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On the Curvature of Homogeneous Functions

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Abstract

Consider a quasiconcave, upper semicontinuous and homogeneous of degree γ function f. This paper shows that the reciprocal of the degree of homogeneity, $1/\gamma$, can be interpreted as a measure of the degree of concavity of f. As a direct implication of this result, it is also shown that f is harmonically concave if $\gamma \leq -1$ or $\gamma \geq 0$, concave if $0 \leq \gamma \leq 1$ and logconcave if $\gamma \geq 0$. Some relevant applications to economic theory are given. For example, it is shown that a quasiconcave and homogeneous production function is concave if it displays nonincreasing returns to scale and logconcave if it displays increasing returns to scale.

Key Words: Concavity; Homogeneity; Logconcavity; $\rho-$concavity; Quasiconcavity

JEL classification: C60

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1 Results

Consider a function $f : \mathcal{X} \subset \mathbb{R}^K \mapsto \mathbb{R}_{++}$, where $\mathcal{X}$ is convex and nonempty.[1] Suppose that f is quasiconcave, upper semicontinuous at every point of $\mathcal{X}$ and positively homogeneous (note that f is positive). Crouzeix [7] shows that f indeed is a concave function.[2]

In this paper, I generalize Crouzeix's [7] result to positive functions which are homogeneous of any degree (and not just positively homogeneous).[3] In particular, the main result given in Theorem 1.1 shows that the *reciprocal* of the degree of homogeneity can be interpreted as a measure of the degree of concavity of a homogeneous function. Thus, for positive functions, Crouzeix's result appear as a special case in Theorem 1.1.

As a direct implication of Theorem 1.1, it follows that the shape restriction of a quasiconcave, upper semicontinuous and homogeneous function of any degree can be further strengthened. The general

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[1] $\mathbb{R}_{++} \equiv \{y \in \mathbb{R} \mid y > 0\}$.

[2] Strictly speaking, Crouzeix [7] proves this result in terms of convexity. Here, I phrase the results in terms of concavity because I apply the results to economics in which functions are more commonly assumed (quasi)concave in order to guarantee a maximum. Moreover, it should be noted that Crouzeix [7] also shows that f is a concave function when f is nonpositive for every x that belongs to the relative interior of $\mathcal{X}$ (i.e., the interior of $\mathcal{X}$ in the affine hull of $\mathcal{X}$).

[3] I discuss the case $f = 0$ in Remark 1.3 but leave the remaining case $f < 0$ to future research.

results in this paper establish a strong relationship between the degree of homogeneity and curvature of a function and are exhaustive in the sense that they cover any degree of homogeneity.

Before stating the main result, we consider some preliminary definitions:

Definition 1.1 A function $f : X \subset R^K \mapsto R$ is homogeneous of degree γ if it can be written as:

$$f(tx) = t^{\gamma} f(x),$$

for any number $t > 0$.

Martos [14] appears to have developed the concept of $\rho-$concavity.[4] Caplin and Nalebuff [6] introduced $\rho-$concavity to the economics literature. The definition and description of $\rho-$concavity given next closely follows Caplin and Nalebuff [6].

Definition 1.2 (Caplin and Nalebuff [6]) Consider $\rho \in [-\infty, \infty]$. For $\rho > 0$, a nonnegative function f, with convex support $\mathcal{X} \subset \mathbb{R}^K$, is called $\rho-$concave if for all $x_1, x_2 \in \mathcal{X}$ and any $\lambda \in [0, 1]$:

$$f(\lambda x_1 + (1-\lambda) x_2) \geq [\lambda f(x_1)^{\rho} + (1-\lambda) f(x_2)^{\rho}]^{\frac{1}{\rho}}. \tag{1}$$

For $\rho < 0$ the condition is exactly as above except when $f(x_1) f(x_2) = 0$, in which case there is no restriction other than $f(\lambda x_1 + (1-\lambda) x_2) \geq 0$. Finally, the definition is extended to include $\rho = \infty, 0, -\infty$ through continuity arguments.

Caplin and Nalebuff [6] discuss implications and limiting cases. For $\rho > 0$, Definition 1.2 states that f^{ρ} is concave, while for $\rho < 0$, $-f^{\rho}$ is concave. The following limiting cases hold:

- If $\rho = \infty$ then f is uniform (constant) on its support, i.e., $\lim_{\rho \to \infty} [\lambda f(x_1)^{\rho} + (1-\lambda) f(x_2)^{\rho}]^{\frac{1}{\rho}} = \max\{f(x_1), f(x_2)\}$,

- If $\rho = 1$ then we obtain the standard definition of concavity, i.e., $[\lambda f(x_1)^{\rho} + (1-\lambda) f(x_2)^{\rho}]^{\frac{1}{\rho}} = \lambda f(x_1) + (1-\lambda) f(x_2)$.

- If $\rho = 0$ then f is logconcave, i.e., $\lim_{\rho \to 0} [\lambda f(x_1)^{\rho} + (1-\lambda) f(x_2)^{\rho}]^{\frac{1}{\rho}} = f(x_1)^{\lambda} f(x_2)^{(1-\lambda)}$ (by L'Hopital's rule), in which case, $\log f(\lambda x_1 + (1-\lambda) x_2) \geq \lambda \log f(x_1) + (1-\lambda) \log f(x_2)$.

- If $\rho = -1$ then f is harmonically concave, i.e., $f(\lambda x_1 + (1-\lambda) x_2) \geq \left[\lambda \frac{1}{f(x_1)} + (1-\lambda) \frac{1}{f(x_2)}\right]^{-1} = f(x_1) f(x_2) [\lambda f(x_2) + (1-\lambda) f(x_1)]^{-1}$ if $\lambda f(x_2) + (1-\lambda) f(x_1) > 0$ and $f(\lambda x_1 + (1-\lambda) x_2) \geq 0$ if $f(x_1) f(x_2) = 0$. Any positive function f is harmonically concave if and only if $1/f$ is convex. This variant of concavity appear less known than the other limiting cases.[5]

- If $\rho = -\infty$ then f is quasiconcave, i.e., $\lim_{\rho \to -\infty} [\lambda f(x_1)^{\rho} + (1-\lambda) f(x_2)^{\rho}]^{\frac{1}{\rho}} = \min\{f(x_1), f(x_2)\}$.

As illustrated by the limiting cases, higher values of ρ correspond to more stringent variants of concavity. To see this, define $M_{\rho}(y_1, y_2) = [\lambda y_1^{\rho} + (1-\lambda) y_2^{\rho}]^{\frac{1}{\rho}}$ for any $y_1, y_2 \geq 0$, and note that M_{ρ} is what Hardy, Littlewood and Pólya [10, p.13, Eq.(2.2.5)] calls the weighted mean of order ρ. Since M_{ρ} is

[4] Balogh and Ewerhart [3] survey the origins of $\rho-$concavity and give Martos [14] credit of having introduced $\rho-$concavity. Martos [14] is written in Hungarian, where $\rho-$concavity is originally called $\omega-$concavity. Balogh and Ewerhart [3] give translations of relevant sections in Martos [14] into English. ρ (or ω) $-$concavity has also been referred to as $\alpha-$concavity in the mathematics literature (e.g., [13], [17]).

[5] But see e.g., [13].

monotonic in ρ, we have that if $\sigma \leq (<)\, \rho$ then $M_\sigma \leq (<)\, M_\rho$.[6] Now, suppose that f is $\rho-$concave, in which case $f(\lambda x_1 + (1-\lambda)\, x_2) \geq [\lambda f(x_1)^\rho + (1-\lambda)\, f(x_2)^\rho]^{\frac{1}{\rho}}$. Set $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Then, for any σ with $\rho \geq \sigma$, f satisfies $f(\lambda x_1 + (1-\lambda)\, x_2) \geq [\lambda f(x_1)^\rho + (1-\lambda)\, f(x_2)^\rho]^{\frac{1}{\rho}} = M_\rho(y_1, y_2) \geq M_\sigma(y_1, y_2) = [\lambda f(x_1)^\sigma + (1-\lambda)\, f(x_2)^\sigma]^{\frac{1}{\sigma}}$, which implies that f also is $\sigma-$concave. Thus, by monotonicity of M_ρ, a $\rho-$concave function is always $\sigma-$concave for all $\sigma \leq \rho$ (See e.g., Kennington [13, Property 2, p.689]). Hence, ρ parameterizes (and can be interpreted as a measure of) the degree of concavity of a function.

Illustrating this on the limiting cases given above, any concave function ($\rho = 1$) is also logconcave ($\rho = 0$), and any logconcave function is always harmonically concave ($\rho = -1$). Of course, any concave, logconcave or harmonically concave function is, by definition, quasiconcave ($\rho = -\infty$). Moreover, since concave, logconcave and harmonically concave functions are $1-$concave, $0-$concave and $-1-$concave, respectively, any $\rho-$concave function, f, with $\rho \geq 1$ is concave, any f with $\rho \geq 0$ is logconcave, and any f with $\rho \geq -1$ is harmonically concave.

Theorem 1.1 *Consider a nonconstant function $f : \mathcal{X} \subset \mathbb{R}^K \mapsto \mathbb{R}_{++}$, where $\mathcal{X}$ is convex and nonempty. Suppose that f is quasiconcave, upper semicontinuous at every point of $\mathcal{X}$ and homogeneous of degree γ. Then f is $\rho-$concave with $\rho = \frac{1}{\gamma}$.*

Proof of Theorem 1.1 Suppose the function $f : \mathcal{X} \mapsto \mathbb{R}_{++}$ is nonconstant, upper semicontinuous, quasiconcave and homogeneous of degree γ. Consider any two points $x_1, x_2 \in \mathcal{X}$. Without loss of generality, it suffices to consider the case $\lambda \in (0,1)$.[7] Define $y_1^\gamma = f(x_1)$ and $y_2^\gamma = f(x_2)$ such that $y_1 = f(x_1)^{\frac{1}{\gamma}}$ and $y_2 = f(x_2)^{\frac{1}{\gamma}}$. Homogeneity of degree γ implies:

$$f\left(\frac{x_1}{y_1}\right) = \frac{1}{y_1^\gamma} f(x_1) = \frac{1}{f(x_1)} f(x_1) = 1,$$

and analogously $f\left(\frac{x_2}{y_2}\right) = 1$. By quasiconcavity, we have, for any $\alpha \in [0,1]$:

$$f\left(\alpha \frac{x_1}{y_1} + (1-\alpha)\frac{x_2}{y_2}\right) \geq \min\left\{f\left(\frac{x_1}{y_1}\right), f\left(\frac{x_2}{y_2}\right)\right\} = 1.$$

Set

$$\alpha = \frac{\lambda y_1}{\lambda y_1 + (1-\lambda)\, y_2}.$$

[6] Specifically, by Jensen's inequality, $\partial M_\rho(y_1, y_2)/\partial\rho \geq 0$. See also Hardy, Littlewood and Pólya [10, Ch.2.9, p.26]. And see e.g., [5, Chapter III] where weighted power means and their properties are studied thoroughly.

[7] Indeed, the cases $\lambda = 0$ and $\lambda = 1$ trivially satisfies the definition of $\rho-$concavity regardless of whether the function is quasi-concave, homogeneous or upper semicontinuous. To see this consider $\lambda = 0$ ($\lambda = 1$ follows analogously), in which case the left-hand side of (1) in Definition 1.2 becomes $f(\lambda x_1 + (1-\lambda)\, x_2) = f(x_2)$ while the right-hand side becomes $\left[\lambda f(x_1)^{\frac{1}{\gamma}} + (1-\lambda)\, f(x_2)^{\frac{1}{\gamma}}\right]^\gamma = f(x_2)$, which of course trivially satisfies (1).

Substituting α and by homogeneity, we have:

$$\begin{aligned}
1 &\leq f\left(\alpha\frac{x_1}{y_1}+(1-\alpha)\frac{x_2}{y_2}\right)\\
&= f\left(\left(\frac{\lambda y_1}{\lambda y_1+(1-\lambda)y_2}\right)\frac{x_1}{y_1}+\left(1-\left(\frac{\lambda y_1}{\lambda y_1+(1-\lambda)y_2}\right)\right)\frac{x_2}{y_2}\right)\\
&= f\left(\left(\frac{\lambda y_1}{\lambda y_1+(1-\lambda)y_2}\right)\frac{x_1}{y_1}+\left(\frac{(1-\lambda)y_2}{\lambda y_1+(1-\lambda)y_2}\right)\frac{x_2}{y_2}\right)\\
&= f\left(\frac{\lambda x_1+(1-\lambda)x_2}{\lambda y_1+(1-\lambda)y_2}\right)\\
&= \frac{1}{[\lambda y_1+(1-\lambda)y_2]^{\gamma}}f(\lambda x_1+(1-\lambda)x_2).
\end{aligned}$$

Since $y_1>0$ and $y_2>0$ (which implies $[\lambda y_1+(1-\lambda)y_2]^{\gamma}>0$) we can multiply both sides of the inequality by $[\lambda y_1+(1-\lambda)y_2]^{\gamma}$ to obtain:

$$f(\lambda x_1+(1-\lambda)x_2)\geq[\lambda y_1+(1-\lambda)y_2]^{\gamma}.$$

Substituting $y_1=f(x_1)^{\frac{1}{\gamma}}$ and $y_2=f(x_2)^{\frac{1}{\gamma}}$, we then get:

$$f(\lambda x_1+(1-\lambda)x_2)\geq\left[\lambda f(x_1)^{\frac{1}{\gamma}}+(1-\lambda)f(x_2)^{\frac{1}{\gamma}}\right]^{\gamma}.$$

Thus, by Definition 1.2 it follows that f is $\rho-$concave with $\rho=\frac{1}{\gamma}$. □

Theorem 1.1 shows that the reciprocal of the degree of homogeneity, $1/\gamma$, can be interpreted as a measure of the degree of concavity of a homogeneous function given by the parameter ρ, i.e., $\rho=\frac{1}{\gamma}$. Thus, Theorem 1.1 establishes a direct relationship between the degree of homogeneity and curvature of a homogeneous function. The next result is a direct implication of Theorem 1.1 and generalize the results in Crouzeix [7].

Corollary 1.1 *Consider a nonconstant function $f:\mathcal{X}\subset\mathbb{R}^K\mapsto\mathbb{R}_{++}$, where $\mathcal{X}$ is convex and nonempty. Suppose that f is quasiconcave, upper semicontinuous at every point of $\mathcal{X}$ and homogeneous of degree γ. Then f is: (i) harmonically concave if $\gamma\leq-1$ or $\gamma\geq0$; (ii) concave if $0\leq\gamma\leq1$ and (iii) logconcave if $\gamma\geq0$.*

Proof of Corollary 1.1 Theorem 1.1 establishes that f is a $\rho-$concave function with $\rho=\frac{1}{\gamma}$. For (i), recall that since harmonically concave functions are $-1-$concave, $\rho\geq-1$ implies that f is harmonically concave. We have $\gamma\leq-1$ or $\gamma\geq0\Longleftrightarrow1/\gamma\geq-1$. Thus, $\rho=\frac{1}{\gamma}\geq-1$ so f is harmonically concave. For (ii), recall that since concave functions are $1-$concave, $\rho\geq1$ implies that f is concave. We have $0\leq\gamma\leq1\Longleftrightarrow1/\gamma\geq1$. Thus, $\rho=\frac{1}{\gamma}\geq1$ so f is concave. Finally, for (iii), recall that since logconcave functions are $0-$concave, $\rho\geq0$ implies that f is logconcave. We have $\gamma\geq0\Longleftrightarrow1/\gamma\geq0$. Thus, $\rho=\frac{1}{\gamma}\geq0$ so f is logconcave. □

Crouzeix [7] established concavity in the case $\gamma=1$ (which is a special case of (ii) in Corollary 1.1). That $0\leq\gamma<1$ also guarantees concavity is well-established. However, cases (i) and (iii) are new. Note that both harmonic concavity and logconcavity are significantly stronger shape restrictions than quasiconcavity and possess several nice properties (not shared by quasiconcavity).[8]

To close this section, we give some remarks pertinent to Theorem 1.1.

[8] See Section 3.5 in [4] for a detailed discussion of the properties of logconvex/logconcave functions and [2] about logconcave/logconvex probability distributions.

Remark 1.1 Theorem 1.1 can be further strengthened whenever $\gamma > 0$ to read: Suppose that f is nonconstant, quasiconcave, upper semicontinuous and homogeneous of degree $\gamma > 0$. Then f is $\rho-$concave with $\rho \leq \frac{1}{\gamma}$.

Remark 1.2 Theorem 1.1 covers nonconstant functions. The extension to constant functions is trivial, since if f is constant on its entire support, then f must be $\rho-$concave with $\rho = \infty$.

Remark 1.3 Theorem 1.1 covers positive functions. However, the extension to nonnegative functions is nontrivial. Indeed, suppose for some $x_1, x_2 \in \mathcal{X}$ that $f(x_1) = 0$ and $f(x_2) > 0$. By homogeneity of degree $\gamma > 0$,

$$\left[\lambda f(x_1)^{\frac{1}{\gamma}} + (1-\lambda) f(x_2)^{\frac{1}{\gamma}}\right]^{\gamma} = (1-\lambda)^{\gamma} f(x_2) = f((1-\lambda) x_2).$$

But suppose $x_1, x_2 > 0$ and that f is strictly decreasing, in which case $f(\lambda x_1 + (1-\lambda) x_2) < f((1-\lambda) x_2)$. As such, f cannot be $\rho-$concave with $\rho = \frac{1}{\gamma}$. Thus, to establish $\rho-$concavity of any nonnegative f such that $\rho = \frac{1}{\gamma}$ requires imposing additional properties on f besides the ones stated in Theorem 1.1.

2 Applications to Economics

In this section, I provide some relevant applications of Theorem 1.1 and Corollary 1.1 to economic theory. Balogh and Ewerhart [2] provide a more exhaustive list of applications of generalized concavity/convexity, and discuss some applications of generalized concavity/convexity in the operations research literature.

Production economics A firm's production process is frequently modelled using the concept of production functions. Specifically, a production function is a positive and nonconstant function that maps a convex set of (nonnegative) input quantities to a unique (positive) quantity of output. Most often, the production function is assumed to satisfy certain properties such as continuity and monotonicity. This usually also includes assuming that the function satisfies some kind of shape restriction often in the form of quasiconcavity. Another standard assumption is that the production function is homogeneous, which is commonly imposed to ensure that the function displays either decreasing, constant or increasing returns to scale.

Definition 2.1 A production function, $f : \mathcal{X} \subset \mathbb{R}^K \mapsto \mathbb{R}_{++}$, displays:

- decreasing returns to scale when $f(tx) < tf(x)$ for any number $t > 1$;
- constant returns to scale when $f(tx) = tf(x)$ for any number $t > 0$;
- increasing returns to scale when $f(tx) > tf(x)$ for any number $t > 1$.

Hence, returns to scale is a measure of the effect of proportionately varying the inputs to the change in output in the long run. If the change in inputs gives exactly the same effect in the output level then we say that the production technology displays constant returns to scale; if the effect in output is lower than the proportionate change in input then the technology displays decreasing returns to scale, and if the effect in output is larger then the technology displays increasing returns to scale. Although there are strong reasons to believe that mature firms are characterized by nonincreasing (i.e., decreasing or constant) returns to scale, there is compelling empirical evidence that many smaller firms in various industries (e.g., electric, gas, motor vehicles and equipment, chemicals, tobacco) are characterized by increasing returns to scale (e.g., [16]).

Proposition 2.1 *Suppose that $f : \mathcal{X} \subset \mathbb{R}^K \mapsto \mathbb{R}_{++}$ is an upper semicontinuous, quasiconcave and homogeneous of degree $\gamma > 0$ production function. Then f displays:*

1) decreasing returns to scale if and only if $\gamma < 1$ for any number $t > 1$;

2) constant returns to scale if and only if $\gamma = 1$ for any number $t > 0$;

3) increasing returns to scale if and only if $1 < \gamma$ for any number $t > 1$.

Proof of Proposition 2.1 2) is obvious. For 1), suppose that f has the property of decreasing returns to scale, i.e., $f(tx) < tf(x)$. Then for any number $t > 1$ and since f is positive and homogeneous,

$$tf(x) > f(tx) = t^{\gamma} f(x) \Leftrightarrow t > t^{\gamma} \Leftrightarrow \ln t > \gamma \ln t \Leftrightarrow 1 > \gamma.$$

Conversely, for any $t > 1$, suppose that $1 < \gamma$. This implies $t^{\gamma} < t$. Then, by homogeneity and since f is positive:

$$f(tx) = t^{\gamma} f(x) < tf(x),$$

which corresponds to decreasing returns to scale in Definition 2.1. Case 3) follows analogously. □

It is a standard textbook exercise to show that a production function displaying constant returns to scale is concave on its entire support; See e.g., Theorem 21.15 in [18].[9] The next result is a generalization to any form of returns to scale whenever f is a homogeneous production function, and follows immediately from Corollary 1.1 and Proposition 2.1.

Corollary 2.1 *Suppose that $f : \mathcal{X} \subset \mathbb{R}^K \mapsto \mathbb{R}_{++}$ is an upper semicontinuous, quasiconcave and homogeneous of degree $\gamma > 0$ production function.*

(1) If f displays nonincreasing returns to scale then f is concave.

(2) If f displays increasing returns to scale then f is logconcave.

Auction theory and mechanism design Many important results in mechanism design and auction theory rest on the assumption that the underlying type distribution is regular, which means that the virtual valuation of auction bids, defined as

$$b(x) = x - \frac{1 - F(x)}{f(x)},$$

is strictly increasing in the type x, where f and F, respectively, are the density and distribution functions of the type distribution. $b(x)$ is an additively separable function, i.e., $b(x) = b_1(x) + b_2(x)$, where $b_1(x) = x$ is linear (i.e., monotonic) and $b_2(x) = -(1 - F(x))/f(x)$ is minus the reciprocal of the so called hazard rate, $f(x)/(1 - F(x))$. Thus, b is monotonic if b_2 is monotonic (since the sum of monotonic functions is monotonic). One approach commonly used in the literature to ensure monotonicity of b_2 is to apply conditions under which the hazard rate is monotonic (as b_2 and the hazard rate will share the same monotonicity property). A sufficient condition to guarantee monotonicity of the hazard rate is that f is drawn from a logconcave probability distribution.[10] However, as pointed out by Ewerhart [9], logconcavity is an overly restrictive condition to ensure monotonicity of the hazard rate. Ewerhart

[9] This also follows as a special case of the results in Theorem 3.1 in Jehle and Reny [11] and [8,15].

[10] See e.g., Bagnoli and Bergstrom [2] for conditions on distributions which guarantee logconcavity.

[9] gives a weaker condition by showing that monotonicity holds for probability distributions that are $\rho-$concave with $\rho > -\frac{1}{2}$, which he calls "strong $\left(-\frac{1}{2}\right)-$concavity". By the properties of $\rho-$concavity, we know that any strongly $\left(-\frac{1}{2}\right)-$concave function is always logconcave ($\rho = 0$). Thus, the class of logconcave functions is nested within the class of strong $\left(-\frac{1}{2}\right)-$concave functions, and Ewerhart [9] gives several examples of distributions satisfying regularity but are not logconcave. A direct application of Theorem 1.1 shows that Ewerhart's condition for regularity applies to homogeneous functions of degree $\frac{1}{\gamma} > -\frac{1}{2}$, which holds for any $\gamma < -2$ and (trivially) for any $\gamma > 0$.

Voting The median voter theorem in political economy aims at explaining why politicians anywhere on the left-right spectrum tend to navigate towards the center in order to attract as many voters as possible. Under certain assumptions, the median voter theorem roughly says that the candidate preferred by the "median voter" always wins and that this outcome is a Nash equilibrium. However, this result does not apply to elections in which candidates differ in more than one dimension. Caplin and Nalebuff [6] provides a multidimensional extension of the median voter theorem by giving conditions under which the preferred candidate of the "mean voter" wins according to a 64%-majority rule. Their result relies on an application of the Prékopa-Borell theorem, which says that $\rho-$concavity of a probability distribution function implies that the cumulative distribution of this probability distribution is $\rho'-$concave with $\rho' = \rho/(1 + n\rho)$, where n is the dimension of the probability distribution and ρ must satisfy $\rho \geq -1/n$. Using this result, Caplin and Nalebuff [6] shows that for all logconcave distributions (given $n \to \infty$), the candidate most preferred by the mean voter is unbeatable under a 64%-majority rule. By a direct application of Theorem 1.1, Caplin and Nalebuff's condition applies to homogeneous functions of degree $1 < \gamma \leq \infty$.

Cournot competition Cournot competition is an economic theory commonly employed to describe competition in a market where firms compete over the amount of output they will produce. It is assumed that each firm on the market produces a homogenous product and maximizes firm profits by choosing how much to produce. All firms choose their output simultaneously taking as given the quantity of every other competing firm. Anderson and Renault [1] derive bounds on the ratios of deadweight loss and consumer surplus to producer surplus under Cournot competition. These bounds are derived from the curvature of market demand which is parametrizied by $\rho-$concavity and $\rho-$convexity. As market demand gets "more concave", the share of producer surplus in overall surplus increases, consumer surplus relative to producer surplus decreases and the ratio of deadweight loss to producer surplus also decreases. Theorem 1.1 can be applied to the results in Anderson and Renault [1] if market demand is assumed to be homogeneous of degree γ. In particular, several of their results rest on the assumption that demand is both $\rho-$concave and $\rho'-$convex with $\rho' \geq \rho > -1$, which implies that demand is harmonically concave ($\rho = -1$). A direct application of Theorem 1.1 then shows that this assumption holds for homogeneous functions of degree $\gamma < -1$ or $\gamma > 0$.

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