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# STOCHASTIC BEHAVIOUR OF DETERMINISTIC SYSTEMS

by

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**1.** In the traditional use of mathematical models showing what is vaguely called chaotic behaviour, the number of variables is very large. This may refer to turbulence in aerodynamics or to economic models. The very influencial discovery of E. N. Lorenz is that this behaviour is not necessarily caused by the size of the system. Very erratic and unstable behaviour can be observed already in small systems. In his case the system was a 3-variable differential equation and similar constructions can also be made in two variables.

At this point the subject became accessible for mathematical analysis and the purpose if this report is to describe some of the mechanism and definitions which have emerged with focus on some aspects in which I have been personally involved.

In most applications one is interested in the long time behaviour of solutions of

$$\frac{dx}{dt} = F(x), \quad x(0) = x_0,$$

where $x = (x_1, \dots, x_d)$ is a point in $\mathbf{R}^d$. The erratic behaviour of $x(t), t \to \infty$, is related to the *non-linearity* of $F(x)$. If we denote $x(1) = x_1$, the equation gives a mapping $T : x_0 \to x_1$, and the value of $x$ at time $n, x_n$, is obtained by iterating this mapping $T\,n$ times: $x_n = T^n(x_0)$. This is a more convenient frame for a mathematical discussion and we shall here only discuss such iterations.

**2. The one-variable case.** It turns out that many characteristic phenomena occur already for mappings $T$ of an interval, $(-1, 1)$ say, into itself. Characteristic for applications is that $T$ also depends on one (ore several) parameters $a$. As a model case one can consider

$$T(x) = 1 - ax^2, \quad 0 < a \leq 2. \tag{1}$$

Of special importance is

$$\frac{dT^n(x_0)}{dx} = \prod_0^{n-1} T'(x_j) \left( = \prod_0^{n-1} (-2ax_j) \right). \tag{2}$$

We shall use $D_n = |T^{n'}(x_0)|$. The chaotic behaviour is related to the growth of $D_n$. Clearly, this expresses the *sensitivity of initial conditions* since by the meanvalue theorem for two orbits:

$$|x_n - x_n'| = D_n(\xi)|x_0 - x_0'|, \quad x_0 \leq \xi \leq x_0'.$$

It is interesting to observe that the same sensitivity automatically occurs in the dependence on the parameter $a$.

THEOREM 1. *Suppose* $|\frac{\partial T}{\partial a}| \leq B$ *and* $|\frac{\partial T^n}{\partial x}| \geq M_n(x)$. *Set*

$$Q_n = \left| \frac{\frac{\partial T^n}{\partial x}}{\frac{\partial T^n}{\partial a}} \right|.$$

*If for some* $N$

$$Q_N > 2B \sum_N^\infty \frac{1}{M_n}$$

*then for* $n > N$

$$\frac{1}{2} Q_N < Q_n < \frac{3}{2} Q_N.$$

In the computer simulation of (1) the behaviour of the system for $a > a_0 \sim 1.4$ shows chaotic regions mixed with small intervals $I_j$ so that for $a \in I_j$ the system has an attractive cycle $x_0, x_1 \dots, x_m = x_0$, so that

$$\left| T^{m'}(x_0) \right| < 1.$$

These intervals become smaller and smaller as $a \to 2$ and it seems that "chaotic" behaviour would more and more be the rule. We first need to make the concepts precise.

**3. Invariant measure.** Let $T$ be a mapping of some space $S$ into itself. A positive measure $\mu$ on $S$ is invariant under $T$ if for measurable sets $E, \mu(T^{-1}(E)) = \mu(E)$. Invariant measures exist very generally. Take any point $x_0$ and consider the orbit $x_\nu = T^\nu(x_0), \nu = 1, 2, \dots$. Let $\mu_n$ be the measure obtained by putting the mass $\frac{1}{n+1}$ at $x_\nu, \nu = 0, \dots, n$. Then if for all $\nu, \mu, x_\nu \neq x_\mu$, i.e. if we don't have a cycle

$$\left| \mu_n(T^{-1}(E) - \mu_n(E) \right| \leq \frac{2}{n}$$

and if we can choose convergent subsequences, as we can in our applications, we obtain an invariant limit.

Intuitively, our dynamical system would be considered chaotic if an invariant measure charges "many" points. If we e.g. have an attractive cycle of length $N, \mu$ would be a pointmass $1/N$ at each point of the cycle. We can measure disorder in three ways.

**I. Entropy.** If $X$ is a stochastic variable taking a finite number of values with probabilities $p_1 \dots p_N$, we call

$$H = -\sum_1^N p_\nu \log p_\nu$$

the entropy of the stochastic variable. It measures the *uncertainty* i.e. the amount of information we get on the average by making an observation.

For a continuous variable we can divide the space into intervals $I_j$ of length $\leq \varepsilon$

$$-\sum \mu(I_j) \log \mu(I_j) = H_\varepsilon.$$

The intervals should also be adapted to the dynamical system so we consider

$$\{I_j\} = \bigcup_{\nu=1}^{n} \{T^{-\nu}(J_1, \ldots, J_p)\}$$

for fixed intervals $J$. The corresponding entropy is $H_n$ and

$$\lim \frac{H_n}{n} = H[J].$$

exists. We can now take

$$H = \sup_{[J]} H[J]$$

and this is the entropy of $\mu$.

**II. Liaponov exponents.** If $\mu$ is invariant as above one can prove that

$$\lambda(x) = \lim_{n=\infty} \frac{\log D_n(x)}{n}$$

exists a.e $(\mu)$. $\lambda(x) > 0$ is obviously an indication of chaotic behaviour of the system.

**III. Absolute continuity of $\mu$.** An extreme case of order is a finite attractive cycle. Disorder is related to the orbits being distributed on a large set. A natural way of describing this is to say that $d\mu = \theta(x)dx$, $\theta(x)$ integrable with respect to ordinary Lebesgue measure.

There is a beautiful connection between the concepts above.

**A.** $H \leq \int \max(\lambda, 0)\, d\mu(x)$ always.

**B.** $\mu$ is absolutely continuous iff there is equality in **A.**

A natural definition is now: $T$ is chaotic if there is an invariant measure which is absolutely continuous. In the example (1) it is unique but for higher dimensional cases, the uniqueness is not satisfactorily known.

**Parameter dependence — a model.**

In physical or economic problems the models depend naturally on usually many parameters such as gravitation constants or viscosity. As mentioned earlier the behaviour of orbits is sensitively depending on these parameters in the chaotic case. One can set up a stochastic model which captures the essential features of the system even when the model is not uniformly expanding, i.e $|T^{n'}(x)| \geq c > 1$ for some $n$ does not hold. A typical such example is (1) above.

**Stochastic model — 1 variable.**

Let $D_\nu(\omega)$, $D_0 = 1$, be random variables and $\nu_i(\omega)$ stopping times $\nu_1 \equiv 1$. If $n = \nu_i$ is a stopping time we choose $t_i$ at random, $0 < t_i < 1$, with uniform distribution and define

$$D_{n+j} = D_{n-1}(2at_i) \cdot D_j, j = 0, 1, \ldots, k \tag{3}$$

where $k$ is the smallest integer for which

$$D_k t_i^2 > 1$$

and we set $\nu_{i+1} = \nu_i + k$. One can prove the following

THEOREM 2. *If $a > \frac{1}{2}e$ then with positive probability*

$$D_n \geq e^{\lambda n}, \quad n = 0, 1, \ldots$$

*for some* $\lambda > 0$.

**4.** To understand how this relates to (1) we first note that in order to prove any of **I, II, III** above it is sufficient that $|T^{n'}(1)| \geq c^n > 1, 0$ being the (only critical) point where $T'(x) = 0$ and $1 = T(0)$. If $T^n(0) = \xi_n$ we have

$$\left|T^{n'}(1)\right| = D_n^* = \prod_1^{n-1} 2a|\xi_\nu(a)|$$

and the formula (2) means that after a return at $\xi_\mu$ very close to zero (at distance $t_i$) we repeat the original orbit $\xi_1, \ldots, \xi_\nu$ as long as

$$\left|\xi_\mu - \xi_{j+\mu}\right| \sim D_j^* t_i^2 << 1.$$

The probability space is simply the parameters $a$.

This can in fact be made completely rigorous and we have

THEOREM 3. *(1) has chaotic behaviour for a set of positive measure of parameters a.*

The same argument works for any family of functions $f(x, a)$ instead of $1 - ax^2$ with a finite number of extrem points.

**5. $n$-dimensional case.**

5.1. For an understanding of the degree of complexity that arises in higher-dimensional systems let us first recall two examples:

1. Consider a mapping defined in a dough-nut shaped domain $D$ in $\mathbb{R}^3$ which twists the domain by streching so that it makes two loops inside $D$ without intersecting ifself. $D_1 = T(D) \subset D$ and $T$ is invertible. When $T$ is iterated we obtain $D_n = T(D_{n-1}) \subset D_{n-1}$ and $D_n$ converges to complicated Cantorlike set $A$. This $A$ is a *stange attractor*. $A$ has a neighborhood $\Delta$ so that $T^n(\Delta) \to A$. This is an example of what is called Anasov maps which are characterized by the uniform streching in certain directions and uniform contractions in other directions . These maps have the important property of being *stable:* small changes in the maps do not change the general behavior.

2. Consider now the map

$$x' = 1 - ax^2 + y$$

$$y' = bx \tag{3}$$

$a_0 < a < 2, 0 < b < b_0$. For this map $T$ (the Hénon map) there is also a domain $D$ with $T(D) \subset D$. However for certain parameter values strange attractors emerge in computer simulations, for other values one obtains a finite attractive cycle. These maps are not uniformly expanding/contracting and are not stable. This situation is in some sense more common.

5.2. **Invariant measures** We consider a smooth $1-1$ mapping of a bounded domain $D \subset \mathbb{R}^n$ into itself. We can repeat the construction in §2 and can obtain an invariant measure $\mu$.

When we want to study the expansion of the map we must now distinguish different directions $v$. We fix $x \in D$ and decompose $v = v_1 + \cdots + v_m$ into components. This decomposition can for almost all $x$ (relatively $\mu$) be chosen so that

$$\lim_{n \to \pm\infty} \frac{\log \|DT^n(x) v_i\|}{n} = \lambda_i(x)$$

exist. $\lambda_i$ are called Liaponov exponents. Call the dimensions of the corresponding subspaces $d_i(x)$ $\sum d_i = n$. Anasov mappings are characterized by the existence and continuity of the decomposition for all $x$.

Also in this case one can define an entropy $H$ and both results $A$ and $B$ in §2 hold

$$H \leq \int \sum d_\iota(x) \max(\lambda_\iota, (x)_0)\, d\mu(x)$$

and equality is equivalent to $\mu$ being absolutely continuous in the directions where $\lambda_i > 0$. This must be given a technical content which I omit here. For the Hénon map one can see these directions as the tangents of the curves which emerge in the simulations.

For higher dimensional systems one must expect a much more complicated pattern then we observed in 1 dimension, with possible coexistence of attractiv periodic orbits and strange attractors, all with different domains of attractor. The very existence of a strange attractors for a system such as the map (3) is in spite of the computer pictures not clear. The rest of this report will be devoted to a description of the dynamics of (3).

This mapping (3) has a fixed point in $x > 0, y > 0$ which is of saddle type, i.e. there is an expanding direction with eigenvalue $\lambda_1 \sim 2$ and one contracting with eigenvalue $\sim -\frac{b}{2}$. Tangent to these are the unstable and the stable manifolds $W^u$ resp $W^s$. What the computer picture shows are the leaves of $W^u$. For certain parameters values $(a, b)$ there is an attractive periodic orbit and all points are drawn to this (e.g. $a = 1.3$, $b = 0.3$).

The main problem about chaotic behaviour is to eliminate the posibility that the picture shows a very long periodic orbit combined with round off errors from the computation. One should not exaggerate the difference between these two cases from a practical point of view, but already a periodic point of cycle length 100 creates so much round off errors that it would not be detected by a normal computer study.

A *model* for mappings such as (4) can be defined in analogy with the model for 1 variable. The difference is that we have to study a family of functions $D_n(y; w)$ where $y$ belongs to a thin Cantor set $E$. At every stoppingtime (4) has to be replaced by

$$D_{n+j}(y; w) = D_{n-1}(y; w)\, 2at_i(y)\, D_j(z; w) \tag{4'}$$

for some choice of $z$ and

$$D_n(y) = D_m(y')$$

if $y, y' \in E$ and $D_m|y - y'| < 1$.

We dont give the details but it can also here be proved that the model has a uniform Liaponov exponent $\lambda$ with positive probability.

For the Hénon map itself it can be proved that a strange attractor exists for a set of positive measure $(a, b)$. What this means is that most orbits are dense on the unstable manifold $W^u$. The relation to the model is that $y$ stands for the leaves of $W^u$. On each leaf there is a "critical" point and $t$ in $(4')$ measures the distance to this critical point.

## 6. References.

For a general theory of dynamical systems, see [1], [2], [3]. For the 1-dimensional theory, see [4]. Theorem 1 is proved in [6]. For more on entropy see [7]. The general theory of I, II, III is based on [8] and proved in [9] (A) was proved by Margulis and Ruelle. For the stochastic model, see [10]. Theorem 3 is due to Jakobson [11], see also [5]. The Hénon map, see [6].

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