, and influence initiates' attitudes, values and beliefs. Proposition 6 formalizes this intuition in terms of network density.

If we interpret the network as a city, then Proposition 6 has an interesting implication for urban economy. It says that the denser is the city, the more assimilated are the ethnic minorities and the more productive they are. That more agglomeration leads to more productivity is a standard result in urban economics (Combes and Gobillon, 2015). That agglomeration leads to more assimilation is a new theoretical result, which, to the best of our knowledge, has not been tested empirically.

5. Welfare analysis and subsidy policies

Let us first determine the first-best outcome of this economy and see if it corresponds to that of the subgame perfect Nash equilibrium determined in the previous section.

5.1. First-best analysis

As in the previous section, the first-best outcome is determined in two stages. First, the planner chooses the level of assimilation for all the ethnic minorities (i.e. choose $\alpha_1, \dots, \alpha_n$) to maximize

⁸ We define network density as follows. Consider an alternative social network g' , $g' \neq g$ such that for all i, j , $g'_{ij} = 1$ if $g_{ij} = 1$. It is conventional to refer to g and g' as nested networks, and to denote their relationship as $g \subset g'$. As discussed in Ballester et al. (2006), the network g' has a denser structure of network links: some agents who are not directly connected in g are directly connected in g' .

total welfare, and then chooses the level of efforts of all individuals (i.e. choose x_1, \dots, x_{q+n}) to maximize total welfare, where the latter is given by $\sum_{i=1}^{i=q+n} U_i(x_i, \mathbf{x}_{-i}, g)$. Observe that the assimilation decisions are made for the n minority individuals while the effort decisions are made for all the $q+n$ individuals.

Let us first solve the second stage. The planner solves the following problem:

$$\begin{aligned} \max_{x_1, \dots, x_{q+n}} \mathcal{W} &= \max_{x_1, \dots, x_{q+n}} \sum_{i=1}^{i=q+n} U_i(x_i, \mathbf{x}_{-i}, g) \\ &= \max_{x_1, \dots, x_{q+n}} \left\{ \sum_{i=1}^{i=q+n} \left[\alpha_i x_i - \frac{1}{2} x_i^2 \right] + \theta \sum_{i=1}^{i=q+n} \sum_{j=1}^{q+n} g_{ij} x_i x_j \right\} \end{aligned}$$

First-order condition gives for each $i = 1, \dots, q+n$:⁹

$$\alpha_i - x_i + \theta \sum_j g_{ij} x_j + \theta \sum_j g_{ji} x_j = 0$$

which implies that (since $g_{ij} = g_{ji}$):¹⁰

$$x_i^O = \alpha_i + 2\theta \sum_j g_{ij} x_j \quad (7)$$

Using (2), we easily see that:

$$x_i^O = x_i^* + \theta \sum_j g_{ij} x_j \quad (8)$$

where x_i^* is the Nash equilibrium efforts given in (2). This means that individuals are exerting *too little* effort at the Nash equilibrium as compared to the social optimum outcome. In other words,

$$x_i^O = \sum_{j=1}^{q+n} \sum_{k=0}^{+\infty} (2\theta)^k g_{ij}^{[k]} \alpha_j \quad (9)$$

Let us now solve the first stage of the maximization problem. The planner has to decide the level of assimilation of all ethnic minority individuals in the economy. There is clearly a trade off for the assimilation process. On the one hand, the planner wants ethnic minorities to be assimilated since, because of strategic complementarities in efforts, they will exert more effort (their productivity is $\alpha > \alpha - t$) and thus obtain a higher utility. On the other hand, there is a fixed cost c of assimilation that reduces their utility.

Using (5) from Section 3.2, for α_i , the equilibrium utility can now be written as:

$$U_i^*(x_i^O, \mathbf{x}_{-i}^O, g) = \frac{1}{2} [x_i^O(\alpha_i, \mathbf{x}_{-i}, g)]^2 = \frac{1}{2} [b_{\alpha_i}(g, 2\theta)]^2 \quad (10)$$

where $x_i^O(\alpha_0, \mathbf{x}_{-i}^O, g)$ and $x_i^O(\alpha_1, \mathbf{x}_{-i}^O, g)$ are the optimal effort of individual i (see (7)) if she is *oppositional* or not assimilated ($\alpha_i^* = \alpha - t$) and if she is *assimilated* ($\alpha_i^* = \alpha$), respectively. Denote $m_{i,j}^O = \sum_{k=0}^{+\infty} (2\theta)^k g_{ij}^{[k]}$. Then, if individual i chooses to assimilate ($\alpha_i^* = \alpha$), her *optimal* utility is equal to:

$$\begin{aligned} U_i^O(x_i^O(\alpha, \mathbf{x}_{-i}, g), \mathbf{x}_{-i}^O, g) \\ = \frac{1}{2} \left[\alpha m_{i,i}^O + \alpha \sum_{j \in \mathcal{A} - \{i\}} m_{i,j}^O + (\alpha - t) \sum_{j \in \mathcal{O} - \{i\}} m_{i,j}^O \right]^2 - c \end{aligned}$$

If individual i chooses to be *oppositional* ($\alpha_0 = \alpha - t$), her equilibrium utility is given by:

$$U_i^O(x_i^O(\alpha - t, \mathbf{x}_{-i}, g), \mathbf{x}_{-i}^O, g)$$

$$= \frac{1}{2} \left[(\alpha - t) m_{i,i}^O + \alpha \sum_{j \in \mathcal{A} - \{i\}} m_{i,j}^O + (\alpha - t) \sum_{j \in \mathcal{O} - \{i\}} m_{i,j}^O \right]^2$$

Thus, individual i will assimilate if and only if $U_i^O(x_i^O(\alpha, \mathbf{x}_{-i}, g), \mathbf{x}_{-i}^O, g) > U_i^O(x_i^O(\alpha - t, \mathbf{x}_{-i}, g), \mathbf{x}_{-i}^O, g)$, which is equivalent to:

$$t(2\alpha - t)(m_{i,i}^O)^2 + 2t \left[\alpha \sum_{j \in \mathcal{A} - \{i\}} m_{i,j}^O + (\alpha - t) \sum_{j \in \mathcal{O} - \{i\}} m_{i,j}^O \right] m_{i,i}^O > 2c$$

Define

$$\Phi^O(m_{i,i}^O) \equiv t(2\alpha - t)(m_{i,i}^O)^2 + 2t \left(\alpha \sum_{j=1, j \neq i}^{q+n} m_{i,j}^O \right) m_{i,i}^O \quad (11)$$

which is the incentive function for a given ethnic individual $i \in N$ to choose to assimilate or not when all other minority (and, of course, majority) agents are assimilated. When $\Phi^O(m_{i,i}^O) > 2c$, then clearly all ethnic minorities choose to be assimilated.

As a result, the first best allocation in terms of assimilation patterns is still given by Proposition 3 but where $\Phi^A(m_{i,i})$ is replaced by $\Phi^O(m_{i,i}^O)$. Since $\Phi^O(m_{i,i}^O) > \Phi^A(m_{i,i})$, there will be more assimilation at the social optimum compared to the subgame-perfect Nash equilibrium.

Proposition 7 (First best allocations). *The first best allocations are such that each individual $i = 1, q+n$ provides an effort x_i^O given by (9) and the ethnic minority individuals assimilate according to Proposition 3 where $\Phi^A(m_{i,i})$ is replaced by $\Phi^O(m_{i,i}^O)$, which is determined by (11).*

Compared to the efficient outcomes, the equilibrium interaction efforts are too low because each agent ignores the positive impact of her effort on the efforts of others, that is, each agent ignores the positive externality arising from complementarity in effort choices. Because efforts are too low, workers tend to assimilate less compared to the efficient outcomes. As a result, individuals are exerting too little effort and do not assimilate enough at the Nash equilibrium as compared to the social optimum outcomes. The market equilibrium is therefore not efficient, mainly because of externalities in the effort choices, which lead to lower efforts and less assimilation decisions. Interestingly, there is no efficiencies in the assimilation choices. Indeed, given a cost of assimilation c , all workers will optimally make their assimilation decisions according to Proposition 3.

Let us illustrate this result by an example. Consider the star-shaped network depicted in Fig. 1 and denote $\tilde{c}(t, \theta, \alpha) \equiv \frac{t(1-2\theta)(1+2\theta)^2[2\alpha-(1-2\theta)t]}{2(1-8\theta^2)^2}$. We have the following result:

Proposition 8 (Optimal assimilation choices in a star network). *Consider the star-shaped network depicted in Fig. 1 where individual 1 belongs to the majority group and individuals 2 and 3 are minority workers. Assume $\theta < 1/(2\sqrt{2})$. Then the first best allocations are such that:*

(i) *If $c < \tilde{c}(t, \theta, \alpha)$, there exists a unique Assimilation allocation where all agents are assimilated, i.e. $\mathcal{A} = \{1, 2, 3\}$ and $\mathcal{O} = \emptyset$, and where the optimal efforts are given by:*

$$\begin{pmatrix} x_1^O \\ x_2^O \\ x_3^O \end{pmatrix} = \frac{\alpha}{(1-8\theta^2)} \begin{pmatrix} 1+4\theta \\ 1+2\theta \\ 1+2\theta \end{pmatrix} \quad (12)$$

(ii) *If $c > \tilde{c}(t, \theta, \alpha)$, there exists a unique Oppositional allocation where all minority workers are oppositional, i.e. $\mathcal{A} = \{1\}$ and $\mathcal{O} = \{2, 3\}$, and where the optimal efforts are equal to:*

$$\begin{pmatrix} x_1^O \\ x_2^O \\ x_3^O \end{pmatrix} = \frac{1}{(1-8\theta^2)} \begin{pmatrix} \alpha(1+4\theta) - 4t\theta \\ \alpha(1+2\theta) - t \\ \alpha(1+2\theta) - t \end{pmatrix} \quad (13)$$

⁹ It is easily checked that there is a unique maximum for each x_i .

¹⁰ The superscript O refers to the “social optimum” outcome while a star refers to the “Nash equilibrium” outcome.

Compared to Proposition 4 where we calculated the (subgame perfect) Nash equilibrium of this network, we see that there is a discrepancy between the equilibrium and assimilation choices. In particular, the first-best outcomes lead to more assimilation and higher efforts. For example, when

$$\frac{t(1-\theta)(1+\theta)^2[2\alpha-(1-\theta)t]}{2(1-2\theta^2)^2} < c < \frac{t(1-2\theta)(1+2\theta)^2[2\alpha-(1-2\theta)t]}{2(1-8\theta^2)^2}$$

it is optimal from the planner's viewpoint that all ethnic minorities are assimilated, i.e. $\mathcal{A} = \{1, 2, 3\}$ and $\mathcal{O} = \emptyset$, while, at the subgame-perfect Nash equilibrium, none of them are assimilated, i.e. $\mathcal{A} = \{1\}$ and $\mathcal{O} = \{2, 3\}$.

Let us give some specific parameter values. In Section 4.1, for the star network described in Fig. 1, we have shown that, if $\alpha = 6$, $t = 1$ and $\theta = 0.2$, then: (i) if $c < 7.62$, there exists a unique Assimilation equilibrium where $\mathcal{A} = \{1, 2, 3\}$ and $\mathcal{O} = \emptyset$ and where efforts are given by: $x_1^* = 9.13$ and $x_2^* = x_3^* = 7.83$; (ii) if $c > 7.62$, there exists a unique Oppositional equilibrium where $\mathcal{A} = \{1\}$ and $\mathcal{O} = \{2, 3\}$ and where efforts are given by: $x_1^* = 8.70$ and $x_2^* = x_3^* = 6.74$. This is the subgame-perfect Nash equilibrium.

Let us now determine the first-best allocation for the same parameters. Using Proposition 8, we have: (i) if $c < 14.5$, there exists a unique Assimilation allocation ($\mathcal{A} = \{1, 2, 3\}$ and $\mathcal{O} = \emptyset$) where efforts are given by: $x_1^0 = 15.88$ and $x_2^0 = x_3^0 = 12.35$; (ii) if $c > 14.5$, there exists a unique Oppositional allocation ($\mathcal{A} = \{1\}$ and $\mathcal{O} = \{2, 3\}$) where efforts are equal to: $x_1^0 = 14.71$ and $x_2^0 = x_3^0 = 10.88$. We see clearly that there is more assimilation in the first-best as opposed to the Nash equilibrium and that efforts are higher. For example, when $7.62 < c < 14.5$, then the first best is such that all ethnic minorities are assimilated while, at the subgame-perfect Nash equilibrium, none of them are assimilated.

5.2. Restoring the first-best allocations: effort subsidies

Since there are externalities that are not taken into account when agents exert efforts and since assimilation choices depend on these efforts, it seems natural to consider effort subsidies in order to restore the first-best allocations.¹¹

5.2.1. General analysis

Letting S_i^0 denote the optimal subsidy to per effort, comparison of (7) and (8) implies:

$$S_i^0 = \theta \sum_j g_{ij} x_j \tag{14}$$

or in matrix form

$$\mathbf{S}^0 = \theta \mathbf{G} \mathbf{x}$$

If we add one stage before the effort game is played, the planner will announce the optimal subsidy S_i^0 to each agent i such that:

$$U_i = (\alpha_i + S_i^0)x_i - \frac{1}{2}x_i^2 + \theta \sum_j g_{ij} x_i x_j = \alpha_i x_i - \frac{1}{2}x_i^2 + 2\theta \sum_j g_{ij} x_i x_j$$

By doing so, the planner will restore the first best. Observe that the optimal subsidy is such that

$$\mathbf{x}^0 = (\mathbf{I} - \theta \mathbf{G})^{-1} (\boldsymbol{\alpha} + \mathbf{S}^0) \mathbf{1} = (\mathbf{I} - 2\theta \mathbf{G})^{-1} \boldsymbol{\alpha}$$

where $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)^T$, which means that

$$x_i^0 = \sum_{j=1}^{q+n} \sum_{k=0}^{+\infty} \theta^k g_{ij}^{[k]} (\alpha_j + S_j^0) = \sum_{j=1}^{q+n} \sum_{k=0}^{+\infty} 2^k \theta^k g_{ij}^{[k]} \alpha_j$$

and thus

$$U_i^0 = \frac{1}{2} [b_{\alpha+S^0}(\theta, \mathbf{g})]^2 = \frac{1}{2} [b_{\alpha}(2\theta, \mathbf{g})]^2$$

In particular, the optimal subsidy is given by:

$$S_i^0 = \theta \sum_j g_{ij} x_j^0 = \theta \sum_{j=1}^{q+n} \sum_{l=1}^{q+n} \sum_{k=0}^{+\infty} 2^k \theta^k g_{ij} g_{jl}^{[k]} \alpha_l \tag{15}$$

What is interesting here is that the planner will give a larger subsidy to more central agents (independently of their ethnicity) in the network. Since we assumed that the individuals from the majority group are more central than the individuals from the minority group, then the planner will subsidize more the activities (in terms of education or work) of the majority group.

Once each individual i has received the subsidy S_i^0 , everybody will provide the optimal effort x_i^0 . Then, the assimilation decisions will be optimal from a welfare viewpoint since agents will now make their assimilation decisions based on $\Phi^0(m_{i,i}^0)$ and not on $\Phi^{\mathcal{A}}(m_{i,i})$.

Proposition 9 (Optimal level of subsidies). *If the planner proposes a subsidy $S_i^0 = \theta \sum_j g_{ij} x_j$ to each individual i , then the first-best outcomes can be restored both in terms of effort and assimilation choices. In that case, it is optimal for the planner to give higher subsidies to more central agents in the network.*

5.2.2. A star network

Let us illustrate our results with an example. Let us now show that, by giving a subsidy of S_i^0 to each agent i , we can restore the first best in terms of both effort and assimilation choices.

Proposition 10 (Optimal level of subsidies in a star network). *Consider the star-shaped network depicted in Fig. 1 where individual 1 belongs to the majority group and individuals 2 and 3 are minority workers. Assume $\theta < 1/(2\sqrt{2})$.*

(i) *If condition $c < \tilde{c}(t, \theta, \alpha)$, then, if the planner proposes the following effort subsidies,*

$$S_1^0 = \frac{2\alpha\theta(1+2\theta)}{(1-8\theta^2)} \text{ and } S_2^0 = S_3^0 = \frac{\alpha\theta(1+4\theta)}{(1-8\theta^2)}$$

the first-best outcomes in terms of efforts and assimilation choices are achieved.

(ii) *If $c > \tilde{c}(t, \theta, \alpha)$, then, if the planner proposes the following subsidies,*

$$S_1^0 = \frac{2\theta[\alpha(1+2\theta)-t]}{(1-8\theta^2)} \text{ and } S_2^0 = S_3^0 = \frac{\theta[\alpha(1+4\theta)-4t\theta]}{(1-8\theta^2)}$$

the first-best outcomes in terms of efforts and assimilation choices are achieved.

As above, assume that $\alpha = 6$, $t = 1$ and $\theta = 0.2$. Let us now calculate the level of subsidies that are needed to restore the first best.

(i) When $c < 14.5$, if $S_1^0 = 4.94$ and $S_2^0 = S_3^0 = 3.18$, then workers will exert the first best efforts $x_1^0 = 15.88$ and $x_2^0 = x_3^0 = 12.35$ and all ethnic minority workers will be assimilated so that $\mathcal{A} = \{1, 2, 3\}$ and $\mathcal{O} = \emptyset$.

(ii) When $c > 14.5$, if $S_1^0 = 4.35$ and $S_2^0 = S_3^0 = 2.94$, then workers will exert the first best efforts $x_1^0 = 14.71$ and $x_2^0 = x_3^0 = 10.88$ and all ethnic minority workers will be oppositional so that $\mathcal{A} = \{1\}$ and $\mathcal{O} = \{2, 3\}$.

6. Integrated versus segregated communities

So far, we have assumed that the majority group and the minority group always interact with each other, i.e. they belong to

¹¹ We don't need to consider assimilation subsidies (i.e. subsidizing c) since the assimilation decisions are socially efficient.



Fig. 3. Two separated communities.

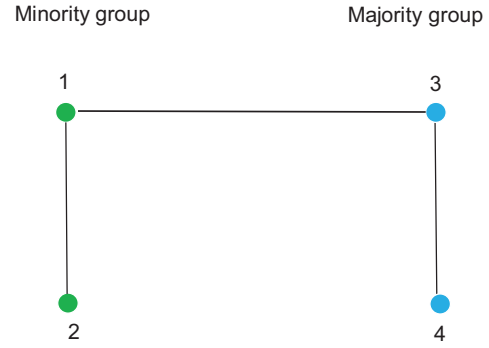


Fig. 4. Bridge between the two communities.

the same social network. We know that this not always true in the real world because, for example, minority and majority individuals do not go to the same school, the same college or do not work in the same workplace or do not live in the same area (segregation). Let us consider the case when this is not the case and analyze its consequences on assimilation behavior. To illustrate the result, let us have the following two *complete* networks where the two communities are totally separated.

In Fig. 3, circle (white) nodes correspond to the majority group while square (black) nodes correspond to the minority group. Since minority workers 1 and 2 are of the same type (i.e. same centrality), there can only be two equilibria: an Assimilation Equilibrium for which $\mathcal{A} = \{1, 2\}$ and $\mathcal{O} = \emptyset$, and an Oppositional Equilibrium for which $\mathcal{A} = \emptyset$ and $\mathcal{O} = \{1, 2\}$. Thus, if $\theta < 1$, we have:

$$\mathbf{M} = \frac{1}{(1 - \theta^2)} \begin{pmatrix} 1 & \theta \\ \theta & 1 \end{pmatrix}$$

and

$$\begin{aligned} \Phi^A(m_{11}) &= \Phi^A(m_{22}) = t(2\alpha - t)(m_{11})^2 + 2t\alpha m_{12}m_{11} \\ &= t(2\alpha - t)(m_{22})^2 + 2t\alpha m_{21}m_{22} = \frac{t[2\alpha(1 + \theta) - t]}{(1 - \theta^2)^2} \end{aligned} \tag{16}$$

As a result, (i) if $c < \frac{t[2\alpha(1+\theta)-t]}{2(1-\theta^2)^2}$, there is a unique Assimilation Equilibrium while if $c > \frac{t[2\alpha(1+\theta)-t]}{2(1-\theta^2)^2}$, there is a unique Oppositional Equilibrium. The planner would certainly want that both groups interact more since this will increase peer effects and thus total productivity in the economy. Imagine that the planner “creates” a bridge between the two communities so that the two networks are now as follows:

The planner can create such a bridge by moving minority individuals to more mixed areas (such as Moving to Opportunity programs) or by integrating minority individuals to schools or colleges (such as the Affirmative Action or busing policies) or by helping them find a job in “majority” types of jobs. Let us analyze the consequence of this bridge on the assimilation process. Observe first that since the minorities are now of two types (in terms of centralities), there will be three equilibria. If $\theta < \frac{2}{1+\sqrt{5}} = 0.618$, we have:¹²

$$\mathbf{M}_B = \frac{1}{(1 - 3\theta^2 + \theta^4)}$$

¹² The subscript *B* indicates that there is a bridge between the two communities.

$$\times \begin{pmatrix} 1 - \theta^2 & \theta(1 - \theta^2) & \theta & \theta^2 \\ \theta(1 - \theta^2) & 1 - 2\theta^2 & \theta^2 & \theta^3 \\ \theta & \theta^2 & 1 - \theta^2 & \theta(1 - \theta^2) \\ \theta^2 & \theta^3 & \theta(1 - \theta^2) & 1 - 2\theta^2 \end{pmatrix}$$

and

$$\begin{aligned} \Phi_B^A(m_{22}) &= t(2\alpha - t)(m_{22})^2 + 2t\alpha(m_{21} + m_{23} + m_{24})m_{22} \\ &= \frac{t(1 - 2\theta^2)[(2\alpha - t)(1 - 2\theta^2) + 2\alpha\theta(1 + \theta)]}{(1 - 3\theta^2 + \theta^4)^2} \end{aligned} \tag{17}$$

As a result, if

$$c < \frac{t(1 - 2\theta^2)[(2\alpha - t)(1 - 2\theta^2) + 2\alpha\theta(1 + \theta)]}{2(1 - 3\theta^2 + \theta^4)^2}$$

there is a unique Assimilation Equilibrium for which $\mathcal{A} = \{1, 2\}$ and $\mathcal{O} = \emptyset$. If

$$\begin{aligned} &\frac{t(1 - 2\theta^2)[(2\alpha - t)(1 - 2\theta^2) + 2\alpha\theta(1 - \theta^2)]}{2(1 - 3\theta^2 + \theta^4)^2} \\ &< c < \frac{t(1 - \theta^2)[(1 - \theta^2)(2\alpha - t - 2t\theta) + 2\alpha\theta(2 + \theta - \theta^2)]}{2(1 - 3\theta^2 + \theta^4)^2} \end{aligned}$$

there exists a unique Assimilation-Oppositional equilibrium for which $\mathcal{A} = \{1\}$ and $\mathcal{O} = \{2\}$. Finally, if

$$c > \frac{t(1 - \theta^2)[(1 - \theta^2)(2\alpha - t - 2t\theta) + 2\alpha\theta(2 + \theta - \theta^2)]}{2(1 - 3\theta^2 + \theta^4)^2}$$

there exists a unique Oppositional equilibrium for which $\mathcal{A} = \emptyset$ and $\mathcal{O} = \{1, 2\}$.

It is clear that there is much more assimilation when there is a bridge between the two communities (Fig. 4) than when there is not (Fig. 3). To see that, take again the same parameter values as above: $\alpha = 6$, $t = 1$ and $\theta = 0.2$. If there is no bridge, we have that, if $c < 7.27$, there is a unique Assimilation equilibrium while if $c > 7.27$, there is a unique Oppositional equilibrium. When there is a bridge, we have that: (i) if $c < 7.69$, there is a unique Assimilation equilibrium; (ii) if $7.69 < c < 9.49$, there is a unique Assimilation-Oppositional equilibrium; (iii) if $c > 9.49$, there is a unique Oppositional equilibrium. So, for example, if $7.27 < c < 7.69$, then in the network with separated communities (Fig. 3), all minority workers are oppositional while, in the network with the bridge between the communities (Fig. 4), individual 1 is assimilated while individual 2 is oppositional. This is a general result that is true for any network. It is due to the fact that, by creating a bridge between the two communities, positive externalities in terms of effort productivity increase. Indeed, if we look at individual 2, we can check that what she receives from individual 1 (i.e. m_{21}) increases from $\frac{\theta}{1 - \theta^2}$

(before the bridge) to $\frac{\theta(1-\theta^2)}{(1-3\theta^2+\theta^4)}$ (after the bridge is created). Similarly, her own centrality m_{22} (i.e. self-loops) increases from $\frac{1}{(1-\theta^2)}$ to $\frac{1-2\theta^2}{(1-3\theta^2+\theta^4)}$. More generally, when creating a bridge between the two communities, the function $\Phi^A(m_{ij})$ that gives the incentive to assimilate always increases for both individuals because the centralities of the individuals from the majority group, i.e. m_{13} and m_{14} for individual 1 and m_{23} and m_{24} for individual 2, are added to this function (compare (16) and (17)). We have therefore the following general result:

Proposition 11 (Bridge and assimilation). *Assume $\theta\rho(\mathbf{G}) < 1$ and consider any network of $q+n$ agents. Creating bridge(s) between the majority and the minority group always (weakly) increases assimilation.*

This proposition demonstrates that assimilation is easier when ethnic minorities interact with individuals from the majority group. This is observed empirically. Using data from the European Community Household Panel, De Palo et al. (2006) study the assimilation of immigrants in the host country society by analyzing a question on the number of interactions between the respondent and his/her neighbor, as related to the number of interactions with friends or relatives not living with them. Their results suggest that non-EU migrants tend to socialize less with natives, even after controlling for individual characteristics. They also find that migrants' behavior tends to slowly assimilate to that of native migrants (ethnic minorities). They suggest that education has a significant impact on the type of social activities undertaken by the individuals. More-educated people tend to relate somewhat less with close neighbors, but socialize more intensively with the broader community.

More generally, Proposition 11 shows that, by creating bridges between communities all workers are better off in terms of productivities and thus utilities. This means that, for example, if a planner provides a job to a minority worker that is usually reserved to majority workers (thus creating a bridge between the communities), it does not just increase the productivity and utility of this worker but it also increases the productivity of other minority workers from the same community. In other words, helping one minority individual benefits the whole community she belongs to because of the spillovers it generates. Since network structure plays an important role here, it would always be more efficient to create a bridge link between the most central individual from the minority group and the most central individual from the majority group. This is because we have positive complementarities in efforts so that more central agents provide more positive externalities than less central agents. This is, in turn, positively affects the assimilation behavior of minorities because of the effect on their productivity. The potential gains from bridging different parts of a network were important in the early work of Granovetter (1974) and are central to the notion of structural holes developed by Burt (1994). The latter postulates that social capital is created by a network in which people can broker connections between otherwise disconnected segments. In recent years, a number of empirical studies have shown that individuals or organizations who bridge 'structural holes' in networks gain significant payoff advantages.¹³

¹³ See Burt (1994) and Mehra et al. (2003) for influence of structural positions on promotions and performance evaluation, Podolny and Baron (1997) for work on network positions and mobility, and Ahuja (2000) for the influence of a firm's position in inter-organizational networks on its innovativeness and overall performance. See also Burt (2004) who explores the influence of individual position in social networks in shaping the generation of creative ideas.

7. Network formation and assimilation choices

So far, we made two important assumptions that allowed us to have general results: (i) the network is fixed and (ii) the assimilation choice is binary. Because of these two assumptions, in Proposition 3, we were able to obtain very general results in terms of networks. However, the main results of Proposition 3 (assimilation increases in the number of central minority agents as well as in the density of the network) are due to the assumption that assimilation leads to increases in marginal utility irrespective of the assimilation choices of other agents in the second stage. In this section, we would like to relax this assumption and propose a more general framework.

First, we would like to endogenize the network. It is, however, well-known that non-cooperative games of network formation with nominal lists of intended links are plagued by coordination problems Myerson (1991), Jackson and Wolinsky (1996), Jackson (2008), Cabrales et al. (2011). This is why it is extremely hard, if not impossible, to provide a full-fledged characterization of all possible stable networks. We are, however, able to provide a full-characterization of equilibria that have a complete-network structure and then study some specific networks that convey information about the different assimilation patterns that may arise in equilibrium.

Second, we will define the assimilation of an ethnic minority individual by the fraction of her friends who are from the majority group and not as a binary variable as before. In other words, the higher is the number of friends an ethnic minority has, the more assimilated is this person. This is a standard way of measure of assimilation in empirical research. Indeed, researchers have used inter-ethnic marriage (Furtado and Theodoropoulos, 2010; Furtado, 2015)) or inter-ethnic friendships (Sigelman and Welch, 1993; Berry, 1997; Ryder et al., 2000; Kao and Joyner, 2004; Patacchini and Zenou, 2016; De Marti and Zenou, 2017) as a measure of assimilation of ethnic minorities.

The timing is now as follows. In the first stage, all individuals form links with individuals from both communities. This corresponds to both the assimilation choice and the network formation. In the second stage, all individuals play the education effort game described in Section 3.1. As usual, we solve the model by backward induction.

7.1. Model

As above, there are two communities, the majority group (type W) and the ethnic minority group (type B). Each agent belongs exclusively to one of the two communities, B or W . The type of individual i is denoted by $\tau(i) \in \{B, W\}$. Let $n^B = n$ denote the number of B (minority) individuals in the population and let $n^W = q$ denote the number of G (majority) individuals in the population. The network g is a set of all individuals $Q+N$, where $Q = \{1, \dots, q\}$, $q \geq 2$ is the set of majority individuals while $N = \{1, \dots, n\}$, $n \geq 2$ is set of minority individuals. there are $q+n$ individuals in the economy.

We denote the link of two connected individuals, i and j , by ij . The set of i 's direct contacts is: $\mathcal{N}_i(g) = \{j \neq i \mid g_{ij} = 1\}$, which is of size $n_i(g)$. The direct contacts of individual i of the same type is $\mathcal{N}_i^{\tau(i)}(g) = \{j \neq i, \tau(i) = \tau(j) \mid g_{ij} = 1\}$, and we denote the cardinality of this set by $n_i^{\tau(i)}(g)$.

Preferences. As in the previous section, the utility of i is given by (1):

$$U_i(x_i, \mathbf{x}_{-i}, g) = \alpha x_i - \frac{1}{2} x_i^2 + \theta \sum_{j=1}^{q+n} g_{ij} x_i x_j - \sum_{j \in \mathcal{N}_i(g)} c_{ij}(g)$$

where $\alpha > 0$ is the same for all agents. Indeed, for simplicity, we assume that there is no more premium in terms of produc-

tivity for assimilated individuals (captured by t in the previous section). We will see below what are the benefits from assimilation. Furthermore, this utility function in the second stage is still given by (1) but we add the total cost of having links (this does not affect the second-stage game). In other words, all individuals pay the costs $c_{ij} > 0$ for maintaining their *direct* relationships. In Proposition 1, we have shown that, if $\theta \rho(\mathbf{G}) < 1$ and $\alpha = \alpha \mathbf{1}$, then there is a unique Nash interior equilibrium given by:

$$\mathbf{x}^* = \alpha(\mathbf{I} - \theta \mathbf{G})^{-1} \mathbf{1} = \alpha \mathbf{b}(g, \theta)$$

or

$$x_i^*(\alpha_i, \alpha_{-i}, g) = \alpha b_i(g, \theta)$$

where $b_i(g, \theta)$ is now the *unweighted* Katz–Bonacich centrality (since α is the same for all agents). As a result, the equilibrium utility of each individual $i = B, W$ is now given by (see (5)):

$$\begin{aligned} U_i^*(x_i^*, \mathbf{x}_{-i}^*, g) &= \frac{1}{2} [x_i^*(\alpha_i, \alpha_{-i}, g)]^2 - \sum_{j \in N_i(g)} c_{ij}(g) \\ &= \frac{\alpha^2}{2} [b_i(g, \theta)]^2 - \sum_{j \in N_i(g)} c_{ij}(g) \end{aligned} \quad (18)$$

For the exposition, denote $U_i^*(g) \equiv U_i^*(x_i^*, \mathbf{x}_{-i}^*, g)$. To characterize linking costs, given a network g , we define the *rate of exposure* of individual i to her own community $\tau(i)$ as:

$$e_i^{\tau(i)}(g) = \frac{n_i^{\tau(i)}(g)}{n_i(g)}. \quad (19)$$

This ratio $e_i^{\tau(i)}(g)$ measures the fraction of *same-type friends* of individual i in network g since $n_i^{\tau(i)}(g)$ is the number of i 's same-type friends in network g while $n_i(g)$ is the total number of i 's friends in network g , independently of their type. We can now introduce the cost structure. Let c and C be strictly positive constants. We assume that:¹⁴

$$c_{ij}(g) = \begin{cases} c & \text{if } \tau(i) = \tau(j) \\ c_{ij} = c + e_i^{\tau(i)} e_j^{\tau(j)} C & \text{if } \tau(i) \neq \tau(j) \end{cases} \quad (20)$$

There are thus different costs, depending with whom a connection is made. Since $C > 0$ and the rates of exposure are non-negative, the main feature of this cost structure is that it is always *more costly* to form a friendship relationship with someone from the other community than with someone from the same community. In particular, if an individual i of type $\tau(i)$ forms a friendship relationship with an individual j of type $\tau(j)$, with $\tau(i) \neq \tau(j)$, then the cost is increasing in their respective rates of exposure to their own communities. In other words, it is always easier for ethnic minorities to interact with ethnic minorities and likewise for whites, and that the inter-ethnic relationships strongly depend on how “exposed” individuals are to their own community, i.e. how many same-race friends they have.¹⁵ These difficulties in inter-ethnic relationships can be due to language barriers or more generally to different social norms and cultures, as highlighted in the Introduction and Section 2. This is also what has been found empirically. For example, using data from the German Socio-Economic Panel for the period 1996–2011, Facchini et al. (2015) document that migrants with a German friend are more similar to natives than those



Fig. 5. Line network with minority and majority individuals.

without a local companion along several important dimensions, including engagement in social activities, concerns about the economy, interest in politics and broad policy issues like the environment, crime and xenophobia.

As a result, we will say that the higher is the percentage of type- W friends for a type- B individual, the more assimilated is this person. We do not need, as in the previous section, to define who is assimilated and who is not. We could clearly do it by saying that, if a minority individual has at least 50 percent of her friends who are of the other type, then she is considered as assimilated. But this is an arbitrary definition and depends on the size of each population.

In this new formulation, the cost of *assimilation* is now given by c_{ij} , which is the cost of forming a link with someone from both communities. Here the benefits of assimilation is captured in a different way compared to the previous section. If a person from the minority group (type B) have many links with persons from the majority group (type W), she will be considered as more assimilated and the benefits will be that her exposure rate increases and, as a result, her cost c_{ij} of forming new links with someone from the majority group will decrease.¹⁶

Example. To illustrate our cost function (20), consider the network depicted in Fig. 5 where circle (white) nodes correspond to the majority group (type W) while square (black) nodes correspond to the minority group (type B). Now, assume that only links 12 and 34 exist and that individual 2 (type B) considers forming a link with individual 3 (type W). In that case, the cost of connecting individual 2 to individual 3 is:

$$c_{23}(g) = c + \frac{n_2^{\tau(1)}(g)}{n_2(g)} \frac{n_3^{\tau(3)}(g)}{n_3(g)} C = c + C$$

since $n_2^{\tau(2)}(g) = n_3^{\tau(3)}(g) = 1$ (number of same-type friends of 2 and 3, respectively) and $n_2(g) = n_3(g) = 1$ (total number of 1/s and 3/s friends independently of type, before the link 32 is formed), which implies that $e_1^{\tau(1)}(g) = e_3^{\tau(3)}(g) = 1$. It is clear that, in this example, the cost of the link 23 is maximum since both individuals 2 and 3 have not been exposed at all to people from the other community when considering forming the link 23. If for example, we add an extra link with someone of type W for individual 2, then the cost $c_{23}(g)$ decreases and is given by:

$$c_{23}(g) = c + \frac{1}{2} C$$

since $e_2^{\tau(2)}(g)$, the rate of exposure of individual 2 to her own community, decreases from 1 to 1/2.

7.2. Network stability

In games played in a network, individuals' payoffs depend on the network structure. In our case, this dependency is established in expression (18), that encompasses both the benefits and costs attributed to an individual given her position in the network of

¹⁴ Assuming that the cost of cross-type links is decreasing in the *absolute* number of friends of the other type rather than the *proportion* of friends will make the algebra more complicated but you will not affect the main result of this section (Proposition 12).

¹⁵ Eguia (2015) and De Marti and Zenou (2017) also present a theory in which the cost of assimilation is endogenous and depends on the choice of others. Their models are, however, very different from ours.

¹⁶ As stated above, we could introduce the benefits of assimilation in terms of productivity t as in the previous section but then we would need some arbitrary definition of assimilation, i.e. percentage of type- W friends above which someone is assimilated. We could also assume that the benefits from a link with a majority individual are higher than with a minority worker. This would add more notations but will not change any of our results.

relationships. Any equilibrium notion introduces some stability requirements. The notion of pairwise-stability, introduced by Jackson and Wolinsky (1996), provides a widely used solution concept in networked environments. Denote by $g + ij$ the network g for which the link $ij \notin g$ has been added and by $g - ij$ the network g for which the link $ij \in g$ has been removed.

Definition 1. A network g is **pairwise stable** if and only if:

- (i) for all $ij \in g$, $U_i(g) \geq U_i(g - ij)$ and $U_j(g) \geq U_j(g - ij)$
- (ii) for all $ij \notin g$, if $U_i(g) < U_i(g + ij)$ then $U_j(g) > U_j(g + ij)$.

In words, a network is pairwise-stable if (i) no player gains by cutting an existing link, and (ii) no two players not yet connected both gain by creating a direct link with each other. Pairwise-stability thus only checks for one-link deviations. It requires that any mutually beneficial link be formed at equilibrium but does not allow for multi-link severance. In other words, it takes into account the individual incentives to create and sever links and the necessary mutual consent between both sides for a link to be formed.

We will use this equilibrium concept throughout this section. Thus, network g is an equilibrium network whenever it is pairwise stable.

7.3. Equilibrium

Let us now characterize the pairwise-stable equilibria for which each community is fully intra-connected (i.e. each individual is linked to all other individuals within the same community) and for which they are totally separated (no links between communities). Denote

$$\Gamma(\theta, n^W, n^B) \equiv \frac{\alpha^2(\theta[1 - \phi(n^W - 1)] + \theta^2)(2(1 + \theta)[1 - \phi(n^B - 1)] + \theta[1 - \phi(n^W - 1)] + \theta)}{2(1 + \theta)^2[1 - \phi(n^B - 1)]^2[1 - \phi(n^W - 1)]^2}$$

and

$$\hat{c}(\theta, \alpha, z) \equiv \frac{\alpha^2\theta(2 + 3\theta + \theta^2z)[1 - \theta(z - 2)]}{2(1 + \theta)^4[1 - \theta(z - 1)]^2}$$

We have the following result:¹⁷

Proposition 12. Assume $\theta < 1/(n - 1)$ and $n^B < n^W$. Denote by $m \equiv n + q$ the total number of individuals in the economy.

(i) If

$$c + \frac{(n^B - 1)(n^W - 1)^2}{(m - 1)^2(m - 2)} C < \hat{c}(\theta, \alpha, m) \tag{21}$$

there exists a unique pairwise-stable network such that the minority and the majority communities are completely integrated, i.e. both communities are completely connected.

(ii) If $c < \hat{c}(\theta, \alpha, n^B)$ and

$$\Gamma(\theta, n^W, n^B) < c + \frac{(n^B - 1)(n^W - 1)}{n^B n^W} C \tag{22}$$

there exists a unique pairwise-stable network such that the minority and the majority communities are completely separated, i.e. there is no link between the two communities.

This is an interesting and general result that gives different conditions under which the network displays complete integration, i.e. each community is fully intra-connected and both communities are fully inter-connected, and under which the community are

totally separated. As can be seen from the proof of this proposition, these are results that are difficult to establish. This is because, when someone considers either to delete or to add a link, she will change the Katz–Bonacich centralities of all agents in the network and will also change her exposure rate and thus her costs of inter-ethnic links.

Consider (i). We only need to check deviations from deleting links since nobody can form new links. However, we need to check four deviations: a link deleted with someone from the same community and with someone from the other community for both types W and B agents. Consider, for example, the case when a type- B wants to sever a link with a type- W individual. First, one has to calculate the loss in terms of benefits, that is the (negative) change in terms of Katz–Bonacich centrality of this agent, which depends on the Katz–Bonacich centralities of all agents in the network. Second, one also needs to calculate the gains in terms of costs. When a type- B agent deletes a link with a type- W individual, there is a direct gain from this deletion but there is also a loss with all the other type- W agents since the exposure rate of the type- B agent is now higher and thus the costs to maintain links with all type- W agents are higher. This is what is captured by condition (21), which depends on the size of each community, n^W and n^B , and the costs c and C (which have to be low enough) and on θ , the strength of interactions.

If we now consider (ii), the results are even more complicated to obtain since we need to consider deviations from deleting a link for each agent but also the possibility for both a type- W and a type- B individual to create a bridge link with the other community. Condition (21) is satisfied if both c and C are low enough while conditions (22) and $c < \hat{c}(\theta, \alpha, n^B)$ are satisfied for intermediate values of c and high values of C . This is because we need to guarantee that both link creation and deletion are not possible.

Observe that α , which captures labor-market opportunities, has an interesting impact on assimilation choices and network structure. Indeed, in condition (21) in Proposition 12, we show that, when α is large enough, complete integration and the complete network are the only equilibrium outcomes. So, if ethnic minorities face a favorable labor-market environment, they will be more likely to be integrated and to form many links so that the complete network is likely to be an equilibrium outcome.

There is an interesting literature in psychology (see, in particular, (Phinney, 1990; Berry, 1997; Ryder et al., 2000)) that proposes a broader concept of self-identification in a two-dimensional framework, where identifications with two different cultures are not necessary mutually exclusive. Berry (1997) presents four distinct strategies for how individuals relate to two cultures. *Assimilation* is a weak identification with the culture of origin and a strong identification with the alternative culture. *Integration* is achieved when an individual combines strong dedication to the culture of origin and large commitment to the other culture. *Marginalization* is a weak dedication to both cultures. Finally, *separation* is an exclusive commitment to the culture of origin. Fig. 6 summarizes these four different possibilities in a two-dimensional space.

As a result, in Proposition 12 (i), we have shown that ethnic minorities are *integrated* since they have not only a strong identification to the majority culture (a large fraction of their friends are of type W) but also to their own culture (a large fraction of their friends are of type B). In (ii), we have shown that the ethnic minorities are *separated* since they have a weak identification to the majority culture but a strong identification to their own culture. In other words, Proposition 12 shows under which condition we can have two opposite behaviors in terms of assimilation choices.

An important difference with the model of the previous section (Proposition 3) is that, now, it is not true anymore that assimilation leads to increases in marginal utility irrespective of the assimilation choices of other agents. Here, the choice of friends is

¹⁷ The proof of all the propositions in this section can be found in Appendix 3.

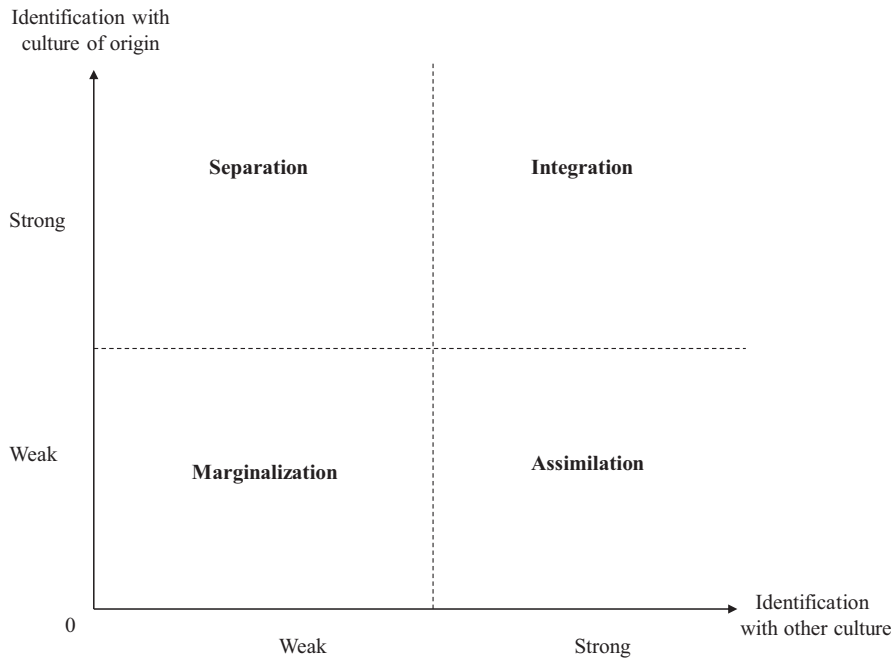


Fig. 6. Different identifications for ethnic minorities.

endogenous and the assimilation decision (percentage of friends of type *W*) strongly depends on the assimilation choices of other individuals from the majority and the minority group. This is because social distance expresses the force underlying the cost structure. Two agents are closer in the social space, the more each of them is exposed to the other community. And, the closer they are in the social space, the easier it is for them to interact. In our model, *this social distance is endogenous* and depends on the respective choice of peers.

7.4. Examples

It is difficult to have more general results because we need to calculate the marginal gain of forming or deleting links and this depends on the difference in Katz–Bonacich centralities. We can, however, give some conditions under which some specific networks are pairwise stables. Let us consider the examples of Section 4.

7.4.1. Star-shaped network

Consider the star-shaped network *g* depicted in Fig. 1 where individual 1, who belongs to the *majority group*, holds a central position, while individuals 2 and 3, who belong to the *minority group*, are peripherals.

Proposition 13 (Assimilation choices in a star-shaped network). *Consider the star-shaped network depicted in Fig. 1 where individual 1 belongs to the majority group and individuals 2 and 3 are minority workers. Assume $\theta < 1/2$. Then this network is an equilibrium (pairwise-stable) network if*

$$\frac{\theta(2 - \theta - 4\theta^2)}{(1 - 2\theta - 2\theta^2 + 4\theta^3)^2} < \frac{2c}{\alpha^2} < \frac{\theta^2(2 - 3\theta^2)}{(1 - \theta - 2\theta^2 + 2\theta^3)^2} \tag{23}$$

As in the previous section (Proposition 4), this proposition shows the role of *c*, α and θ in the assimilation choices. If *c* and α have intermediate values, then the star network of Fig. 1 is an equilibrium network because individual 1 (type *W*) has no incentive to delete a link while individual 2 or 3 (type *B*) has no incentive to delete or create a link. In this equilibrium, both minor-

ity individuals are assimilated (see Fig. 5) since they have a strong identification with the host majority culture since all their friends are from the majority group but a very weak identification with their own culture (no friend from the same community). As in Proposition 4, increasing θ , the intensity of social interactions, will always lead to more assimilation (i.e. more friends for the minority group) because they will be more interactions between all agents. This is not what condition (23) is capturing since if θ is too large, then individuals 2 and 3 would form a link and the network depicted in Fig. 1 will not anymore be pairwise stable.

7.4.2. Complete network

Consider now the complete network *g* depicted in Fig. 2 where individual 1, who belongs to the *majority group*, holds a central position, while individuals 2 and 3, who belong to the *minority group*, are peripherals.

Proposition 14 (Assimilation choices in a complete network). *Consider the complete network depicted in Fig. 2 where individual 1 belongs to the majority group and individuals 2 and 3 are minority workers. Assume $\theta < 1/2$. Then this network is an equilibrium (pairwise-stable) network if*

$$\frac{2c}{\alpha^2} < \frac{\theta(2 - \theta - 4\theta^2)}{(1 - 2\theta - 2\theta^2 + 4\theta^3)^2} \tag{24}$$

Condition (24) implies that the lower is *c* or the higher is α , the more likely that the complete network described in Fig. 2 will emerge. Moreover, when $\theta < 1/2$, it is easily verified that the right-hand side of condition (24) is increasing in θ so that the higher is θ , the more valuable are social interactions in the network and the more likely the complete network will be formed.

Observe that, for these two specific networks of Figs. 1 and 2, the inter-ethnic cost is always equal to the intra-ethnic cost, i.e. *c*, because individual 1 from the majority group has no friend from the majority group and her exposure rate was always zero. This considerably simplifies the proof of Propositions 13 and 14. Of course, in richer networks, this will not be the case (see e.g. Proposition 12). Consider, for example, the network of Fig. 5. If agent 2 (type *B*) wants to sever her link with agent 3 (type *W*),

the gain is not anymore c , as in Propositions 13 and 14, but $c + C/4$ since, by deleting link 23, the exposure rate of individual 2 to her own community increases from 1/2 to 1 and the total cost of her links decreases from $2c + C/4$ (costs of the links 12 and 13) to c (cost of the link 12).

This proposition is interesting because it provides interesting policy implications. Indeed, if the government could decrease c , and, more generally C (see Proposition 12) the intra- and inter-ethnic cost of interactions, for example, by having language courses to immigrants,¹⁸ or increase α , the workers' productivity, or θ , social interactions, then assimilation will increase.

If we interpret the network as a city, then these results provide some interesting implications in terms of formation of cities. Indeed, a star network could be interpreted as a monocentric city while the complete network could be viewed as a polycentric city Fujita and Thisse (2013). Then, Proposition 13 says that, for a monocentric city to emerge, the cost c of interactions and the labor-market opportunities α should have intermediate values. On the contrary, Proposition 14 states that c has to be low enough while α has to be high enough for a polycentric city to emerge in equilibrium. This is related to the literature of city formation where it is usually shown that the cost of interactions between agents has to be high enough for a monocentric to emerge (Fujita and Ogawa, 1980; Zenou, 2000).

8. Concluding remarks

In this paper, we have proposed a simple two stage-model where ethnic minorities choose whether to assimilate or not to the majority culture in the first stage while, in the second stage, both ethnic minorities and majority workers decide how much effort to exert in some activity. The network is taken as given. We show that the more central agents in the network tend to assimilate more than the less central agents. We also show that there is more assimilation in denser networks and the Nash equilibrium is not efficient because of positive externalities created by effort choices. We then study a policy that can restore the first best by looking at effort subsidies. We also show that bridging networks between the majority and the majority group is always beneficial for assimilation behaviors.

We then extend our model to include network formation and, instead of having a binary choice of assimilation, we have a more "continuous" definition of assimilation, which is endogenously determined by the percentage of friends from the majority group each minority worker has. Using the concept of pairwise stability for equilibrium networks, we provide conditions for which the ethnic minorities either assimilate to or separated themselves from the majority group.

As a result, there is a trade off between the two models. In the first one, which has a simple view of the assimilation process and for which the network is exogenously given, we can derive several general results. In the second one, where the position in the network of each individual as well as the number of friends of both types are endogenous, we can only provide some partial results, especially for specific networks such as the complete ones. However, both of them lead to the policy conclusion that integrating the two communities (by either moving minorities from areas with many minority families to areas with many majority fami-

lies, as, for example, in the MTO programs in the United States,¹⁹ and/or by reducing the cost of assimilation and the cost of interaction between communities) leads to more assimilation. In particular, both show that there will be more assimilation if there are more inter-ethnic links in the network. Therefore, our results suggest that ethnic enclaves may not be good for assimilation while providing host country language proficiency could be a more effective policy.²⁰ They also suggest that targeting the most central individuals among the minority group (such as community leaders) could be an efficient policy if they are themselves assimilated because this will generate positive externalities in terms of assimilation behaviors and outcomes for minority individuals in the network.

Appendix A. Alternative formulation of the utility function

In the main text, the utility function is given by (1), which means that the effects of social networks on individual's returns from effort depend only on the effort of an individual who is directly linked to the individual and does not depend on the level of assimilation (i.e., α_i). We assume that, when an individual is not assimilated, she has lower returns from own effort level because of e.g., poor language skills that affect the productivity of education or on the job skill. In this Appendix, we extend our analysis by assuming that she also has lower benefits from links to majority individuals than when she is assimilated. Put differently, instead of (1), we now assume

$$U_i(x_i, \mathbf{x}_{-i}, \mathbf{g}) = \alpha_i x_i - \frac{1}{2} x_i^2 + \alpha_i \theta \sum_{j=1}^{q+n} g_{ij}^{inter} x_i x_j + \theta \sum_{j=1}^{q+n} g_{ij}^{intra} x_i x_j \quad (25)$$

where $\mathbf{G}^{inter} = (g_{ij}^{inter})$ is the adjacency matrix for only the inter-type links, i.e. only between majority (W) and minority (B) individuals, and by $\mathbf{G}^{intra} = (g_{ij}^{intra})$ is the adjacency matrix for only the intra-type links, i.e. only between majority (W) and majority (W) individuals and between minority (B) and minority (B) individuals. Obviously, $\mathbf{G}^{inter} + \mathbf{G}^{intra} = \mathbf{G}$. For simplicity, we order the players so that the q first players are of type W and the next n ones are of type B , so that the first q rows of the \mathbf{G} matrix correspond to the q type- W players and the last n rows corresponds to the type- B players. We also assume that, among the n players, the first ones are assimilated and the last ones are oppositional. This ranking is clearly without loss of generality.

This specification (25) implies that a minority individual has stronger spillover effects from majority individuals when she is assimilated than when she is not. The best-reply functions for each $i = 1, \dots, q+n$ are now given by:

$$x_i = \alpha_i + \theta \alpha_i \sum_{j=1}^{q+n} g_{ij}^{inter} x_j + \theta \sum_{j=1}^{q+n} g_{ij}^{intra} x_j$$

In matrix form, we have:

$$\mathbf{x} = \boldsymbol{\alpha} + \theta \mathbf{A} \mathbf{G}^W \mathbf{x} + \theta \mathbf{G}^B \mathbf{x}$$

where

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \boldsymbol{\alpha} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} \alpha_1 & 0 & \dots & 0 \\ 0 & \alpha_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \alpha_n \end{pmatrix}$$

¹⁸ The law on veiling, by preventing young Muslim women to wear the veil in public areas, could also be seen as a way to help young ethnic minorities to assimilate to the majority culture. This is a controversial law and its effects on assimilation are not clear. For example, Carvalho (2013) models the veiling among Muslim women as a form of cultural resistance, which inhibits the transmission of secular values. His theory predicts that veiling is highest when individuals from highly religious communities interact in highly secular environments.

¹⁹ The Moving to Opportunity (MTO) programs do that by giving housing vouchers to poor family, usually blacks or Hispanics, to help them move to richer areas. See Ludwig et al. (2001) and Kling et al. (2005).

²⁰ Interestingly, this is also the conclusion of Angelini et al. (2015) who empirically investigate the association between assimilation with a host culture and immigrants' subjective well-being in Germany.

Denote by $\rho(\mathbf{G})$ the spectral radius of \mathbf{G} . We have the following result:

Proposition 15. *If $\theta\rho(\mathbf{A}\mathbf{G}^{Inter} + \mathbf{G}^{Intra}) < 1$, the peer effect game with payoffs (25) has a unique interior Nash equilibrium in pure strategies given by:*

$$\mathbf{x} = (\mathbf{I} - \theta\mathbf{A}\mathbf{G}^{Inter} - \theta\mathbf{G}^{Intra})^{-1}\boldsymbol{\alpha} \tag{26}$$

Proof: We need to show that $\mathbf{I} - \mathbf{B}$ is non-singular (i.e. invertible), where $\mathbf{B} \equiv \theta\mathbf{A}\mathbf{G}^{Inter} + \theta\mathbf{G}^{Intra}$. We know that $\mathbf{I} - \mathbf{B}$ is non-singular if $\theta\rho(\mathbf{A}\mathbf{G}^{Inter} + \mathbf{G}^{Intra}) < 1$ (see, e.g. Meyer (2000), page 618). To prove the interiority of the solution, we can use exactly the same arguments as in the proof of Theorem 1 in Ballester et al. (2006). \square

Corollary 1. *A sufficient condition for the Nash equilibrium (26) to exist, to be unique and to be interior is: $\theta(\alpha + 1)\rho(\mathbf{G}) < 1$.*

Proof: Let us start with two lemmas.

Lemma 1. *If \mathbf{G} is an $n \times n$ Hermitian matrix, then*

$$\rho(\mathbf{G}) = \sup_{\|\mathbf{x}\|=1} \mathbf{x}^T \mathbf{G} \mathbf{x} \tag{27}$$

Proof: The Rayleigh–Ritz quotient is defined as:

$$R(\mathbf{G}, \mathbf{x}) = \frac{\mathbf{x}^T \mathbf{G} \mathbf{x}}{\mathbf{x}^T \mathbf{x}}$$

where \mathbf{x}^T is transpose of \mathbf{x} . This is equal to

$$R(\mathbf{G}, \mathbf{x}) = \frac{\mathbf{x}^T \mathbf{G} \mathbf{x}}{\mathbf{x}^T \mathbf{x}} = \frac{\sum_{i=1}^n \sum_{j=1}^n x_i x_j g_{ij}}{\mathbf{x}^T \mathbf{x}}$$

That is

$$R(\mathbf{G}, \mathbf{x}) = \frac{\mathbf{x}^T \mathbf{G} \mathbf{x}}{\mathbf{x}^T \mathbf{x}} = \frac{\frac{\mathbf{x}^T \mathbf{G} \mathbf{x}}{\|\mathbf{x}\|_2^2}}{\frac{\mathbf{x}^T \mathbf{x}}{\|\mathbf{x}\|_2^2}} = \frac{\left(\frac{\mathbf{x}}{\|\mathbf{x}\|_2}\right)^T \mathbf{G} \left(\frac{\mathbf{x}}{\|\mathbf{x}\|_2}\right)}{\left(\frac{\mathbf{x}}{\|\mathbf{x}\|_2}\right)^T \left(\frac{\mathbf{x}}{\|\mathbf{x}\|_2}\right)}$$

where $\|\mathbf{x}\|_2 \equiv \sqrt{(\sum_{i=1}^n |x_i|^2)}$ is the Euclidian norm (or vector 2-norm). We want to compute

$$\sup_{\mathbf{x} \in \mathbb{R}^n \setminus \{0\}} R(\mathbf{G}, \mathbf{x}) = \sup_{\mathbf{x} \in \mathbb{R}^n \setminus \{0\}} \frac{\left(\frac{\mathbf{x}}{\|\mathbf{x}\|_2}\right)^T \mathbf{G} \left(\frac{\mathbf{x}}{\|\mathbf{x}\|_2}\right)}{\left(\frac{\mathbf{x}}{\|\mathbf{x}\|_2}\right)^T \left(\frac{\mathbf{x}}{\|\mathbf{x}\|_2}\right)}$$

Define $\mathbf{y} \equiv \frac{\mathbf{x}}{\|\mathbf{x}\|_2}$ so that $\mathbf{y}^T \mathbf{y} = 1$, then

$$\sup_{\mathbf{x} \in \mathbb{R}^n \setminus \{0\}} R(\mathbf{G}, \mathbf{x}) = \sup_{\mathbf{y} \in \mathbb{R}^n \setminus \{0\}} \{\mathbf{y}^T \mathbf{G} \mathbf{y} \mid \|\mathbf{y}\|_2 = 1\}$$

We want to show that $\sup_{\mathbf{y} \in \mathbb{R}^n \setminus \{0\}} \{\mathbf{y}^T \mathbf{G} \mathbf{y} \mid \|\mathbf{y}\|_2 = 1\} = \rho(\mathbf{A})$. Observe that the function $\mathbf{y} \rightarrow \mathbf{y}^T \mathbf{G} \mathbf{y}$ is continuous with compact domain ($n - 1$ dimensional sphere). Every continuous function attains a maximum and a minimum on a compact set. There is thus a \mathbf{y} so that the sup is a maximum. The problem is to find the critical points of the function $\mathbf{y} \rightarrow \mathbf{y}^T \mathbf{G} \mathbf{y}$ subject to the constraint $\|\mathbf{y}\|_2^2 = \mathbf{y}^T \mathbf{y} = 1$, i.e. to find the critical points of the following Lagrangian

$$\mathcal{L}(\mathbf{y}) = \mathbf{y}^T \mathbf{G} \mathbf{y} - \mu(\mathbf{y}^T \mathbf{y} - 1)$$

where μ is a Lagrange multiplier. The stationary points of $\mathbf{y} \rightarrow \mathcal{L}(\mathbf{y})$ are given by: $\frac{d\mathcal{L}(\mathbf{y})}{d\mathbf{y}} = \mathbf{0}$. We obtain:

$$\frac{d\mathcal{L}(\mathbf{y})}{d\mathbf{y}} = (\mathbf{G} + \mathbf{G}^T)\mathbf{y} - \mu(\mathbf{I} + \mathbf{I}^T)\mathbf{y} = \mathbf{0}$$

Using the fact that \mathbf{G} is symmetric, i.e. $\mathbf{G} = \mathbf{G}^T$, then

$$\frac{d\mathcal{L}(\mathbf{y})}{d\mathbf{y}} = 2\mathbf{G}\mathbf{y} - 2\mu\mathbf{y} = \mathbf{0}$$

which is equivalent to

$$\mathbf{G}\mathbf{y} = \mu\mathbf{y}$$

This means that \mathbf{y} is an eigenvector and ρ the associated eigenvalue. Therefore, the eigenvectors $\mathbf{y}_1, \dots, \mathbf{y}_n$ of \mathbf{G} are the critical points of the Rayleigh Quotient and their corresponding eigenvalues ρ_1, \dots, ρ_n , with $\rho_1 \equiv \rho \geq \dots \geq \rho_n$, are the stationary values of $R(\mathbf{G}, \mathbf{x})$.

At the stationay points, we have

$$\mathbf{y}^T \mathbf{G} \mathbf{y} = \mathbf{y}^T \rho \mathbf{y} = \mu \mathbf{y}^T \mathbf{y} = \rho$$

This implies that

$$\sup_{\mathbf{x} \in \mathbb{R}^n \setminus \{0\}} R(\mathbf{G}, \mathbf{x}) = \sup_{\mathbf{y} \in \mathbb{R}^n \setminus \{0\}} \{\mathbf{y}^T \mathbf{G} \mathbf{y} \mid \|\mathbf{y}\|_2 = 1\} = \rho(\mathbf{G})$$

We have thus shown that $\rho(\mathbf{G}) = \sup_{\|\mathbf{y}\|=1} \mathbf{y}^T \mathbf{G} \mathbf{y}$. \square

Lemma 2. *If \mathbf{G}_1 and \mathbf{G}_2 are two $n \times n$ symmetric matrices, then*

$$\rho(\mathbf{G}_1 + \mathbf{G}_2) \leq \rho(\mathbf{G}_1) + \rho(\mathbf{G}_2) \tag{28}$$

Proof: In Lemma 1, we have shown that

$$\rho(\mathbf{G}_1) = \sup_{\|\mathbf{y}\|=1} \mathbf{y}^T \mathbf{G}_1 \mathbf{y}$$

which implies that

$$\rho(\mathbf{G}_2) = \sup_{\|\mathbf{y}\|=1} \mathbf{y}^T \mathbf{G}_2 \mathbf{y}$$

and

$$\rho(\mathbf{G}_1 + \mathbf{G}_2) = \sup_{\|\mathbf{y}\|=1} \mathbf{y}^T (\mathbf{G}_1 + \mathbf{G}_2) \mathbf{y}$$

We need now to show that

$$\rho(\mathbf{G}_1 + \mathbf{G}_2) \leq \rho(\mathbf{G}_1) + \rho(\mathbf{G}_2)$$

Using the sub-additivity of the sup function, we have

$$\begin{aligned} \rho(\mathbf{G}_1 + \mathbf{G}_2) &= \sup_{\|\mathbf{y}\|=1} \mathbf{y}^T (\mathbf{G}_1 + \mathbf{G}_2) \mathbf{y} \\ &= \sup_{\|\mathbf{y}\|=1} \{\mathbf{y}^T \mathbf{G}_1 \mathbf{y} + \mathbf{y}^T \mathbf{G}_2 \mathbf{y}\} \\ &\leq \sup_{\|\mathbf{y}\|=1} \mathbf{y}^T \mathbf{G}_1 \mathbf{y} + \sup_{\|\mathbf{y}\|=1} \mathbf{y}^T \mathbf{G}_2 \mathbf{y} \\ &= \rho(\mathbf{G}_1) + \rho(\mathbf{G}_2) \end{aligned}$$

which is the statement of the lemma. \square

In Proposition 1, we have shown that $\mathbf{I} - \mathbf{B}$ is non-singular if $\theta\rho(\mathbf{A}\mathbf{G}^{Inter} + \mathbf{G}^{Intra}) < 1$. Observe that since \mathbf{A} is a diagonal matrix, then $\rho(\mathbf{A}) = \bar{\alpha}$, where $\bar{\alpha} = \max\{\alpha_1, \dots, \alpha_n\}$. This is because the diagonal entries of \mathbf{A} are the eigenvalues of \mathbf{A} . Since here the α s can only take two values, α and $\alpha - t$, $\bar{\alpha} = \alpha$ and thus $\rho(\mathbf{A}) = \alpha$. We also know that the largest eigenvalue of a symmetric matrix is equal to the matrix norm. In particular (see Meyer, 2000),

$$\rho(\mathbf{A}\mathbf{G}) \leq \|\mathbf{A}\mathbf{G}\| \leq \|\mathbf{A}\| \|\mathbf{G}\| = \rho(\mathbf{A})\rho(\mathbf{G}) = \alpha\rho(\mathbf{G})$$

Therefore,

$$\rho(\theta\mathbf{A}\mathbf{G}^{Inter}) = \theta\rho(\mathbf{A}\mathbf{G}^{Inter}) \leq \theta\alpha\rho(\mathbf{G}^{Inter}) \tag{29}$$

Since \mathbf{G} is symmetric, then both \mathbf{G}^{Inter} and \mathbf{G}^{Intra} also are symmetric. We can use Lemma 2, which states that (given $\theta > 0$):

$$\rho(\theta\mathbf{A}\mathbf{G}^{Inter} + \theta\mathbf{G}^{Intra}) \leq \rho(\theta\mathbf{A}\mathbf{G}^{Inter}) + \rho(\theta\mathbf{G}^{Intra})$$

Using (29), we have:

$$\begin{aligned} \rho(\theta\mathbf{A}\mathbf{G}^{Inter} + \theta\mathbf{G}^{Intra}) &\leq \rho(\theta\mathbf{A}\mathbf{G}^{Inter}) + \rho(\theta\mathbf{G}^{Intra}) \\ &\leq \alpha\theta\rho(\mathbf{G}^{Inter}) + \theta\rho(\mathbf{G}^{Intra}) \end{aligned}$$

i.e.

$$\rho(\theta\mathbf{A}\mathbf{G}^{Inter} + \theta\mathbf{G}^{Intra}) \leq \alpha\theta\rho(\mathbf{G}^{Inter}) + \theta\rho(\mathbf{G}^{Intra}) \tag{30}$$

For $\mathbf{A} = \{a_{ij}\}$ and $\mathbf{B} = \{b_{ij}\}$, we say $\mathbf{A} \leq \mathbf{B}$ if $a_{ij} \leq b_{ij}$ for all i, j . In our context, this means that $0 \leq \mathbf{G}^{Inter} \leq \mathbf{G}$ and $0 \leq \mathbf{G}^{Intra} \leq \mathbf{G}$ (since $g_{ij}^{Inter} \leq g_{ij}$ and $g_{ij}^{Intra} \leq g_{ij}$ for all i, j). As a result, using Theorem I*, page 600 in [Debreu and Herstein \(1953\)](#), we have: $\rho(\mathbf{G}^{Inter}) \leq \rho(\mathbf{G})$ and $\rho(\mathbf{G}^{Intra}) \leq \rho(\mathbf{G})$. This implies that:

$$\alpha\theta\rho(\mathbf{G}^{Inter}) + \theta\rho(\mathbf{G}^{Intra}) \leq \theta(\alpha + 1)\rho(\mathbf{G})$$

which, using (30), leads to

$$\rho(\theta\mathbf{AG}^{Inter} + \theta\mathbf{G}^{Intra}) \leq \theta(\alpha + 1)\rho(\mathbf{G}) \tag{31}$$

Remember that the condition for $\mathbf{I} - \theta\mathbf{AG}^{Inter} - \theta\mathbf{G}^{Intra}$ to be non-singular was given by ([Proposition 1](#)):

$$\theta\rho(\mathbf{AG}^{Inter} + \mathbf{G}^{Intra}) < 1$$

Given (31), a sufficient condition for this condition to hold is

$$\theta(\alpha + 1)\rho(\mathbf{G}) < 1$$

which is the condition given in [Corollary 1](#). \square

This is an interesting result because it connects the adjacency matrix \mathbf{G} to the split structure of peer effects (inter and intra-peer effects) and it is directly comparable to the condition given in [Ballester et al. \(2006\)](#), i.e. $\theta\rho(\mathbf{G}) < 1$, where peer effects were assumed to be the same across all agents.

Example. Consider the following network with 4 players and the following links 12, 13, 23, 24 (the network is undirected so that if ij exists then ji also exists). Players 1 and 2 are of type W , player 3 is of type B but is assimilated, player 4 is of type B and is oppositional. Thus,

$$\mathbf{G} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \mathbf{G}^{Inter} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$\mathbf{G}^{Intra} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \mathbf{A} = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \alpha - t \end{pmatrix}$$

The first-order conditions, $\mathbf{x} = \alpha + \theta\mathbf{AG}^W\mathbf{x} + \theta\mathbf{G}^B\mathbf{x}$, can be written as:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \alpha \\ \alpha \\ \alpha \\ \alpha - t \end{pmatrix} + \theta \begin{pmatrix} \alpha x_3 \\ \alpha(x_3 + x_4) \\ \alpha(x_1 + x_2) \\ (\alpha - t)x_2 \end{pmatrix} + \theta \begin{pmatrix} x_2 \\ x_1 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha + \theta(\alpha x_3 + x_2) \\ \alpha + \theta[\alpha(x_3 + x_4) + x_1] \\ \alpha + \alpha\theta(x_1 + x_2) \\ (\alpha - t) + \theta(\alpha - t)x_2 \end{pmatrix}$$

Assume now that $\alpha = 2$ and $t = 1$. Then, the condition in [Proposition 15](#) can be written as: $\theta < 0.28$ and the sufficient condition in [Corollary 1](#) is given by: $\theta < 1$.

Appendix B. The Katz–Bonacich network centrality measure

The Katz–Bonacich centrality measure (due to [Katz \(1953\)](#), and [Bonacich \(1987\)](#)) has proven to be extremely useful in game theoretical applications ([Ballester et al., 2006](#); [Dequiedt and Zenou, 2014](#)). \mathbf{G} is the $n \times n$ adjacency matrix that keeps track of the direct connections in the network. Let \mathbf{G}^k be the k th power of \mathbf{G} , with elements $g_{ij}^{[k]}$, where k is an integer. The matrix \mathbf{G}^k keeps track of the indirect connections in the network: $g_{ij}^{[k]} \geq 0$ gives the number of walks or paths of length $k \geq 1$ from i to j in the network g . In particular, $\mathbf{G}^0 = \mathbf{I}$. Consider the matrix $\mathbf{M} = \sum_{k=0}^{+\infty} \theta^k \mathbf{G}^k$. The elements of this matrix, m_{ij} , count the number of walks of all lengths

from i to j in the network g , where walks of length k are weighted by θ^k . The unweighted Katz–Bonacich centrality of agent i , denoted, $b_i(g, \theta)$ is equal to the sum of the elements of the i th row of \mathbf{M} :

$$b_i(g, \theta) = \sum_{j=1}^n m_{ij} = \sum_{j=1}^n \sum_{k=0}^{+\infty} \theta^k g_{ij}^{[k]}. \tag{32}$$

Note that the Katz–Bonacich centrality is well-defined if θ is low enough. In matrix form, the vector of Katz–Bonacich centralities can be written as:

$$\mathbf{b}(g, \theta) = \mathbf{M}\mathbf{1} = [\mathbf{I}_n - \theta\mathbf{G}]^{-1}\mathbf{1}_n, \tag{33}$$

where \mathbf{I}_n is the $n \times n$ identity matrix and $\mathbf{1}_n$ is the n -vector of ones. We can also define the weighted Katz–Bonacich centrality of agent i as:

$$b_{\alpha_i}(g, \theta) = \sum_{j=1}^n \sum_{k=0}^{+\infty} \theta^k g_{ij}^{[k]} \alpha_j, \tag{34}$$

where the weight attached to the walks from i to j is α_j . For any n -dimensional vector α , i.e. $\alpha = (\alpha_1, \dots, \alpha_n)^T$, where T stands for the transpose vector, the matrix equivalent of (34) is given by:

$$\mathbf{b}_{\alpha}(g, \theta) = \mathbf{M}\alpha = [\mathbf{I}_n - \theta\mathbf{G}]^{-1}\alpha$$

Denote by g_n^c , with corresponding adjacency matrix \mathbf{G}_n^c , the complete network with n agents. We have the following result:

Lemma 3. Assume $\theta < 1/(n - 1)$. For the complete network with n agents, the weighted Katz–Bonacich centrality is equal to:

$$\mathbf{b}_{\alpha}(g_n^c, \theta) = \frac{1}{1 + \theta} \alpha + \frac{\theta \sum_{i=1}^n \alpha_i}{(1 + \theta)[1 - \theta(n - 1)]} \mathbf{1}_n \tag{35}$$

while the unweighted Katz–Bonacich centrality is given by:

$$\mathbf{b}(g_n^c, \theta) = \frac{\alpha}{1 - \theta(n - 1)} \mathbf{1}_n \tag{36}$$

Proof. First, the largest eigenvalue of a complete graph is $n - 1$ and therefore the condition for the Katz–Bonacich to be well-defined is $\theta < 1/(n - 1)$. Second, observe that the adjacency matrix of the complete network with n agents, \mathbf{G}_n^c is equal to:

$$\mathbf{G}_n^c = \mathbf{1}_n \mathbf{1}_n^T - \mathbf{I}_n$$

where $\mathbf{1}_n^T$ is the transpose of the vector $\mathbf{1}_n$. This is because \mathbf{G}_n^c has a zero diagonal and a 1 everywhere else. As a result, the weighted vector of Katz–Bonacich centralities is given by:

$$\begin{aligned} \mathbf{b}_{\alpha}(g_n^c, \theta) &= [\mathbf{I}_n - \theta\mathbf{G}_n^c]^{-1}\alpha \\ &= [\mathbf{I}_n - \theta(\mathbf{1}_n \mathbf{1}_n^T - \mathbf{I}_n)]^{-1}\alpha \\ &= [(1 + \theta)\mathbf{I}_n + (-\theta\mathbf{1}_n \mathbf{1}_n^T)]^{-1}\alpha \end{aligned}$$

Now, using the Sherman and Morrison’s formula (see [Bartlett \(1951\)](#), p. 107), we have:

$$\begin{aligned} &[(1 + \theta)\mathbf{I}_n + (-\theta\mathbf{1}_n \mathbf{1}_n^T)]^{-1} \\ &= [(1 + \theta)\mathbf{I}_n]^{-1} - \frac{[(1 + \theta)\mathbf{I}_n]^{-1}(-\theta\mathbf{1}_n \mathbf{1}_n^T)[(1 + \theta)\mathbf{I}_n]^{-1}}{1 + (-\theta\mathbf{1}_n^T)[(1 + \theta)\mathbf{I}_n]^{-1}\mathbf{1}_n} \\ &= \frac{1}{1 + \theta} \mathbf{I}_n + \frac{\frac{\theta}{(1 + \theta)^2} \mathbf{1}_n \mathbf{1}_n^T}{1 - \frac{\theta}{1 + \theta} n} \\ &= \frac{1}{1 + \theta} \mathbf{I}_n + \frac{\theta}{(1 + \theta)^2} \frac{1 + \theta}{1 - \theta(n - 1)} \mathbf{1}_n \mathbf{1}_n^T \\ &= \frac{1}{1 + \theta} \left(\mathbf{I}_n + \frac{\theta}{1 - \theta(n - 1)} \mathbf{1}_n \mathbf{1}_n^T \right) \end{aligned}$$

Hence,

$$\begin{aligned} \mathbf{b}_\alpha(\mathbf{g}_n^c, \theta) &= [\mathbf{I}_n - \theta \mathbf{G}_n^c]^{-1} \boldsymbol{\alpha} \\ &= [(1 + \theta)\mathbf{I}_n + (-\theta \mathbf{1}_n \mathbf{1}_n^T)]^{-1} \boldsymbol{\alpha} \\ &= \frac{1}{1 + \theta} \left(\mathbf{I}_n + \frac{\theta}{1 - \theta(n-1)} \mathbf{1}_n \mathbf{1}_n^T \right) \boldsymbol{\alpha} \\ &= \frac{1}{1 + \theta} \boldsymbol{\alpha} + \frac{\theta \sum_{i=1}^{i=n} \alpha_i}{(1 + \theta)[1 - \theta(n-1)]} \mathbf{1}_n \end{aligned}$$

If agents are ex ante identical so that $\boldsymbol{\alpha} = \alpha \mathbf{1}_n$, then we obtain the unweighted vector of Katz–Bonacich centralities:

$$\begin{aligned} \mathbf{b}(\mathbf{g}_n^c, \theta) &= \frac{\alpha}{1 + \theta} \mathbf{1}_n + \frac{\theta n \alpha}{(1 + \theta)[1 - \theta(n-1)]} \mathbf{1}_n \\ &= \frac{\alpha}{1 - \theta(n-1)} \mathbf{1}_n \end{aligned}$$

This completes the proof. \square

Appendix C. Proofs

Proof of Proposition 12:

We denote by $m = n + q = n^B + n^W$ the total number of individuals in the economy.

(i) We want to show that the network for which each agent has $m - 1$ links (the two communities are totally integrated) is pairwise stable. We don't need to check for link-formation deviation since nobody can create a link because the total network is complete. However, each person can delete a link. When an individual i deletes a link, the loss is equal to: $B_i^*(\mathbf{g}_m^c - ij) - B_i^*(\mathbf{g}_m^c)$, where \mathbf{g}_m^c denotes the complete network with m agents, $\mathbf{g}_m^c - ij$, the network \mathbf{g}_m^c where the link ij has been removed and $B_i^*(\mathbf{g})$, the positive part (benefits) of the equilibrium utility function (18), i.e. $B_i^*(\mathbf{g}) \equiv \frac{\alpha^2}{2} [b_i(\mathbf{g}, \theta)]^2$. We have:

$$\begin{aligned} B_i^*(\mathbf{g}_m^c - ij) - B_i^*(\mathbf{g}_m^c) &= \frac{\alpha^2}{2} [b_i(\mathbf{g}_m^c - ij, \theta)]^2 - \frac{\alpha^2}{2} [b_i(\mathbf{g}_m^c, \theta)]^2 \\ &= \frac{\alpha^2}{2} [b_i(\mathbf{g}_m^c - ij, \theta) - b_i(\mathbf{g}_m^c, \theta)] \\ &\quad \times [b_i(\mathbf{g}_m^c - ij, \theta) + b_i(\mathbf{g}_m^c, \theta)] \end{aligned}$$

Let us first calculate $b_i(\mathbf{g}_m^c - ij, \theta) - b_i(\mathbf{g}_m^c, \theta)$, which the (negative) change in the Katz–Bonacich centrality for individual i when she removes the link ij from the complete graph with n agents. Denote by \mathbf{G}_m^{c-ij} the adjacency matrix of the network $\mathbf{g}_m^c - ij$. We have

$$\mathbf{G}_m^{c-ij} = \mathbf{1}_m \mathbf{1}_m^T - \mathbf{e}_i \mathbf{e}_j^T - \mathbf{I}_m$$

where \mathbf{e}_i is the m -vector who has i th component equal to 1 and the rest 0. Remember from Appendix 2 that the weighted Katz–Bonacich centrality is equal to:

$$\mathbf{b}_\alpha(\mathbf{g}, \theta) = \mathbf{M} \boldsymbol{\alpha} = [\mathbf{I}_m - \theta \mathbf{G}]^{-1} \boldsymbol{\alpha}$$

As a result, we need to calculate $[\mathbf{I}_m - \theta \mathbf{G}_m^{c-ij}]^{-1}$. We have:

$$\begin{aligned} [\mathbf{I}_m - \theta \mathbf{G}_m^{c-ij}]^{-1} &= [\mathbf{I}_m - \theta (\mathbf{1}_m \mathbf{1}_m^T - \mathbf{e}_i \mathbf{e}_j^T - \mathbf{I}_m)]^{-1} \\ &= [((1 + \theta)\mathbf{I}_m + (-\theta \mathbf{1}_m \mathbf{1}_m^T)) + \theta \mathbf{e}_i \mathbf{e}_j^T]^{-1} \end{aligned}$$

Using the Sherman and Morrison's formula (see Bartlett (1951), p. 107), we obtain:

$$\begin{aligned} & [((1 + \theta)\mathbf{I}_m + (-\theta \mathbf{1}_m \mathbf{1}_m^T)) + \theta \mathbf{e}_i \mathbf{e}_j^T]^{-1} \\ &= [(1 + \theta)\mathbf{I}_m + (-\theta \mathbf{1}_m \mathbf{1}_m^T)]^{-1} \\ &\quad - \frac{[(1 + \theta)\mathbf{I}_m + (-\theta \mathbf{1}_m \mathbf{1}_m^T)]^{-1} (\theta \mathbf{e}_i \mathbf{e}_j^T) [(1 + \theta)\mathbf{I}_m + (-\theta \mathbf{1}_m \mathbf{1}_m^T)]^{-1}}{1 + (\theta \mathbf{1}_m^T) [(1 + \theta)\mathbf{I}_m + (-\theta \mathbf{1}_m \mathbf{1}_m^T)]^{-1} \mathbf{1}_m} \end{aligned}$$

$$\begin{aligned} &= \frac{1}{1 + \theta} \left[\mathbf{I}_m + \frac{\theta}{1 - \theta(m-1)} \mathbf{1}_m \mathbf{1}_m^T \right] \\ &\quad - \frac{\frac{\theta}{(1 + \theta)^2} \left[\mathbf{I}_m + \frac{\theta}{1 - \theta(m-1)} \mathbf{1}_m \mathbf{1}_m^T \right] \mathbf{e}_i \mathbf{e}_j^T \left[\mathbf{I}_m + \frac{\theta}{1 - \theta(m-1)} \mathbf{1}_m \mathbf{1}_m^T \right]}{1 + \frac{\theta}{1 + \theta} \mathbf{1}_m^T \left[\mathbf{I}_m + \frac{\theta}{1 - \theta(m-1)} \mathbf{1}_m \mathbf{1}_m^T \right] \mathbf{1}_m} \\ &= \frac{1}{1 + \theta} \left[\mathbf{I}_m + \frac{\theta}{1 - \theta(m-1)} \mathbf{1}_m \mathbf{1}_m^T \right] \\ &\quad - \frac{\frac{\theta}{(1 + \theta)^2} \left[\mathbf{e}_i \mathbf{e}_j^T + \frac{\theta}{1 - \theta(m-1)} (\mathbf{1}_m \mathbf{e}_j^T + \mathbf{e}_i \mathbf{1}_m^T) + \frac{\theta^2}{(1 - \theta(m-1))^2} \mathbf{1}_m \mathbf{1}_m^T \right]}{1 + \frac{\theta}{1 + \theta} \left[m + \frac{\theta}{1 - \theta(m-1)} m^2 \right]} \end{aligned}$$

Observe that the denominator of the second term is equal to:

$$\begin{aligned} & 1 + \frac{\theta}{1 + \theta} \left[m + \frac{\theta}{1 - \theta(m-1)} m^2 \right] \\ &= 1 + \frac{m\theta}{1 + \theta} \left[1 + \frac{m\theta}{1 - \theta(m-1)} \right] \\ &= \frac{1 + \theta}{1 - \theta(m-1)} \end{aligned}$$

Therefore,

$$\begin{aligned} & [\mathbf{I}_m - \theta \mathbf{G}_m^{c-ij}]^{-1} \\ &= \frac{1}{1 + \theta} \left[\mathbf{I}_m + \frac{\theta}{1 - \theta(m-1)} \mathbf{1}_m \mathbf{1}_m^T \right] \\ &\quad - \frac{\frac{\theta}{(1 + \theta)^2} \left[\mathbf{e}_i \mathbf{e}_j^T + \frac{\theta}{1 - \theta(m-1)} (\mathbf{1}_m \mathbf{e}_j^T + \mathbf{e}_i \mathbf{1}_m^T) + \frac{\theta^2}{(1 - \theta(m-1))^2} \mathbf{1}_m \mathbf{1}_m^T \right]}{\frac{1 + \theta}{1 - \theta(m-1)}} \end{aligned}$$

From the proof of Lemma 3, we know that:

$$[\mathbf{I}_m - \theta \mathbf{G}_m^c]^{-1} = \frac{1}{1 + \theta} \left(\mathbf{I}_m + \frac{\theta}{1 - \theta(m-1)} \mathbf{1}_m \mathbf{1}_m^T \right)$$

As a result,

$$\begin{aligned} & [\mathbf{I}_m - \theta \mathbf{G}_m^{c-ij}]^{-1} = [\mathbf{I}_m - \theta \mathbf{G}_m^c]^{-1} \\ &\quad - \frac{\frac{\theta}{(1 + \theta)^2} \left[\mathbf{e}_i \mathbf{e}_j^T + \frac{\theta}{1 - \theta(m-1)} (\mathbf{1}_m \mathbf{e}_j^T + \mathbf{e}_i \mathbf{1}_m^T) + \frac{\theta^2}{(1 - \theta(m-1))^2} \mathbf{1}_m \mathbf{1}_m^T \right]}{\frac{1 + \theta}{1 - \theta(m-1)}} \end{aligned}$$

Thus

$$\begin{aligned} & [\mathbf{I}_m - \theta \mathbf{G}_m^{c-ij}]^{-1} - [\mathbf{I}_m - \theta \mathbf{G}_m^c]^{-1} \\ &= - \frac{\frac{\theta}{(1 + \theta)^2} \left[\mathbf{e}_i \mathbf{e}_j^T + \frac{\theta}{1 - \theta(m-1)} (\mathbf{1}_m \mathbf{e}_j^T + \mathbf{e}_i \mathbf{1}_m^T) + \frac{\theta^2}{(1 - \theta(m-1))^2} \mathbf{1}_m \mathbf{1}_m^T \right]}{\frac{1 + \theta}{1 - \theta(m-1)}} \\ &= - \frac{\theta [1 - \theta(m-1)]}{(1 + \theta)^3} \left[\mathbf{e}_i \mathbf{e}_j^T + \frac{\theta}{1 - \theta(m-1)} (\mathbf{1}_m \mathbf{e}_j^T + \mathbf{e}_i \mathbf{1}_m^T) \right. \\ &\quad \left. + \frac{\theta^2}{(1 - \theta(m-1))^2} \mathbf{1}_m \mathbf{1}_m^T \right] \end{aligned}$$

while

$$\begin{aligned} & [\mathbf{I}_m - \theta \mathbf{G}_m^{c-ij}]^{-1} + [\mathbf{I}_m - \theta \mathbf{G}_m^c]^{-1} \\ &= \frac{2}{1 + \theta} \left(\mathbf{I}_m + \frac{\theta}{1 - \theta(m-1)} \mathbf{1}_m \mathbf{1}_m^T \right) \\ &\quad - \frac{\theta [1 - \theta(m-1)]}{(1 + \theta)^3} \left[\mathbf{e}_i \mathbf{e}_j^T + \frac{\theta}{1 - \theta(m-1)} (\mathbf{1}_m \mathbf{e}_j^T + \mathbf{e}_i \mathbf{1}_m^T) \right. \\ &\quad \left. + \frac{\theta^2}{(1 - \theta(m-1))^2} \mathbf{1}_m \mathbf{1}_m^T \right] \end{aligned} \tag{37}$$

Consequently, we have:

$$\begin{aligned} & \mathbf{b}_\alpha(g_m^c - ij, \theta) - \mathbf{b}_\alpha(g_m^c, \theta) \\ &= [\mathbf{I}_m - \theta \mathbf{G}_m^{c-ij}]^{-1} \alpha - [\mathbf{I}_m - \theta \mathbf{G}_m^c]^{-1} \alpha \\ &= \left([\mathbf{I}_m - \theta \mathbf{G}_m^{c-ij}]^{-1} - [\mathbf{I}_m - \theta \mathbf{G}_m^c]^{-1} \right) \alpha \\ &= -\frac{\theta[1-\theta(m-1)]}{(1+\theta)^3} \left[\alpha_j \mathbf{e}_i + \frac{\theta}{1-\theta(m-1)} \left(\alpha_j \mathbf{1}_m + \left(\sum_{i=1}^{i=m} \alpha_i \right) \mathbf{e}_i \right) \right. \\ & \quad \left. + \frac{\theta^2 \sum_{i=1}^{i=m} \alpha_i}{(1-\theta(m-1))^2} \mathbf{1}_m \right] \\ &= -\frac{\theta[1-\theta(m-1)]}{(1+\theta)^3} \left[\left(\frac{\alpha_j \theta}{1-\theta(m-1)} + \frac{\theta^2 \sum_{i=1}^{i=m} \alpha_i}{[1-\theta(m-1)]^2} \right) \mathbf{1}_m \right. \\ & \quad \left. + \left(\alpha_j + \frac{\theta \sum_{i=1}^{i=m} \alpha_i}{1-\theta(m-1)} \right) \mathbf{e}_i \right] \end{aligned}$$

When $\alpha = \alpha \mathbf{1}_n$, for $\alpha > 0$, the difference in terms of the unweighted Katz–Bonacich centrality is given by:

$$\begin{aligned} & \mathbf{b}(g_m^c - ij, \theta) - \mathbf{b}(g_m^c, \theta) \\ &= \alpha [\mathbf{I}_m - \theta \mathbf{G}_m^{c-ij}]^{-1} \mathbf{1}_m - \alpha [\mathbf{I}_m - \theta \mathbf{G}_m^c]^{-1} \mathbf{1}_m \\ &= -\frac{\theta[1-\theta(m-1)]}{(1+\theta)^3} \left[\left(\frac{\alpha \theta}{1-\theta(m-1)} + \frac{\theta^2 m \alpha}{[1-\theta(m-1)]^2} \right) \mathbf{1}_n \right. \\ & \quad \left. + \left(\alpha + \frac{\theta m \alpha}{1-\theta(m-1)} \right) \mathbf{e}_i \right] \\ &= -\frac{\theta[1-\theta(m-1)]}{(1+\theta)^3} \left[\frac{\alpha \theta (1+\theta)}{[1-\theta(m-1)]^2} \mathbf{1}_n + \frac{\alpha (1+\theta)}{1-\theta(m-1)} \mathbf{e}_i \right] \\ &= -\frac{\alpha \theta}{(1+\theta)^2} \left[\frac{\theta}{1-\theta(m-1)} \mathbf{1}_m + \mathbf{e}_i \right] \end{aligned}$$

As a result, when $\alpha = \alpha \mathbf{1}_n$, for individual i , the difference in terms of the unweighted Katz–Bonacich centrality is equal to:

$$\begin{aligned} b_i(g_m^c - ij, \theta) - b_i(g_m^c, \theta) &= -\frac{\alpha \theta}{(1+\theta)^2} \left[\frac{\theta}{1-\theta(m-1)} + 1 \right] \\ &= -\frac{\alpha \theta}{(1+\theta)^2} \left[\frac{1-\theta(m-2)}{1-\theta(m-1)} \right] \end{aligned}$$

Finally, the difference of the unweighted Katz–Bonacich centralities for which $\alpha = 1$ for all agents is given by:

$$b_i(g_m^c - ij, \theta) - b_i(g_m^c, \theta) = -\frac{\theta}{(1+\theta)^2} \left[\frac{1-\theta(m-2)}{1-\theta(m-1)} \right]$$

Let us now calculate $b_i(g_m^c - ij, \theta) + b_i(g_m^c, \theta)$. Using (37), we have:

$$\begin{aligned} & [\mathbf{I}_m - \theta \mathbf{G}_m^c]^{-1} + [\mathbf{I}_m - \theta \mathbf{G}_m^{c-ij}]^{-1} \\ &= \frac{2}{1+\theta} \left[\mathbf{I}_m + \frac{\theta}{1-\theta(m-1)} \mathbf{1}_m \mathbf{1}_m^T \right] \\ & \quad - \frac{\frac{\theta}{(1+\theta)^2} \left[\mathbf{e}_i \mathbf{e}_i^T + \frac{\theta}{1-\theta(m-1)} (\mathbf{1}_m \mathbf{e}_i^T + \mathbf{e}_i \mathbf{1}_m^T) + \frac{\theta^2}{(1-\theta(m-1))^2} \mathbf{1}_m \mathbf{1}_m^T \right]}{\frac{1+\theta}{1-\theta(m-1)}} \end{aligned}$$

Therefore,

$$\begin{aligned} & \mathbf{b}_\alpha(g_m^c, \theta) + \mathbf{b}_\alpha(g_m^c - ij, \theta) \\ &= [\mathbf{I}_m - \theta \mathbf{G}_m^c]^{-1} \alpha + [\mathbf{I}_m - \theta \mathbf{G}_m^{c-ij}]^{-1} \alpha \\ &= \left(\frac{2}{1+\theta} \left(\mathbf{I}_m + \frac{\theta}{1-\theta(m-1)} \mathbf{1}_m \mathbf{1}_m^T \right) \right) \alpha \\ & \quad - \left(\frac{\frac{\theta}{(1+\theta)^2} \left[\mathbf{e}_i \mathbf{e}_i^T + \frac{\theta}{1-\theta(m-1)} (\mathbf{1}_m \mathbf{e}_i^T + \mathbf{e}_i \mathbf{1}_m^T) + \frac{\theta^2}{(1-\theta(m-1))^2} \mathbf{1}_m \mathbf{1}_m^T \right]}{\frac{1+\theta}{1-\theta(m-1)}} \right) \alpha \\ &= \frac{2}{1+\theta} \left(\alpha + \frac{\theta \sum_{i=1}^{i=m} \alpha_i}{1-\theta(m-1)} \mathbf{1}_m \right) \\ & \quad - \frac{\theta[1-\theta(m-1)]}{(1+\theta)^3} \left[\left(\frac{\alpha_j \theta}{1-\theta(m-1)} + \frac{\theta^2 \sum_{i=1}^{i=m} \alpha_i}{[1-\theta(m-1)]^2} \right) \mathbf{1}_m \right. \\ & \quad \left. + \left(\alpha_j + \frac{\theta \sum_{i=1}^{i=m} \alpha_i}{1-\theta(m-1)} \right) \mathbf{e}_i \right] \end{aligned}$$

When $\alpha = \alpha \mathbf{1}_m$, for $\alpha > 0$,

$$\begin{aligned} & \mathbf{b}(g_m^c, \theta) + \mathbf{b}(g_m^c - ij, \theta) \\ &= \frac{2\alpha}{1+\theta} \left(\mathbf{1}_m + \frac{m\theta}{1-\theta(m-1)} \mathbf{1}_m \right) \\ & \quad - \frac{\alpha \theta}{(1+\theta)^2} \left[\frac{\theta}{1-\theta(m-1)} \mathbf{1}_m + \mathbf{e}_i \right] \end{aligned}$$

As a result, when $\alpha = \alpha \mathbf{1}_m$, for individual i , the sum in terms of the unweighted Katz–Bonacich centrality is equal to:

$$\begin{aligned} & b_i(g_m^c, \theta) + b_i(g_m^c - ij, \theta) \\ &= \frac{2\alpha}{1+\theta} \left(1 + \frac{m\theta}{1-\theta(m-1)} \right) - \frac{\alpha \theta}{(1+\theta)^2} \left[\frac{\theta}{1-\theta(m-1)} + 1 \right] \\ &= \frac{2\alpha}{1+\theta} \left[\frac{1+\theta}{1-\theta(m-1)} \right] - \frac{\alpha \theta}{(1+\theta)^2} \left[\frac{1-\theta(m-2)}{1-\theta(m-1)} \right] \\ &= \frac{2\alpha}{1-\theta(m-1)} - \frac{\alpha \theta}{(1+\theta)^2} \left[\frac{1-\theta(m-2)}{1-\theta(m-1)} \right] \\ &= \frac{\alpha [2+3\theta+\theta^2 m]}{(1+\theta)^2 [1-\theta(m-1)]} \end{aligned}$$

Finally, the sum of the unweighted Katz–Bonacich centralities for which $\alpha = 1$ for all agents is given by:

$$b_i(g_m^c, \theta) + b_i(g_m^c - ij, \theta) = \frac{2+3\theta+\theta^2 m}{(1+\theta)^2 [1-\theta(m-1)]}$$

Therefore, the loss for individual i of deleting a link ij is:

$$\begin{aligned} & B_i^*(g_m^c - ij) - B_i^*(g_m^c) \\ &= \frac{\alpha^2}{2} [b_i(g_m^c - ij, \theta) - b_i(g_m^c, \theta)] [b_i(g_m^c - ij, \theta) + b_i(g_m^c, \theta)] \\ &= -\frac{\alpha^2}{2} \frac{\theta}{(1+\theta)^2} \left[\frac{1-\theta(m-2)}{1-\theta(m-1)} \right] \frac{[2+3\theta+\theta^2 m]}{(1+\theta)^2 [1-\theta(m-1)]} \\ &= -\frac{\alpha^2 \theta (2+3\theta+\theta^2 m) [1-\theta(m-2)]}{2(1+\theta)^4 [1-\theta(m-1)]^2} \end{aligned}$$

What about the gain in costs?

When a type- B individual deletes a link with a type- W , her direct gain from this deletion is (remember that $m = n^W + n^B$):

$$c + \left(\frac{n^B - 1}{n - 1}\right) \left(\frac{n^W - 1}{m - 1}\right) C = c + \frac{(n^B - 1)(n^W - 1)}{(m - 1)^2} C$$

Now, when she deletes this link, her exposure rate changes and thus we need to calculate the loss in terms of costs with the remaining $n^W - 1$ individuals of type W . Before the deletion of this link, the total cost of having a link with the $n^W - 1$ individuals of type W was:

$$\begin{aligned} & (n^W - 1) \left[c + \left(\frac{n^B - 1}{m - 1}\right) \left(\frac{n^W - 1}{m - 1}\right) C \right] \\ &= (n^W - 1) \left[c + \frac{(n^B - 1)(n^W - 1)}{(m - 1)^2} C \right] \end{aligned}$$

After the deletion of this link, the total cost of having a link with the $n^W - 1$ individuals of type W is:

$$\begin{aligned} & (n^W - 1) \left[c + \left(\frac{n^B - 1}{m - 2}\right) \left(\frac{n^W - 1}{m - 1}\right) C \right] \\ &= (n^W - 1) \left[c + \frac{(n^B - 1)(n^W - 1)}{(m - 1)(m - 2)} C \right] \end{aligned}$$

As a result, the total gain in terms of cost for a type- B individual who deletes a link with a type- W is given by:

$$\begin{aligned} & c + \frac{(n^B - 1)(n^W - 1)}{(m - 1)^2} C \\ &+ \left\{ (n^W - 1) \left[c + \frac{(n^B - 1)(n^W - 1)}{(m - 1)^2} C - c - \frac{(n^B - 1)(n^W - 1)}{(m - 1)(m - 2)} C \right] \right\} \\ &= c + \frac{(n^B - 1)(n^W - 1)}{(m - 1)^2} C \\ &- \left\{ (n^W - 1) \left[c + \frac{(n^B - 1)(n^W - 1)}{(m - 1)(m - 2)} C - c - \frac{(n^B - 1)(n^W - 1)}{(m - 1)^2} C \right] \right\} \\ &= c + \frac{(n^B - 1)(n^W - 1)}{(m - 1)^2} C - \frac{(n^W - 1)^2 (n^B - 1)}{(m - 1)^2 (m - 2)} C \\ &= c + (n^B - 1)(n^W - 1) \left[\frac{n - 2 - n^W + 1}{(m - 1)^2 (m - 2)} \right] C \\ &= c + \frac{(n^B - 1)^2 (n^W - 1)}{(m - 1)^2 (m - 2)} C \end{aligned}$$

Since this is positive, there is a gain in terms of costs.

Proceeding in the same way, it is easily verified that the total gain in terms of cost for a type- W individual who deletes a link with a type- B is given by:

$$\begin{aligned} & c + \frac{(n^B - 1)(n^W - 1)^2}{(m - 1)^2 (m - 2)} C \\ & \text{Since } n^B < n^W, \\ & c + \frac{(n^B - 1)^2 (n^W - 1)}{(m - 1)^2 (m - 2)} C < c + \frac{(n^B - 1)(n^W - 1)^2}{(m - 1)^2 (m - 2)} C \end{aligned} \quad (38)$$

We also need to consider the gains in terms of cost when a type- B individual deletes a link with another type- B individual. Her direct gain from this deletion is c .

Now, when she deletes this link, her exposure rate changes and thus we need to calculate the loss in terms of costs with the remaining n^W individuals of type W . Before the deletion of this link, the total cost of having a link with the n^W individuals of type W was:

$$\begin{aligned} & n^W \left[c + \left(\frac{n^B - 1}{m - 1}\right) \left(\frac{n^W - 1}{m - 1}\right) C \right] \\ &= (n^W - 1) \left[c + \frac{(n^B - 1)(n^W - 1)}{(m - 1)^2} C \right] \end{aligned}$$

After the deletion of this link, the total cost of having a link with the n^W individuals of type W is:

$$\begin{aligned} & n^W \left[c + \left(\frac{n^B - 1}{m - 2}\right) \left(\frac{n^W - 1}{m - 1}\right) C \right] \\ &= (n^W - 1) \left[c + \frac{(n^B - 1)(n^W - 1)}{(m - 1)(m - 2)} C \right] \end{aligned}$$

As a result, the total gain in terms of cost for a type- B individual who deletes a link with another type- B individual is given by:

$$\begin{aligned} & c + n^W \left[c + \frac{(n^B - 1)(n^W - 1)}{(m - 1)^2} C - c - \frac{(n^B - 1)(n^W - 1)}{(m - 1)(m - 2)} C \right] \\ &= c - n^W \left[c + \frac{(n^B - 1)(n^W - 1)}{(m - 1)(m - 2)} C - c - \frac{(n^B - 1)(n^W - 1)}{(m - 1)^2} C \right] \\ &= c - \frac{n^W (n^B - 1)(n^W - 1)}{(m - 1)^2 (m - 2)} C \end{aligned} \quad (39)$$

Proceeding in the same way, it is easily verified that the total gain in terms of cost for a type- W individual who deletes a link with a type- B is given by:

$$c - \frac{n^B (n^B - 1)(n^W - 1)}{(m - 1)^2 (m - 2)} C \quad (40)$$

We need to take the highest value between (38), (39) and (40). It is clearly:

$$\begin{aligned} & c + \frac{(n^B - 1)(n^W - 1)^2}{(m - 1)^2 (m - 2)} C \\ & \text{As a result, if} \\ & c + \frac{(n^B - 1)(n^W - 1)^2}{(m - 1)^2 (m - 2)} C < \frac{\alpha^2 \theta (2 + 3\theta + \theta^2 m) [1 - \theta(m - 2)]}{2(1 + \theta)^4 [1 - \theta(m - 1)]^2} \end{aligned}$$

then nobody in the network (type W or type B) will want to delete a link and the complete network with m individuals is pairwise stable. This is condition (21).

(ii) We want now to show that the network for which each community is separated and each community forms a complete network, i.e. each agent of type W has $n^W - 1$ links and each agent of type B has $n^B - 1$ links, is pairwise stable.

First, we need to check that nobody wants to delete a link with someone from the same community. Proceeding exactly as in (i), given that $n^B < n^W$, the condition for which nobody wants to sever a link is:

$$c < \frac{\alpha^2 \theta (2 + 3\theta + \theta^2 n^B) [1 - \theta(n^B - 2)]}{2(1 + \theta)^4 [1 - \theta(n^B - 1)]^2}$$

and

$$c < \frac{\alpha^2 \theta (2 + 3\theta + \theta^2 n^W) [1 - \theta(n^W - 2)]}{2(1 + \theta)^4 [1 - \theta(n^W - 1)]^2}$$

It is easily verified that

$$\frac{\alpha^2\theta(2 + 3\theta + \theta^2m)[1 - \theta(m - 2)]}{2(1 + \theta)^4[1 - \theta(m - 1)]^2}$$

is increasing in m . As a result, since $n^B < n^W$,

$$\begin{aligned} & \frac{\alpha^2\theta(2 + 3\theta + \theta^2n^B)[1 - \theta(n^B - 2)]}{2(1 + \theta)^4[1 - \theta(n^B - 1)]^2} \\ & < \frac{\alpha^2\theta(2 + 3\theta + \theta^2n^W)[1 - \theta(n^W - 2)]}{2(1 + \theta)^4[1 - \theta(n^W - 1)]^2} \end{aligned}$$

Thus, if

$$c < \frac{\alpha^2\theta(2 + 3\theta + \theta^2n^B)[1 - \theta(n^B - 2)]}{2(1 + \theta)^4[1 - \theta(n^B - 1)]^2} \quad (41)$$

then nobody in her own network (type W or type B) will want to delete a link with someone from the same community.

Now, we need to check for link-formation deviation between the two communities.

Connecting two complete components

Let $q > 1$ and $n > 1$. Assume that

$$\mathbf{G}_{qn}^C = \begin{pmatrix} \mathbf{1}_q \mathbf{1}_q^\top - \mathbf{I}_q & \mathbf{0}_q \mathbf{0}_n^\top \\ \mathbf{0}_n \mathbf{0}_q^\top & \mathbf{1}_n \mathbf{1}_n^\top - \mathbf{I}_n \end{pmatrix}$$

where $\mathbf{0}_q$ is the q -vector of zeros and \mathbf{G}_{qn}^C is a $(q + n) \times (q + n)$ matrix. Observe that \mathbf{G}_{qn}^C , a block-diagonal matrix, is the adjacency matrix of two complete separated networks, one with all individuals of type W ($q = n^W$) and one with all individuals of type B ($n = n^B$) and with no connection between the two networks.

Observe that $m = q + n$. Then, we would like to calculate $\mathbf{b}(\mathcal{g}_{qn}^C, \theta) = (\mathbf{I}_m - \theta \mathbf{G}_{qn}^C)^{-1}$, the Katz–Bonacich centrality of each agent in each network. We have:

$$\begin{aligned} & (\mathbf{I}_m - \theta \mathbf{G}_{qn}^C)^{-1} \\ & = \begin{pmatrix} (1 + \theta)\mathbf{I}_q - \theta \mathbf{1}_q \mathbf{1}_q^\top & \mathbf{0}_q \mathbf{0}_n^\top \\ \mathbf{0}_n \mathbf{0}_q^\top & (1 + \theta)\mathbf{I}_n - \theta \mathbf{1}_n \mathbf{1}_n^\top \end{pmatrix}^{-1} \\ & = \begin{pmatrix} ((1 + \theta)\mathbf{I}_q - \theta \mathbf{1}_q \mathbf{1}_q^\top)^{-1} & \mathbf{0}_q \mathbf{0}_n^\top \\ \mathbf{0}_n \mathbf{0}_q^\top & ((1 + \theta)\mathbf{I}_n - \theta \mathbf{1}_n \mathbf{1}_n^\top)^{-1} \end{pmatrix} \\ & = \begin{pmatrix} \frac{1}{1 + \theta}(\mathbf{I}_q - \frac{\theta}{1 - \theta(q-1)} \mathbf{1}_q \mathbf{1}_q^\top) & \mathbf{0}_q \mathbf{0}_n^\top \\ \mathbf{0}_n \mathbf{0}_q^\top & \frac{1}{1 + \theta}(\mathbf{I}_n - \frac{\theta}{1 - \theta(n-1)} \mathbf{1}_n \mathbf{1}_n^\top) \end{pmatrix} \\ & = \frac{1}{1 + \theta} \mathbf{I}_m + \frac{\theta}{1 + \theta} \begin{pmatrix} \frac{1}{1 - \theta(q-1)} \mathbf{1}_q \mathbf{1}_q^\top & \mathbf{0}_q \mathbf{0}_n^\top \\ \mathbf{0}_n \mathbf{0}_q^\top & \frac{\theta}{1 - \theta(n-1)} \mathbf{1}_n \mathbf{1}_n^\top \end{pmatrix} \end{aligned}$$

Now consider adding a (bridge) link to the network \mathbf{G}_{qn}^C between an individual from network q (type W), say individual 1, and an individual from network m (type B), say individual $q + n = m$, which results in a new network: $\mathbf{G}_{qn}^C + \mathbf{e}_1 \mathbf{e}_m^\top$. We have:

$$\begin{aligned} & (\mathbf{I}_m - \theta(\mathbf{G}_{qn}^C + \mathbf{e}_1 \mathbf{e}_m^\top))^{-1} \\ & = ((\mathbf{I}_m - \theta \mathbf{G}_{qn}^C) - \theta \mathbf{e}_1 \mathbf{e}_m^\top)^{-1} \\ & = (\mathbf{I}_m - \theta \mathbf{G}_{qn}^C)^{-1} + \frac{\theta(\mathbf{I}_m - \theta \mathbf{G}_{qn}^C)^{-1} \mathbf{e}_1 \mathbf{e}_m^\top (\mathbf{I}_m - \theta \mathbf{G}_{qn}^C)^{-1}}{1 - \theta \mathbf{e}_m^\top (\mathbf{I}_m - \theta \mathbf{G}_{qn}^C)^{-1} \mathbf{e}_1} \end{aligned}$$

where

$$\theta(\mathbf{I}_m - \theta \mathbf{G}_{qn}^C)^{-1} \mathbf{e}_1 \mathbf{e}_m^\top (\mathbf{I}_m - \theta \mathbf{G}_{qn}^C)^{-1}$$

$$\begin{aligned} & = \begin{bmatrix} \frac{1}{1 + \theta} \mathbf{I}_m + \frac{\theta}{1 + \theta} \begin{pmatrix} \frac{1}{1 - \theta(q-1)} \mathbf{1}_q \mathbf{1}_q^\top & \mathbf{0}_q \mathbf{0}_n^\top \\ \mathbf{0}_n \mathbf{0}_q^\top & \frac{\theta}{1 - \theta(n-1)} \mathbf{1}_n \mathbf{1}_n^\top \end{pmatrix} \\ \times \mathbf{e}_m^\top \begin{bmatrix} \frac{1}{1 + \theta} \mathbf{I}_m + \frac{\theta}{1 + \theta} \begin{pmatrix} \frac{1}{1 - \theta(q-1)} \mathbf{1}_q \mathbf{1}_q^\top & \mathbf{0}_q \mathbf{0}_n^\top \\ \mathbf{0}_n \mathbf{0}_q^\top & \frac{\theta}{1 - \theta(n-1)} \mathbf{1}_n \mathbf{1}_n^\top \end{pmatrix} \end{bmatrix} \\ & = \begin{bmatrix} \left(\frac{1}{1 + \theta}\right) \mathbf{e}_1 + \frac{\theta}{1 + \theta} \begin{pmatrix} \frac{1}{1 - \theta(q-1)} \mathbf{1}_q \\ \mathbf{0}_n \end{pmatrix} \\ \times \begin{bmatrix} \left(\frac{1}{1 + \theta}\right) \mathbf{e}_m^\top + \frac{\theta}{1 + \theta} \begin{pmatrix} \frac{1}{1 - \theta(n-1)} \mathbf{1}_n^\top \\ \mathbf{0}_q^\top \end{pmatrix} \end{bmatrix} \end{bmatrix} \\ & = \frac{1}{(1 + \theta)^2} \mathbf{e}_1 \mathbf{e}_m^\top + \frac{\theta}{(1 + \theta)^2} \mathbf{e}_1 \begin{pmatrix} \mathbf{0}_q \\ \frac{1}{1 - \theta(n-1)} \mathbf{1}_n \end{pmatrix}^\top \\ & \quad + \frac{\theta}{(1 + \theta)^2} \begin{pmatrix} \frac{1}{1 - \theta(q-1)} \mathbf{1}_q \\ \mathbf{0}_n \end{pmatrix} \mathbf{e}_m^\top \\ & \quad + \frac{\theta^2}{(1 + \theta)^2} \begin{pmatrix} \frac{1}{1 - \theta(q-1)} \mathbf{1}_q \\ \mathbf{0}_n \end{pmatrix} \begin{pmatrix} \mathbf{0}_q \\ \frac{1}{1 - \theta(n-1)} \mathbf{1}_n \end{pmatrix}^\top \end{aligned}$$

and where

$$\begin{aligned} & 1 - \theta \mathbf{e}_m^\top (\mathbf{I}_m - \theta \mathbf{G}_{qn}^C)^{-1} \mathbf{e}_1 \\ & = 1 - \theta \mathbf{e}_m^\top \begin{bmatrix} \frac{1}{1 + \theta} \mathbf{I}_m + \frac{\theta}{1 + \theta} \begin{pmatrix} \frac{1}{1 - \theta(q-1)} \mathbf{1}_q \mathbf{1}_q^\top & \mathbf{0}_q \mathbf{0}_n^\top \\ \mathbf{0}_n \mathbf{0}_q^\top & \frac{\theta}{1 - \theta(n-1)} \mathbf{1}_n \mathbf{1}_n^\top \end{pmatrix} \\ \times \begin{bmatrix} \frac{1}{1 + \theta} \mathbf{e}_1 + \frac{\theta}{1 + \theta} \begin{pmatrix} \frac{1}{1 - \theta(q-1)} \mathbf{1}_q \\ \mathbf{0}_n \end{pmatrix} \end{bmatrix} \\ & = 1 \\ & = 1 \end{aligned}$$

Assume that $\alpha = 1$ for all agents. The block-diagonal structure of \mathbf{G}_{qn}^C implies that the unweighted Katz–Bonacich centrality is equal to (see also Lemma 3):

$$\mathbf{b}(\mathcal{g}_{qn}^C, \theta) = \begin{pmatrix} \frac{1}{1 - \theta(q-1)} \mathbf{1}_q \\ \frac{1}{1 - \theta(n-1)} \mathbf{1}_n \end{pmatrix}$$

We find that:

$$\begin{aligned} & \mathbf{b}(\mathcal{g}_{qn}^C + \mathbf{e}_1 \mathbf{e}_m^\top, \theta) \\ & = (\mathbf{I}_m - \theta(\mathbf{G}_{qn}^C + \mathbf{e}_1 \mathbf{e}_m^\top))^{-1} \mathbf{1}_m \\ & = (\mathbf{I}_m - \theta \mathbf{G}_{qn}^C)^{-1} \mathbf{1}_m \\ & \quad + \theta \left[\frac{1}{(1 + \theta)^2} \mathbf{e}_1 \mathbf{e}_m^\top + \frac{\theta}{(1 + \theta)^2} \mathbf{e}_1 \begin{pmatrix} \mathbf{0}_q \\ \frac{1}{1 - \theta(n-1)} \mathbf{1}_n \end{pmatrix}^\top + \frac{\theta}{(1 + \theta)^2} \begin{pmatrix} \frac{1}{1 - \theta(q-1)} \mathbf{1}_q \\ \mathbf{0}_n \end{pmatrix} \mathbf{e}_m^\top \right] \mathbf{1}_m \\ & \quad + \frac{\theta^2}{(1 + \theta)^2} \begin{pmatrix} \frac{1}{1 - \theta(q-1)} \mathbf{1}_q \\ \mathbf{0}_n \end{pmatrix} \begin{pmatrix} \mathbf{0}_q \\ \frac{1}{1 - \theta(n-1)} \mathbf{1}_n \end{pmatrix}^\top \\ & = \mathbf{b}(\mathcal{g}_{qn}^C, \theta) + \frac{\theta}{(1 + \theta)^2} \mathbf{e}_1 + \frac{n\theta^2}{(1 + \theta)^2[1 - \theta(n - 1)]} \mathbf{e}_1 \\ & \quad + \frac{\theta^2}{(1 + \theta)^2} \begin{pmatrix} \frac{1}{1 - \theta(q-1)} \mathbf{1}_q \\ \mathbf{0}_n \end{pmatrix} + \frac{n\theta^3}{(1 + \theta)^2[1 - \theta(n - 1)]} \begin{pmatrix} \frac{1}{1 - \theta(q-1)} \mathbf{1}_q \\ \mathbf{0}_n \end{pmatrix} \\ & = \mathbf{b}(\mathcal{g}_{qn}^C, \theta) + \frac{\theta}{(1 + \theta)[1 - \theta(n - 1)]} \mathbf{e}_1 \\ & \quad + \frac{\theta^2}{(1 + \theta)[1 - \theta(n - 1)]} \begin{pmatrix} \frac{1}{1 - \theta(q-1)} \mathbf{1}_q \\ \mathbf{0}_n \end{pmatrix} \end{aligned}$$

Denote the new link between 1 and m by $1m$. Then, the change in Katz–Bonacich centrality by adding the directed link $1m$ to G_{qn}^c is given by:

$$\mathbf{b}(g_{qn}^c + 1m, \theta) - \mathbf{b}(g_{qn}^c, \theta) = \frac{\theta}{(1+\theta)[1-\theta(n-1)]} \mathbf{e}_1 + \frac{\theta^2}{(1+\theta)[1-\theta(n-1)]} \begin{pmatrix} \frac{1}{1-\theta(q-1)} \mathbf{1}_q \\ \mathbf{0}_n \end{pmatrix}$$

As a result, for individual 1, we have

$$b_1(g_{qn}^c + 1m, \theta) - b_1(g_{qn}^c, \theta) = \frac{\theta}{(1+\theta)[1-\theta(n-1)]} + \frac{\theta^2}{(1+\theta)[1-\theta(n-1)]} \frac{1}{1-\theta(q-1)} = \frac{\theta[1-\theta(q-1)] + \theta^2}{(1+\theta)[1-\theta(n-1)][1-\theta(q-1)]}$$

Similarly, we can calculate:

$$\begin{aligned} & \mathbf{b}(g_{qn}^c + \mathbf{e}_1 \mathbf{e}_m^T, \theta) + \mathbf{b}(g_{qn}^c, \theta) \\ &= 2\mathbf{b}(g_{qn}^c, \theta) + \frac{\theta}{(1+\theta)[1-\theta(n-1)]} \mathbf{e}_1 \\ & \quad + \frac{\theta^2}{(1+\theta)[1-\theta(n-1)]} \begin{pmatrix} \frac{1}{1-\theta(q-1)} \mathbf{1}_q \\ \mathbf{0}_n \end{pmatrix} \\ &= 2 \begin{pmatrix} \frac{1}{1-\theta(q-1)} \mathbf{1}_q \\ \frac{1}{1-\theta(n-1)} \mathbf{1}_n \end{pmatrix} + \frac{\theta}{(1+\theta)[1-\theta(n-1)]} \mathbf{e}_1 \\ & \quad + \frac{\theta^2}{(1+\theta)[1-\theta(n-1)]} \begin{pmatrix} \frac{1}{1-\theta(q-1)} \mathbf{1}_q \\ \mathbf{0}_n \end{pmatrix} \end{aligned}$$

As a result, for individual 1:

$$b_1(g_{qn}^c + 1m, \theta) + b_1(g_{qn}^c, \theta) = \frac{2}{1-\theta(q-1)} + \frac{\theta[1-\theta(q-1)] + \theta^2}{(1+\theta)[1-\theta(n-1)][1-\theta(q-1)]}$$

Therefore, the gain for individual 1 of adding a bridge link $1m$ is:

$$\begin{aligned} & B_1^*(g_{qn}^c + 1m) - B_1^*(g_{qn}^c) \\ &= \frac{\alpha^2}{2} [b_1(g_{qn}^c + 1m, \theta)]^2 - \frac{1}{2} [b_1(g_{qn}^c, \theta)]^2 \\ &= \frac{\alpha^2}{2} [b_1(g_{qn}^c + 1m, \theta) - b_1(g_{qn}^c, \theta)] [b_1(g_{qn}^c + 1m, \theta) + b_1(g_{qn}^c, \theta)] \\ &= \frac{\alpha^2}{2} \left[\frac{\theta[1-\theta(q-1)] + \theta^2}{(1+\theta)[1-\theta(n-1)][1-\theta(q-1)]} \right] \\ & \quad \times \left[\frac{2}{1-\theta(q-1)} + \frac{\theta[1-\theta(q-1)] + \theta^2}{(1+\theta)[1-\theta(n-1)][1-\theta(q-1)]} \right] \end{aligned}$$

Since $q = n^W$ and $n = n^B$, we have:

$$B_1^*(g_{n^W n^B}^c + 1m) - B_1^*(g_{n^W n^B}^c) = \frac{\alpha^2(\theta[1-\theta(n^W-1)] + \theta^2)(2(1+\theta)[1-\theta(n^B-1)] + \theta[1-\theta(n^W-1)] + \theta)}{2(1+\theta)^2[1-\theta(n^B-1)]^2[1-\theta(n^W-1)]^2}$$

As a result, individual 1 will not form a bridge link between individual 1 (from the type- W network) and individual m if and only if

$$B_1^*(g_{n^W n^B}^c + 1m) - B_1^*(g_{n^W n^B}^c) < c + \frac{(n^B - 1)(n^W - 1)}{n^B n^W} C$$

which is equivalent to:

$$\frac{\alpha^2(\theta[1-\phi(n^W-1)] + \theta^2)(2(1+\theta)[1-\phi(n^B-1)] + \theta[1-\phi(n^W-1)] + \theta)}{2(1+\theta)^2[1-\phi(n^B-1)]^2[1-\phi(n^W-1)]^2} < c + \frac{(n^B - 1)(n^W - 1)}{n^B n^W} C \tag{42}$$

Thus if (41) and (42) hold, then the separated network is pairwise stable. \square

Proof of Proposition 13

Let us denote by $B_i^*(g)$, the positive part (benefits) of the equilibrium utility function (18), i.e. $B_i^*(g) \equiv \frac{\alpha^2}{2} [b_i(g, \theta)]^2$.

Let us show that the network described in Fig. 1 is an equilibrium network.

Let us start with link deletion:

• Any minority agent (individual 2 or 3) has no incentives to sever a link with individual 1 (type W) if

$$c < \frac{\alpha^2}{2} \left[\left(\frac{1+\theta}{1-2\theta^2} \right)^2 - 1 \right] \tag{43}$$

Indeed, the utility of individual 2 (or 3) before the link is severed is:

$$U_2^*(g^S) = \frac{\alpha^2}{2} [b_2(g^S, \theta)]^2 - c_{21}(g^S)$$

As a result, if $\theta < 1/\sqrt{2}$, we have:

$$\begin{pmatrix} b_1(g^S, \theta) \\ b_2(g^S, \theta) \\ b_3(g^S, \theta) \end{pmatrix} = \frac{1}{1-2\theta^2} \begin{pmatrix} 1+2\theta \\ 1+\theta \\ 1+\theta \end{pmatrix}$$

and thus,²¹ if $\theta < 1/\sqrt{2}$,

$$\begin{aligned} B_2^*(g^I) - B_2^*(g^S) &= \frac{\alpha^2}{2} \left[1 - \left(\frac{1+\theta}{1-2\theta^2} \right)^2 \right] \\ &= -\frac{\alpha^2 \theta (2+5\theta-4\theta^3)}{2(1-2\theta^2)^2} < 0 \end{aligned}$$

Let us now determine the gains in terms of costs. It is equal to: c , which is the cost for individual 2 or 3 of having a link with individual 1. As a result, if

$$\frac{\alpha^2}{2} \left[1 - \left(\frac{1+\theta}{1-2\theta^2} \right)^2 \right] + c < 0$$

then individual 2 or 3 will never delete a link with individual 1. This is condition (43).

• The majority agent (individual 1) will not sever a link with a minority agent (individual 2 or 3) if

$$c < \frac{\alpha^2}{2} \left[\left(\frac{1+\theta}{1-2\theta^2} \right)^2 - \frac{1}{(1-\theta)^2} \right] \tag{44}$$

Let us first calculate the loss of deleting such a link. We have seen that, before deleting this link, if $\theta < 1/\sqrt{2}$, we have:

$$\begin{pmatrix} b_1(g^S, \theta) \\ b_2(g^S, \theta) \\ b_3(g^S, \theta) \end{pmatrix} = \frac{1}{1-2\theta^2} \begin{pmatrix} 1+2\theta \\ 1+\theta \\ 1+\theta \end{pmatrix}$$

After deleting the link (for example link 12), the network is a dyad with one link 13, and, if $\theta < 1$,

$$\begin{pmatrix} b_1(g^S, \theta) \\ b_2(g^S, \theta) \end{pmatrix} = \frac{1}{1-\theta} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

As a result, if $\theta < 1/\sqrt{2}$, then²²

$$\begin{aligned} B_1^*(g^D) - B_1^*(g^S) &= \frac{\alpha^2}{2} \left[\frac{1}{(1-\theta)^2} - \left(\frac{1+\theta}{1-2\theta^2} \right)^2 \right] \\ &= -\frac{\theta^2(2-3\theta^2)}{(1-\theta-2\theta^2+2\theta^3)^2} < 0 \end{aligned}$$

²¹ The superscript S denotes the star network while the superscript I denotes the isolated (empty) network.

²² The superscript D denotes the dyad network, i.e. the complete network with two agents.

Let us now determine the gains in terms of costs. It is equal to: $2c - c = c$. As a result, if

$$\frac{\alpha^2}{2} \left[\frac{1}{(1-\theta)^2} - \left(\frac{1+\theta}{1-2\theta^2} \right)^2 \right] + c < 0$$

then individual 1 will never delete a link with individual 2 or 3. This is condition (44).

It is easily verified that, if $\theta < 1/\sqrt{2}$, conditions (43) and (44) reduce to:

$$c < \frac{\alpha^2}{2} \left[\left(\frac{1+\theta}{1-2\theta^2} \right)^2 - \frac{1}{(1-\theta)^2} \right] \quad (45)$$

since

$$\frac{\alpha^2}{2} \left[\left(\frac{1+\theta}{1-2\theta^2} \right)^2 - \frac{1}{(1-\theta)^2} \right] < \frac{\alpha^2}{2} \left[\left(\frac{1+\theta}{1-2\theta^2} \right)^2 - 1 \right]$$

Let us now consider *link creation*:

- The majority agent (individual 1) cannot create more links since she is already linked to all agents in the network.
- The minority agent (individual 2) will not create a link with the minority agent 3 if

$$c > \frac{\alpha^2}{2} \left[\frac{1}{(1-2\theta)^2} - \left(\frac{1+\theta}{1-2\theta^2} \right)^2 \right] \quad (46)$$

We have seen that, before the creation of the link 23, if $\theta < 1/\sqrt{2}$, we have:

$$\begin{pmatrix} b_1(g^S, \theta) \\ b_2(g^S, \theta) \\ b_3(g^S, \theta) \end{pmatrix} = \frac{1}{1-2\theta^2} \begin{pmatrix} 2\theta+1 \\ \theta+1 \\ \theta+1 \end{pmatrix}$$

If the link 23 is created so that the network is complete, then, if $\theta < 1/2$,

$$\begin{pmatrix} b_1(g^C, \theta) \\ b_2(g^C, \theta) \\ b_3(g^C, \theta) \end{pmatrix} = \frac{1}{1-2\theta} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

As a result, if $\theta < 1/2$, the gain of creating the link 23 is equal to (for agent 2 or 3):

$$\begin{aligned} B_2^*(g^C) - B_2^*(g^S) &= \frac{\alpha^2}{2} \left[\frac{1}{(1-2\theta)^2} - \left(\frac{1+\theta}{1-2\theta^2} \right)^2 \right] \\ &= \frac{\alpha^2}{2} \frac{\theta(2-\theta-4\theta^2)}{(1-2\theta-2\theta^2+4\theta^3)^2} > 0 \end{aligned}$$

When the link 23 is created, the cost increases by c since the total cost for individual 3 before creating the link 23 is: c while the total cost for individual 3 after creating the link 23 is: $2c$. As a result, individual 2 will never create a link with individual 3 if

$$\frac{\alpha^2}{2} \left[\frac{1}{1-2\theta} - \frac{1+\theta}{1-2\theta^2} \right] - c < 0$$

This is condition (46).

We are thus left with two conditions (45) and (46). Combining them gives

$$\frac{1}{(1-2\theta)^2} - \left(\frac{1+\theta}{1-2\theta^2} \right)^2 < \frac{2c}{\alpha^2} < \left(\frac{1+\theta}{1-2\theta^2} \right)^2 - \frac{1}{(1-\theta)^2} \quad (47)$$

If $\theta < 1/\sqrt{2}$, it is easily verified that

$$\frac{1}{(1-2\theta)^2} - \left(\frac{1+\theta}{1-2\theta^2} \right)^2 < \left(\frac{1+\theta}{1-2\theta^2} \right)^2 - \frac{1}{(1-\theta)^2}$$

By simplifying (47), we obtain condition (23). \square

Proof of Proposition 14: Let us denote by $B_i^*(g)$, the positive part (benefits) of the equilibrium utility function (18), i.e. $B_i^*(g) \equiv \frac{\alpha^2}{2} [b_i(g, \theta)]^2$. Since we have a complete network, we can only consider the deviations due to link deletion and not link creation (nobody can form new links). Let us show that the network described in Fig. 2 is an equilibrium network.

• Any minority agent (take individual 2 without loss of generality) has no incentive to sever a link with individual 2 (type B) if

$$c < \frac{\alpha^2}{2} \left[\frac{1}{(1-2\theta)^2} - \left(\frac{1+\theta}{1-2\theta^2} \right)^2 \right] \quad (48)$$

Indeed, before the link 23 is severed, the network is complete, and, if $\theta < 1/2$,

$$\begin{pmatrix} b_1(g^C, \theta) \\ b_2(g^C, \theta) \\ b_3(g^C, \theta) \end{pmatrix} = \frac{1}{1-2\theta} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

When the link 23 is severed, the network is a star, and, if $\theta < 1/\sqrt{2}$, we have:

$$\begin{pmatrix} b_1(g^S, \theta) \\ b_2(g^S, \theta) \\ b_3(g^S, \theta) \end{pmatrix} = \frac{1}{1-2\theta^2} \begin{pmatrix} 1+2\theta \\ 1+\theta \\ 1+\theta \end{pmatrix}$$

Thus, if $\theta < 1/\sqrt{2}$,

$$\begin{aligned} B_2^*(g^S) - B_2^*(g^C) &= \frac{\alpha^2}{2} \left[\left(\frac{1+\theta}{1-2\theta^2} \right)^2 - \frac{1}{(1-2\theta)^2} \right] \\ &= -\frac{\theta(2-\theta-4\theta^2)}{(1-2\theta-2\theta^2+4\theta^3)^2} < 0 \end{aligned}$$

Let us now determine the gains in terms of costs. It is clearly equal to c . As a result, if

$$\frac{\alpha^2}{2} \left[\left(\frac{1+\theta}{1-2\theta^2} \right)^2 - \frac{1}{(1-2\theta)^2} \right] + c < 0$$

then individual 2 or 3 will never delete a link with individual 1. This is condition (48).

• Any minority agent (take individual 2 without loss of generality) has no incentive to sever a link with individual 1 (type W) if condition (48) holds. Indeed, before the link 12 is severed, the network is complete, and, if $\theta < 1/2$,

$$\begin{pmatrix} b_1(g^C, \theta) \\ b_2(g^C, \theta) \\ b_3(g^C, \theta) \end{pmatrix} = \frac{1}{1-2\theta} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

When the link 12 is severed, we have star but it is individual 3 who is the star. If $\theta < 1/\sqrt{2}$,

$$\begin{pmatrix} b_1^S(g, \theta) \\ b_2^S(g, \theta) \\ b_3^S(g, \theta) \end{pmatrix} = \frac{1}{1-2\theta^2} \begin{pmatrix} 1+\theta \\ 1+\theta \\ 1+2\theta \end{pmatrix}$$

Therefore,

$$\begin{aligned} B_2^*(g^S) - B_2^*(g^C) &= \frac{\alpha^2}{2} \left[\left(\frac{1+\theta}{1-2\theta^2} \right)^2 - \frac{1}{(1-2\theta)^2} \right] \\ &= -\frac{\theta(2-\theta-4\theta^2)}{(1-2\theta-2\theta^2+4\theta^3)^2} < 0 \end{aligned}$$

Let is now determine the gains in terms of costs. It is clearly equal to c . As a result, if

$$\frac{\alpha^2}{2} \left[\left(\frac{1+\theta}{1-2\theta^2} \right)^2 - \frac{1}{(1-2\theta)^2} \right] + c < 0$$

then individual 2 or 3 will never delete a link with individual 1. This is condition (48).

• The majority agent (individual 1) will not sever a link with a minority agent (individual 2 or 3) if condition (48) holds. The intuition is as before since, when the link 12 is severed, there is a star network where individual 1 is a peripheral agent. So the loss is as before and the gain is still c .

Thus condition (48) guarantees that the complete network of Fig. 2 is pairwise stable. By developing (48), we obtain (24). \square

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