RESEARCH INSTITUTE OF
INDUSTRIAL ECONOMICS

IFN Working Paper No. 1327, 2020

Testing for Weak Separability and Utility Maximization with Incomplete Adjustment

Per Hjertstrand, James L. Swofford and
Gerald A. Whitney

Research Institute of Industrial Economics
P.O. Box 55665
SE-102 15 Stockholm, Sweden
info@ifn.se
www.ifn.se

Testing for Weak Separability and Utility Maximization with Incomplete Adjustment*

May 2023

Per Hjertstrand
Research Institute of Industrial Economics

James L. Swofford
Department of Economics and Finance, University of South Alabama

Gerald A. Whitney
Department of Economics and Finance, University of New Orleans

JEL Codes: C14; C60; D01; D10

Key Words: Aggregation; Incomplete Adjustment; Revealed Preference; Utility maximization;

ABSTRACT

We develop models for weakly separable utility maximization with incomplete adjustment. By allowing for incomplete adjustment, these models account for the decision maker's inability to instantaneously adjust to optimal allocations of demanded goods and assets. This is especially important when modelling preferences for durable goods and financial/monetary assets. We propose computationally attractive nonparametric revealed preference procedures to test the models using observed data on prices and quantities. An empirical application shows that it is important to account for incomplete adjustment in consumer demand models of durable consumption goods.

* Hjertstrand thank Torsten Söderbergs Stiftelse for financial support. Email: per.hjertstrand@ifn.se (P. Hjertstrand); jswoffor@southalabama.edu (J. Swofford); gwhitney@uno.edu (G. Whitney).

1. INTRODUCTION

That decision makers cannot instantaneously adjust to their optimal equilibrium demand for goods and assets has been recognized ever since Böhm-Bawerk (1888). In an equilibrium growth model, Kydland and Prescott (1982) argue theoretically and establish empirically that incomplete adjustment plays a vital role when modelling the aggregate economy. Swofford and Whitney (1988), Fleissig and Swofford (1996), Jones et al. (2005), Elger et al. (2008) and Jha and Longjam (2006) are examples of studies showing that various microeconomic decisions including those on capital, monetary and financial assets, and consumer durables are characterized by incomplete adjustment. Koyck (1954) and Almon (1965) developed distributed lag models to allow for incomplete adjustment in regression analysis. In light of this literature, incomplete adjustment can arise because of a number of factors, such as habit persistence (formation), adjustment costs, information asymmetries, the formation of expectations, or a combination of reasons.

Given the nature of durable goods and in keeping with the empirical evidence on durable goods in previous studies it seems natural to assume that it is primarily durable goods and services which provide a stream of utility over periods that may cause incomplete adjustment. However, this does not preclude nondurable goods and services to also cause incomplete adjustment. For example, consumers facing a price level shock may take more than one period to adjust their money balances.[1]

In this paper, we propose models of weak separability and utility maximization that allow for incomplete adjustment. We show how these models can be implemented and tested empirically using consumer choice data. Allowing for incomplete adjustment in consumer demand models

[1] To simplify the following exposition, we will from now on simply refer to the goods causing incomplete adjustment as “durable goods” and those goods not causing incomplete adjustment as “nondurable goods”.

may have important implications in several fields of economics. For example, it is common by assumption to omit durable goods in empirical demand analysis, implying that consumption of nondurable goods and services in these models are independent of how much the consumer spends on durables. In an empirical application, we find that this may be an invalid assumption using detailed panel data over Spanish households.

Utility maximization is a core concept in economics and an underlying assumption for all rational choice theory. Utility maximization is a necessary condition for weak separability, which is another key concept in economics. A group of goods is said to be weakly separable from all other goods if the quantities demanded of the other goods are independent of the marginal rates of substitution between any pair of goods in the separable group. This implies that the quantities demanded of the goods inside the separable block depend solely on the prices of the goods that form the block. In contrast, if a group of goods were not weakly separable, then it would be possible for the prices of all goods outside the block to enter into the demand functions for the group of goods being considered.

Utility maximization and weak separability can be tested using either parametric consumer demand models or nonparametric revealed preference methods.[2] As argued by Varian (1982, p.945) "[the parametric] procedure will be satisfactory only when the postulated parametric forms are good approximations to the 'true' demand functions. Since this hypothesis is not directly testable, it must be taken on faith." Thus, given that some functional form must be imposed in the parametric approach, any test of weak separability (utility maximization) may be regarded as a joint test of the hypothesis of weak separability (utility maximization) and the hypothesis that correct functional form and error structures have been employed.

[2] Some examples of parametric demand models are the translog (Christensen et al. 1975), almost ideal demand model (Deaton and Muellbauer, 1980), or quadratic almost ideal demand model (Banks et al., 1993).

Instead, we implement our models of weak separability and utility maximization with incomplete adjustment using nonparametric revealed preference methods. These methods build upon the empirical revealed preference theory for utility maximization and weak separability proposed by Afriat (1967,1969) and Varian (1983). These methods do not require any *ad hoc* assumptions regarding functional form and there is no parameter estimation. Thus, the hypothesis of weak separability (utility maximization) is not confounded in any subsidiary hypothesis from estimation. The main drawback of these methods is that they are, contrary to the parametric approach, non-stochastic and are therefore not tests in the "statistical sense".

The nonparametric revealed preference approach offers a natural way of allowing for incomplete adjustment since it can be modelled as a constraint on a good or a constraint placed upon expenditures on a particular subset of goods (as we do here). Since the nature of the constraint need not be specified in the revealed preference approach, no particular adjustment process needs to be specified a priori. In contrast, although incomplete adjustment in parametric models allows for additional flexibility, it must rest on some kind of parametric assumption as to how incomplete adjustment enters the estimating equations.[3]

Swofford and Whitney (1994) introduced a model of incomplete adjustment assuming that all goods causing incomplete adjustment (i.e., durables) exclusively form the weakly separable block of goods. Consequently, this assumption implies that all remaining goods (not causing incomplete adjustment, i.e., nondurables) exclusively form the nonseparable block of goods. However, this is a strong assumption since it does not allow to test for weak separability on hypothesized structures where durables and nondurables simultaneously enter the separable and nonseparable blocks of goods. In contrast, our models of incomplete adjustment relax this

[3] See e.g., Anderson and Blundell (1982) and Serletis (1991) for two early studies incorporating incomplete adjustment in parametric models.

assumption and does not place any restriction on how durables and nondurables enter the separable and nonseparable blocks.[4] This allows to test for weak separability with incomplete adjustment on any hypothesized structure.

Varian (1983) and Fleissig and Whitney (2003) proposed and implemented computationally efficient nonparametric revealed preference procedures for weak separability that are sufficient but not necessary.[5] Cherchye, Demuynck, De Rock and Hjertstrand (2015) proposed and implemented a computationally efficient mixed integer linear programming procedure for weak separability that is both necessary and sufficient. However, none of these procedures allow for incomplete adjustment.[6]

Swofford and Whitney (1994) derived necessary and sufficient revealed preference restrictions for their restricted model of incomplete adjustment which come in the form of a set of nonlinear inequalities. They then propose a procedure to test for weak separability based on solving a nonlinear optimization problem where the nonlinear revealed preference restrictions act as constraints. There are n^2 nonlinear constraints in this problem, where n denotes the number of observations, implying that the number of nonlinear constraints grow quadratically with the number of observations. Hence, this can be a computationally difficult problem in practice even for data sets with rather few observations. Moreover, this nonlinear problem only gives locally optimal solutions of the minimal amount of incomplete adjustment required to rationalize the data (given that the problem has a feasible solution). However, since these may not be globally optimal there may exist other feasible solutions which produce lower levels of incomplete

[4] Our model nests Swofford and Whitney's (1994) model as a special case.
[5] Thus, if for some hypothesized structure sufficiency fails, the structure cannot be ruled out as being weakly separable, as some other sufficient condition might still hold.
[6] Crawford (2010) give necessary and sufficient revealed preference conditions for habit formation, i.e., when lags of the demanded quantities enter as arguments in the utility function. Our framework is conceptually different from the habits model since it is based on a notion of disequilibrium.

adjustment. Hence, any solution to Swofford and Whitney's (1994) nonlinear procedure gives an upper bound on the minimal amount of incomplete adjustment required to rationalize the data.

We derive necessary and sufficient revealed preference restrictions for our model of incomplete adjustment. Using insights from Cherchye et al. (2015), we show how these restrictions can be reformulated as a set of linear inequalities, allowing the restrictions to act as linear constraints in an optimization program. Moreover, we show how the minimal amount of incomplete adjustment can be calculated by minimizing a quadratic objective function. Combining this objective function with the linear constraints implied by the revealed preference restrictions give a mixed integer quadratic programming (MIQP) problem which solves for the minimal amount of incomplete adjustment such that the data can be rationalized by weak separability and incomplete adjustment. The problem is mixed integer because some variables in the problem only take binary values.

Compared to Cherchye et al.'s (2015) procedure with complete adjustment, our procedure is only marginally more computationally involved since it contains n additional parameters and is based on solving a quadratic objective function as opposed to a linear objective in Cherchye et al. (2015). In fact, Cherchye et al.'s (2015) problem and our problem share the same computational complexity since both belong to the np-complete class of problems.

Our procedure based on the solving a MIQP problem is computationally simpler to solve than Swofford and Whitney's (1994) nonlinear problem since it consists of linear constraints, and as such, can be applied to larger data sets. From a theoretical point of view, the key motivation for adopting an integer programming approach is that this is a widely accepted and well-known approach to handle np-complete problems.

As previously mentioned, in our empirical application, we address the question of whether nondurable goods and services can be modelled independently from durable goods and services. This is a common assumption in consumer demand modelling, and made to simplify the analysis since a consumer derives utility from the flow of services that the durables provide over time, but is usually never tested prior to the analysis. Thus, modelling durables is inheritably difficult without including some form of adjustment process in the model.

If expenditure on nondurables is independent of the amount consumed of durable goods and services, then it effectively imposes the restriction that nondurables are weakly separable from durables. Thus, the methods and models introduced in here allow us to test this restriction. For this purpose, we use the Encuesta Continua de Presupuestos Familiares (abbreviated ECPF) which is a panel survey data set ranging from 1985-1997 over disaggregated Spanish household expenditures on 25 durable and nondurable consumption goods and services (We use data on 1,585 households). This data set is ideal for our purposes because it is the only consumer survey data set that has a panel structure containing exhaustive and complete information on household expenditures over more than four quarters. Since this allow us to apply our test for weak separability with incomplete adjustment on data from each individual household, we avoid making any assumption of a representative agent. Moreover, by modelling every household individually, we avoid making any preference homogeneity assumption between households, which means that we can effectively account for any form of heterogeneity between households (even any form of unobserved heterogeneity). Thus, using the models and methods introduced here enables us to optimally exploit the panel structure of the ECPF.

We find that none of the households satisfy weak separability with complete adjustment, while only about 8% of the households satisfy weak separability with incomplete adjustment.

This indicates that the assumption that nondurables are weakly separable from durables is questionable. Hence, this suggests that models of consumption behavior should contain nondurables as well as durable goods and services and that these categories should be modelled in common since consumption on durables affect optimal expenditure on nondurables.

The paper is organized as follows. The next section introduces our theoretical models of weak separability and utility maximization with incomplete adjustment. In a parametric example, we also illustrate the effects of incomplete adjustment on expenditure and optimal choices. Section 3 derives nonparametric revealed preference restrictions for our models and discusses implementation issues and the computational complexity of the models. Using a parametric example, we also show the ability of our procedures to detect habit formation. Section 4 contains our empirical application and Section 5 concludes.

We include an online appendix, where we first recall the standard weak separability model with complete adjustment (Appendix A). Then, we provide a proof of our main theoretical result (Appendix B), show that our models are empirically refutable (Appendix C), and provide a generalization of our results to homothetic weak separability and homothetic utility maximization (Appendix D). Appendix E provides an additional application of our models and methods, where we derive economically valid monetary aggregates using data on financial and monetary assets, leisure and consumption expenditures from Hjertstrand et al. (2016).

2. INCOMPLETE ADJUSTMENT

2.1 Weak Separability with Incomplete Adjustment

Suppose that there are k goods and assets split into two mutually exclusive blocks.[7] Let $\boldsymbol{x}$ and $\boldsymbol{m}$ denote column vectors of the quantity data of the first and second blocks. Let $\boldsymbol{p}$ denote a row vector of the prices for the $\boldsymbol{x}$-goods and $\boldsymbol{r}$ denote a row vector of the prices for the $\boldsymbol{m}$-goods.

We assume each block may contain durable and nondurable goods and that the durable goods may be subject to incomplete adjustment. That is, we refer to the goods the decision maker (DM) may fail to fully adjust as durable goods. Let $\boldsymbol{x_D}$ and $\boldsymbol{x_{ND}}$ denote the durable and nondurable goods in $\boldsymbol{x}$. Let $\boldsymbol{p_D}$ and $\boldsymbol{p_{ND}}$ be the prices of $\boldsymbol{x_D}$ and $\boldsymbol{x_{ND}}$. Analogously, let $\boldsymbol{m_D}$ and $\boldsymbol{m_{ND}}$ denote the durable and nondurable goods in $\boldsymbol{m}$, and let $\boldsymbol{r_D}$ and $\boldsymbol{r_{ND}}$ be the prices of $\boldsymbol{m_D}$ and $\boldsymbol{m_{ND}}$.

Suppose there are n observations on the prices and quantities. The i^{th} observation on the prices and quantities is denoted $\boldsymbol{p^i} = (\boldsymbol{p^i_D}, \boldsymbol{p^i_{ND}})$, $\boldsymbol{r^i} = (\boldsymbol{r^i_D}, \boldsymbol{r^i_{ND}})$, $\boldsymbol{x^i} = (\boldsymbol{x^i_D}, \boldsymbol{x^i_{ND}})$ and $\boldsymbol{m^i} = (\boldsymbol{m^i_D}, \boldsymbol{m^i_{ND}})$ for $i = 1, \ldots, n$. The DM's utility function, $u(\boldsymbol{x}, \boldsymbol{m}) = u(\boldsymbol{x_D}, \boldsymbol{x_{ND}}, \boldsymbol{m_D}, \boldsymbol{m_{ND}})$, is weakly separable in the $\boldsymbol{m}$ goods if there exists a function U and a sub-utility function V so u can be written as $u(\boldsymbol{x_D}, \boldsymbol{x_{ND}}, \boldsymbol{m_D}, \boldsymbol{m_{ND}}) = U(\boldsymbol{x_D}, \boldsymbol{x_{ND}}, V(\boldsymbol{m_D}, \boldsymbol{m_{ND}}))$. This specification allows for durable and nondurable goods in both the weakly separable block, $\boldsymbol{m}$, and in other goods, $\boldsymbol{x}$.

Weak separability with incomplete adjustment requires that at each observation the DM solves an overall utility maximization problem and a sub-utility maximization problem involving a sub-set of the goods. Thus, the DM solves the overall utility maximization problem:

$$\max_{\{\boldsymbol{x_D}, \boldsymbol{x_{ND}}, \boldsymbol{m_D}, \boldsymbol{m_{ND}}\}} U(\boldsymbol{x_D}, \boldsymbol{x_{ND}}, V(\boldsymbol{m_D}, \boldsymbol{m_{ND}})), \tag{1}$$

subject to the two budget constraints which are defined in (2) and (3). The first constraint is the standard budget constraint restricting total outlay for all goods:

$$\boldsymbol{p}^i_D \boldsymbol{x_D} + \boldsymbol{p}^i_{ND} \boldsymbol{x_{ND}} + \boldsymbol{r}^i_D \boldsymbol{m_D} + \boldsymbol{r}^i_{ND} \boldsymbol{m_{ND}} \leq Y^i, \tag{2}$$

[7] For compactness, we refer to all goods and assets simply as "goods", and the two types of goods as nondurable and durable goods, respectively.

where Y^i is total expenditure on all goods. The second restriction imposes a constraint on expenditure for all durable goods:

$$\boldsymbol{p}_D^i \boldsymbol{x_D} + \boldsymbol{r}_D^i \boldsymbol{m_D} \leq Y_D^i, \tag{3}$$

where Y_D^i denotes total expenditure on the durable goods $(\boldsymbol{x_D}, \boldsymbol{m_D})$.

The sub-utility maximization problem solved by the DM is given by:

$$\max_{\{\boldsymbol{m_D}, \boldsymbol{m_{ND}}\}} V(\boldsymbol{m_D}, \boldsymbol{m_{ND}}), \tag{4}$$

subject to the two budget constraints defined in (5) and (6). The first constraint puts a restriction on total outlay for all separable goods $(\boldsymbol{m})$:

$$\boldsymbol{r}_D^i \boldsymbol{m_D} + \boldsymbol{r}_{ND}^i \boldsymbol{m_{ND}} \leq E^i, \tag{5}$$

where E^i is total expenditure on the separable goods. The second restriction imposes a constraint on expenditure for all durable goods in the separable block:

$$\boldsymbol{r}_D^i \boldsymbol{m_D} \leq E_D^i, \tag{6}$$

where E_D^i denotes total expenditure on the durable goods $(\boldsymbol{m_D})$.

The DM's Lagrangian based on (1)-(3) is:

$$L_U = U\big(\boldsymbol{x_D}, \boldsymbol{x_{ND}}, V(\boldsymbol{m_D}, \boldsymbol{m_{ND}})\big) + \tau^i\big[Y^i - \boldsymbol{p}_D^i \boldsymbol{x_D} - \boldsymbol{p}_{ND}^i \boldsymbol{x_{ND}} - \boldsymbol{r}_D^i \boldsymbol{m_D} - \boldsymbol{r}_{ND}^i \boldsymbol{m_{ND}}\big]$$

$$+\Omega^i\big[Y_D^i - \boldsymbol{p}_D^i \boldsymbol{x_D} - \boldsymbol{r}_D^i \boldsymbol{m_D}\big]$$

$$= U\big(\boldsymbol{x_D}, \boldsymbol{x_{ND}}, V(\boldsymbol{m_D}, \boldsymbol{m_{ND}})\big) + \tau^i\big[Y^i - \boldsymbol{p}_D^i \boldsymbol{x_D} - \boldsymbol{p}_{ND}^i \boldsymbol{x_{ND}} - \boldsymbol{r}_D^i \boldsymbol{m_D} - \boldsymbol{r}_{ND}^i \boldsymbol{m_{ND}}\big]$$

$$+\tau^i \left(\frac{\Omega^i}{\tau^i}\right)\big[Y_D^i - \boldsymbol{p}_D^i \boldsymbol{x_D} - \boldsymbol{r}_D^i \boldsymbol{m_D}\big] \tag{7}$$

where $\tau^i = \partial U / \partial Y^i$ and $\Omega^i = \partial U / \partial Y_D^i$ are the Lagrange multipliers. If Ω^i is negative then expenditure on the durable goods is greater than desired and if Ω^i is positive, expenditure on the durable goods is less than desired.

Thus, Ω^i may be viewed as a measure of the deviation from the optimal level of expenditure on durables. However, this number is not invariant to monotonic transformations and may therefore be difficult to interpret. But normalizing Ω^i with respect to marginal utility of total expenditure for all goods, τ^i, give a number that is easier to interpret. This normalized number Ω^i/τ^i represents the increment of overall utility from spending an additional dollar on the durable goods $\boldsymbol{x_D}$ and $\boldsymbol{m_D}$ relative to the marginal utility of total expenditure for all goods. Since Ω^i/τ^i is a ratio of marginal utilities, it is ordinal by construction and invariant to any monotonic transformation. Hence, we interpret Ω^i/τ^i as overall amount of incomplete adjustment, and define:

$$\mathrm{IA}_U^i = \frac{\partial U\big(\boldsymbol{x_D}, \boldsymbol{x_{ND}}, V(\boldsymbol{m_D}, \boldsymbol{m_{ND}})\big)/\partial Y_D^i}{\partial U\big(\boldsymbol{x_D}, \boldsymbol{x_{ND}}, V(\boldsymbol{m_D}, \boldsymbol{m_{ND}})\big)/\partial Y^i} = \frac{\Omega^i}{\tau^i}, \tag{8}$$

for all observations $i = 1, \dots, n$.

Using this definition, the first-order conditions of the Lagrangian, L_U, in (7) are:

$$\frac{\partial U}{\partial \boldsymbol{x_{ND}^i}} = \tau^i \boldsymbol{p}_{ND}^i, \tag{9}$$

$$\frac{\partial U}{\partial \boldsymbol{x_D^i}} = \tau^i \left(1 + \frac{\Omega^i}{\tau^i}\right) \boldsymbol{p}_D^i = \tau^i \big(1 + \mathrm{IA}_U^i\big) \boldsymbol{p}_D^i, \tag{10}$$

$$\frac{\partial U}{\partial \boldsymbol{m_{ND}^i}} = \frac{\partial U}{\partial V} \frac{\partial V}{\partial \boldsymbol{m_{ND}^i}} = \tau^i \boldsymbol{r}_{ND}^i, \tag{11}$$

$$\frac{\partial U}{\partial \boldsymbol{m_D^i}} = \frac{\partial U}{\partial V} \frac{\partial V}{\partial \boldsymbol{m_D^i}} = \tau^i \left(1 + \frac{\Omega^i}{\tau^i}\right) \boldsymbol{r}_D^i = \tau^i \big(1 + \mathrm{IA}_U^i\big) \boldsymbol{r}_D^i, \tag{12}$$

The Lagrangian of the sub-utility maximization problem in (4)-(6) is:

$$L_V = V(\boldsymbol{m_D}, \boldsymbol{m_{ND}}) + \mu^i \big[E^i - \boldsymbol{r}_D^i \boldsymbol{m_D} - \boldsymbol{r}_{ND}^i \boldsymbol{m_{ND}}\big] + \Theta^i \big[E_D^i - \boldsymbol{r}_D^i \boldsymbol{m_D}\big]$$

$$= V(\boldsymbol{m_D}, \boldsymbol{m_{ND}}) + \mu^i \big[E^i - \boldsymbol{r}_D^i \boldsymbol{m_D} - \boldsymbol{r}_{ND}^i \boldsymbol{m_{ND}}\big] + \mu^i \left(\frac{\Theta^i}{\mu^i}\right) \big[E_D^i - \boldsymbol{r}_D^i \boldsymbol{m_D}\big], \tag{13}$$

where $\mu^i = \partial V/\partial E^i$ and $\Theta^i = \partial V/\partial E_D^i$ are Lagrange multipliers. Thus, Θ/μ represents the increment of sub-group utility from spending an additional dollar on the durable goods in the sub-group relative to the marginal utility of total expenditure of the goods in the sub-group. Hence, we interpret Θ/μ as the amount of incomplete adjustment for the sub-group, and define:

$$\mathrm{IA}_V^i = \frac{\partial V(\boldsymbol{m_D}, \boldsymbol{m_{ND}})/\partial E_D^i}{\partial V(\boldsymbol{m_D}, \boldsymbol{m_{ND}})/\partial E^i} = \frac{\Theta^i}{\mu^i}, \tag{14}$$

for all observations $i = 1, \dots, n$.

The first-order conditions of the Lagrangian, L_V, in (13) are:

$$\frac{\partial V}{\partial \boldsymbol{m_{ND}^i}} = \mu^i \boldsymbol{r}_{ND}^i, \tag{15}$$

$$\frac{\partial V}{\partial \boldsymbol{m_D^i}} = \mu^i \left(1 + \frac{\Theta^i}{\mu^i}\right) \boldsymbol{r}_D^i = \mu^i \left(1 + \mathrm{IA}_V^i\right) \boldsymbol{r}_D^i. \tag{16}$$

The first-order condition (12) of L_U give the prices of the durable goods in the separable block, $\boldsymbol{r_D}$, at which the optimal quantities, $\boldsymbol{m_D}$ are demanded in terms of the overall utility maximization problem. Analogously, (16) gives a set of equivalent conditions in terms of the sub-utility maximization problem. These prices, denoted by $\left(1 + \mathrm{IA}_U^i\right)\boldsymbol{r}_D^i$ in (12) and $\left(1 + \mathrm{IA}_V^i\right)\boldsymbol{r}_D^i$ in (16) are "virtual prices" of the constrained goods $\boldsymbol{m_D}$, and give the prices at which the bundle $\boldsymbol{m_D}$ is demanded in equilibrium. Clearly, these virtual prices must be the same, implying that $\left(1 + \mathrm{IA}_U^i\right)\boldsymbol{r}_D^i = \left(1 + \mathrm{IA}_V^i\right)\boldsymbol{r}_D^i$. Thus, $\mathrm{IA}_U^i = \mathrm{IA}_V^i$, and we define the virtual prices as $\tilde{\boldsymbol{r}}_D^i = \left(1 + \mathrm{IA}_U^i\right)\boldsymbol{r}_D^i = \left(1 + \mathrm{IA}_V^i\right)\boldsymbol{r}_D^i = \left(1 + \mathrm{IA}^i\right)\boldsymbol{r}_D^i$, where $\mathrm{IA}^i = \mathrm{IA}_U^i = \mathrm{IA}_V^i$.

Analogously, the virtual prices for the durable goods in the non-separable block, $\tilde{\boldsymbol{p}}_D^i = \left(1 + \mathrm{IA}_U^i\right)\boldsymbol{p}_D^i = \left(1 + \mathrm{IA}^i\right)\boldsymbol{p}_D^i$ in the first-order condition (10) of L_U, give the prices, $\tilde{\boldsymbol{p}}_D^i$, at which the optimal bundle $\boldsymbol{x_D}$ is demanded in equilibrium.

We close this section with some remarks. First, given the virtual prices $\tilde{\boldsymbol{p}}_D^i$ and $\tilde{\boldsymbol{r}}_D^i$ and if we consider the first-order conditions (9)-(12) and (15)-(16), by substituting (15) into (11), and (16) into (12) we have:

$$\frac{\partial U\big(\boldsymbol{x_D}, \boldsymbol{x_{ND}}, V(\boldsymbol{m_D}, \boldsymbol{m_{ND}})\big)}{\partial V\big(\boldsymbol{m_D^i}, \boldsymbol{m_{ND}^i}\big)} = \frac{\tau^i}{\mu^i}. \tag{17}$$

Second, our model of incomplete adjustment defined by the overall utility and sub-utility maximization problems in (1)-(3) and (4)-(6) is identical to a model that solves the following overall utility maximization problem:

$$\max_{\{\boldsymbol{x_D}, \boldsymbol{x_{ND}}, \boldsymbol{m_D}, \boldsymbol{m_{ND}}\}} U\big(\boldsymbol{x_D}, \boldsymbol{x_{ND}}, V(\boldsymbol{m_D}, \boldsymbol{m_{ND}})\big) \quad \textit{subject to}$$

$$\tilde{\boldsymbol{p}}_D^i \boldsymbol{x_D} + \boldsymbol{p}_{ND}^i \boldsymbol{x_{ND}} + \tilde{\boldsymbol{r}}_D^i \boldsymbol{m_D} + \boldsymbol{r}_{ND}^i \boldsymbol{m_{ND}} \leq \tilde{Y}^i, \tag{18}$$

and the following sub-utility maximization problem

$$\max_{\{\boldsymbol{m_D}, \boldsymbol{m_{ND}}\}} V(\boldsymbol{m_D}, \boldsymbol{m_{ND}}) \quad \textit{subject to}$$

$$\tilde{\boldsymbol{r}}_D^i \boldsymbol{m_D} + \boldsymbol{r}_{ND}^i \boldsymbol{m_{ND}} \leq \tilde{E}^i. \tag{19}$$

This model is simply the standard weakly separable utility maximization model with the usual prices on the durable goods replaced by the virtual prices, and where total outlay for all goods and the goods in the sub-group are restricted by the "virtual expenditures" $\tilde{Y}^i = \tilde{\boldsymbol{p}}_D^i \boldsymbol{x_D^i} + \boldsymbol{p}_{ND}^i \boldsymbol{x_{ND}^i} + \tilde{\boldsymbol{r}}_D^i \boldsymbol{m_D^i} + \boldsymbol{r}_{ND}^i \boldsymbol{m_{ND}^i}$ and $\tilde{E}^i = \boldsymbol{r}_{ND}^i \boldsymbol{m_{ND}^i} + \tilde{\boldsymbol{r}}_D^i \boldsymbol{m_D^i}$.

Third, the first-order conditions allow us to define and classify weak separability with complete or incomplete adjustment: We say that the $\boldsymbol{m}$-goods are weakly separable with *complete adjustment* if the first-order conditions (9)-(12) and (15)-(16) hold with $\mathrm{IA}^i = 0$ for all $i = 1, \ldots, n$. In this case, they reduce to the first-order conditions from the standard weakly separable utility maximization model.[8] The $\boldsymbol{m}$-goods are said to be weakly separable with *incomplete adjustment* if the first-order conditions hold with $\mathrm{IA}^i \neq 0$ for some $i = 1, \dots, n$ ($\mathrm{IA}^i > -1$). The $\boldsymbol{m}$-goods are *not* weakly separable if the first-order conditions fail to hold for any value $\mathrm{IA}^i > -1$.

Fourth, incomplete adjustment has a clear interpretation in terms of the relative difference between virtual and observed expenditure on the durables. Recall that $Y_D^i = \boldsymbol{p}_D^i \boldsymbol{x}_D^i + \boldsymbol{r}_D^i \boldsymbol{m}_D^i$ denotes observed expenditure on the durables at observation $i = 1, \dots, n$ and let $\tilde{Y}_D^i = \tilde{\boldsymbol{p}}_D^i \boldsymbol{x}_{\boldsymbol{D}}^{\boldsymbol{i}} + \tilde{\boldsymbol{r}}_D^i \boldsymbol{m}_D^i = \left(1 + \mathrm{IA}^i\right)\boldsymbol{p}_D^i \boldsymbol{x}_D^i + \left(1 + \mathrm{IA}^i\right)\boldsymbol{r}_D^i \boldsymbol{m}_D^i = \left(1 + \mathrm{IA}^i\right)\left(\boldsymbol{p}_D^i \boldsymbol{x}_D^i + \boldsymbol{r}_D^i \boldsymbol{m}_D^i\right) = \left(1 + \mathrm{IA}^i\right)Y_D^i$ denote the virtual expenditure on the durable goods. At observation $i = 1, \dots, n$, we have

$$\frac{\tilde{Y}_D^i - Y_D^i}{Y_D^i} = \frac{\left(1 + \mathrm{IA}^i\right)Y_D^i - Y_D^i}{Y_D^i} = \mathrm{IA}^i. \tag{20}$$

Thus, the number $100 \times \mathrm{IA}^i$ gives the required percentage adjustment of expenditure on the durable goods in order for the optimal bundle of durable goods to be demanded in equilibrium.

Finally, we note that our model nests Swofford and Whitney's (1994) model of incomplete adjustment when there are neither any durable goods in $\boldsymbol{x}$ nor any nondurable goods in $\boldsymbol{m}$, i.e., $\boldsymbol{x_D} = \boldsymbol{m_{ND}} = \emptyset$.[9] Conversely, if there are neither any nondurable goods in $\boldsymbol{x}$ nor any durable goods in $\boldsymbol{m}$, i.e., $\boldsymbol{x_{ND}} = \boldsymbol{m_D} = \emptyset$, then by interchanging the prices and quantities of the durable

[8] Thus, for the standard weakly separable utility maximization model (i.e., with complete adjustment), the first-order conditions (16), (10) and (12) becomes: $\partial V(\boldsymbol{m_D}, \boldsymbol{m_{ND}})/\partial \boldsymbol{m_D^i} = \mu^i \boldsymbol{r}_D^i$, $\partial U\left(\boldsymbol{x_D}, \boldsymbol{x_{ND}}, V(\boldsymbol{m_D}, \boldsymbol{m_{ND}})\right)/\partial \boldsymbol{x_D^i} = \tau^i \boldsymbol{p}_D^i$ and $\partial U\left(\boldsymbol{x_D}, \boldsymbol{x_{ND}}, V(\boldsymbol{m_D}, \boldsymbol{m_{ND}})\right)/\partial \boldsymbol{m_D^i} = \tau^i \boldsymbol{r}_D^i$ (Varian, 1983, p.105). Note that we obtain the same first-order conditions when there are only nondurable goods in $\boldsymbol{x}$ and $\boldsymbol{m}$, i.e., $\boldsymbol{x_D} = \boldsymbol{m_D} = \emptyset$.

[9] As discussed in the Introduction, Swofford and Whitney's (1994) model may be rather restrictive in empirical work. Indeed, the application in online appendix E, where we derive economically valid monetary aggregates contain several relevant hypothesized structures where durables and nondurables are both part of the separable and nonseparable blocks of goods.

goods with the prices and quantities of the nondurable goods, the testable condition becomes that of Swofford and Whitney (1994).[10]

2.2 Utility maximization with incomplete adjustment

In this section, we generalize our results and introduce a model of utility maximization with incomplete adjustment. Suppose that the durable goods, $\boldsymbol{x_D}$ and $\boldsymbol{m_D}$, are chosen with incomplete adjustment according to the standard utility maximization problem:

$$\max_{\{\boldsymbol{x_D},\boldsymbol{x_{ND}},\boldsymbol{m_D},\boldsymbol{m_{ND}}\}} u(\boldsymbol{x_D},\boldsymbol{x_{ND}},\boldsymbol{m_D},\boldsymbol{m_{ND}}) \quad \textit{subject to}$$

$$\boldsymbol{p}_D^i\boldsymbol{x_D} + \boldsymbol{p}_{ND}^i\boldsymbol{x_{ND}} + \boldsymbol{r}_D^i\boldsymbol{m_D} + \boldsymbol{r}_{ND}^i\boldsymbol{m_{ND}} \le Y^i \tag{21}$$

$$\boldsymbol{p}_D^i\boldsymbol{x_D} + \boldsymbol{r}_D^i\boldsymbol{m_D} \le Y_D^i. \tag{22}$$

The restrictions (21)-(22) have the same interpretations as the restrictions (2)-(3) in the weak separability model. Analogous to that model, we define the amount of incomplete adjustment as:

$$\mathrm{IA}^i = \frac{\partial u(\boldsymbol{x_D},\boldsymbol{x_{ND}},\boldsymbol{m_D},\boldsymbol{m_{ND}})/\partial Y_D^i}{\partial u(\boldsymbol{x_D},\boldsymbol{x_{ND}},\boldsymbol{m_D},\boldsymbol{m_{ND}})/\partial Y^i} = \frac{\Omega^i}{\tau^i},$$

for all $i = 1, \dots, n$.

The first-order conditions can be used to show that this model is equivalent to the standard utility maximization problem:

$$\max_{\{\boldsymbol{x_D},\boldsymbol{x_{ND}},\boldsymbol{m_D},\boldsymbol{m_{ND}}\}} u(\boldsymbol{x_D},\boldsymbol{x_{ND}},\boldsymbol{m_D},\boldsymbol{m_{ND}}) \quad \textit{subject to}$$

$$\tilde{\boldsymbol{p}}_D^i\boldsymbol{x_D} + \boldsymbol{p}_{ND}^i\boldsymbol{x_{ND}} + \tilde{\boldsymbol{r}}_D^i\boldsymbol{m_D} + \boldsymbol{r}_{ND}^i\boldsymbol{m_{ND}} \le \tilde{Y}^i, \tag{23}$$

where $\tilde{\boldsymbol{p}}_D^i$ and $\tilde{\boldsymbol{r}}_D^i$ are the virtual prices and $\tilde{Y}^i$ is virtual expenditure.

2.3 Parametric example

[10] In one special case, our model does not have any testable implications. Suppose that there are only durable goods in $\boldsymbol{x}$ and $\boldsymbol{m}$. As such, the first-order conditions (9)-(12) and (15)-(16) implies that it is not possible to separately identify the degree of incomplete adjustment, IA, from the Lagrange multipliers. Hence, in this case, the model with incomplete adjustment is observationally equivalent to a model with complete adjustment.

In this section, we illustrate the results in Section 2.1 and 2.2 by showing the effects of incomplete adjustment in a parametric example. Suppose there are only nondurable goods in the $\boldsymbol{x}$- block, $\boldsymbol{x} = \boldsymbol{x_{ND}} = (x_1, x_2)$, and only durable goods in the $\boldsymbol{m}$- block, $\boldsymbol{m} = \boldsymbol{m_D} = (m_1, m_2)$.[11] The prices for the two $\boldsymbol{x}$-goods are denoted $\boldsymbol{p} = \boldsymbol{p_{ND}} = (p_1, p_2)$ and the prices for the two $\boldsymbol{m}$-goods are denoted $\boldsymbol{r} = \boldsymbol{r_D} = (r_1, r_2)$. Suppose that the utility function is:

$$u(\boldsymbol{x}, \boldsymbol{m}) = u(\boldsymbol{x_{ND}}, \boldsymbol{m_D}) = x_1 m_1 m_2 + \sqrt{x_2 m_1 m_2}.$$

This utility function is weakly separable in the (durable) $\boldsymbol{m}$-goods since it can be written as:

$$u(\boldsymbol{x}, \boldsymbol{m}) = u(\boldsymbol{x_{ND}}, \boldsymbol{m_D}) = U\big(\boldsymbol{x_{ND}}, V(\boldsymbol{m_D})\big) = x_1 V(\boldsymbol{m_D}) + \sqrt{x_2 V(\boldsymbol{m_D})},$$

where $V(\boldsymbol{m_D}) = m_1 m_2$. Solving the sub-utility maximization problem in (4)-(6) gives the conditional demands, $\widetilde{m}_1 = E/r_1$ and $\widetilde{m}_2 = E/r_2$ The reduced form is obtained by plugging in these solutions:

$$U\left(\boldsymbol{x_{ND}}, \tilde{V}(\boldsymbol{r_D}, E)\right) = x_1 \tilde{V}(\boldsymbol{r_D}, E) + \sqrt{x_2 \tilde{V}(\boldsymbol{r_D}, E)} = x_1 \frac{(E)^2}{4r_1 r_2} + \sqrt{x_2 \frac{(E)^2}{4r_1 r_2}},$$

where $\tilde{V}(\boldsymbol{r_D}, E) = \widetilde{m}_1 \widetilde{m}_2 = (E)^2/4r_1 r_2$ is the indirect sub-utility function corresponding to the sub-utility function $V(\boldsymbol{m_D})$. Solving the reduced form problem gives the optimal (unconditional) demand functions $(\bar{x}_1, \bar{x}_2)$ and the optimal allocation of sub-expenditure $\bar{E}$:

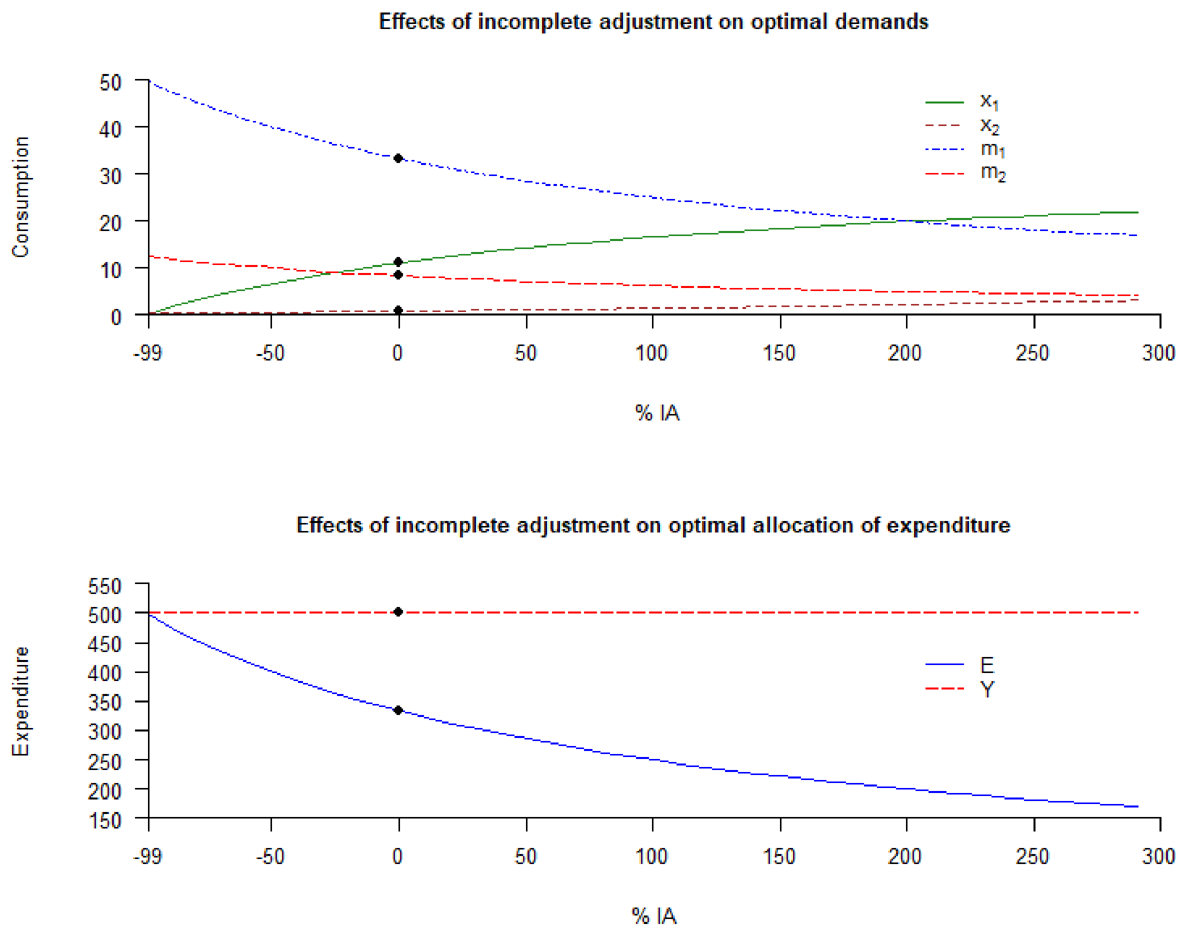
$$\bar{x}_1 = \frac{4p_2(Y)^3(3 + \text{IA}) - 8p_2(Y)^3 - (p_1)^2 r_1 r_2 (3 + \text{IA})^2}{4p_1 p_2 (Y)^2 (3 + \text{IA})},$$

$$\bar{x}_2 = (3 + \text{IA})^2 \frac{(p_1)^2 r_1 r_2}{4(p_2)^2 (Y)^2},$$

$$\bar{E} = \frac{2Y}{3 + \text{IA}}.$$

[11] Thus, this setup corresponds to Swofford and Whitney's (1994) model of incomplete adjustment.

Figure 1: Optimal demands and expenditures as functions of incomplete adjustment in the parametric example. with p_1 =15, p_2 =0.5, r_1 =5, r_2 =20, and Y =500. Filled points are optimal demands and expenditures with complete adjustment.



Plugging in the optimal allocation of sub-expenditure into the conditional demand functions $(\tilde{m}_1, \tilde{m}_2)$ gives the optimal (unconditional) sub-utility demand functions, $\bar{m}_1 = Y/(3 + \mathrm{IA})r_1$ and $\bar{m}_2 = Y/(3 + \mathrm{IA})r_2$.

The upper graph in Figure 1 plots the optimal demands $(\bar{x}_1, \bar{x}_2, \bar{m}_1, \bar{m}_2)$ for different values of percentage incomplete adjustment, defined as $\%\,\mathrm{IA} = 100 \times \mathrm{IA}$. We see that the demands vary quite considerably even for small amounts of incomplete adjustment (filled points correspond to optimal demands with complete adjustment, $\%\,\mathrm{IA} = 0$). The lower plot in Figure 1 shows total expenditure (Y) and sub-expenditure for the separable $\boldsymbol{m}$-goods (E) for different values of % IA. When the amount of incomplete adjustment is negative, expenditure on the durable goods (E) is greater than desired and increases for lower values of %IA. When the amount of incomplete adjustment is positive, expenditure on the durable goods is less than desired and will eventually approach zero in the limit as IA $\rightarrow \infty$. Again, there are large differences in the optimal allocations.

3. TEST-PROCEDURES

In this section, we propose computationally efficient non-parametric revealed preference procedures to implement the models outlined in previous sections.

3.1 Testing for weak separability with incomplete adjustment

3.1.1 Rationalizability and revealed preference characterization

By the maximization problems (18) and (19), we know that the weak separability model with incomplete adjustment is equivalent to the standard weakly separable utility maximization model with complete adjustment where observed expenditures and prices on durables are replaced by the virtual expenditures and prices. Varian (1983) and Cherchye et al. (2015) give a complete revealed preference characterization of the weakly separable utility maximization model.[12] Thus, we can directly apply these results to provide an analogous characterization of the weak separability model with incomplete adjustment (by replacing observed expenditures and prices with virtual expenditures and prices). The following definition of rationalizability with incomplete adjustment is a straightforward adaptation of Varian's notion of rationalizability for the standard weak separability model:

[12] Online appendix A recalls Varian's (1983) and Cherchye et al.'s (2015) revealed preference characterization of the (standard) weak separability model with complete adjustment.

Definition 1. *A data set* $\mathbb{D} = \{(\boldsymbol{p}_{\boldsymbol{D}}^{\boldsymbol{i}}, \boldsymbol{p}_{\boldsymbol{ND}}^{\boldsymbol{i}}), (\boldsymbol{r}_{\boldsymbol{D}}^{\boldsymbol{i}}, \boldsymbol{r}_{\boldsymbol{ND}}^{\boldsymbol{i}}); (\boldsymbol{x}_{\boldsymbol{D}}^{\boldsymbol{i}}, \boldsymbol{x}_{\boldsymbol{ND}}^{\boldsymbol{i}}), (\boldsymbol{m}_{\boldsymbol{D}}^{\boldsymbol{i}}, \boldsymbol{m}_{\boldsymbol{ND}}^{\boldsymbol{i}})\}_{i=1,\dots,n}$ *can be rationalized by a weakly separable utility function and incomplete adjustment if*

$$U(\boldsymbol{x_D}, \boldsymbol{x_{ND}}, V(\boldsymbol{m_D}, \boldsymbol{m_{ND}})) \leq U\left(\boldsymbol{x}_{\boldsymbol{D}}^{\boldsymbol{i}}, \boldsymbol{x}_{\boldsymbol{ND}}^{\boldsymbol{i}}, V(\boldsymbol{m}_{\boldsymbol{D}}^{\boldsymbol{i}}, \boldsymbol{m}_{\boldsymbol{ND}}^{\boldsymbol{i}})\right),$$

for all $(\boldsymbol{x_D}, \boldsymbol{x_{ND}}, \boldsymbol{m_D}, \boldsymbol{m_{ND}}) \in \mathbb{R}_+^{\boldsymbol{k}}$ *such that* $\boldsymbol{p}_{\boldsymbol{ND}}^{\boldsymbol{i}}\boldsymbol{x_{ND}} + \tilde{\boldsymbol{p}}_{\boldsymbol{D}}^{\boldsymbol{i}}\boldsymbol{x_D} + \boldsymbol{r}_{\boldsymbol{ND}}^{\boldsymbol{i}}\boldsymbol{m_{ND}} + \tilde{\boldsymbol{r}}_{\boldsymbol{D}}^{\boldsymbol{i}}\boldsymbol{m_D} \leq \boldsymbol{p}_{\boldsymbol{ND}}^{\boldsymbol{i}}\boldsymbol{x}_{\boldsymbol{ND}}^{\boldsymbol{i}} + \tilde{\boldsymbol{p}}_{\boldsymbol{D}}^{\boldsymbol{i}}\boldsymbol{x}_{\boldsymbol{D}}^{\boldsymbol{i}} + \boldsymbol{r}_{\boldsymbol{ND}}^{\boldsymbol{i}}\boldsymbol{m}_{\boldsymbol{ND}}^{\boldsymbol{i}} + \tilde{\boldsymbol{r}}_{\boldsymbol{D}}^{\boldsymbol{i}}\boldsymbol{m}_{\boldsymbol{D}}^{\boldsymbol{i}}$, *where* $\tilde{\boldsymbol{p}}_{\boldsymbol{D}}^{\boldsymbol{i}} = (1 + \mathrm{IA}^i)\boldsymbol{p}_D^i$ *and* $\tilde{\boldsymbol{r}}_{\boldsymbol{D}}^{\boldsymbol{i}} = (1 + \mathrm{IA}^i)\boldsymbol{r}_D^i$ *for all* $i = 1, \dots, n$.

This definition states that since the DM has chosen the observed bundle $(\boldsymbol{x}_{\boldsymbol{D}}^{\boldsymbol{i}}, \boldsymbol{x}_{\boldsymbol{ND}}^{\boldsymbol{i}}, \boldsymbol{m}_{\boldsymbol{D}}^{\boldsymbol{i}}, \boldsymbol{m}_{\boldsymbol{ND}}^{\boldsymbol{i}})$ to any other affordable bundle $(\boldsymbol{x_D}, \boldsymbol{x_{ND}}, \boldsymbol{m_D}, \boldsymbol{m_{ND}})$ at prices $(\tilde{\boldsymbol{p}}_{\boldsymbol{D}}^{\boldsymbol{i}}, \boldsymbol{p}_{\boldsymbol{ND}}^{\boldsymbol{i}}, \tilde{\boldsymbol{r}}_{\boldsymbol{D}}^{\boldsymbol{i}}, \boldsymbol{r}_{\boldsymbol{ND}}^{\boldsymbol{i}})$**,** the utility she enjoys from consuming $(\boldsymbol{x}_{\boldsymbol{D}}^{\boldsymbol{i}}, \boldsymbol{x}_{\boldsymbol{ND}}^{\boldsymbol{i}}, \boldsymbol{m}_{\boldsymbol{D}}^{\boldsymbol{i}}, \boldsymbol{m}_{\boldsymbol{ND}}^{\boldsymbol{i}})$ must be weakly higher than the utility she would enjoy from consuming any other affordable bundle $(\boldsymbol{x_D}, \boldsymbol{x_{ND}}, \boldsymbol{m_D}, \boldsymbol{m_{ND}})$. Thus, the notion of rationalizability with incomplete adjustment is equivalent to saying that every observed bundle $(\boldsymbol{x}_{\boldsymbol{D}}^{\boldsymbol{i}}, \boldsymbol{x}_{\boldsymbol{ND}}^{\boldsymbol{i}}, \boldsymbol{m}_{\boldsymbol{D}}^{\boldsymbol{i}}, \boldsymbol{m}_{\boldsymbol{ND}}^{\boldsymbol{i}})$ is the optimal solution to the maximization problem (1)-(6).

Varian (1983) and Cherchye et al. (2015) derived necessary and sufficient revealed preference conditions for when a data set $\mathbb{D}$ can be rationalized by the weak separability model. By directly applying these results, we obtain the following necessary and sufficient revealed preference conditions for when $\mathbb{D}$ can be rationalized by the weak separability model with incomplete adjustment.[13]

Theorem 1. *Consider the data set* $\mathbb{D} = \{(\boldsymbol{p}_{\boldsymbol{D}}^{\boldsymbol{i}}, \boldsymbol{p}_{\boldsymbol{ND}}^{\boldsymbol{i}}), (\boldsymbol{r}_{\boldsymbol{D}}^{\boldsymbol{i}}, \boldsymbol{r}_{\boldsymbol{ND}}^{\boldsymbol{i}}); (\boldsymbol{x}_{\boldsymbol{D}}^{\boldsymbol{i}}, \boldsymbol{x}_{\boldsymbol{ND}}^{\boldsymbol{i}}), (\boldsymbol{m}_{\boldsymbol{D}}^{\boldsymbol{i}}, \boldsymbol{m}_{\boldsymbol{ND}}^{\boldsymbol{i}})\}_{i=1,\dots,n}$. *Statements (a)-(c) are equivalent:*

(a) $\mathbb{D}$ *can be rationalized by a weakly separable utility function and incomplete adjustment.*

(b) There exist numbers $V^i, U^i, \mu^i > 0, \tau^i > 0$ *and* $\mathrm{IA}^i > -1$ *such that (for all* $i, j = 1, \dots, n$*):*

$$V^j \leq V^i + \mu^i \boldsymbol{r}_{ND}^i(\boldsymbol{m}_{\boldsymbol{ND}}^{\boldsymbol{j}} - \boldsymbol{m}_{\boldsymbol{ND}}^{\boldsymbol{i}}) + \mu^i(1 + \mathrm{IA}^i)\boldsymbol{r}_D^i(\boldsymbol{m}_{\boldsymbol{D}}^{\boldsymbol{j}} - \boldsymbol{m}_{\boldsymbol{D}}^{\boldsymbol{i}}), \quad (24)$$

[13] A proof of Theorem 1 is given in online appendix B.

$$U^j \leq U^i + \tau^i \boldsymbol{p}^i_{ND}\left(\boldsymbol{x^j_{ND}} - \boldsymbol{x^i_{ND}}\right) + \tau^i\left(1 + \mathrm{IA}^i\right)\boldsymbol{p}^i_D\left(\boldsymbol{x^j_D} - \boldsymbol{x^i_D}\right) + \frac{\tau^i}{\mu^i}\left(V^j - V^i\right). \tag{25}$$

(c) There exist numbers V^i, W^i, $\mu^i > 0$ and $\mathrm{IA}^i > -1$ such that (for all $i, j = 1, \dots, n$):

$$V^j \leq V^i + \mu^i \boldsymbol{r}^i_{ND}\left(\boldsymbol{m^j_{ND}} - \boldsymbol{m^i_{ND}}\right) + \mu^i\left(1 + \mathrm{IA}^i\right)\boldsymbol{r}^i_D\left(\boldsymbol{m^j_D} - \boldsymbol{m^i_D}\right), \tag{26}$$

$$if\ \mu^i \boldsymbol{p}^i_{ND}\left(\boldsymbol{x^i_{ND}} - \boldsymbol{x^j_{ND}}\right) + \mu^i\left(1 + \mathrm{IA}^i\right)\boldsymbol{p}^i_D\left(\boldsymbol{x^i_D} - \boldsymbol{x^j_D}\right) + \left(V^i - V^j\right) \geq 0$$

$$then\ \ W^i \geq W^j, \tag{27}$$

$$if\ \mu^i \boldsymbol{p}^i_{ND}\left(\boldsymbol{x^i_{ND}} - \boldsymbol{x^j_{ND}}\right) + \mu^i\left(1 + \mathrm{IA}^i\right)\boldsymbol{p}^i_D\left(\boldsymbol{x^i_D} - \boldsymbol{x^j_D}\right) + \left(V^i - V^j\right) > 0$$

$$then\ \ W^i > W^j. \tag{28}$$

This theorem exhausts the empirical content of the weak separability model with incomplete adjustment, and is a straightforward adaptation of the necessary and sufficient revealed preference conditions for the standard weak separability model.

Statements (b) and (c) are testable conditions that can be implemented to test whether the data $\mathbb{D}$ can be rationalized with incomplete adjustment. Varian (1983, p.105) provides a simple intuition for the inequalities in statement (b) using concavity arguments. However, the inequalities (25) in statement (b) are not very attractive from a computational perspective since they contain the non-linear term $\tau^i\left(V^j - V^i\right)/\mu^i$; we return to this issue in Section 3.1.3. Demuynck and Hjertstrand (2019, p.197-198) gives a simple intuition for statement (c) based on quasi-concave arguments and contrast it to Varian’s intuition of the inequalities in statement (b). However, as currently stated, the inequalities (27) and (28) in statement (c) are also non-linear because of the term $\mu^i\left(1 + \mathrm{IA}^i\right)$. The inequalities can be linearized by defining the scalars:

$$\Psi^i = \mu^i\left(1 + \mathrm{IA}^i\right), \tag{29}$$

in which case the amount of incomplete adjustment can be expressed as:

$$\mathrm{IA}^i = \frac{\Psi^i}{\mu^i} - 1. \tag{30}$$

Substituting this into (27) and (28), we obtain the inequalities:

$$V^j \leq V^i + \mu^i \boldsymbol{r}_{ND}^i\left(\boldsymbol{m}_{\boldsymbol{ND}}^{\boldsymbol{j}} - \boldsymbol{m}_{\boldsymbol{ND}}^{\boldsymbol{i}}\right) + \Psi^i \boldsymbol{r}_D^i\left(\boldsymbol{m}_{\boldsymbol{D}}^{\boldsymbol{j}} - \boldsymbol{m}_{\boldsymbol{D}}^{\boldsymbol{i}}\right), \tag{31}$$

$$if\ \mu^i \boldsymbol{p}_{ND}^i\left(\boldsymbol{x}_{\boldsymbol{ND}}^{\boldsymbol{i}} - \boldsymbol{x}_{\boldsymbol{ND}}^{\boldsymbol{j}}\right) + \Psi^i \boldsymbol{p}_D^i\left(\boldsymbol{x}_{\boldsymbol{D}}^{\boldsymbol{i}} - \boldsymbol{x}_{\boldsymbol{D}}^{\boldsymbol{j}}\right) + \left(V^i - V^j\right) \geq 0 \quad then \quad W^i \geq W^j, \tag{32}$$

$$if\ \mu^i \boldsymbol{p}_{ND}^i\left(\boldsymbol{x}_{\boldsymbol{ND}}^{\boldsymbol{i}} - \boldsymbol{x}_{\boldsymbol{ND}}^{\boldsymbol{j}}\right) + \Psi^i \boldsymbol{p}_D^i\left(\boldsymbol{x}_{\boldsymbol{D}}^{\boldsymbol{i}} - \boldsymbol{x}_{\boldsymbol{D}}^{\boldsymbol{j}}\right) + \left(V^i - V^j\right) > 0 \quad then \quad W^i > W^j. \tag{33}$$

These inequalities are linear and therefore considerably easier to implement in empirical applications. Thus, we will base our revealed preference procedure on the linear inequalities (31)-(33).

3.1.2 The mixed-integer quadratic programming problem

In this section, we construct a practical test of the inequalities (31)-(33). This test is based on solving a mixed integer quadratic programming problem. In constructing such a procedure there are two issues that need to be addressed. First, note that (31) consists of a set of weak inequalities which are linear in the unknowns μ^i and Ψ^i. Since there is one inequality for each pair of observations $i, j = 1, \dots, n$, there are in total n^2 inequalities in (31). Similarly, the inequalities (32) and (33) are linear in the unknowns μ^i and Ψ^i, and also consists of one inequality for each pair of observations. Thus, there are in total n^2 inequalities each of (32) and (33). However, these inequalities are logical statements and are therefore not practically operational since we need to capture the logical relation between the right- and left-hand sides of them. One of the main contributions of Cherchye et al. (2015) was to show that these logical statements can be expressed as a set of linear inequalities that can be solved in straightforward manner using numerical optimization packages in mathematical and statistical software. Specifically, Cherchye et al. (2015) showed that (32) and (33) are equivalent to the existence of binary numbers $X^{ij} \in \{0,1\}$ for all $i, j = 1, \ldots, n$ such that the following (linear) inequalities hold:[14]

$$W^i - W^j - X^{ij} \leq -\varepsilon, \tag{c.1}$$

$$\left(X^{ij} - 1\right) \leq W^i - W^j, \tag{c.2}$$

$$\mu^i \boldsymbol{p}_{ND}^i \left(\boldsymbol{x}_{\boldsymbol{ND}}^{\boldsymbol{i}} - \boldsymbol{x}_{\boldsymbol{ND}}^{\boldsymbol{j}}\right) + \Psi^i \boldsymbol{p}_D^i \left(\boldsymbol{x}_{\boldsymbol{D}}^{\boldsymbol{i}} - \boldsymbol{x}_{\boldsymbol{D}}^{\boldsymbol{j}}\right) + \left(V^i - V^j\right) - X^{ij} A^i \leq -\varepsilon, \tag{c.3}$$

$$\left(X^{ij} - 1\right) A^j \leq \mu^j \boldsymbol{p}_{ND}^j \left(\boldsymbol{x}_{\boldsymbol{ND}}^{\boldsymbol{i}} - \boldsymbol{x}_{\boldsymbol{ND}}^{\boldsymbol{j}}\right) + \Psi^j \boldsymbol{p}_D^j \left(\boldsymbol{x}_{\boldsymbol{D}}^{\boldsymbol{i}} - \boldsymbol{x}_{\boldsymbol{D}}^{\boldsymbol{j}}\right) + \left(V^i - V^j\right), \tag{c.4}$$

$$0 \leq V^i \leq 1, \tag{c.5}$$

$$0 \leq W^i \leq 1 - \varepsilon, \tag{c.6}$$

$$\varepsilon \leq \mu^i \leq 1, \tag{c.7}$$

$$\varepsilon \leq \Psi^i \leq 1, \tag{c.8}$$

Conditions (c.1) and (c.2) reproduce the right-hand side of inequalities (32) and (33), while (c.3) and (c.4) reproduce the left-hand sides of the inequalities. The binary variables X^{ij}capture the logical relationship in (32) and (33) and equal one if and only if $W^i \geq W^j$. Moreover, ε is a small positive number and A^i is a fixed number larger than $\boldsymbol{p}_{ND}^i \boldsymbol{x}_{\boldsymbol{ND}}^{\boldsymbol{i}} + \boldsymbol{p}_D^i \boldsymbol{x}_{\boldsymbol{D}}^{\boldsymbol{i}} + 1$. Importantly, notice that (c.1)-(c.4) are weak inequalities and linear in the unknowns which allow for them to act as linear constraints in optimization programs (which is precisely the role they serve here). Note that there is one inequality for each pair of observations in (c.1)-(c.4), making a total of n^2 inequalities for each set of inequalities and $4n^2$ inequalities in total for (c.1)-(c.4). Finally, notice that the sign restrictions (c.5)-(c.8) are just harmless normalizations since the inequalities (32) and (33) are homogeneous of degree zero in the unknowns.

[14] Actually, Cherchye et al.'s (2015) showed the equivalence between (32) and (33) and (c.1)-(c.8) in the special case $\Psi^i = \mu^i$. However, the extension to cover the case $\Psi^i \neq \mu^i$ is trivial and we therefore omit a proof.

The second issue to be addressed concerns how to compute the incomplete adjustment. Recall from (29) that $\Psi^i = \mu^i(1 + \text{IA}^i)$, which by solving for IA^i gives (30), i.e., $\text{IA}^i = \Psi^i/\mu^i - 1$. Thus, the amount of incomplete adjustment is minimized whenever Ψ^i and μ^i are as close as possible to each other.[15] One approach to find the total minimal amount of incomplete adjustment is therefore to minimize the sum of squared deviations between Ψ^i and μ^i. This is an optimization problem in the quadratic norm and bounded below by zero. Combining this objective function with the linear constraints given by (31) and (c.1)-(c.8) gives us the following optimization problem with respect to the variables $\{V^i, W^i, \mu^i, \Psi^i, X^{ij}\}_{i,j=1,\dots,n}$:

OP:

$$\Xi = \min \sum_{i=1}^{n} \left(\Psi^i - \mu^i\right)^2 \qquad \textit{subject to the inequalities in } (31) \textit{ and } (\text{c}.1) - (\text{c}.8).$$

This optimization problem finds the minimal total amount of incomplete adjustment such that the $\boldsymbol{m}$-goods are weakly separable from all other goods. Notice that OP solves a quadratic objective function subject to linear constraints where some variables (X^{ij}) only takes binary values. Thus, OP is a so-called mixed integer (binary) quadratic programming (MIQP) problem.

Suppose that there exists a feasible solution to OP and let $\{\widehat{\Psi}^i, \hat{\mu}^i\}_{i=1,\dots,n}$ be the optimal solutions from OP. If $\widehat{\Psi}^i = \hat{\mu}^i$ for all $i = 1, \dots, n$ such that $\hat{\Xi} = \sum_{i=1}^{n}\left(\widehat{\Psi}^i - \hat{\mu}^i\right)^2 = 0$, then $\widehat{\text{IA}}^i = \widehat{\Psi}^i/\hat{\mu}^i - 1 = 0$ for all $i = 1, \dots, n$, in which case we say that the $\boldsymbol{m}$-goods are weakly separable with *complete adjustment*. But if $\widehat{\Psi}^i \neq \hat{\mu}^i$ such that $\hat{\Xi} > 0$ then $\widehat{\text{IA}}^i \neq 0$ for at least one i and we say that the $\boldsymbol{m}$-goods are weakly separable with *incomplete adjustment*. On the other hand, if no feasible solution to OP exists, then the $\boldsymbol{m}$-goods are *not* weakly separable.

[15] In particular, $\text{IA}^i = 0$ if and only if $\Psi^i = \mu^i$.

3.1.3 Practical implementation and computational complexity

Solving the MIQP problem OP gives a deterministic procedure as it does not attach any stochastic elements to the data.[16] As such, OP gives a binary response to whether or not the data can be rationalized by a weakly separable utility function with incomplete adjustment. This binary response has a one-to-one mapping with the feasibility of the problem OP since a data set can be rationalized by weak separability with incomplete adjustment if and only if there exists a feasible solution to OP. Of course, this hinges on that the inequalities (31) and (c.1)-(c.8) are empirically refutable, i.e., that we can rule out the possibility that there always exist numbers $\{V^i, W^i, \mu^i, \Psi^i, X^{ij}\}_{i,j=1,\ldots,n}$ such that (31) and (c.1)-(c.8) holds for any choice of data. In online appendix C we verify theoretically that the inequalities (31) and (c.1)-(c.8) are refutable by constructing a data set which violates a necessary condition for (31) and (c.1)-(c.8).

Next, some words on practical implementation. OP is a mixed integer quadratic programming (MIQP) problem, which formally amounts to optimizing a quadratic function over points in a polyhedral set where some of the components are restricted to be integral. This class of problems has received considerable attention in the mathematical programming and operations research literatures, and has, for example, been used in applications to portfolio optimization, machine learning and quantile regression modelling.[17] There exist algorithms designed to solve MIQP problems such as branch-and-bound, branch-and-cut and outer approximation (Naik, 2018). For our empirical application in Section 4, we use the function

[16] As discussed in the introduction, the procedures outlined in this paper builds on a large literature on revealed preference tests that are intrinsically deterministic; See Demuynck and Hjertstrand (2019) for a recent overview of this literature.

[17] See e.g., Bienstock (1996) for an early study on MIQP problems. See Chen and Lee (2018) for a recent application of MIQP problems to GMM estimation of instrumental variable quantile regression models in econometrics.

cplexmiqp from the commercial state-of-the-art solver CPLEX to solve the problem OP.[18] CPLEX uses a branch-and-bound type algorithm and has been shown to be one of the most efficient solvers for MIQP problems that is publicly available (Rimmi et al., 2017; Bliek 2014).

Finally, some words on the computational complexity of the problem OP. The worst-case computational complexity in MIQP problems grows exponentially with the number of binary decision variables. It is well-known that such MIQP problems belong to the class of np-hard problems.[19]

Cherchye et al. (2015) showed that the problem of verifying whether a data set is rationalizable by a weakly separable utility function with complete adjustment is an np-complete problem. Since weak separability with complete adjustment is a special case of the model with incomplete adjustment and every instance of the restricted problem is an instance of the more general problem with incomplete adjustment with the n extra parameters $\{\Psi^i\}_{i=1,\dots,n}$ set equal to $\{\mu^i\}_{i=1,\dots,n}$, verifying whether a data set is rationalizable by a weakly separable utility function with incomplete adjustment is also an np-complete problem. This means it is highly unlikely that the problem of verifying whether a data set is rationalizable by a weakly separable utility

[18] In particular, we used IBM ILOG CPLEX optimization studio 12.10 in Matlab. Of course, the codes to replicate all empirical results are available upon request from the authors.

[19] The theory of computational complexity classifies decision problems according to the time it takes to come to an answer "yes" or "no" for a specific instance of a given decision problem (in our case the question of whether or not the data is rationalizable by weak separability/utility maximization and incomplete adjustment). The two most important classes of decision problems are called p (polynomial) and np (nondeterministic polynomial). The class p contains all decision problems that are solvable in polynomial time, while the class np contains all problems that might be difficult to solve (i.e., it might take exponential time to find a solution), but are easy to verify (i.e., in polynomial time). A decision problem which is at least as difficult to solve as any problem in the class np is called np-hard. A decision problem is np-complete if it is both np-hard and in np. np-complete problems are among the most difficult problems in the class np, and all known solution methods applicable to np-complete problems suffer from exponential worst-time complexity. See e.g., Garey and Johnson (1979) and Papadimitriou (1994) for textbook treatments on the theory of computational complexity.

function and incomplete adjustment can be achieved by means of an efficient algorithm, e.g., linear programming.

However, despite this (though not surprising) result, the MIQP problem OP is considerably computationally easier to solve than Swofford and Whitney's (1994) implementation of the nonlinear inequalities in statement (b) of Theorem 1, which required solving a complex non-linear optimization problem with non-linear constraints. More precisely, the optimand in any such nonlinear problem may be badly behaved with saddle points and local optima. This presents an additional problem since local optima may not be globally optimal (unless some additional concavity assumptions are true). Generally, finding a global optimum requires a fine grid search over the set of initial values. Even a very fine grid search cannot exclude that weak separability is rejected while the assumption effectively holds. From a theoretical point of view, the core motivation for adopting an integer programming approach like the one proposed here is that this is a widely accepted and well-known approach to handle np-complete problems.

3.2 Testing for utility maximization with incomplete adjustment

In this section we present an analogous test-procedure for utility maximization with incomplete adjustment. We also illustrate how well the procedure can detect habit formation in a parametric example.

3.2.1 A quadratic programming problem

A data set $\mathbb{D} = \{(\boldsymbol{p}_D^i, \boldsymbol{p}_{ND}^i), (\boldsymbol{r}_D^i, \boldsymbol{r}_{ND}^i); (\boldsymbol{x}_D^i, \boldsymbol{x}_{ND}^i), (\boldsymbol{m}_D^i, \boldsymbol{m}_{ND}^i)\}_{i=1,\dots,n}$ can be *rationalized* by a utility function and incomplete adjustment if

$$U(\boldsymbol{x}_D, \boldsymbol{x}_{ND}, \boldsymbol{m}_D, \boldsymbol{m}_{ND}) \le U(\boldsymbol{x}_D^i, \boldsymbol{x}_{ND}^i, \boldsymbol{m}_D^i, \boldsymbol{m}_{ND}^i),$$

for all $(\boldsymbol{x}_D, \boldsymbol{x}_{ND}, \boldsymbol{m}_D, \boldsymbol{m}_{ND}) \in \mathbb{R}_+^{\boldsymbol{k}}$ such that $\boldsymbol{p}_{ND}^i \boldsymbol{x}_{ND} + \widetilde{\boldsymbol{p}}_D^i \boldsymbol{x}_D + \boldsymbol{r}_{ND}^i \boldsymbol{m}_{ND} + \tilde{\boldsymbol{r}}_D^i \boldsymbol{m}_D \le \boldsymbol{p}_{ND}^i \boldsymbol{x}_{ND}^i + \widetilde{\boldsymbol{p}}_D^i \boldsymbol{x}_D^i + \boldsymbol{r}_{ND}^i \boldsymbol{m}_{ND}^i + \tilde{\boldsymbol{r}}_D^i \boldsymbol{m}_D^i$, where $\widetilde{\boldsymbol{p}}_D^i = (1 + \mathrm{IA}^i)\boldsymbol{p}_D^i$ and $\tilde{\boldsymbol{r}}_D^i = (1 + \mathrm{IA}^i)\boldsymbol{r}_D^i$ for all $i = 1, \dots, n$

Theorem 2 states the non-parametric revealed preference characterization for utility maximization with incomplete adjustment.

Theorem 2. *A data set* $\mathbb{D} = \{(\boldsymbol{p_D^i}, \boldsymbol{p_{ND}^i}), (\boldsymbol{r_D^i}, \boldsymbol{r_{ND}^i}); (\boldsymbol{x_D^i}, \boldsymbol{x_{ND}^i}), (\boldsymbol{m_D^i}, \boldsymbol{m_{ND}^i})\}_{i=1,\dots,n}$ *can be rationalized by a utility function and incomplete adjustment if and only if there exist numbers* U^i*,* $\tau^i > 0$ *and* $\mathrm{IA}^i > -1$ *such that (for all* $i, j = 1, \dots, n$*):*

$$U^j \leq U^i + \tau^i \boldsymbol{p}_{ND}^i(\boldsymbol{x_{ND}^j} - \boldsymbol{x_{ND}^i}) + \tau^i(1 + \mathrm{IA}^i)\boldsymbol{p}_D^i(\boldsymbol{x_D^j} - \boldsymbol{x_D^i}) + \tau^i \boldsymbol{r}_{ND}^i(\boldsymbol{m_{ND}^j} - \boldsymbol{m_{ND}^i}) + \tau^i(1 + \mathrm{IA}^i)\boldsymbol{r}_D^i(\boldsymbol{m_D^j} - \boldsymbol{m_D^i}). \tag{34}$$

By defining the scalar $\Omega^i = \tau^i(1 + \mathrm{IA}^i) > 0$, we can express the amounts of incomplete adjustment as:

$$\mathrm{IA}^i = \frac{\Omega^i}{\tau^i} - 1. \tag{35}$$

This gives the following linearized condition:

$$U^j \leq U^i + \tau^i \boldsymbol{p}_{ND}^i(\boldsymbol{x_{ND}^j} - \boldsymbol{x_{ND}^i}) + \Omega^i \boldsymbol{p}_D^i(\boldsymbol{x_D^j} - \boldsymbol{x_D^i}) + \tau^i \boldsymbol{r}_{ND}^i(\boldsymbol{m_{ND}^j} - \boldsymbol{m_{ND}^i}) + \Omega^i \boldsymbol{r}_D^i(\boldsymbol{m_D^j} - \boldsymbol{m_D^i}). \tag{36}$$

Note that the inequalities (36) reduce to the well-known Afriat inequalities whenever $\Omega^i = \tau^i$ holds for all $i = 1, \dots, n$. We follow the same approach as for the weak separability procedure to find the total minimal amount of incomplete adjustment by minimizing the sum of squared deviations between Ω^i and τ^i. Thus, combining this objective function with the linear constraints given by (36) gives us the following quadratic optimization problem with respect to the variables $\{U^i, \tau^i, \Omega^i\}_{i,j=1,\dots,n}$:

OPU:

$$\Theta = \min \sum_{i=1}^{n} (\Omega^i - \mu^i)^2 \qquad \textit{subject to the inequalities in } (36).$$

This optimization problem finds the minimal total amount of incomplete adjustment such that the data can be rationalized by a utility function and incomplete adjustment. Notice that OPU solves a quadratic objective function subject to linear constraints which makes the problem a quadratic programming (QP) problem.

3.2.2 Habit formation and incomplete adjustment

Pollak (1970) introduced models of consumer behavior based on habit formation. Consider the following simple model:[20]

$$U(\boldsymbol{x^i}) = \sum_{j=1}^{k} a_j \, log\left(x_j^i - b_j^i\right), \tag{37}$$

where $b_j^i = \beta_j x_j^{i-1}$ with $0 \leq \beta_j < 1$ for all goods $j = 1, \dots, k$. The parameter β_j governs the "degree" of habit formation and is called the "habit formation coefficient" by Pollak (1970). This model reduces to a standard Cobb-Douglas utility function whenever $\beta_j = 0$ for all $j = 1, \dots, k$.

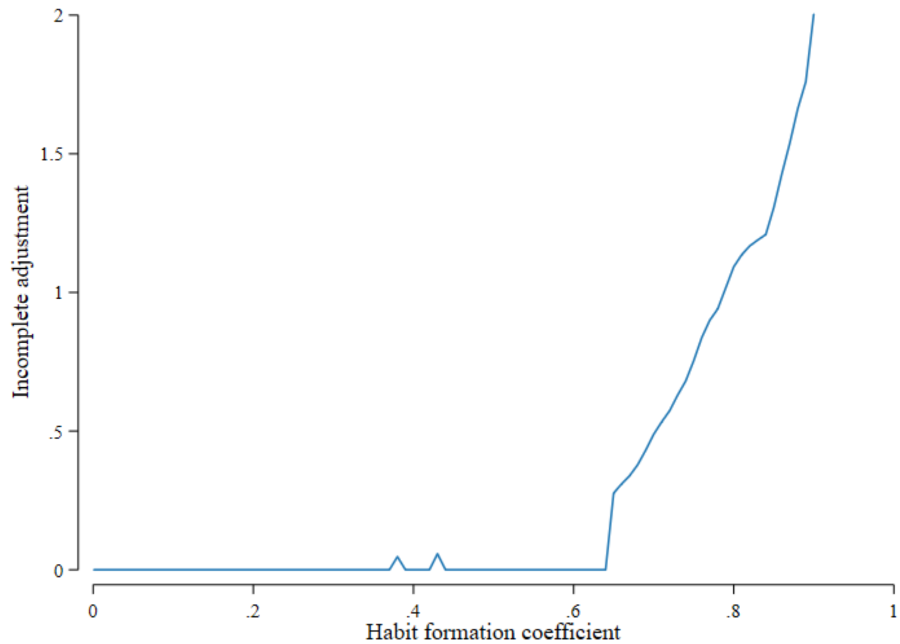
Maximizing the utility function (37) with respect to the budget constraint $\boldsymbol{p^i x} \leq Y^i$ for every $i = 1, \dots, n$ yield the demand functions:

$$h_j\left(\boldsymbol{p^i}, Y^i\right) = \frac{a_j}{p_j^i} Y^i + \beta_j x_j^{i-1} - \frac{a_j}{p_j^i} \sum_{j=1}^{k} p_j^i \, \beta_j x_j^{i-1}, \tag{38}$$

for all goods $j = 1, \dots, k$. These are short-run demand functions which are locally linear in income. Since the b's in (37) are linear in past consumption and since current consumption depends linearly on the b's, present consumption of each good is a linear function of past consumption of all goods. These assumptions of habit formation imply that consumption in the previous period influences current preference and demand, but that consumption in the more

[20] For brevity, we write $\boldsymbol{x}$ to denote all goods (nondurables as well as durables).

Figure 2: Amount of incomplete adjustment for the habit formation model



distant past only affects present consumption through indirect effects. Moreover, since the β's are positive, there is a positive relation between past and current consumption of each good.[21]

We take prices and expenditure for the 18 goods and assets in the application to quarterly financial and monetary data ($n = 47$) presented in online appendix E. Using these prices and expenditures, we generate consumption data from the demand functions (38) at each node in an equally-spaced grid for $\beta_j \in [0,0.9]$ with increments 0.01. The α-parameters are set to the mean budget shares for the goods and assets in the financial and monetary data. Moreover, we use the initial condition $x_j^0 = 10$ for all $j = 1, \dots, k$ and employ the same classification of durables and nondurables as in the Table G in online appendix E.[22] For each data set of generated consumption data and prices (which only differ by the choice of β_j), we solve the problem OPU

[21] Negative correlation can be allowed for by instead imposing $[\beta_j] < 1$.
[22] Thus, we set $\beta_j = 0$ for all nondurables.

and calculate the aggregated percentage root mean squared incomplete adjustment, $\text{RMSE}_{\%\text{IA}}$, from (41).

Figure 2 plots the $\text{RMSE}_{\%\text{IA}}$ against the habit formation coefficient β_j. As expected, we see that the amount of incomplete adjustment increases as the degree of habit formation increases. Notably, the calculated amount of incomplete adjustment is zero up to $\beta_j = 0.64$.

4. APPLICATION

In this section, we show how our models and methods provide additional insights in the analysis of consumer choice data previously not possible with tests unable to allow for incomplete adjustment.

It is common in empirical demand analysis to assume that nondurable goods and services are weakly separable from durable goods and services. This is done for tractability as it allows the researcher to model nondurables in demand systems without reference to durables which may otherwise be problematic since a consumer does not derive utility directly from spending on durables, but rather from the flow of services they provide over time. In this application, we test this often-maintained hypothesis with the following hypothesized structure:

$U\big(\text{“Durables“}, V(\text{“Nondurables“})\big)$.

4.1 Data and setup

To test if nondurables are weakly separable from durables, we use the Encuesta Continua de Presupuestos Familiares (ECPF), which is a Spanish panel survey data set ranging from 1985-1997 over disaggregated household expenditures on durable and nondurable consumption goods and services.[23] This is a unique data set for three reasons. First, households report very detailed

[23] These data were obtained from Crawford (2010) and have previously been used in revealed preference applications. For example, Cherchye et al. (2015) used the ECPF to test if households' utility functions are weakly separable in food categories.

and complete information on income and expenditures of a large number of disaggregated categories on nondurable, durable and services (25 in total). Second, the structure of the ECPF is more convenient than the most widely used consumer surveys because these do not share the panel structure of the ECPF.[24] Third, ECPF collects data quarterly. To our knowledge, the ECPF is the only large panel data set that gives information on household expenditures over more than four quarters providing information on expenditures for between five and eight quarters per household (Browning and Collado, 2001).

It is the panel structure of ECPF that makes these data ideal for our purposes. Since we can apply our test for weak separability with incomplete adjustment on data from each individual household, we avoid making any assumption of a representative agent, and also avoid making any preference homogeneity assumption between households. Thus, our procedure for weak separability effectively accounts for any form of heterogeneity between households (even any form of latent or unobserved heterogeneity), and we are therefore able to optimally exploit the panel structure of the ECPF. This is a clear advantage over standard parametric analysis, since any such analysis has to be conditional on some pooling and heterogeneity assumptions.

In order to ensure that our test have the highest power possible, we exclude households with less than $n = 8$ observations which give us in total data on 1,585 households. Hence, we run our test for weak separability with incomplete adjustment 1,585 times, i.e., we solve the MIQP problem OP once for each household.

The data in ECPF consists of 25 goods and services and are specified in more detail in Table 1. Table 2 contain descriptive statistics of the prices, budget shares and total expenditure for the

[24] For example, the UK Family Expenditure Survey (FES) has independent waves, and in the American Consumer Expenditure Survey (CEX), even though households are interviewed in four consecutive quarters, the information on income is only recorded in the 1st and 4th interview (Collado, 2001).

Table 1: Goods in ECPF data and classification according to COICOP.

Durable goods	COICOP
Durables at home (e.g., furniture and appliances)	Durable
Small durables at home	Durable
Durable medicines (e.g., spectacles, crutches and wheelchairs)	Durable
Cars	Durable
Durables at home (e.g., tv and music)	Durable
Small durables (e.g., books, toys and CDs)	Durable
Personal small durables (e.g., hair-dryer, shavers, lighters and suitcases)	Durable
Nondurable and semidurable goods	
Food and non-alcoholic drinks at home	Nondurable
Alcohol	Nondurable
Restaurants and bars	Nondurable
Tobacco	Nondurable
Nondurables at home (e.g., cleaning products)	Nondurable
Nondurable medicines	Nondurable
Petrol	Nondurable
Personal nondurables (e.g., toothpaste and soap)	Nondurable
Clothing and footwear	Nondurable
Energy at home (e.g., heating by electricity)	Nondurable
Services	
Services at home (e.g., heating not electricity, water and furniture repair)	Nondurable
Personal services	Nondurable
House rent (includes imputed rent)	Nondurable
Transportation	Nondurable
Travelling	Nondurable
Leisure (e.g., cinema, theatre and clubs for sports)	Nondurable
Education	Durable
Medical services	Durable

data in Table 1. We begin by classifying all goods and services as durables, nondurables and semidurables, and services using UN's "classification of individual consumption according to purpose" (COICOP). See the last column in Table 1 for the classification of each good.[25] Following standard practice, we classify education and medical expenditures as durables goods

[25] See https://unstats.un.org/unsd/classifications/Econ/Download/In%20Text/CPCprov_english.pdf.

Table 2: Descriptive statistics of prices, budget shares and total expenditure for all durables, semidurables, nondurables and services in the ECPF data.

	Prices				Budget shares			
	Mean	Min	Max	s.d	Mean	Min	Max	s.d
Durables at home	0.96	0.79	1.12	0.09	0.02	0	0.73	0.06
Small durables at home	0.98	0.83	1.19	0.1	0.01	0	0.52	0.03
Durable medicines	1.00	0.87	1.15	0.08	0.00	0	0.38	0.01
Cars	1.02	0.81	1.29	0.12	0.03	0	0.87	0.09
Durables at home	1.01	0.95	1.08	0.03	0.01	0	0.47	0.02
Small durables	0.96	0.74	1.18	0.13	0.02	0	0.58	0.03
Personal small durables	0.98	0.79	1.20	0.11	0.01	0	0.45	0.02
Food & non-alcoholics at home	0.98	0.79	1.17	0.10	0.26	0	0.91	0.11
Alcohol	0.95	0.65	1.26	0.16	0.01	0	0.20	0.02
Restaurants and bars	0.93	0.63	1.24	0.18	0.08	0	0.72	0.08
Tobacco	0.98	0.67	1.63	0.25	0.02	0	0.37	0.02
Non-durables at home	0.97	0.86	1.12	0.07	0.02	0	0.19	0.02
Non-durable medicines	0.98	0.88	1.09	0.05	0.01	0	0.34	0.02
Petrol	0.96	0.76	1.26	0.15	0.04	0	0.40	0.04
Personal non-durables	0.97	0.81	1.18	0.11	0.01	0	0.27	0.02
Clothing and footwear	0.95	0.72	1.14	0.11	0.1	0	0.71	0.09
Energy at home	0.96	0.74	1.22	0.14	0.03	0	0.52	0.02
Services at home	0.92	0.59	1.29	0.20	<0.00	0	0.41	0.02
Personal services	0.92	0.62	1.27	0.19	0.01	0	0.66	0.02
House rent	0.96	0.68	1.28	0.16	0.19	0	0.85	0.11
Transportation	0.91	0.54	1.25	0.21	0.06	0	0.94	0.07
Travelling	0.95	0.60	1.35	0.18	0.01	0	0.54	0.03
Leisure	0.95	0.69	1.19	0.14	0.02	0	0.53	0.03
Education	0.96	0.67	1.39	0.20	0.03	0	0.62	0.04
Medical services	0.89	0.51	1.27	0.22	0.01	0	0.78	0.04

	Mean	Min	Max	s.d
Total expenditure	695215.17	93598.50	9220404.19	427890.56

because they increase the stock of human capital and can therefore be "consumed" over a long time-span.[26]

[26] The reason for why these services should be regarded as durables is elaborated by the ILO, which in their consumer price index manual writes: "For some analytical purposes, it may be appropriate to treat certain kind of services such as education and health, as the service equivalent to durable goods. Expenditures on such services can be viewed as investments that augment the stock of human capital. Another characteristic that education and health services share with durable goods is that they are often so expensive that their purchase has to be financed by borrowing or by running down other assets" (ILO consumer price index manual, 2004, chapter 3.25).

Recall from (20) that $\mathrm{IA}^i = (\tilde{Y}_D^i - Y_D^i)/Y_D^i$, where Y_D^i and $\tilde{Y}_D^i$ are the observed and virtual expenditure on the durables in observation $i = 1, \ldots, n$, respectively. Thus, the number $\%\ \mathrm{IA}^i = 100 \times \mathrm{IA}^i$ can be interpreted as the required percentage adjustment of expenditure on the durable goods in order for the optimal bundle of durable goods to be demanded in equilibrium. We report the results using the three following summary statistics based on $\%\ \mathrm{IA}^i$:

The *mean percentage incomplete adjustment*:

$$\%\ \overline{\mathrm{IA}} = \frac{1}{n}\sum_{i=1}^{n} \%\ \mathrm{IA}^i. \tag{39}$$

The *maximal absolute percentage incomplete adjustment*:

$$\mathrm{Max}_{|\%\mathrm{IA}|} = max_{i=1,\ldots,n}\{|\%\ \mathrm{IA}^i|\}. \tag{40}$$

The *root mean squared percentage incomplete adjustment*:

$$\mathrm{RMSE}_{\%\mathrm{IA}} = \sqrt{\sum_{i=1}^{n} (\%\ \mathrm{IA}^i)^2}\,. \tag{41}$$

4.2 Results

In Table 3 we present results of the fraction of households satisfying utility maximization with and without complete adjustment. We find that data from 1,444 households (91%) can be rationalized with complete adjustment, and a further 140 households (9%) can be rationalized with incomplete adjustment. Hence, the data on one household fail the rationalizability conditions.

The top panel in Figure 3 presents histograms of the various summary statistics (39)-(41) over the 140 households satisfying utility maximization with incomplete adjustment. We see that there is some heterogeneity between households. While some most households can be

Table 3: Summary statistics of pass rates

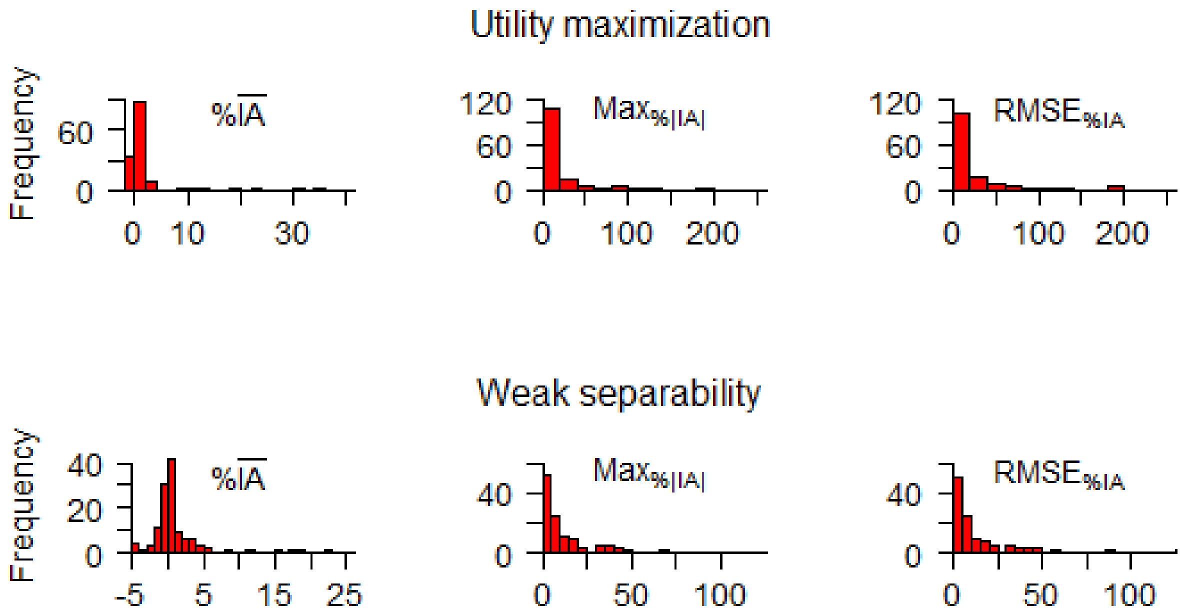
Total number of households:	1,585
# households satisfying utility maximization with complete adjustment:	1,444 (91%)
# households satisfying utility maximization with incomplete adjustment:	140 (9%)
# households satisfying weak separability with complete adjustment:	0 (0%)
# households satisfying weak separability with incomplete adjustment:	125 (8%)

rationalized by similar amounts of incomplete adjustment, there are a few households that require a considerably higher amount to be rationalized.

The left panel in Table 4, denoted, “Utility maximization“, provides summary statistics given by (39)-(41) over the 140 households that satisfy utility maximization with incomplete adjustment. These results confirm the findings from Figure 3 and show that there is some heterogeneity between households, but that a large majority of households can be rationalized with similar amounts of incomplete adjustment.

Next, we consider the results from the weak separability tests. The last two rows in Table 3 present results of the fraction of households satisfying weak separability. None of the households satisfy weak separability with complete adjustment and only 125 households or 8% satisfy weak separability with incomplete adjustment. The lower panel in Figure 3 presents histograms of the various summary statistics (39)-(41) over the 125 households satisfying weak separability with

Figure 3: Histograms of the summary statistics (39)-(41) satisfying utility maximization (140 households) and weak separability (125 households) with incomplete adjustment



incomplete adjustment. The results are similar to the results for utility maximization with incomplete adjustment in the top panel: Many households can be rationalized with similar amounts of incomplete adjustment, but there are some households that require considerably larger amounts of incomplete adjustment to be rationalized. However, compared to the utility maximization model, there is less heterogeneity among households in terms of minimal required incomplete adjustment.

The right panel in Table 4, denoted “Weak separability“, provides the summary statistics given by (39)-(41) over the 125 households that satisfy weak separability with incomplete adjustment, and confirm our findings from Figure 3.

Finally, we perform some robustness analysis by calculating the power against uniformly random behavior. This alternative hypothesis of rational behavior as a measure of power in revealed preference tests was proposed by Bronars (1987), based on Becker (1962). Bronars’

Table 4: Summary statistics over households satisfying utility maximization (140 households) and weak separability (125 households) with incomplete adjustment

	Utility maximization			Weak separability		
Statistic	% $\overline{\text{IA}}$	$\text{Max}_{\lvert\%\text{IA}\rvert}$	$\text{RMSE}_{\%\text{IA}}$	% $\overline{\text{IA}}$	$\text{Max}_{\lvert\%\text{IA}\rvert}$	$\text{RMSE}_{\%\text{IA}}$
Mean	3.92	36.77	40.67	0.86	19.95	22.50
Min	-0.50	0.09	0.12	-33.80	<0.00	<0.00
Q_1	9.95e-04	3.26	4.43	-0.53	2.24	2.24
Median	0.07	6.35	8.31	0.10	7.23	7.23
Q_3	0.40	17.59	22.24	1.00	19.01	20.26
Max	230.86	1.88e+03	1.88e+03	80.56	416.88	474.96
Std. dev.	21.70	171.42	172.15	8.85	44.62	51.23

Note: Q_1: First quartile, Q_3: Third quartile and Std. dev.: standard deviation.

procedure consists of first generating uniformly distributed budget shares on the $k-1$ unit simplex, and then solve for the uniformly random quantities corresponding to the observed prices and expenditure.[27] This is done many times, and for every simulation, we test if the simulated quantity data and observed price data pass the tests for weakly separable utility maximization with incomplete adjustment. As a benchmark, we also apply tests for utility maximization (GARP; See Varian 1982) and weak separability with complete adjustment (the procedure proposed by Cherchye et al. 2015). The power for all models lies in the unit interval, i.e., $0 \leq \text{power} \leq 1$.

To get a sense of how well these models perform in relation to each other, we also calculate the predictive success (PS) for all the models proposed by Beatty and Crawford (2011), based on an idea of Selten (1991). PS combines the pass rate of a model with the discriminatory power

[27] At every observation, the uniformly random budgets shares are generated as k random variables from the Dirichlet distribution with parameters set to $(1, \ldots, 1)$.

against uniformly random behavior. PS is calculated as the difference between the pass rate (0 or 1) and $(1 - \text{power})$ or $\text{PS} = \text{Pass Rate} - (1 - \text{power})$. By construction, $-1 \leq \text{PS} \leq 1$ and negative values indicates that the model inadequately describes the data. If the model provides a poor fit of the representative consumer behavior (pass rate 0) while at the same time the power is low (i.e., the model is difficult to reject empirically), then PS = -1. Conversely, a high and positive PS-value indicates a successful model or the model is able to explain the observed behavior (pass rate equals 1) and the power is high. One interpretation of a predictive success close to zero is that the theory in question performs about as well as random behavior.

Table 5 present summary statistics over all 1,585 households. The power of the utility maximization models with complete and incomplete adjustment is essentially zero for a large majority of the households. Since the utility maximization model with complete adjustment is nested within the incomplete adjustment-model, the power of the latter is by definition as least as high as the power of the latter.[28] The low power of the tests for utility maximization may not be that surprising in light of that inflation in Spain was high over the sample period (1985-1997). With growing nominal expenditures over time there will be limited budget hyperplane intersections, in which case the tests are not able to detect violations. We find support of this explanation from the descriptive statistics in Table 2. While there are large differences in total expenditure as measured by the standard deviation, there seem to be small differences in prices of the goods and services, implying that there may be limited budget hyperplane intersections.[29]

Consider next the weak separability models with complete and incomplete adjustment. For all households, the power of the model with complete adjustment equals 1. However, as already

[28] In other words, a violation of utility maximization with incomplete adjustment implies that utility maximization with complete adjustment is also violated, but the opposite is not necessarily true.
[29] Measurement errors may also be a factor when explaining the growing level of expenditure over the sample period.

Table 5: Power and predictive success

	Utility maximization		Weak separability	
Statistic	Power	PS	Power	PS
Mean	<0.00 (0.02)	-0.91 (-0.07)	0.20 (1)	-0.72 (0)
Min	0 (0)	-1 (-1)	0.20 (1)	-0.80 (0)
Q_1	0 (0)	-1 (0)	0.20 (1)	-0.80 (0)
Median	0 (0.01)	-1 (<0.00)	0.20 (1)	-0.80 (0)
Q_3	0 (0.03)	-1 (0.03)	0.21 (1)	-0.79 (0)
Max	<0.00 (0.15)	<0.00 (0.15)	0.21 (1)	0.21 (0)
Std. dev.	<0.00 (0.02)	0.28 (0.28)	<0.00 (0)	0.27 (0)

Note: The numbers inside parenthesis refer to the power and PS for GARP (utility maximization) and Cherchye's et al.'s (2015) procedure (weak separability). The number of Monte Carlo simulations in the power and PS calculations is set to 500.

noted from Table 3, not a single household is consistent with this model, which implies that the predictive success equals zero for every household. The power for the weak separability model with incomplete adjustment equals about 20% for almost every household.[30] However, since the pass rate equals one for only 125 households (See Table 3), the predictive success is close to -1 for the large majority of households, i.e., those failing the rationalizability conditions (92%). In contrast, the predictive success for households that pass the rationalizability conditions is the highest among all models considered, and for these households it seems reasonable to say that the model explains observed consumer behavior fairly well. However, since this only accounts for 8% of the total number of households, the model is considerably less successful in explaining observed consumer behavior on the aggregate level.

[30] We draw the same Dirichlet numbers in the power calculations for every household. Since prices for goods does not differ much between households, this explains why all households essentially have the same power.

The results in Tables 3, 4 and 5 strongly indicate that the commonly made assumption that nondurables are weakly separable from durables is questionable. Put differently, our panel data results suggest that nondurable and durable goods and services should be modelled in common in empirical demand and consumption analysis.

4.3 Monetary and real sector aggregates for the U.S.

We also applied our models and methods to U.S. data on consumption, monetary and financial assets. A detailed description of the data and results are contained in online appendix E. In this section, we only briefly discuss the data and summarize the results.

We use aggregated U.S. data on three categories of consumptions goods and leisure and 15 categories of monetary and financial goods, ranging from 2000Q1 to 2011Q3, that was previously analyzed in Hjertstrand et al. (2016). This sample period covers a little more than the first decade of the twenty-first century, which was characterized by large economic turmoil. While the early part of the sample period was part of what is called the Great Moderation, this came to an end with the start of the Great recession which lasted from December 2007 until June 2009 and also contained a financial crisis with bankruptcies of several large banks. This was followed by a period of slow economic recovery. The Fed's response to the crisis and the recession was unprecedented. It aggressively pursued expansionary policies using both conventional and unconventional tools, which are not representative of the Fed's monetary policies in the long run. So, while the standard caution about extending results beyond the sample period apply, in this instance it should be emphasized and the results summarized next should be viewed with precaution keeping the nature of the sample period in mind.

All of the monetary and real sector aggregates identified in Hjertstrand et al. (2016) including (i) M1; (ii) a modern analog to what Friedman and Schwartz (1963) called money; (iii) the old

FED aggregate L; and (iv) four real sector goods were also found by the procedure set forth in this paper to be weakly separable from all other goods and thus to form valid economic aggregates. In contrast to Hjertstrand et al. (2016), by allowing for incomplete adjustment, our procedure also finds that one additional real sector aggregate is weakly separable from leisure and all monetary and financial assets. All other hypothesized structures that were rejected in Hjertstrand et al. (2016), including M2, M3 and MZero, are still rejected by our more flexible procedure.

5. CONCLUSIONS

This paper has introduced models and procedures to test if observed consumer choice data on goods and assets can be rationalized by weakly separable utility maximization. A novel feature of our methods is that they allow for incomplete adjustment which means that expenditure on a subset of goods might not adjust completely within one period, and thus be suboptimal. Incomplete adjustment might arise because of various dynamic effects such as habit persistence (formation), adjustment costs, information asymmetries, the formation of expectations, or a combination of reasons. The standard weak separability model (with complete adjustment) appears as a special case. Overall, allowing for incomplete adjustment grants researchers more degrees of freedom to detect permissible categories of goods and assets.

Weak separability is a core assumption in, for examples, aggregation theory, index number theory and demand system analysis but is usually implicitly assumed rather than explicitly tested. A key benefit for future research is that the procedures put forward in this paper are computationally feasible which facilitates simultaneous analysis of consumer choice data using parametric and nonparametric models, performing simulation studies, conducting extensive robustness analysis etc.

We provide a generalization of our results to test if demand data can be rationalized by utility maximization with incomplete adjustment. We can also think of several other generalizations of our results that might be relevant for future research. In online appendix D, we give one such generalization to homothetic weak separability and homothetic utility maximization with incomplete adjustment. It seems well-established that homotheticity is a strong and possibly unrealistic assumption but can nevertheless be a convenient assumption in empirical consumer analysis. Incomplete adjustment adds additional flexibility to these models which may make them empirically tractable under parsimonious specifications. Other generalizations along the same lines would include imposing quasilinearity instead of homotheticity on the utility functions or allowing for measurement errors in the data in combination with incomplete adjustment.

Our methods can be used to answer several empirical questions. For example, it is common in empirical consumer demand analyses to (implicitly) assume that nondurable goods and services are weakly separable from durable goods and services (rather than explicitly testing it). Using detailed panel data over households, we find that this may be an invalid assumption.

REFERENCES

Afriat, Sydney N. (1967). “The construction of utility functions from expenditure data,” International Economic Review 8, 67–77.

Afriat, Sydney N. (1969). “The Construction of Separable Utility Functions from Expenditure Data,” Mimeo. University of North Carolina.

Almon, Shirley, (1965). "The distributed lag between capital appropriations and net expenditures," *Econometrica* 33, 1965, 178-196.

Anderson, Gordon J. and Richard W. Blundell (1982). “Estimation and hypothesis testing in dynamic singular equation systems,” *Econometrica* 50, 1559-l 571.

Banks, James, Richard W. Blundell and Arthur Lewbel (1997). “Quadratic Engel curves and consumer demand,” *Review of Economics and Statistics* 79, 527–539.

Becker, Gary S. (1962). “Irrational Behavior and Economic Theory,” *Journal of Political Economy* 70, 1–13.

Beatty, Timothy K.M. and Ian Crawford (2011). "How Demanding Is the Revealed Preference Approach to Demand," *American Economic Review* 101, 2782–2795.

Bienstock, Daniel (1996). "Computational study of a family of mixed-integer quadratic programming problems," *Mathematical Programming* 74, 121-140.

Bliek, Christian, Pierre Bonami and Andrea Lodi (2014). "Solving Mixed-Integer Quadratic Programming problems with IBM-CPLEX: a progress report," Proceedings of the Twenty-Sixth RAMP Symposium Hosei University, Tokyo.

Bronars, Stephan G. (1987). "The power of nonparametric tests of preference maximization," *Econometrica* 55, 693–698.

Böhm-Bawerk, E. von, (1888). *Positive Theory of Capital*, translated by William Smart, London: Macmillan.

Browning, Martin and M. Dolores Collado (2001). "The Response of Expenditures to Anticipated Income Changes". *American Economic Review* 91, 681-692.

Chen, Le-Yu and Sokbae Lee (2018). "Exact Computation of GMM Estimators for Instrumental Variable Quantile Regression Models," *Journal of Applied Econometrics* 33, 553-567

Cherchye, Laurens, Thomas Demuynck, Bram De Rock and Per Hjertstrand (2015). "Revealed Preference Tests for Weak Separability: an integer programming approach," *Journal of Econometrics* 186, 129–141.

Christensen, Laurits R., Dale W. Jorgenson and Lawrence J. Lau (1975). "Transcendental logarithmic utility functions," *American Economic Review* 65, 367–383.

Collado M. Dolores, (2001). "Separability and Aggregate Shocks in the Life-cycle Model of Consumption: Evidence from Spain," *Oxford Bulletin of Economics and Statistics* 60, 227-247

Crawford, Ian (2010). "Habits revealed," *Review of Economic Studies* 77, 1382-1402.

Deaton, Angus and John Muellbauer (1980). "An almost ideal demand system," *American Economic Review* 70, 312–326.

Demuynck Thomas and Per Hjertstrand (2019). "Samuelson's approach to revealed preference theory: Some recent advances". In R. Cord, R. Anderson, R. and W.A. Barnett (Eds.), *Paul Samuelson Master of Modern Economics. Remaking Economics: Eminent Post-war Economists*. Palgrave Macmillan: London.

Elger, Thomas, Barry E. Jones, David L. Edgerton and Jane Binner (2008). "A note on the optimal level of monetary aggregation in the UK," *Macroeconomic Dynamics* 12, 117-131.

Fleissig, Adrian R. and James L. Swofford (1996). "Dynamic asymptotically ideal demand models and finite approximation," *Journal of Business and Economic Statistics* 15, 482-492.

Fleissig, Adrian R. and Gerald A. Whitney (2003). "A New PC-Based Test for Varian's Weak Separability Conditions," *Journal of Business and Economic Statistics* 21, 133-144.

Garey, Michael R., Daniel S. Johnson (1979). *Computers and Intractability*, Bell Telephone Laboratories, Inc.

Hjertstrand, Per, James L. Swofford and Gerald Whitney (2016). "Mixed Integer Programming Revealed Preference Tests of Utility Maximization and Weak Separability of Consumption Leisure and Money," *Journal of Money Credit and Banking* 48, 1547-1561.

ILO (2004). *Consumer Price Index Manual: Theory and Practice*. International Labour Organization, Geneva. Downloadable at: www.ilo.org/wcmsp5/groups/public/---dgreports/---stat/documents/presentation/wcms_331153.pdf

Jha, Raghbendra and Ibotombi Longjam (2006). "Structure of financial savings during Indian economic reforms," *Empirical Economics* 31, 861-869.

Jones, Barry E., Donald H. Dutkowsky and Thomas Elger, (2005). "Sweep programs and Optimal Monetary Aggregation," *Journal of Banking & Finance* 29, 483-508.

Koyck, Leendert M. (1954). *Distributed lags and investment analysis*. Amsterdam: North-Holland.

Kydland, Finn E., and Edward C. Prescott (1982). "Time to Build and Aggregate Fluctuations," *Econometrica*, vol. 50, no. 6, pp. 1345–1370.

Naik, Vihangkumar V. (2018). *Mixed-Integer Quadratic Programming Algorithms for Embedded Control and Estimation*, Ph.D. thesis, IMT School for Advanced Studies Lucca, Lucca, Italy.

Papadimitriou, Christos H. (1994). *Computational Complexity*. Addison-Wesley, Longman.

Pollak, Robert A. (1970). "Habit Formation and Dynamic Demand Functions," *Journal of Political Economy* 78, 745-763

Rimmi, Anand, Divya Aggarwal and Vijay Kumar (2017). "A comparative analysis of optimization solvers," *Journal of Statistics & Management Systems* 20, 623–635

Selten, Reinhard (1991). "Properties of a Measure of Predictive Success," *Mathematical Social Sciences* 21, 153–167.

Serletis, Apostolos (1991). "The demand for Divisia money in the United States: A dynamic flexible demand system," *Journal of Money, Credit and Banking* 23, 35-52.

Swofford, James L. and Gerald A. Whitney (1988). "A Comparison of Nonparametric Tests of Weak Separability for Annual and Quarterly Data on Consumption, Leisure, and Money," *Journal of Business and Economic Statistics* 6, 241-246.

Swofford, James L. and Gerald A. Whitney (1994). "A Revealed Preference Test for Weakly Separable Utility Maximization with Incomplete Adjustment," *Journal of Econometrics* 60, 235-249.

Varian, Hal R. (1982). "The nonparametric approach to demand analysis," Econometrica 50, 945–974.

Varian, Hal R. (1983). "Nonparametric Tests of Consumer Behavior," *Review of Economic Studies*, 50, 99-110.